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Algorithmic trading in a microstructural limit order book model

Frédéric ABERGEL * Côme HURÉ † Huyên PHAM ‡

April 18, 2019

Abstract

We propose a microstructural modeling framework for studying optimal market making policies in a FIFO (first in first out) limit order book (order book). In this context, the limit orders, market orders, and cancel orders arrivals in the order book are modeled as Cox point processes with intensities that only depend on the state of the order book. These are high-dimensional models which are realistic from a micro-structure point of view and have been recently developed in the literature. In this context, we consider a market maker who stands ready to buy and sell stock on a regular and continuous basis at a publicly quoted price, and identifies the strategies that maximize her P&L penalized by her inventory.

We apply the theory of Markov Decision Processes and dynamic programming method to characterize analytically the solutions to our optimal market making problem. The second part of the paper deals with the numerical aspect of the high-dimensional trading problem. We use a control randomization method combined with quantization method to compute the optimal strategies. Several computational tests are performed on simulated data to illustrate the efficiency of the computed optimal strategy. In particular, we simulated an order book with constant/ symmetric/ asymmetrical/ state dependent intensities, and compared the computed optimal strategy with naive strategies.

Keywords: Limit order book, pure-jump controlled process, high-frequency trading, high-dimensional stochastic control, Markov Decision Process, quantization, local regression

1 Introduction

Most of the markets use a limit order book (order book) mechanism to facilitate trade. Any market participant can interact with the order book by posting either market orders or limit orders. In such type of markets, the market makers play a fundamental role by providing liquidity to other market participants, typically to impatient agents who are willing to cross the bid-ask spread. The profit made by a market making strategy comes from the alternation of buy and sell orders.

From the mathematical modeling point of view, the market making problem corresponds to the choice of an optimal strategy for the placement of orders in the order book. Such a strategy should maximize the expected utility function of the wealth of the market maker up to a penalization of her inventory. In

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the recent literature, several works focused on the problem of market making through stochastic control methods. The seminal paper by Avellaneda and Stoikov [AS07] inspired by the work of Ho and Stoll [HS79] proposes a framework for trading in an order driven market. They modeled a reference price for the stock as a Wiener process, and the arrival of a buy or sell liquidity-consuming order at a distance δ from the reference price is described by a point process with an intensity in an exponential form decreasing with δ . They characterized the optimal market making strategies that maximize an exponential utility function of terminal wealth. Since this paper, other authors have worked on related market making problems. Gueant, Lehalle, and Fernandez-Tapia [GLFT12] generalized the market making problem of [AS07] by dealing with the inventory risk. Cartea and Jaimungal [CJ13] also designed algorithms that manage inventory risk. Fodra and Pham [FP15b] and [FP15a] considered a model designed to be a good compromise between accuracy and tractability, where the stock price is driven by a Markov Renewal Process, and solved the market making problem. Guilbaud and Pham [GP13] also considered a model for the mid-price, modeled the spread as a discrete Markov chain that jumps according to a stochastic clock, and studied the performance of the market making strategy both theoretically and numerically. Cartea and Jaimungal [CJ10] employed a hidden Markov model to examine the intra-day changes of dynamics of the order book. Very recently, Cartea, Penalva, and Jaimungal [CPJ15] and Gueant [Gu16] published monographs in which they developed models for algorithmic trading in different contexts. Abergel and El Aoud [EAA15] extended the framework of Avellaneda and Stoikov to the options market making. A common feature of all these works is that a model for the price or/and the spread is considered, and the order book is then built from these quantities. This approach leads to models that predict well the long-term behavior of the order book. The reason for this choice is that it is generally easier to solve the market making problem when the controlled process is low-dimensional. Yet, some recent works have introduced accurate and sophisticated micro-structural order book models. These models reproduce accurately the short-term behavior of the market data. The focus is on conditional probabilities of events, given the state of the order book and the positions of the market maker. Abergel, Anane, Chakraborti, Jedidi, Muni Toke [Abe+16] proposed models of order book where the arrivals of orders in the order book are driven by Poisson processes or Hawkes processes. Stoikov, Talreja, and Cont [CST07] also modeled the orders arrivals with Poisson processes. Lehalle, Rosenbaum and Huang [HLR15] proposed a queue-reactive model for the order book. In this model the arrivals of orders are driven by Cox point processes with intensities that only depend on the state of the order book (they are not time dependent). Other tractable dynamic models of order-driven market are available (see e.g. Stoikov, Talreja, and Cont [CST07], Rosu [Ros08], Cartea, Jaimungal, Ricci [CJR14]).

In this paper we adopt the micro-structural model of order book in [Abe+16], and solve the associated trading problem. The problem is formulated in the general framework of Piecewise Deterministic Markov Decision Process (PDMDP), see Bauerle and Rieder [BR11]. Given the model of order book, the PDMDP formulation is natural. Indeed, between two jumps, the order book remains constant, so one can see the modeled order book as a point process where the time becomes a component of the state space. As for the control, the market maker fixes her strategy as a deterministic function of the time right after each jump time. We prove that the value function of the market making problem is equal to the value function of an associated non-finite horizon Markov decision process (MDP). This provides a characterization of the value function in terms of a fixed point dynamic programming equation. Jacquier and Liu in [JL18] recently followed a similar idea to solve an optimal liquidation problem, while Baradel et al. [BBEM18] and Lehalle et al. [LOR18] also tackled this problem of reward functional maximization in a micro-structure model of order book framework.

The second part of the paper deals with the numerical simulation of the value functions. The computation is challenging because the micro-structural model used to model the order book leads to a high-dimensional pure jump controlled process, so evaluating the value function is computationally intensive. We rely on control randomization and Markovian quantization methods to compute the value functions. Markovian quantization has been proved to be very efficient for solving control problems associated with high-dimensional Markov processes. We first quantize the jump times and then quantize the state space of the order book. See Pages, Pham, Printemps [PPP04] for a general description of quantization applied to controlled processes. The projections are time-consuming in the algorithm, but Fast approximate nearest neighbors algorithms (see e.g. [ML09]) can be implemented to alleviate the procedure. We borrow the values of intensities of the arrivals of orders for the order book simulations from Huang et al. [HLR15] in order to test our optimal trading strategies.

The paper is organized as follows. The model setup is introduced in Section 2: we present the micro-structural model for the order book, and show how the market maker interacts with the market. In Section 3, we prove the existence and provide a characterization of the value function and optimal trading strategies. In Section 4, we introduce a quantization-based algorithm to numerically solve a general class of discrete-time control problem with finite horizon, and then apply it on our trading problem. We then present some results of numerical tests on simulated order book. Section 5 presents an extension of our model when order arrivals are driven by Hawkes processes, and finally the appendix collects some results used in the paper.

2 Model setup

2.1 Order book representation

We consider a model of the order book inspired by the one introduced in chapter 6 of [Abe+16].

Fix $K \geq 0$. An order book is supposed to be fully described by K limits on the bid side and K limits on the ask side. Denote by pa_t the *best ask* at time t , which is the cheapest price a participant in the market is willing to sell a stock at time t , and by pb_t the *best bid* at time t , which is the highest price a participant in the market is willing to buy a stock at time t . We use the pair of vectors $(\underline{a}_t, \underline{b}_t) = (a_t^1, \dots, a_t^K, b_t^1, \dots, b_t^K)$

- a_t^i is the number of shares available i ticks away from pb_t ,
- $-b_t^i$ is the number of shares available i ticks away from pa_t ,

to describe the order book. The vector \underline{a}_t and \underline{b}_t describe the ask and the bid sides at time t . The quantities a_t^i , $1 \leq i \leq K$, live in the discrete space $q\mathbb{N}$ where $q \in \mathbb{R}^*$ is the minimum order size on each specific market (*lot size*). The quantities b_t^i , $1 \leq i \leq K$, live in the discrete space $-q\mathbb{N}$. By convention, the a^i are non-negative, and the b^i are non-positive for $0 \leq i \leq K$. The tick size ϵ represents the smallest interval between price levels.

In the sequel we assume that the orders arrivals have the same size $q = 1$, and set the tick size to $\epsilon = 1$ for simplicity.

Constant boundary conditions are imposed outside the moving frame of size $2K$ in order to guarantee that both sides of the LOB are never empty: we assume that all the limits up to the K -th ones are equal

to a_∞ in the ask side, and equal to b_∞ in the bid side.

We shall assume some conditions on the structure of the orders arrivals in the order book.

(Harrivals) The orders arrivals from general market participants (market orders, limit orders and cancel orders) occur according to Markov jump processes which intensities only depends on the state of the order book. Moreover, we assume that the all the intensities are at most linear w.r.t. the couple $(\underline{a}, \underline{b})$ and are constant between two events.

