



**HAL**  
open science

# Gauduchon's form and compactness of the space of divisors

Daniel Barlet

► **To cite this version:**

| Daniel Barlet. Gauduchon's form and compactness of the space of divisors. 2017. hal-01515717v1

**HAL Id: hal-01515717**

**<https://hal.science/hal-01515717v1>**

Preprint submitted on 28 Apr 2017 (v1), last revised 15 Sep 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Gauduchon's form and compactness of the space of divisors

Daniel Barlet\*

28/04/17

*En hommage à E. Bishop*

**ABSTRACT.** We show that in a holomorphic family of compact complex connected manifolds parametrized by an irreducible complex space  $S$ , assuming that on a dense Zariski open set  $S^*$  in  $S$  the fibres satisfy the  $\partial\bar{\partial}$ -lemma, then the algebraic dimension of each fibre in this family is at least equal to the minimal algebraic dimension of the fibres in  $S^*$ . For instance, if each fibre in  $S^*$  are Moishezon, then all fibres are Moishezon.

**KEY WORDS.** Family of compact complex manifolds, Algebraic dimension, relative codimension 1 cycle-space,  $\partial\bar{\partial}$ -lemma.

## 1 Introduction

In this short article we give a rather elementary proof of the following result which answers a “classical” question.

**Theorem 1.0.1** *Let  $\pi : \mathcal{X} \rightarrow D$  be a smooth holomorphic family of compact complex connected manifolds parametrized by the unit disc  $D$  in  $\mathbb{C}$ . Assume that for each  $s \in D^*$  the fiber  $X_s$  of  $\pi$  at  $s$  satisfies the  $\partial\bar{\partial}$ -lemma. Let*

$$a := \text{Inf}\{\text{alg}(X_s), s \in D^*\}$$

*where  $\text{alg}(X)$  denotes the algebraic dimension of the compact complex connected manifold  $X$ . Then we have  $\text{alg}(X_0) \geq a$ .*

---

\*Institut Elie Cartan, Algèbre et Géométrie,  
Université de Lorraine, CNRS UMR 7502 and Institut Universitaire de France.

As a special case (when  $a = n := \dim X_s$ ) we obtain that if for any  $s \in D^*$  each  $X_s$  is a Moishezon manifold, then  $X_0$  is also a Moishezon manifold.

The previous theorem gives easily the following corollary (see [B.15] for details).

**Corollary 1.0.2** *Let  $\pi : \mathcal{X} \rightarrow S$  be a holomorphic family of compact complex connected manifolds parametrized by a reduced and irreducible complex space  $S$ . Let  $S^*$  be a dense Zariski open set in  $S$ . Assume that for each  $s \in S^*$  the fiber  $X_s$  of  $\pi$  at  $s$  satisfies the  $\partial\bar{\partial}$ -lemma. Let  $a := \text{Inf}\{\text{alg}(X_s), s \in S^*\}$ . Then for any  $s \in S$  we have  $\text{alg}(X_s) \geq a$ . ■*

Note that, as in [B.15] the proof shows that the minimum of the algebraic dimension is obtained at the general point<sup>1</sup> in  $S$ .

In the first section we show that the existence of a Gauduchon metric on a compact complex connected manifold  $X$  implies the compactness of the connected components of the space of divisors in  $X$ . This new proof of this classical result is the key of our proof for the theorem above which appears as a relative version of it.

## 2 The absolute case

Let me begin by two simple lemmas.