Under **(Harrivals)**, let us define

- λ^{M^+} the intensity of the buy-to-market orders flow M_t^+ ,
- λ^{M^-} the intensity of the sell-to-market orders flow M_t^- ,
- $\lambda_i^{L^+}$, $i \in \{1, \dots, K\}$, the intensity of the sell orders flow L_i^+ at the i^{th} limit of the ask side,
- $\lambda_i^{L^-}$, $i \in \{1, \dots, K\}$, the intensity of the buy orders flow L_i^- at the i^{th} limit of the bid side,
- $\lambda_i^{C^+}$, $i \in \{1, \dots, K\}$, the intensity of the cancel orders flow C_i^+ at the i^{th} limit of the ask side,
- $\lambda_i^{C^-}$, $i \in \{1, \dots, K\}$, the intensity of the cancel orders flow C_i^- at the i^{th} limit of the bid side,

and let λ^L , λ^C , λ^M be such that

$$\begin{aligned} \sum_{i=0}^K \lambda(L_i^\pm)(z) &\leq \lambda^L(|\underline{a}| + |\underline{b}|), \\ \sum_{i=0}^K \lambda(C_i^\pm)(z) &\leq \lambda^C(|\underline{a}| + |\underline{b}|), \\ \lambda(M^-)(z) + \lambda(M^+)(z) &\leq \lambda^M(|\underline{a}| + |\underline{b}|), \end{aligned}$$

for all state $(\underline{a}, \underline{b})$ of the LOB. We remind that λ^L , λ^C , λ^M are well-defined under assumption **(Harrivals)**.

Remark 2.1. *The linear conditions on the intensities are required to prove that the control problem is well-posed.*

Remark 2.2. *We generalize the structure of the orders arrivals in section 5 by modeling them as Hawkes processes with exponential kernel.*

We provide in figure 1 a graphical representation of an LOB that may help to get more familiar with the introduced notations.

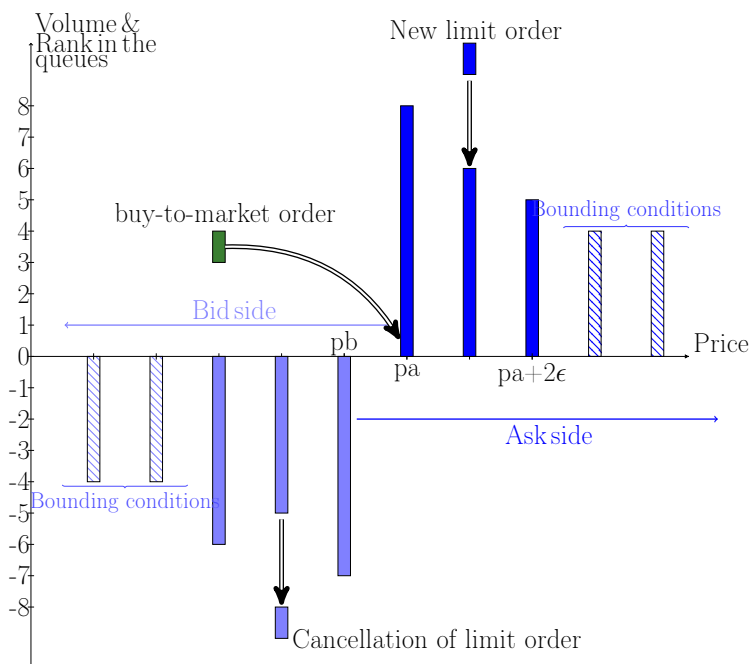


Figure 1 – Order book dynamics: in this example, $K = 3$, $q = 1$, $a_\infty = 4$, $b_\infty = -4$, $\underline{a} = (8, 6, 5)$, $\underline{b} = (-7, -5, -6)$. The spread is equal to 1. At any time, the order book can receive limit orders, market orders or cancel orders.

2.2 Market maker strategies

We assume that the market is governed by a *FIFO* (*First In First Out*) rule, which means that each limit of the order book is a queue where the first order in the queue is the first one to be executed. We consider a market maker who stands ready to send buy and sell limit orders on a regular and continuous basis at quoted prices. A usual assumption in stochastic control in order to characterize value function as solution of HJB equation is to constrain the control space to be compact. In this spirit, we shall make the following assumption on the market maker's decisions.

(Hcontrol) Assume that at any time, the total number of limit orders placed by the marker maker does not exceed a fixed (possibly large) integer \bar{M} .

2.2.1 Controls and strategies of the market maker

The market maker can choose at any time to keep, cancel or take positions in the order book (as long as she does not hold more than \bar{M} positions in the order book). Her positions are fully described by the following \bar{M} -dimensional vectors $\underline{ra}_t, \underline{rb}_t, \underline{na}_t, \underline{nb}_t$ where \underline{ra} (resp. \underline{rb}) records the limits in which the market maker's sell (resp. buy) orders are located; and \underline{na} (resp. \underline{nb}) records the ranks in the queues of each market maker's sell (resp. buy) orders. In order to guarantee that the strategy of the market maker is predictable w.r.t. the natural filtration generated by the orders arrivals processes, we shall make the following assumption.

(Harrivals2) The intensities do not depend on the control. Moreover, the market maker does not cross the spread.

We discuss in Appendix how to control the intensity, by transferring the control on the probability measure, see Section A

To simplify the theoretical analysis, we also make the following assumption: **(Harrivals3)** Assume that the market maker does not change her strategy between two orders arrivals of the order book. In other words, the market maker makes a decision right after one of the order arrivals processes L^\pm, C^\pm, M^\pm jumps, and keep it until the next the jump of an order arrival.

Note that assumption **(Harrivals3)** is mild if the order book jumps frequently, since the market maker can change her decisions frequently in such a case.

We provide in figure 2 a graphical representation of the controlled LOB. Notice that the market maker interacts with the order book by placing orders at some limits. The latter have ranks that evolve after each orders arrivals.

Denote by $(T_n)_{n \in \mathbb{N}}$ the sequence of jump times of the order book. We denote by \mathbb{A} the set of the admissible strategies, defined as the predictable processes $(\underline{ra}_t, \underline{rb}_t)_{t \leq T}$ such that:

- for all $n \in \mathbb{N}$, $(\underline{ra}_t, \underline{rb}_t) \in \{0, \dots, K\}^{\bar{M}} \times \{0, \dots, K\}^{\bar{M}}$ are constant on $(T_n, T_{n+1}]$
- $ra_*, rb_* \geq a0$

where, for every vector \underline{a} : $a_* = \min_{0 \leq i \leq K} \{a_i \text{ s.t. } a_i \neq -1\}$; and: $a0 = \operatorname{argmin}_{0 \leq i \leq K} (a_i \text{ s.t. } a_i > 0)$. The control is the double vector of the positions of the \bar{M} market maker's orders in the order book. By convention, we set: $ra_i(t) = -1$ if the i th market maker's order is not placed in the order book.

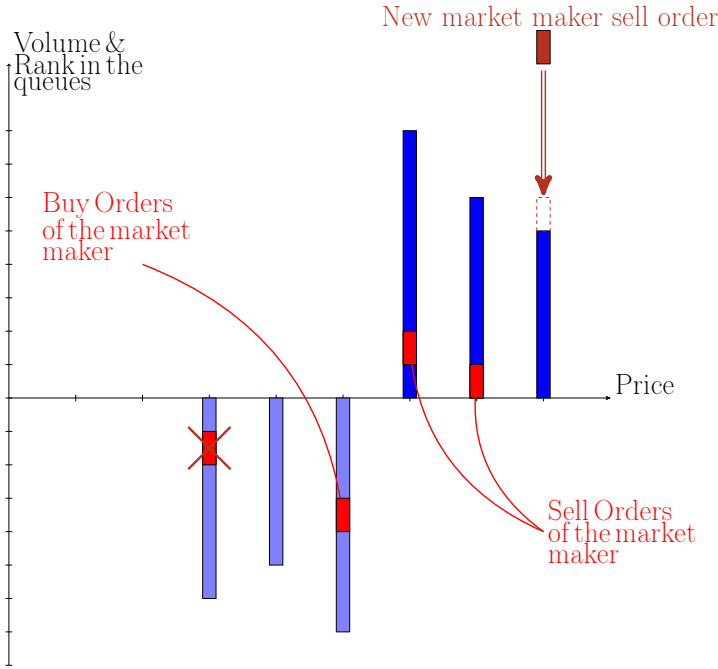


Figure 2 – Example of market maker’s placements and decisions she might make. In this example: her positions are $\underline{ra} = (0, 1, -1, \dots)$, $\underline{rb} = (0, 2, -1, \dots)$. The ranks vectors associated are $\underline{na} = (2, 1, -1, \dots)$ and $\underline{nb} = (4, 2, -1, \dots)$. After each order arrival, she can send new limit orders, cancel some positions, or just keep the latter unchanged.

2.2.2 Controlled order book

We describe the controlled order book by the following state process Z :

$$Z_t := (X_t, Y_t, \underline{a}_t, \underline{b}_t, \underline{na}_t, \underline{nb}_t, pa_t, pb_t, \underline{ra}_t, \underline{rb}_t),$$

where, at time t :

- X_t is the cash held by the market maker on a zero interest account.
- Y_t is the inventory of the market maker, i.e. it is the (signed) number of shares held by the market maker.
- pa_t is the ask price, i.e. the cheapest price a general market participant is willing to sell stock.
- pb_t is the bid price, i.e. the highest price a general market participant is willing to buy stock.
- $\underline{a}_t = (a_1(t), \dots, a_K(t))$ (resp. $\underline{b}_t = (b_1(t), \dots, b_K(t))$) describes the ask (resp. bid) side: $i \in \{1, \dots, K\}$, $a_i(t)$ is the sum of all the general market participants’ sell orders which are i ticks away from the bid (resp. ask) price.
- \underline{ra}_t (resp. \underline{ra}_t) describes the market maker’s orders in the ask (resp. bid) side: for $i \in \{1, \dots, \bar{M}\}$, $\underline{ra}_t(i)$ is the number of ticks between the i -th market maker’s sell (resp. bid) order and the bid (resp. ask) price.

ask) price. By convention, we set $\underline{ra}_t(i) = -1$ (resp. $\underline{rb}_t(i) = -1$) if the i -th sell (resp. buy) order of the market maker is not placed in the order book. As a result $\underline{ra}_t(i), \underline{rb}_t(i) \in \{0, \dots, K\} \cup \{-1\}$.

- \underline{na}_t (resp. \underline{nb}_t) describes the ranks of the market maker's orders in the ask (resp. bid) side. For $i \in \{1, \dots, \bar{M}\}$, $\underline{na}_t(i) \in \{-1, \dots, |a| + \bar{M}\}$ (resp. $\underline{nb}_t(i) \in \{-1, \dots, |b| + \bar{M}\}$) is the rank of the i -th sell (resp. buy) orders of the market maker in the queue. By convention, we assume that $\underline{na}_t(i) = -1$ (resp. $\underline{nb}_t(i) = -1$) if the i -th sell (resp. buy) order of the market maker is not placed in the order book.

The dynamics of (Z_t) has been computed in the case where the set of admissible strategies is restricted to those where the market maker only makes orders at the two best limits in the bid and ask sides. We present the computations in Section B in the Appendix, for the case where the market maker can only send limit orders at the best-bid and best-ask. We only present the numerical results, in Section 4.4, in the case where the market maker can send limit orders at the two best limits in the ask and bid sides.

3 Existence and characterization of the optimal strategy

3.1 Definition and well-posedness of the value function

We denote by V the value function for the market-making problem, defined as follows:

$$V(t, z) = \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,z}^\alpha \left[\int_t^T f(\alpha_s, Z_s) ds + g(Z_T) \right], \quad (t, z) \in [0, T] \times E, \quad (3.1)$$

where:

- \mathbb{A} is the set of the admissible strategies, defined in Section 2.2.1.
- f and g are respectively the running and terminal reward functions. A usual definition for g is the market maker's wealth function, possibly with an inventory penalization, i.e. $g : z \mapsto x + L(y) - \eta y^2$ where L^1 returns the amount earned from the immediate liquidation of the inventory; where η is the penalization parameter of the latter; and where we remind that y stands for the (signed) market maker's inventory.
- $\mathbb{E}_{t,z}^\alpha$ stands for the expectation conditioned by $Z_t = z$ and when strategy $\alpha = (\alpha_s)_{t \leq s < T}$ is followed on $[t, T]$.

¹ L is defined as follows:

$$L(z) = \begin{cases} \sum_{k=0}^{j-1} [a_k(pa + k\epsilon)] + (y - a_0 - \dots - a_{j-1})(pa + j\epsilon) & \text{if } y < 0 \\ -\sum_{k=0}^{j-1} [b_k(pb - k\epsilon)] + (y + b_0 + \dots + b_{j-1})(pb - j\epsilon) & \text{if } y > 0 \\ 0 & \text{if } y = 0, \end{cases}$$

for all state $z = (x, y, \underline{a}, \underline{b}, \underline{na}, \underline{nb}, pa, pb, \underline{ra}, \underline{rb})$ of order book, where:

$$j = \begin{cases} \min \{j \mid \sum_{i=0}^j a_i > -y\} & \text{if } y < 0 \\ \min \{j \mid \sum_{i=0}^j |b_i| > y\} & \text{if } y > 0. \end{cases}$$

We shall assume conditions on the rewards to insure the well-posedness of the market-making problem.

(Hrewards) The expected running reward is uniformly upper-bounded w.r.t. the strategies in \mathbb{A} , i.e.

$$\sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,z}^{\alpha} \left[\int_t^T f^+(Z_s, \alpha_s) ds \right] < +\infty$$

holds. The terminal reward $g(Z_T)$ is a.s. no more than linear with respect to the number of events up to time T , denoted by N_T in the sequel, i.e. there exists a constant $c_1 > 0$ such as $g(Z_T) \leq c_1 N_T$, a.s..

Remark 3.1. Under Assumption **(Hcontrols)**, Assumption **(Hrewards)** holds when g is defined as the wealth of the market maker plus an inventory penalization. In particular, we have $g(Z_T) \leq N_T \bar{M}$, where \bar{M} is the maximal number of orders that can be sent by the market maker, which holds a.s. since the best profit the market maker can make is when her buy (resp. sell) limit orders are all executed, and then the price keeps going to the right (resp. left) direction. Hence the second condition of **(Hrewards)** holds with $c_1 = \bar{M}$.

The following Lemma 3.1 tackles the well-posedness of the control problem.

Lemma 3.1. Under **(Hrewards)** and **(Hcontrols)**, the value function is well-defined, i.e.

$$\sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,z}^{\alpha} \left[g(Z_T) + \int_t^T r(\alpha_s, Z_s) ds \right] < +\infty,$$

where, as defined previously, $\mathbb{E}_{t,z}^{\alpha}[\cdot]$ stands for the expectation conditioned by the event $\{Z_t = z\}$, assuming that strategy $\alpha \in \mathbb{A}$ is followed in $[t, T]$.

Proof. Denote by $(N_t)_t$ the sum of all the arrivals of orders up to time t . Under **(Hrewards)**, we can bound $\mathbb{E}_{t,z}^{\alpha} \left[\int_t^T f(\alpha_s, Z_s) ds + g(Z_T) \right]$, the reward functional at time t associated to a strategy $\alpha \in \mathbb{A}$, as follows:

$$\begin{aligned} \mathbb{E}_{t,z}^{\alpha} \left[\int_t^T f(\alpha_s, Z_s) ds + g(Z_T) \right] &\leq \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,z}^{\alpha} [g(Z_T)] + \sup_{\alpha \in \mathbb{A}} \mathbb{E}^{\alpha} \left[\int_t^T f^+(Z_s, \alpha_s) ds \right] \\ &\leq c_1 \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,0}^{\alpha} [N_T] + \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,z}^{\alpha} \left[\int_t^T f^+(Z_s, \alpha_s) ds \right], \end{aligned} \quad (3.2)$$

where once again, for all general process M and all $m \in E$, $\mathbb{E}_{t,m}^{\alpha}[M_T]$ stands for the expectation of M_T conditioned by $M_t = m$ and assuming that the market maker follows strategy $\alpha \in \mathbb{A}$ in $[t, T]$. Let us show that the first term in the r.h.s. of (3.2) is bounded. On one hand, we have:

$$\mathbb{E}_{t,0}^{\alpha} [N_T] \leq \|\lambda\|_{\infty} \int_0^T \mathbb{E} (|a|_t + |b|_t) dt, \quad (3.3)$$

where $\|\lambda\|_{\infty} := \lambda^L + \lambda^C + \lambda^M$ is a bound on the intensity rate of N_t . On the other hand, there exists a constant $c_2 > 0$ such that $d(|a| + |b|)_t \leq c_2 dL_t$ so that: $\mathbb{E}_{t,|a|_0+|b|_0}^{\alpha} [|a|_t + |b|_t] \leq |a|_0 + |b|_0 + c_3 \int_0^t \mathbb{E} [|a|_s + |b|_s] ds$. Applying Gronwall's inequality, we then get:

$$\mathbb{E}_{t,|a|_0+|b|_0}^{\alpha} [|a|_t + |b|_t] \leq (|a|_0 + |b|_0) e^{c_3 t}. \quad (3.4)$$

Plugging (3.4) into (3.3) finally leads to:

$$\mathbb{E}_{t,0}^\alpha [N_T] \leq c_4 e^{c_3 T}$$

wit c_3 and $c_4 > 0$ that do not depends on α , which proves that the first term in the r.h.s. of (3.2) is bounded. Also, its second term in the r.h.s. of (3.2) is bounded under **(Hrewards)**. Hence, the reward functional is bounded uniformly in α , which proves that the value function of the considered market-making problem is well-defined. \square

3.2 Markov Decision Process formulation of the market-making problem

In this section, we aim first at reformulating the market-making problem as a Markov Decision Process (MDP), and secondly deriving a characterization of the value function as solution of a Bellman equation.