**Lemma 2.0.3** *Let  $M$  be a connected reduced complex space and let  $T : M \rightarrow \mathbb{R}^+$  be a continuous function on  $M$ . Assume that  $T$  is pluri-harmonic on the smooth part  $M'$  of  $M$  and that the function  $T$  achieves its minimum at a point  $x_0 \in M$ . Then  $T$  is constant on  $M$ .*

PROOF. This is an elementary exercise. ■

**Lemma 2.0.4** *Let  $X$  be a compact reduced complex space and let  $\omega$  be a continuous real  $(n, n)$ -form on  $X$  which is strictly positive in the Lelong sense. Let  $\Gamma$  be a connected component of the reduced complex space of compact  $n$ -cycles in  $X$  and define*

$$\theta : \Gamma \rightarrow \mathbb{R}^+ \quad \text{by} \quad C \mapsto \theta(C) := \int_C \omega.$$

*Then the continuous function  $\theta$  achieves its minimum on  $\Gamma$ .*

PROOF. Note first that  $\theta$  is continuous thanks to the prop. IV 3.2.1 of [B-M 1]. Let  $\alpha := \min\{\theta(C), C \in \Gamma\}$ . Then the subset  $A := \{C \in \Gamma / \theta(C) \leq \alpha + 1\}$  is a compact subset in  $\Gamma$  thanks to Bishop's theorem [Bi.64], because it is a closed subset in  $\Gamma$  such that each cycle in  $A$  has bounded volume (relative to  $\omega$ ) and support in

---

<sup>1</sup>this means on the complement of a countable union of closed nowhere dense analytic subsets in  $S$ .

the compact space  $X$  (see th. IV 2.7.23 in [B-M 1]). Then  $\theta$  achieves its minimum on  $A$ . ■

**Proposition 2.0.5** *Let  $X$  be a compact reduced complex space and let  $\omega$  be a  $\mathcal{C}^2$  real  $(n, n)$ -form on  $X$  which is strictly positive in the Lelong sense and satisfies  $\partial\bar{\partial}\omega = 0$  on  $X$ . Then any connected component of the reduced complex space  $\mathcal{C}_n(X)$  of compact  $n$ -cycles in  $X$  is compact.*

PROOF. The function  $C \rightarrow \theta(C) := \int_C \omega$  is continuous on  $\mathcal{C}_n(X)$  and  $\partial\bar{\partial}$ -closed in the sense of currents; as it achieves its minimum on any connected component  $\Gamma$  of  $\mathcal{C}_n(X)$  thanks to lemma 2.0.4 we conclude that it is constant on each  $\Gamma$ , thanks to lemma 2.0.3, and then any such  $\Gamma$  is compact (see [B-M 1] ch.IV th. 2.7.20). ■

REMARK. As any compact complex manifold  $X$  of dimension  $n$  admits a Gauduchon metric (see [G.77]), the proposition above gives a proof of the fact that the space of divisors of such a  $X$  has always compact connected components, which is a well known classical result (see [C.82] and [Fu.82]).

### 3 The relative case

Let now consider a holomorphic family  $\pi : \mathcal{X} \rightarrow D$  of compact complex connected manifolds of dimension  $n$  parametrized by the unit disc in  $\mathbb{C}$ . So  $\mathcal{X}$  is a smooth complex manifold of dimension  $n + 1$ , and fix a smooth relative Gauduchon form  $\omega$  on  $\mathcal{X}$ . Then  $\omega$  is a smooth  $\pi$ -relative differential form on  $\mathcal{X}$  on type  $(n - 1, n - 1)$  which is positive definite in the fibres and satisfies  $\partial_{/\pi}\bar{\partial}_{/\pi}\omega \equiv 0$  on  $\mathcal{X}$ .

**Lemma 3.0.6** *For any compact set  $K$  in  $D$  there exists a constant  $C > 0$  such that on  $\pi^{-1}(K)$  we have the following inequality, where  $s$  is the coordinate in  $D$*

$$\left| \frac{\partial\omega}{\partial s} \right| \leq C.\omega \quad \text{and} \quad \left| \frac{\bar{\partial}\omega}{\partial \bar{s}} \right| \leq C.\omega$$

*between  $(n - 1, n - 1)$  relative forms on  $\mathcal{X}$  over  $D$  identified with hermitian forms on the  $(n - 1)$ -th exterior power of vertical tangent space  $\Lambda^{n-1}(T_{\mathcal{X}/\pi})$  restricted to  $\pi^{-1}(K)$ .*