We consider the Markov Decision Process (MDP) characterized by the following information

$$\underbrace{[0, T] \times E}_{\text{state space}}, \quad \underbrace{A_z}_{\text{market maker control}}, \quad \underbrace{\lambda}_{\text{intensity of the jump}}, \quad \underbrace{Q}_{\text{transitions kernel}}, \quad \underbrace{r}_{\text{reward}}$$

such that:

- $E := \mathbb{R} \times \mathbb{N} \times \mathbb{N}^K \times \mathbb{N}^K \times \mathbb{N}^{\bar{M}} \times \mathbb{N}^{\bar{M}} \times \mathbb{N}^{\bar{M}} \times \mathbb{N}^{\bar{M}} \times \mathbb{R} \times \mathbb{R}$ is the state space of (Z_t) . For $z \in E$, $z = (x, y, \underline{a}, \underline{b}, \underline{na}, \underline{nb}, \underline{ra}, \underline{rb}, pa, pb)$ where: x is the cash held by the market maker, y her inventory; \underline{na} (resp. \underline{nb}) is the \bar{M} -dimensional vector of the ranks of the market maker's sell (resp. buy) orders in the queues ; \underline{ra} (resp. \underline{rb}) is the \bar{M} -dimensional vector of the number of ticks the \bar{M} market maker's sell (resp. buy) orders are from the bid (resp. ask) price; pa (resp. pb) is the ask-price (resp. bid-price).
- for every state $z \in E$, denote by A_z the space of the admissible controls which is the set of all the actions the market maker can take when the order book is at state z .

$$A_z = \left\{ \underline{ra}, \underline{rb} \in \{0, \dots, K\}^{\bar{M}} \times \{0, \dots, K\}^{\bar{M}} \mid rb_*, ra_* \geq a0 \right\},$$

where we define $c_* = \min_i \{c_i \mid c_i \neq -1\}$ and $c0 = \operatorname{argmin}_{0 \leq i \leq K} \{c_i > 0\}$ for $\underline{c} \in \mathbb{N}^{\bar{M}}$. The control is the vectors of positions of the market maker's orders. The condition for the control to be admissible comes from the assumption that the market maker is not allowed to cross the spread.

- Given a market-making strategy α , the stochastic evolution is given by a marked point process (T_n, Z_n) where (T_n) is the increasing sequence of jump times of the controlled order book with intensity $\lambda(Z_{n-1})$. Just after the jump at time T_n , the process can jump again, due to the decision of the market maker. Then it remains constant on $]T_n, T_{n+1}[$ since the market maker does not change her strategy between two jumps.

We denote by $\phi^a(z) \in E$ the state of the order book at time t such that $T_n < t < T_{n+1}$, given that $Z_{T_n} = z$ and given that the strategy a has been chosen by the market maker at time T_n .

- In the sequel, we denote $([0, T] \times E)^C := \left\{ (t, z, a) \in E \times \{0, \dots, K\}^{2\bar{M}} \mid t \in [0, T], z \in E, a \in A_z \right\}$, and $E^C := \left\{ (z, a) \in E \times \{0, \dots, K\}^{2\bar{M}} \mid z \in E, a \in A_z \right\}$. Q' is the stochastic kernel from E^C to E that describes the distribution of the jump goals, i.e., $Q'(B|z, u)$ is the probability that the order book jumps in the set B given that it was at state $z \in E$ right before the jump, and the control action $u \in A_z$ has been chosen right after the jump time.

An admissible policy $\alpha = (\alpha_t)$ is entirely characterized by decision functions $f_n : [0, T] \times E \rightarrow A$ such that²

$$\alpha_t = f_n(T_n, Z_n) \text{ for } t \in (T_n, T_{n+1}]$$

By abuse of notation, we denote in the sequel by α the sequence of controls $(f_n)_{n=0}^\infty$. The intensity of the controlled process (Z_t) is:

$$\lambda(z) := \lambda^{M^+}(z) + \lambda^{M^-}(z) + \sum_{1 \leq j \leq K} \lambda^{L_j^+}(z) + \sum_{1 \leq j \leq K} \lambda^{L_j^-}(z) + \sum_{1 \leq j \leq K} \lambda^{C_j^+}(z) + \sum_{1 \leq j \leq K} \lambda^{C_j^-}(z)$$

It does not depend on the strategy α chosen by the market maker since we assumed that the general participants does not "see" the market maker's orders in the order book. The intensity of the order book process only depends on the vectors \underline{a} and \underline{b} .

The transition kernel of the controlled order book, given a state z , is given by:

$$Q'(z'|z, u) = \begin{cases} \frac{\lambda^{M^+}(z)}{\lambda(z)} & \text{if } z' = e^{M^+}(\phi^u(z)) \\ \vdots & \\ \frac{\lambda^{C_K^+}(z)}{\lambda(z)} & \text{if } z' = e^{C_K^+}(\phi^u(z)), \end{cases}$$

where $\phi^u(z)$ is the new state of the controlled order book when decision u as been taken and when the order book was at state z before the decision; $e^{M^+}(z)$ is the new state of the order book right after it received a buy market order, given that it was at state z before the jump; and $e^{C_i^\pm}(z)$ is the new state of the order book right after it received a cancel order from a general market participant on its i^{th} ask/bid limit, given that it was at state z .

Let us fix an admissible policy $\alpha = (f_n)_{n=0}^\infty \in \mathbb{A}$ and take $t \in [0, T]$. Then, for all Borelian B in E , it holds:

$$\begin{aligned} \mathbb{P}(T_{n+1} - T_n \leq t, Z_{n+1} \in B | T_0, Z_0, \dots, T_n, Z_n) &= \lambda(Z_n) \int_0^t e^{-\lambda(Z_n)s} Q'(B | Z_{T_n}, \alpha_{T_n}) ds \\ &= \lambda(Z_n) \int_0^t e^{-\lambda(Z_n)s} Q'(B | Z_{T_n}, f_n(Z_n)) ds, \end{aligned}$$

so that the stochastic kernel Q of the MDP is defined as follows:

$$Q(B \times C | t, z, \alpha) := \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \mathbf{1}_B(t+s) Q'(C | \phi^\alpha(z), \alpha) ds + e^{-\lambda(z)(T-t)} \mathbf{1}_{T \in B, z \in C},$$

²Note that we restrict ourselves to the feedback controls here

for all Borelian sets $B \subset \mathbb{R}_+$ and $C \subset E$, for all $(t, z) \in [0, T] \times E$, and for all $\alpha \in A$.

We denote by $(T_n, Z_n)_{n \in \mathbb{N}}$ the corresponding state of the controlled Markov chain. It remains to define the value function of this reformulated control problem.

Let r be the running reward function $r : [0, T] \times E^C \rightarrow \mathbb{R}$ defined as:

$$r(t, z, a) := -c(z, a)e^{-\lambda(z)(T-t)}(T-t)\mathbf{1}_{t>T} + c(z, a)\left(\frac{1}{\lambda(z)} - \frac{e^{-\lambda(z)(T-t)}}{\lambda(z)}\right) + e^{-\lambda(z)(T-t)}g(z)\mathbf{1}_{t \leq T} \quad (3.5)$$

and let us define the cumulated reward functional associated to the discrete-time Markov Decision Model for an admissible policy $(f_n)_{n=0}^\infty$ as:

$$V_{\infty, (f_n)}(t, z) = \mathbb{E}_{t, z}^{(f_n)} \left[\sum_{n=0}^{\infty} r(T_n, Z_n, f_n(T_n, Z_n)) \right].$$

The value function associated to $(T_n, Z_n)_{n \in \mathbb{N}}$ is then defined as the supremum of the cumulated reward functional over all the admissible controls in \mathbb{A} , i.e.

$$V_\infty(t, z) = \sup_{(f_n)_{n=0}^\infty \in \mathbb{A}} V_{\infty, \alpha}(t, z), \quad (t, z) \in [0, T] \times E, \quad (3.6)$$

Notice that we used the same notation for admissible controls of the MDP and those of the continuous-time control problem.

Proposition 3.1. *The value function of the MDP defined by (3.6) coincides with (3.1), i.e. we have for all $(t, z) \in E'$:*

$$V_\infty(t, z) = V(t, z). \quad (3.7)$$

Proof. Let us show that for all $\alpha = (f_n) \in \mathbb{A}$ and all $(t, z) \in E'$

$$V_\alpha(t, z) = V_\infty^{(f_n)}(t, z). \quad (3.8)$$

Let us first denote by $H_n := (T_0, Z_0, \dots, T_n, Z_n)$. Notice then that for all admissible strategy α :

$$\begin{aligned} V_\alpha(t, z) &= \mathbb{E}_{t, z}^\alpha \left[\sum_{n=0}^{\infty} \mathbf{1}_{T > T_{n+1}} (T_{n+1} - T_n) c(Z_n, \alpha_n) \right. \\ &\quad \left. + \mathbf{1}_{[T_n \leq T < T_{n+1}]} \left(g(Z_T) - \eta Y_T^2 + (T - T_n) c(Z_n, \alpha_n) \right) \right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}_{t, z}^{(f_n)} \left[r(T_n, Z_n, f_n(T_n, Z_n)) \right], \end{aligned} \quad (3.9)$$

where we conditioned by H_n between the first and the second line. We recognize $V_\infty^{(f_n)}$ in the r.h.s. of (3.9), so that the proof of (3.8) is completed.

It remains to take the supremum over all the admissible strategies \mathbb{A} in (3.8) to get (3.7). \square

From Proposition 3.1, we deduce that the value function of the market-making problem is the same as the value function V_∞ of the discrete-time MDP with infinite horizon. We now aim at solving the MDP control problem. To proceed, we first define the maximal reward mapping for the infinite horizon MDP:

$$\begin{aligned} (\mathcal{T}v)(t, z) &:= \sup_{a \in A_z} \left\{ r(t, z, a) + \int v(t', z') Q(t', z' | t, \phi^a(z), a) \right\} \\ &= \sup_{a \in A_z} \left\{ r(t, z, a) + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s, z') Q'(dz' | \phi^a(z), a) ds \right\}, \end{aligned} \quad (3.10)$$

where we recall that:

- $\phi^a(z)$ is the new state of the order book when the market maker follows the strategy α and the order book is at state z before the decision is taken.
- $\lambda(z)$ is the intensity of the order book process given that the order book is at state z .