PROOF. This is just an obvious compactness argument as  $\omega$  is positive definite along fibers by assumption. ■

**Lemma 3.0.7** *Let  $U$  a bounded domain in  $\mathbb{C}$ . Assume that there is a point  $x_0 \in U$  such that any point in  $U$  can be joined to  $x_0$  by a  $\mathcal{C}^1$  path in  $U$  with length uniformly*

bounded by a number  $L$ . For instance, we can restrict ourselves to  $U := D^* \setminus A$  where  $D^*$  is a punctured open disc in  $\mathbb{C}$  and  $A$  a closed discrete subset in  $D^*$ . Let  $\eta : U \rightarrow \mathbb{R}^{+*}$  be a  $\mathcal{C}^1$  function on  $U$  such that

$$\left| \frac{\partial \eta}{\partial s} \right| \leq C \cdot \eta \quad \text{and} \quad \left| \frac{\bar{\partial} \eta}{\partial \bar{s}} \right| \leq C \cdot \eta \quad \text{in } U \quad (\textcircled{a})$$

where  $C > 0$  is a given constant. Then this implies that the function  $\eta$  is uniformly bounded on  $U$  by  $\eta(x_0) \cdot \exp(2CL)$ .

PROOF. Firstly consider the case of an open interval  $]a, b[$  in  $\mathbb{R}$  and let  $L := b - a$ . If  $\eta : ]a, b[ \rightarrow \mathbb{R}^{+*}$  is a  $\mathcal{C}^1$  function such that  $|\eta'(x)| \leq C \cdot \eta(x) \quad \forall x \in ]a, b[$ ; we have

$$-C \leq \frac{\eta'(x)}{\eta(x)} \leq C$$

which gives, after integration on  $[x_0, x] \subset ]a, b[$

$$-C \cdot |x - x_0| \leq \text{Log} \frac{\eta(x)}{\eta(x_0)} \leq C \cdot |x - x_0|$$

and then, for any choice of  $x_0 \in ]a, b[$  the estimate  $\eta(x) \leq \eta(x_0) \cdot \exp(CL)$  is valid. In the case of the bounded domain  $U \subset \mathbb{C}$ , choose for any  $x \in U$  a  $\mathcal{C}^1$  path of length strictly less than  $L$  and consider a parametrization of this path extended a little around  $x_0$  and  $x$  :

$$\begin{aligned} \Phi : ]a, b[ \rightarrow U \quad x := \varphi(t), y := \psi(t) \quad \text{with} \quad \varphi'(t)^2 + \psi'(t)^2 &\equiv 1 \\ \text{and} \quad \Phi(a + \varepsilon) = x_0 \quad \text{and} \quad \Phi(b - \varepsilon) = x \quad \text{with} \quad 0 < \varepsilon \ll 1. \end{aligned}$$

Then apply the one dimensional case to the function  $F(t) := \eta(\Phi(t))$  which satisfies  $|F'(t)| \leq 2C \cdot F(t)$ . This gives  $\eta(x) \leq \eta(x_0) \cdot \exp(2CL) \quad \forall x \in U$ .  $\blacksquare$

PROOF OF THE THEOREM 1.0.1. We shall divide the proof in several steps.

FIRST STEP. For each  $s \in D^*$  we know that  $X_s$  admits a strongly Gauduchon form and then there exists a strongly Gauduchon form for the family over a small disc around  $s$  in the sense of [B.15] def. 3.0.4. So we have a smooth  $d$ -closed  $2(n-1)$ -form such that its  $(n-1, n-1)$  part is positive definite on each fibre. This implies that each connected component of the relative cycle space  $\mathcal{C}_{n-1}(\pi^*)$  is proper on  $D^*$ , where  $\pi^*$  is the restriction of  $\pi$  to  $\pi^{-1}(D^*)$ : in a connected component  $\Gamma$  of the relative cycle space  $\mathcal{C}_{n-1}(\pi^*)$ , for any disc  $\Delta \subset D^*$  all cycles in  $\pi^{-1}(\Delta) \cap \Gamma$  are homologous in  $\pi^{-1}(\Delta)$  because we have a  $\mathcal{C}^\infty$ -relative isomorphism  $\mathcal{X} \simeq D \times X_0$ . So if we dispose of a  $d$ -closed  $2(n-1)$ -smooth form  $\varphi$  on  $\pi^{-1}(\Delta)$  with a positive definite  $(n-1, n-1)$  part on fibres we see that the volume for  $(n-1)$ -cycles in  $\Gamma \cap \pi^{-1}(\Delta)$  relative to the ‘‘volume’’ defined by  $\varphi$  has to be constant, proving the properness on  $\Delta$  of  $\Gamma \cap \pi^{-1}(\Delta)$ .

SECOND STEP. Applying the theorem 1.0.2 of [B.15] we obtain that the minimum of the algebraic dimension of the fibres over  $D^*$  is obtained on a dense (in fact complement of a countable subset) in  $D^*$ . This implies that it is enough, under the hypothesis of the theorem 1.0.1, to prove that the conclusion holds on a small open disc  $D_0$  with center 0 in  $D$  because this “weak” version of this theorem may be apply then to a small disc centered at any point point in  $D^*$  and this gives the theorem 1.0.1.

Thanks to the second step, we may now assume that there exists a positive constant  $C$  such that the inequalities of the the lemma 3.0.7 holds on  $\mathcal{X}$ .

STEP 3. Let  $\Gamma$  be a connected component of  $\mathcal{C}_{n-1}(\pi^*)$  with reduced generic cycle and such that the projection  $p : \Gamma \rightarrow D^*$  is surjective. As the map  $p$  is proper, let  $\tau : \pi^{-1}(D^*) \rightarrow S$  be a Stein factorization (we may assumed that  $S$  is normal) of  $p$  and let  $\tilde{\tau} : S \rightarrow D^*$  the corresponding proper finite surjective map.

Now let  $\omega$  be a smooth  $\pi$ -relative Gauduchon form on  $\mathcal{X}$  and define on  $\Gamma$  the function

$$\theta : \Gamma \rightarrow \mathbb{R}^{+*} \quad C \mapsto \int_C \omega.$$

As the function  $\theta$  is continuous, pluri-harmonic along the fibres of  $p$  as a distribution<sup>2</sup> and proper, it is constant in the fibres of  $\tau$  which are the connected components of fibres of  $p$  over  $D^*$ . So it defines a continuous function  $\tilde{\theta} : S \rightarrow \mathbb{R}^{+*}$ . This situation corresponds to the following diagram:

$$\begin{array}{ccccc}
 & & \Gamma & \xleftarrow{p_1} & G & \xrightarrow{\quad} & \Gamma \times \mathcal{X} \\
 & \searrow \theta & \downarrow \tau & & \searrow p_2 & & \downarrow p_2 \\
 & & S & & & & \mathcal{X} \\
 & & \downarrow \tilde{\tau} & & & & \downarrow \pi \\
 \mathbb{R}^{+*} & \xleftarrow{\tilde{\theta}} & S & \xrightarrow{\tilde{\tau}} & D^* & \xrightarrow{\quad} & D
 \end{array}$$

STEP 4. Let  $\tilde{A}_0 \subset S$  be the set of points  $\sigma \in S$  where the fibre  $\tau^{-1}(\sigma)$  is not contained in the union of the singular set of  $\Gamma$  with the critical set of  $p : \Gamma \rightarrow D^*$ . Then  $\tilde{A}_0$  is a closed analytic subset in  $S$  with no interior point. So  $A_0 := \tilde{\tau}(\tilde{A}_0)$  is a closed discrete subset in  $D^*$ . Let  $A_1 \subset D^*$  be the ramification set of  $\tilde{\tau}$ . It is also closed and discrete in  $D^*$  and finally put  $A := A_0 \cup A_1$ .