We shall tighten assumption **(Hrewards)** in order to guarantee existence and uniqueness of a solution to (3.1), as well as characterizing the latter.

(HrewardsBis): The running and terminal rewards are at most quadratic w.r.t. the state variable, uniformly w.r.t. the control variable, i.e.

1. The running reward f is such that $|c|$ is uniformly bounded by a quadratic in z function, i.e. there exists $c_5 > 0$ such that:

$$\forall (z, a) \in E \times A, \quad |f(z, a)| \leq c_5(1 + |z|^2).$$

2. The terminal reward g has no more than a quadratic growth, i.e. there exists $c_6 > 0$ such that:

$$\forall z \in E, \quad |g(z)| \leq c_6(1 + |z|^2).$$

Remark 3.2. Assumption **(HrewardsBis)** holds in the case where g is the terminal wealth of the market maker plus a penalization of her inventory, and where with no running reward, i.e. $f = 0$.

The main result of this section is the following theorem that gives existence and uniqueness of a solution to (3.1), and moreover characterizes the latter as fixed point of the maximal reward operator defined in (3.10).

Theorem 3.1. \mathcal{T} admits a unique fixed point v which coincides with the value function of the MDP. Moreover we have:

$$v = V_\infty = V.$$

Denote by f^* the maximizer of the operator \mathcal{T} . Then (f^*, f^*, \dots) is an optimal stationary (in the MDP sense) policy.

Remark 3.3. *Theorem 3.1 states that the optimal strategy is stationary in the MDP formulation of the problem, but of course, it is not stationary for the original time-continuous trading problem with finite horizon (3.1), since the time component is not a state variable anymore in the original formulation. Actually, given $n \in \mathbb{N}$ and the state of order book z at that time, the optimal decision to take at time T_n is given by $f^*(T_n, z)$.*

We devoted the next section to the proof of Theorem 3.1.

3.3 Proof of Theorem 3.1

Remind first that we defined in the previous section $E^C := \left\{ (z, a) \in E \times \{0, \dots, K\}^{2\bar{M}} \mid z \in E, a \in A_z \right\}$ and $([0, T] \times E)^C := \left\{ (t, z, a) \in [0, T] \times E \times \{0, \dots, K\}^{2\bar{M}} \mid t \in [0, T], z \in E, a \in A_z \right\}$.

Let us define the bounding functions:

Definition 3.1. *A measurable function $b : E \rightarrow \mathbb{R}_+$ is called a bounding function for the controlled process (Z_t) if there exists positive constants $c_c, c_g, c_{Q'}, c_\phi$ such that:*

1. $|f(z, a)| \leq c_c b(z)$ for all $(z, a) \in E^C$.
2. $|g(z)| \leq c_g b(z)$ for all z in E .
3. $\int b(z') Q'(dz' | z, a) \leq c_{Q'} b(z)$ for all $(z, a) \in E^C$.
4. $b(\phi_t^\alpha(z)) \leq c_\phi b(z)$ for all $(t, z, \alpha) \in ([0, T] \times E)^C$.

Proposition 3.2. *Let b be such that :*

$$\forall z \in E, b(z) := 1 + |z|^2.$$

*Then, b is a bounding function for the controlled process (Z_t) , under Assumption (**HrewardsBis**).*

Proof. Let us check that b defined in Proposition 3.2 satisfies the four assertions in Definition 3.1.

- Assertion 1 and 2 of Definition 3.1 holds under (**HrewardsBis**).
- First notice that $\underline{ra}, \underline{rb}$ are bounded by $\sqrt{\bar{M}}K$ (where we recall that K is the number of limits in each side of the order book, and \bar{M} is the biggest number of limit orders that the market maker is allowed to send in the market). Secondly, $pa' \in B(pa, K), pb' \in B(pb, K)$, where $B(x, r)$ is the ball centered in x with radius $r > 0$, because of the limit conditions that we imposed in our LOB model. And last, we can see that $|\underline{a}'| \leq |\underline{a}| + a_\infty K$. These three bounds are linear w.r.t. z so that assertion 3 holds.
- $\phi^\alpha(z) = z^\alpha$ only differs from z by its $\underline{na}, \underline{nb}$, and $\underline{ra}, \underline{rb}$ components. But $|\underline{na}| \leq \sqrt{\bar{M}}(|\underline{a}| + \bar{M})$ and $|\underline{nb}| \leq \sqrt{\bar{M}}(|\underline{b}| + \bar{M})$ are bounded by a linear function of $(\underline{a}, \underline{b})$, also $|\underline{ra}|$ and $|\underline{rb}|$ are bounded by the universal constant $\sqrt{\bar{M}}K$, so assertion 4 in Definition 3.1 holds.

□

Let us define

$$\Lambda := (4K + 2) \sup \left\{ \frac{\lambda^{M^\pm}}{|\underline{a}| + |\underline{b}|}, \frac{\lambda^{L^\pm}}{|\underline{a}| + |\underline{b}|}, \frac{\lambda^{C^\pm}}{|\underline{a}(z)| + |\underline{b}(z)|} \right\}.$$

Note that Λ is well-defined under **(Harrivals)**.

Proposition 3.3. *If b is a bounding function for (Z_t) , then*

$$b(t, z) := b(z)e^{\gamma(z)(T-t)}, \text{ with } \gamma(z) = \gamma_0(4K + 2)\Lambda(1 + |\underline{a}| + |\underline{b}|) \text{ and } \gamma_0 > 0$$

is a bounding function for the MDP, i.e. for all $t \in [0, T]$, $z \in E$, $a \in A_z$, we have:

$$|r(t, z, a)| \leq c_g b(t, z),$$

$$\int b(s, z') Q(ds, dz' | t, z, a) \leq c_\phi c_Q e^{C(T-t)} \frac{1}{1 + \gamma_0} b(t, z),$$

with $C = \gamma_0 \Lambda K (4K + 2) (|\underline{a}|_\infty + |\underline{b}|_\infty)$.

Proof. Let $z' = (x', y', \underline{a}', \underline{b}', \underline{na}', \underline{nb}', \underline{ra}', \underline{rb}')$ be the state of the order book after an exogenous jump occurs given that it was at state z before the jump. Since $|\underline{a}'| \leq |\underline{a}| + a_\infty K$ and $|\underline{b}'| \leq |\underline{b}| + b_\infty$, where a_∞ and b_∞ are defined as the border conditions of the order book, we have:

$$\gamma(z') \leq \gamma(z) + C, \tag{3.11}$$

with $C = \gamma_0 \Lambda K (4K + 2) (a_\infty + b_\infty)$. Then, we get:

$$\begin{aligned} \int b(s, z') Q(ds, dz' | t, \phi^\alpha(z), \alpha) &= \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int b(t+s, z') Q'(dz' | \phi_s^\alpha(z), \alpha) ds \\ &= \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int b(z') e^{\gamma(z')(T-(t+s))} Q'(dz' | \phi_s^\alpha(z), \alpha) ds \\ &\leq \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int b(z') e^{(\gamma(z)+C)(T-(t+s))} Q'(dz' | \phi_s^\alpha(z), \alpha) ds \\ &\leq \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} e^{(\gamma(z)+C)(T-(t+s))} \int b(z') Q'(dz' | \phi_s^\alpha(z), \alpha) ds \\ &\leq \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} e^{(\gamma(z)+C)(T-(t+s))} c_Q c_\phi b(z) ds \\ &\leq \frac{\lambda(z) c_Q c_\phi}{\lambda(z) + \gamma(z) + C} e^{(\gamma(z)+C)(T-t)} \left(1 - e^{-(T-t)(\lambda(z)+\gamma(z)+C)}\right) b(z) \\ &\leq c_Q c_\phi \frac{\lambda(z)}{\lambda(z) + \gamma(z) + C} e^{C(T-t)} \left(1 - e^{-(T-t)(\lambda(z)+\gamma(z)+C)}\right) b(t, z), \end{aligned}$$

where we applied (3.11) at the third line. It remains to notice that

$$\frac{\lambda(z)}{\lambda(z) + \gamma(z) + C} = \frac{\lambda(z)}{\lambda(z)(1 + \gamma_0) + \gamma_0 \underbrace{[\Lambda(|\underline{a}| + |\underline{b}|) - \lambda(z)]}_{\geq 0}} \leq \frac{1}{1 + \gamma_0},$$

to complete the proof of the Proposition. □

Let us denote by $\|\cdot\|_b$ the *weighted supremum norm* such that for all measurable function $v : E' \rightarrow \mathbb{R}$,

$$\|v\|_b := \sup_{(t,z) \in E'} \frac{|v(t,z)|}{b(t,z)},$$

and define the set:

$$\mathbb{B}_b := \left\{ v : E' \rightarrow \mathbb{R} \mid v \text{ is measurable and } \|v\|_b < \infty \right\}.$$

Moreover let us define

$$\alpha_b := \sup_{(t,z,\alpha) \in E' \times \mathcal{R}} \frac{\int b(s,z') Q(ds, dz' | t, \phi^\alpha(z), \alpha)}{b(t,z)}.$$

From the preceding estimations we can bound α_b as follows:

$$\alpha_b \leq c_{Qc\phi} \frac{1}{1 + \gamma_0} e^{CT},$$

So that, by taking: $\gamma_0 = c_{Qc\phi} e^{CT}$, we get: $\alpha_b < 1$. In the sequel, we then assume w.l.o.g. that $\alpha_b < 1$. Recall that the maximal reward mapping for the MDP has been defined as:

$$\mathcal{T}v : (t,z) \mapsto \sup_{a \in A_z} \left\{ r(t,z,a) + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s,z') Q'(dz' | \phi^a(z), a) ds \right\}$$

It is straightforward to see that:

$$\|\mathcal{T}v - \mathcal{T}w\|_b \leq \alpha_b \|v - w\|_b, \quad (3.12)$$

which implies that \mathcal{T} is contracting, since $\alpha_b < 1$.