We shall show now that the continuous function

$$\eta : D^* \rightarrow \mathbb{R}^{+*}, s \mapsto \eta(s) := \text{Trace}_{\tilde{\tau}}(\tilde{\theta})(s)$$

satisfies the following properties :

- i) The partial derivatives  $\frac{\partial \eta}{\partial s}$  and  $\frac{\bar{\partial} \eta}{\partial \bar{s}}$  are continuous on  $D^* \setminus A$ .

---

<sup>2</sup>meaning that it is  $\partial\bar{\partial}$ -closed as a distribution along the fibres of  $p : \Gamma \rightarrow D^*$ .

ii) They satisfy the inequalities  $|\frac{\partial\eta}{\partial s}| \leq C.\eta$  and  $|\frac{\partial\bar{\eta}}{\partial\bar{s}}| \leq C.\eta$  on  $D^* \setminus A$ , where  $C$  is the constant introduced above (see lemma 3.0.6).

Note that it is enough to prove the properties i) and ii) near each point in  $D^* \setminus A$ . Choose a point  $s_0 \in D^* \setminus A$  and fix an open disc  $\Delta_0 \subset D^* \setminus A$  with center  $s_0$ . Choose now in  $p^{-1}(s_0)$  generic point  $C_0$  and, as  $C_0$  is a smooth point of  $\Gamma$  such that  $p$  has rank 1 at this point, we can find a smooth locally closed curve  $\Sigma$  through  $C_0$  in  $\Gamma$  such that the restriction of  $p$  to  $\Sigma$  induces an isomorphism of  $\Sigma$  onto  $\Delta_1$  where  $\Delta_1 \subset \Delta_0$  is an open disc with center  $s_0$ .

Define, for  $s \in \Delta_1$  the cycle  $Y_s$  as the cycle corresponding to the point  $p_{|\Sigma}^{-1}(s)$  in  $\Sigma \subset \Gamma$ . This defines an analytic family of  $\pi$ -relative  $(n-1)$ -cycles. So the function  $\eta_1$  which is given by  $s \mapsto \int_{Y_s} \omega$  is continuous on  $\Delta_1$  (see the proposition IV 2.3.1 in [B-M 1]) and coincides with  $\frac{1}{\delta}.\eta$  where  $\delta$  is the degree of the map  $\tilde{\tau}$ . We shall compute now the partial derivatives in the distribution sense of this function  $\eta_1$ .

Let  $\mathcal{Y} \subset \Delta_1 \times \mathcal{X}$  be the graph of the analytic family  $(Y_s)_{s \in \Delta_1}$  and let  $\varphi$  be in  $\mathcal{C}_c^\infty(\Delta_1)$ . Then we have

$$\begin{aligned} \langle \eta_1, \frac{\partial\varphi}{\partial s}.ds \wedge d\bar{s} \rangle &= \int_{\mathcal{Y}} \omega \wedge \pi^* \left( \frac{\partial\varphi}{\partial s}.ds \wedge d\bar{s} \right) \\ &= \int_{\mathcal{Y}} \omega \wedge \pi^*(d\varphi \wedge d\bar{s}) = \int_{\mathcal{Y}} \omega \wedge d(\pi^*(\varphi.d\bar{s})) \end{aligned}$$

As our computation is local on  $D^*$  we may assume that the support of  $\varphi$  is a small disc  $\Delta$  around a point  $s$  in  $D^*$ . Now we can assume that we have an open finite covering the compact manifold  $X_s$  by  $\pi$ -relative charts defined on open sets  $\Omega_j$  onto products  $\Delta \times U_j$  where  $U_j$  is a polydisc in  $\mathbb{C}^n$ , such that on  $\Delta \times U_j$  the  $\pi$ -relative form  $\omega|_{\Omega_j}$  is given by a form  $\omega_j$  which is smooth and of type  $(n-1, n-1)$  on  $U_j$  with smooth coefficients in  $\Delta \times U_j$ . Let  $\rho_j$  be a smooth partition of unity relative to the covering  $(\Omega_j$  of  $\pi^{-1}(\Delta)$ . Then we obtain,

$$\langle \eta_1, \frac{\partial\varphi}{\partial s}.ds \wedge d\bar{s} \rangle = - \sum_j \int_{\mathcal{Y}} \rho_j.d(\omega_j) \wedge \pi^*(\varphi.d\bar{s}) + d\rho_j \wedge \omega_j \wedge \pi^*(\varphi.d\bar{s}).$$

But at the generic points of  $\mathcal{Y} \cap (\Delta \times U_j)$  the restriction of a section of  $\Omega_{U_j}^n$  vanishes, so we can replace  $d(\omega_j)$  and  $d\rho_j$  respectively by  $\frac{\partial\omega_j}{\partial s}.ds$  and  $\frac{\partial\rho_j}{\partial s}.ds$  in the previous formula, thanks to the lemma 3.0.8 below. Then, using the vanishing of the sum

$$\sum_j \frac{\partial\rho_j}{\partial s} \equiv 0$$

and the fact that the  $\pi$ -relative forms  $\frac{\partial\omega_j}{\partial s}$  are the images in the charts of the  $\pi$ -relative form  $\frac{\partial\omega}{\partial s}$  which is well defined globally, we obtain:

$$\langle \eta_1, \frac{\partial \varphi}{\partial s} . ds \wedge d\bar{s} \rangle = \int_{X_\Sigma} \frac{\partial \omega}{\partial s} \wedge \pi^*(\varphi . ds \wedge d\bar{s}).$$

As the  $(n-1, n-1)$  relative form  $\frac{\partial \omega}{\partial s}$  is  $\mathcal{C}^\infty$  on  $\mathcal{X}$ , we conclude that the distribution  $\frac{\partial \eta_1}{\partial s}$  is equal to a continuous function. We have the analogous result for  $\frac{\bar{\partial} \eta_1}{\partial \bar{s}}$ . Now using the inequality of the lemma 3.0.6 we obtain, assuming that the test function  $\varphi$  is non negative:

$$|\langle \frac{\partial \eta_1}{\partial s}, i . \varphi . ds \wedge d\bar{s} \rangle| \leq C . \int_{X_\Sigma} \omega \wedge \pi^*(i . \varphi . ds \wedge d\bar{s})$$

which implies the inequality  $|\frac{\partial \eta_1}{\partial s}| \leq C . \eta_1$  on  $\Delta_1$ . The inequality  $|\frac{\bar{\partial} \eta_1}{\partial \bar{s}}| \leq C . \eta_1$  is obtained in a similar way.

So the lemma 3.0.7 applies and the function  $\theta$  on  $\Gamma$  is uniformly bounded. Then the closure of  $\Gamma$  in  $\mathcal{C}_{n-1}(\pi)$  is proper over  $D$ .

**Lemma 3.0.8** *Let  $D \subset \mathbb{C}$  and  $U \subset \mathbb{C}^n$  a disc and a polydisc. Let  $\pi : D \times U \rightarrow D$  the projection and consider an irreducible hypersurface  $H \subset D \times U$  such that the projection  $p : H \rightarrow D$  is open. Let  $\alpha$  be a smooth  $\pi$ -relative  $(n-1)$  holomorphic form on  $D \times U$ . Then we have at the generic point in  $H$  the equality*