Let \mathcal{M} be the set of all the continuous function in \mathbb{B}_b . Since b is continuous, $(\mathcal{M}, \|\cdot\|_b)$ is a Banach space.

\mathcal{T} sends \mathcal{M} to \mathcal{M} . Indeed, for all continuous function v in \mathbb{B}_b , $(t,z,a) \mapsto r(t,z,a) + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s,z') Q'(dz' | \phi^a(z), a) ds$ is continuous on $[0, T] \times E^C$. A_z is finite, so we get the continuity of the application:

$$\mathcal{T}v : (t,z) \mapsto \sup_{a \in A_z} \left\{ r(t,z,a) + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s,z') Q'(dz' | \phi^a(z), a) ds \right\}.$$

Proposition 3.4. *There exists a maximizer for \mathcal{T} , i.e. let $v \in \mathcal{M}$, then there exists a Borelian function $f : [0, T] \times E \rightarrow A$ such that for all $(t,z) \in E'$:*

$$\mathcal{T}v(t,z, f(t,z)) = \sup_{a \in A} \left\{ r(t,z,a) + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s,z') Q'(dz' | \phi^a(z), a) ds \right\}$$

Proof. $D^*(t,z) = \left\{ a \in A \mid \mathcal{T}_a v(t,z) = \mathcal{T}v(t,z) \right\}$ is finite, so it is compact. So $(t,z) \mapsto D^*(t,z)$ is a compact-valued mapping. Since the application $(t,z,a) \mapsto \mathcal{T}_a v(t,z) - \mathcal{T}v(t,z)$ is continuous, we get that $D^* = \left\{ (t,z,a) \in E'^C \mid \mathcal{T}_a v(t,z) = \mathcal{T}v(t,z) \right\}$ is borelian. Applying the measurable selection theorem yields to the existence of the maximizer. (see [BR11] p.352) \square

Lemma 3.2. *The following holds:*

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}_{t,z}^{\alpha} \left[\sum_{k=n}^{\infty} |r(T_k, Z_k)| \right] \leq \frac{\alpha_b^n}{1 - \alpha_b} b(t, z),$$

and in particular, we have:

$$\lim_{n \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \mathbb{E}_{t,z}^{\alpha} \left[\sum_{k=n}^{\infty} |r(t_k, Z_k)| \right] = 0.$$

Proof. By conditioning we get $\mathbb{E}_{t,z}^{\alpha} \left[|r(T_k, Z_k)| \right] \leq c_g \alpha_b^k b(t, z)$ for $k \in \mathbb{N}$, and for all $\alpha \in \mathbb{A}$. It remains to sum this inequality to complete the proof of Lemma 3.2. \square

We can now prove Theorem 3.1.

Proof. We divided the proof of Theorem 3.1 into four steps.

Step 1: Inequality (3.12) and Proposition 3.3 imply that \mathcal{T} is a stable and contracting operator defined on the Banach space \mathcal{M} . Banach's fixed point theorem states that \mathcal{T} admits a fixed point, i.e. there exists a function $v \in \mathbb{M}$ such that $v = \mathcal{T}v$, and moreover we have $v = \lim_{n \rightarrow \infty} \mathcal{T}^n 0$. Notice that $\mathcal{T}^N 0$ coincides with v_0 defined recursively by the following Bellman equation:

$$\begin{cases} v_N &= 0 \\ v_n &= \mathcal{T}v_{n+1} \text{ for } n = N - 1, \dots, 0. \end{cases} \quad (3.13)$$

The solution of the Bellman equation is always larger than the value function of the MDP associated (see e.g. Theorem 2.3.7 p.22 in [BR11]). Then we have: $\mathcal{T}^n 0 \geq \sup_{(f_k)} \mathbb{E}_n^{(f_k)} \left[\sum_{k=0}^{n-1} r(t_k, X_k) \right] =: J_n$, where

J_n is the value function of the MDP with finite horizon n and terminal reward 0, associated to (3.13). Moreover, by Lemma 7.1.4 p.197 in [BR11], we know that $(J_n)_n$ converges as $n \rightarrow \infty$ to a limit that we denote by J . Passing at the limit in the previous inequality we get: $\lim_{n \rightarrow \infty} \mathcal{T}^n 0 \geq J$, i.e.

$$v \geq J. \quad (3.14)$$

Step 2: Let us fix a strategy $\alpha \in \mathbb{A}$, and take $n \in \mathbb{N}$. We denote $J_n(\alpha) := \mathbb{E}_0^{(\alpha_k)} \left[\sum_{k=0}^{n-1} r(t_k, X_k) \right]$, the reward functional associated to the control α on the discrete finite time horizon $\{0, \dots, n\}$. By definition, we have $J_n(\alpha) \leq J_n$. We get by letting $n \rightarrow \infty$: $\lim_{n \rightarrow +\infty} J_n(\alpha) =: J_{\infty}(\alpha) \leq J$. Taking the supremum over all the admissible strategies α finally leads to:

$$V_{\infty} \leq J. \quad (3.15)$$

Step 3: Let us denote by f a maximizer of \mathcal{T} associated to v , which exists, as stated in Proposition 3.4. v is the fixed point of \mathcal{T} so that $v = \mathcal{T}_f^n(v)$, for $n \in \mathbb{N}$. Moreover $v \leq \delta$ where $\delta := \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\sum_{k=0}^{\infty} r^+(Z_k, \alpha_k) \right]$,

so that $\mathcal{T}_f^n(v) \leq \mathcal{T}_f^n 0 + \mathcal{T}_o^n \delta$, where $\mathcal{T}_o^n \delta = \sup_{\alpha} \mathbb{E}_n^{\alpha} \left[\sum_{k=n}^{\infty} r^+(t_k, Z_k) \right]$. Lemma 3.2 implies that $\mathcal{T}_o^n \delta \rightarrow 0$ as $n \rightarrow \infty$. Hence, we get:

$$v \leq J_f. \quad (3.16)$$

Step 4: Conclusion. Since it holds

$$J_f \leq V_\infty, \quad (3.17)$$

we get by combining (3.14), (3.15), (3.16) and (3.17):

$$V_\infty \leq J \leq v \leq J_f \leq V_\infty. \quad (3.18)$$

All the inequalities in (3.18) are then equalities, which completes the proof of Theorem 3.1. \square

4 Numerical Algorithm

In this section, we first introduce an algorithm to numerically solve a general class of discrete-time control problem with finite horizon, and then apply it on the trading problem (3.1).

4.1 Framework

Let us consider a general discrete-time stochastic control problem over a finite horizon $N \in \mathbb{N} \setminus \{0\}$. The dynamics of the controlled state process $Z^\alpha = (Z_n^\alpha)_n$ valued in \mathbb{R}^d is given by

$$Z_{n+1}^\alpha = F(Z_n^\alpha, \alpha_n, \varepsilon_{n+1}), \quad n = 0, \dots, N-1, \quad Z_0^\alpha = z \in \mathbb{R}^d,$$

with $(\varepsilon_n)_n$ is a sequence of i.i.d. random variables valued in some Borel space $(E, \mathcal{B}(E))$, and defined on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_n)_n$ generated by the noise $(\varepsilon_n)_n$ (\mathcal{F}_0 is the trivial σ -algebra), the control $\alpha = (\alpha_n)_n$ is an \mathbb{F} -adapted process valued in $A \subset \mathbb{R}^q$, and F is a measurable function from $\mathbb{R}^d \times \mathbb{R}^q \times E$ into \mathbb{R}^d .

Given a running cost function f defined on $\mathbb{R}^d \times \mathbb{R}^q$, a terminal cost function g defined on \mathbb{R}^d , the cost functional associated to a control process α is

$$J(\alpha) = \mathbb{E} \left[\sum_{n=0}^{N-1} f(Z_n^\alpha, \alpha_n) + g(Z_N^\alpha) \right].$$

The set \mathbb{A} of admissible control is the set of control processes α satisfying some integrability conditions ensuring that the cost functional $J(\alpha)$ is well-defined and finite. The control problem, also called Markov decision process (MDP), is formulated as

$$V_0(x_0) := \sup_{\alpha \in \mathcal{A}} J(\alpha),$$

and the goal is to find an optimal control $\alpha^* \in \mathcal{A}$, i.e., attaining the optimal value: $V_0(z) = J(\alpha^*)$. Notice that problem (4.1)-(4.2) may also be viewed as the time discretization of a continuous time stochastic control problem, in which case, F is typically the Euler scheme for a controlled diffusion process.