$$\partial \alpha|_H = (\frac{\partial \alpha}{\partial s} . ds)|_H.$$

PROOF. Near the generic point in  $H$  we may assume that

$$H := \{(s, t) \in D \times U / t_1 = 0\}$$

because if  $f = 0$  is a local reduced equation of  $H$  near such a point we can assume that  $df \wedge ds \neq 0$  and then we may choose  $s, f$  as first and second coordinates in a neighbourhood. Then we may write

$$\alpha := \sum_{j=1}^n a_j(s, t) . dt_1 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_n$$

and this gives

$$d\partial \alpha = \frac{\partial a_1}{\partial s}(s, t) . ds \wedge dt_2 \wedge \cdots \wedge dt_n + dt_1 \wedge \beta$$

which induces on  $H$  the form

Now

$$\frac{\partial \alpha}{\partial s} \wedge ds := ds \wedge \sum_{j=1}^n \frac{\partial a_j}{\partial s} . dt_1 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_n$$



induces on  $H = \{t_1 = 0\}$  the form

$$\frac{\partial a_1}{\partial s} \cdot ds \wedge dt_2 \wedge \cdots \wedge dt_n$$

and the lemma is proved. ■

FINAL STEP. Consider now any irreducible component  $\tilde{\Gamma}$  of  $\mathcal{C}_{n-1}(\pi)$ . Then either  $\tilde{\Gamma}$  is contained in some  $\mathcal{C}_{n-1}(X_s)$  for some  $s \in D$ , and then it is an irreducible closed subset in  $\mathcal{C}_{n-1}(X_s)$  and then it is compact, or it surjects to  $D^*$  and then  $\tilde{\Gamma} \setminus p^{-1}(0)$  is a closed irreducible subset in  $\mathcal{C}_{n-1}(\pi^*)$  and then we may apply the previous result to conclude that  $\tilde{\Gamma}$  (which is the closure of  $\tilde{\Gamma} \setminus p^{-1}(0)$ ) is proper over  $D$ .

As any irreducible component of  $\mathcal{C}_{n-1}(\pi)$  is proper over  $D$ , the conclusion follows from [B.15]. ■

We conclude by noticing that there exists an analytic family of smooth complex compact surfaces of the class VII (not Kähler) parametrized by a disc such that the central fibre has algebraic dimension 0 and all other fibres have algebraic dimension equal to 1, see [F-P.09].

This shows that in our theorem 1.0.1 some “Kähler type” assumption on the general fibre  $X_s$  cannot be avoided in order that the general algebraic dimension gives a lower bound for the algebraic dimensions of all fibres.

Note that our assumption that the generic fibres satisfy the  $\partial\bar{\partial}$ -lemma (in fact for the type  $(n, n-1)$ ) is a rather weak such assumption.

## 4 Bibliography

1. [B.15] Barlet, D. *Two semi-continuity results for the algebraic dimension of compact complex manifolds*, J. Math. Sci. Univ. Tokyo 22 (2015), no. 1, pp.39-54.
2. [B-M 1] Barlet, D. et Magnússon, J. *Cycles analytiques complexes. I. Théorèmes de préparation des cycles*, Cours Spécialisés 22., Société Mathématique de France, Paris, 2014. 525 pp.
3. [Bi.64] Bishop, E. *Conditions for the analyticity of certain sets*, Mich. Math. Journal (1964) p.289-304.
4. [C.82] Campana, F., *Sur les diviseurs non-polaires d'un espace analytique compact*, J. Reine Angew. Math. 332 (1982), 126-133.
5. [Fu.82] Fujiki, A., *Projectivity of the space of divisors of a normal compact complex space*, Publ. Res. Inst. Math. Sci. 18 (1982), no. 3, 1163-1173.
6. [F-P.09] Fujiki, A. and Pontecorvo, M. *Non-upper continuity of algebraic dimension for families of compact complex manifolds*  
arXiv: 0903.4232v2 [math. AG] .
7. [G.77] Gauduchon, P., *Le théorème de lexcentricité nulle*, C. R. Acad. Sci. Paris, Ser. A 285 (1977), 387-390.