Problem (4.2) is tackled by the dynamic programming approach. For $n = N, \dots, 0$, the value function V_n at time n is characterized as solution of the following backward (Bellman) equation:

$$\begin{cases} V_N(z) = g(z) \\ V_n(z) = \sup_{a \in A} \{ f(z, a) + \mathbb{E}_{n,z}^a [V_{n+1}(Z_{n+1})] \}, \quad z \in \mathbb{R}^d, \end{cases} \quad (4.3)$$

Moreover, when the supremum is attained in the DP formula at any time n by $a_n^*(z)$, we get an optimal control in feedback form given by: $\alpha^* = (a_n^*(Z_n^*))_n$ where $Z^* = Z^{\alpha^*}$ is the Markov process defined by

$$Z_{n+1}^* = F(Z_n^*, a_n^*(Z_n^*), \varepsilon_{n+1}), \quad n = 0, \dots, N-1, \quad Z_0^* = z.$$

There are two usual ways that have been studied in the literature, to solve numerically (4.3): some methods make use of quantization to discretize the state space and approximate the conditional expectations by cubature methods; another way is to rely on MC regress-now or Later methods to regress the value functions V_{n+1} at time n for $n = 0, \dots, N-1$ on basis functions or neural networks. See e.g. [KLP14] for the regress-now and [BP17] for the regress-Later methods for algorithms using basis functions, and e.g. [HPBL18] for regression on neural networks based on regress-now or regress-later techniques.

4.2 Presentation and rate of convergence of the Qknn algorithm

In this section, we present an algorithm based on k-nn estimates for local non-parametric regression of the value function, and optimal quantization to quantize the exogenous noise, in order to numerically solve (4.3).

Let us first introduce some ingredients of the quantization approximation:

- We denote by $\hat{\varepsilon}$ a K -quantizer of the E -valued random variable $\varepsilon_{n+1} \sim \varepsilon_1$, that is a discrete random variable on a grid $\Gamma = \{e_1, \dots, e_K\} \subset E^K$ defined by

$$\hat{\varepsilon} = \text{Proj}_\Gamma(\varepsilon_1) := \sum_{\ell=1}^K e_\ell \mathbf{1}_{\varepsilon_1 \in C_\ell(\Gamma)},$$

where $C_1(\Gamma), \dots, C_K(\Gamma)$ are Voronoi tessellations of Γ , i.e., Borel partitions of the Euclidian space $(E, |\cdot|)$ satisfying

$$C_\ell(\Gamma) \subset \left\{ e \in E : |e - e_\ell| = \min_{j=1, \dots, K} |e - e_j| \right\}.$$

The discrete law of $\hat{\varepsilon}$ is then characterized by

$$\hat{p}_\ell := \mathbb{P}[\hat{\varepsilon} = e_\ell] = \mathbb{P}[\varepsilon_1 \in C_\ell(\Gamma)], \quad \ell = 1, \dots, K.$$

The grid points (e_ℓ) which minimize the L^2 -quantization error $\|\varepsilon_1 - \hat{\varepsilon}\|_2$ lead to the so-called optimal L -quantizer, and can be obtained by a stochastic gradient descent method, known as Kohonen algorithm or competitive learning vector quantization (CLVQ) algorithm, which also provides as a byproduct an estimation of the associated weights (\hat{p}_ℓ) . We refer to [PPP04] for a description of the algorithm, and mention that for the normal distribution, the optimal grids and the weights of the Voronoi tessellations are precomputed on the website <http://www.quantize.maths-fi.com>

- Recalling the dynamics (4.1), the conditional expectation operator is equal to

$$P^{\hat{a}_n^M(z)} W(x) = \mathbb{E}[W(Z_{n+1}^{\hat{a}_n^M}) | Z_n = x] = \mathbb{E}[W(F(z, \hat{a}_n^M(z), \varepsilon_1))], \quad z \in,$$

that we shall approximate analytically by quantization via:

$$\widehat{P}^{\hat{a}_n^M(z)} W(z) := \mathbb{E}[W(F(z, \hat{a}_n^M(z), \hat{\varepsilon}))] = \sum_{\ell=1}^K \hat{p}_\ell W(F(z, \hat{a}_n^M(z), e_\ell)).$$

Let us secondly introduce the notion of training distribution that will be used to build the estimators of value functions at time n , for $n = 0, \dots, N - 1$. Let us consider a measure μ on the state space E . We refer to it in the sequel as the training measure. Let us take a large integer M , and for $n = 0, \dots, N$, introduce $\Gamma_n = \{Z_1^{(1)}, \dots, Z_n^{(M)}\}$, where $(Z_n^{(m)})_{m=1}^M$ is a i.i.d. sequence of r.v. following law μ . Γ_n should be seen as a training sampling to estimate the value function V_n at time n .

The proposed algorithm reads as:

$$\begin{cases} \hat{V}_N^Q(z) &= g(z), & \text{for } z \in \Gamma_N, \\ \hat{Q}_n(z, a) &= \sum_{\ell=1}^K p_\ell \left[f(z, a) + \hat{V}_{n+1}^Q(\text{Proj}_{n+1}(F(z, e_\ell, a))) \right], \\ \hat{V}_n^Q(z) &= \sup_{a \in A} \hat{Q}_n(z, a), & \text{for } z \in \Gamma_n, n = 0, \dots, N - 1. \end{cases} \quad (4.4)$$

where, for $n = 0, \dots, N$, $\text{Proj}_n(z)$ stands for the closest neighbor of $z \in E$ in the grid Γ_n , i.e. the operator $z \mapsto \text{Proj}_n(z)$ is actually the euclidean projection on the grid Γ_n .

Remark 4.1. We could have generalized the operator Proj_n by considering $z \in E \mapsto \hat{z} = \frac{1}{k} \sum_{j=1}^k w_j Z_n^{(j)}$, with the weight w_j such as

$$w_j(z) = \frac{|z - Z_n^{(j)}|}{\sum_{i=1}^k |z - Z_n^{(i)}|},$$

and where $Z_n^{(j)}$ stands for the j^{th} nearest neighbors of z in Γ_n , for $j = 1, \dots, k$. This generalization brings continuity to the estimates.

Others local generalizations of Proj_n , based e.g. kernel methods, are available in the literature, and we refer to [BKS10] for more details.

In the sequel, we refer to (4.4) as the Qknn algorithm.

We shall make the following assumption on the transition probability of $(Z_n)_{0 \leq n \leq N}$, to guarantee the convergence of the Qknn algorithm.

(Htrans) Assume that the transition probability $\mathbb{P}(Z_{n+1} \in A | Z_n = z, a)$ conditioned by $Z_n = z$ when control a is followed at time n admits a density r w.r.t. the training measure μ , which is uniformly bounded and lipschitz w.r.t. the state variable z , i.e. there exists $\|r\|_\infty > 0$ such that for all $z \in E$ and control u taken at time n :

$$|r(y; n, x, a)| \leq \|r\|_\infty \quad \text{and} \quad |r(y; n, x, a) - r(y; n, x', a)| \leq [r]_L |x - x'|$$

and r is defined as follows:

$$\mathbb{P}(Z_{n+1} \in O | Z_n = z, u) = \int_O r(y; n, x, a) d\mu(y).$$

and where we denoted by $[r]_L$ the Lipschitz constant of r w.r.t. x .

Denote by $\text{Supp}(\mu)$ the support of μ . We shall assume smoothness conditions on μ and F to provide a bound on the projection error.

(H μ) We assume $\text{Supp}(\mu)$ to be bounded, and denote by $\|\mu\|_\infty$ the smallest real such that $\text{Supp}(\mu) \subset B(0, \|\mu\|_\infty)$. Moreover, we assume $x \in E \mapsto \mu(B(x, \eta))$ to be Lipschitz, uniformly w.r.t. η , and we denote by $[\mu]_L$ its Lipschitz constant.

(HF) For $x \in E$ and $a \in A$, assume F to be \mathbb{L}^1 -Lipschitz w.r.t. the noise component ε , i.e., there exists $[F]_L > 0$ such that for all $x \in E$ and $a \in A$, for all r.v. ε and ε' , we have:

$$\mathbb{E} [|F(x, a, \varepsilon) - F(x, a, \varepsilon')|] \leq [F]_L \mathbb{E} [|\varepsilon - \varepsilon'|]$$

We now state the main result of this section whose proof is postponed in Appendix C.

Theorem 4.1. *Take $K = M^{2+d}$ points for the optimal quantization of the exogenous noise ε_n , $n = 1, \dots, N$. There exist constants $[\hat{V}_n^Q]_L > 0$, that only depends on the Lipschitz coefficients of f , g and F , such that, under **(Htrans)**, it holds for $n = 0, \dots, N - 1$, as $M \rightarrow +\infty$:*

$$\|\hat{V}_n^Q(X_n) - V_n(X_n)\|_2 \leq \sum_{k=n+1}^N \|r\|_\infty^{N-k} [\hat{V}_k^Q]_L \left(\varepsilon_k^{proj} + [F]_L \varepsilon_k^Q \right) + \mathcal{O} \left(\frac{1}{M^{1/d}} \right), \quad (4.5)$$

where $\varepsilon_k^Q := \|\hat{\varepsilon}_k - \varepsilon_k\|_2$ stands for the quantization error, and

$$\varepsilon_n^{proj} := \sup_{a \in A} \|\text{Proj}_{n+1}(F(X_n, a, \hat{\varepsilon}_n)) - F(X_n, a, \hat{\varepsilon}_n)\|_2$$

stands for the projection error, when decision a is taken at time n .

Remark 4.2. *The constants $[\hat{V}_n^Q]_L > 0$ are defined in (C.8).*

From Theorem 4.1, we can deduce consistency and provide a rate of convergence for the estimator \hat{V}_n^Q , $n = 0, \dots, N - 1$, under some rather tough yet usual compactness conditions on the state space.

Corollary 4.1. *Under **(H μ)** and **(HF)**, the Qknn-estimator \hat{V}_n^Q is consistent for $n = 0, \dots, N - 1$, when taking M^{d+1} points for the quantization; and moreover, we have for $n = 0, \dots, N - 1$, as $M \rightarrow +\infty$:*

$$\|\hat{V}_n^Q(X_n) - V_n(X_n)\|_2 \leq \mathcal{O} \left(\frac{1}{M^{1/d}} \right).$$

Proof. We postpone the proof of Theorem 4.1 to Appendix C. □

4.3 Qknn algorithm applied to the order book control problem (3.1)

We recall the expression of the controlled order book, as described in section 3:

$$Z_t = (X_t, Y_t, \underline{a}_t, \underline{b}_t, \underline{na}_t, \underline{nb}_t, pa_t, pb_t, ra_t, rb_t).$$

In section 3.3, we proved that the value function V is characterized as the unique solution of the Bellman equation (3.10). In this section, some implementation details on the Qknn algorithm are presented in order to numerically solve the market-making problem.

Training set design

Inspired by [FPS18], we use product-quantization method and randomization techniques to build the training set Γ_n on which we project (T_n, Z_n) that lies on $[0, T] \times E$, where T_n and Z_n stands for the n^{th} jump of Z and the state of Z at time t_n , i.e. $Z_n = Z_{T_n}$, for $n \geq 0$. This basic idea of Control Randomization consists in replacing in the dynamics of Z the endogenous control by an exogenous control $(I_{T_n})_{n \geq 0}$, as introduced in [KLP14]. In order to alleviate the notations, we denote by I_n the control taken at time T_n , for $n \geq 0$.

Initialization. Set: $\Gamma_0^E = \{z\}$ and $\Gamma_0^T = \{0\}$.

Randomize the control, using e.g. uniform distribution on A at each time step, and then simulate D randomized processes to generate $(T_n^k, Z_n^k)_{n=0, k=1}^{N, D}$.

For all $n = 1, \dots, N$, set $\Gamma_n^T = \{T_n^k, 1 \leq k \leq D\}$, which stands for the grid associated to the quantization of the n^{th} jump time T_n , and set $\Gamma_n^E = \{Z_n^k, 1 \leq k \leq D\}$ which stands for the grid associated to the quantization of the state Z_n of Z at time T_n .

Remark 4.3. *The way we chose our training sets is often referred to as an exploration strategy in the reinforcement learning literature. Of course, if one has ideas or good guess of where to optimally drive the controlled process, she shouldn't follow an exploration-type strategy to build the training set, but should rather use the guess to build it, which is referred to as the exploitation strategy in the reinforcement learning and the stochastic bandits literature. We refer to [Bal+19] for several other applications of the exploration strategy to build training sets.*

Let F and G be the Borelian functions such that $Z_n = F(Z_{n-1}, d_n, I_n)$ and $T_n = G(T_{n-1}, \epsilon_n, I_n)$, where $\epsilon_n \sim \mathcal{E}(1)$ stands for the temporal noise, and d_n is the state noise, for $n \geq 0$.

Let us fix $N \geq 1$ and consider $(\widehat{T}_n, \widehat{Z}_n)_{n=0}^N$, the dimension-wise projection of $(T_n, Z_n)_{n=0}^N$ on the grids $\Gamma_n^T \times \Gamma_n^E$, $n = 0, \dots, N$, i.e. $\widehat{T}_0 = 0$, $\widehat{Z}_0 = z$, and

$$\begin{cases} \widehat{T}_n = \text{Proj}\left(G(\widehat{T}_{n-1}, \epsilon_n, I_n), \Gamma_n^T\right), \\ \widehat{Z}_n = \text{Proj}\left(F(\widehat{Z}_{n-1}, d_n, I_n), \Gamma_n^E\right), \end{cases} \quad \text{for } n = 1, \dots, N.$$

$(\widehat{T}_n, \widehat{Z}_n, I_n)_{n \in \{0, N\}}$ is a Markov chain, and its probability transition matrix at time $n = 1, \dots, N$ reads:

$$\hat{p}_k^{ij}(a) = \mathbb{P}\left[\hat{t}_k = t_k^j, \widehat{Z}_k = z_k^j \mid \hat{t}_{k-1} = t_{k-1}^i, \widehat{Z}_{k-1} = z_{k-1}^i, I_k = a\right] = \frac{\hat{\beta}_k^{ij}}{\hat{p}_{k-1}^i}, \quad i = 1, \dots, N_{k-1}, j = 1, \dots, N_k, a \in \mathcal{A}$$

where:

$$\hat{p}_{k-1}^{ij} = \mathbb{P}[\hat{t}_{k-1} = t_{k-1}^i, \hat{Z}_{k-1} = z_{k-1}^i] = \begin{cases} \mathbb{P}[F(\hat{t}_{k-2}, \hat{Z}_{k-2}, \epsilon_{k-1}, d_{k-1}) \in C_i(\Gamma_{k-1} \times \Gamma_{k-1}^E)] & \text{if } k \geq 2 \\ 1 & \text{if } k = 1 \end{cases}$$

$$\begin{aligned} \hat{\beta}_k^{ij} &= \mathbb{P}[\hat{t}_{k-1} = t_{k-1}^i, \hat{Z}_{k-1} = z_{k-1}^i, \hat{t}_k = t_k^j, \hat{Z}_k = z_k^j] \\ &= \begin{cases} \mathbb{P}[F_k(\hat{t}_{k-2}, \hat{Z}_{k-2}, \epsilon_{k-1}, d_{k-1}) \in C_i(\Gamma_{k-1} \times \Gamma_{k-1}^E); F_k(\hat{t}_{k-1}, \hat{Z}_{k-1}, \epsilon_k, d_k) \in C_j(\Gamma_k \times \Gamma_k^E)] & \text{if } k \geq 2 \\ 1 & \text{if } k = 1 \end{cases} \end{aligned}$$

and where, for all $i, 0 \leq i \leq D$, for all $k \in \mathbb{N}$, we denoted by $C_i(\Gamma_k \times \Gamma_k^E)$ the Voronoi cell associated to the point (T_k^i, z_k^i) .

Define then $(\hat{T}_n^Q, \hat{Z}_n^Q)_{n=0}^N$ as temporal noise-quantized version of $(\hat{T}_n, \hat{Z}_n, I_n)_{n=0}^N$. Note that we do not need to quantize the spacial noise since this noise already takes a finite number of states. Let $\hat{\epsilon}_n$ be the quantized process associated to ϵ_n . The process $(\hat{T}_n^Q, \hat{Z}_n^Q)_{n=0}^N$ is then defined as follows: $\hat{Z}_0^Q = z, \hat{T}_0^Q = 0$ and $\forall 1 \leq n \leq N$:

$$\begin{cases} \hat{T}_n^Q = \text{Proj}\left(G(\hat{t}_{n-1}, \hat{\epsilon}_n, I_n), \Gamma_n^T\right), \\ \hat{Z}_n^Q = \text{Proj}\left(F(\hat{Z}_{n-1}, d_n, I_n), \Gamma_n^E\right). \end{cases}$$

Denote by $(\hat{V}_n^{Q,(N,D)})_{n=0}^N$ the solution of the Bellman equation associated to $(\hat{T}_n^Q, \hat{Z}_n^Q)_{n=0}^N$:

$$\left(\hat{B}_{N,D}^Q\right) : \begin{cases} \hat{V}_N^{Q,(N,D)} &= 0 \\ \hat{V}_n^{Q,(N,D)}(t, z) &= r(t, z, a) + \sup_{a \in A} \left\{ \mathbb{E}_{t,z}^a \left[\hat{V}_{n+1}^{Q,(N,D)}(\hat{T}_{n+1}^Q, \hat{Z}_{n+1}^Q) \right] \right\}, \text{ for } n = 0, \dots, N, \end{cases}$$

where $\mathbb{E}_{t,z}^a[\cdot]$ stands for the expectation conditioned by the events $\hat{T}_n^Q = t, \hat{Z}_n^Q = z$ and when decision $I_n = a$ is taken at time t .

We wrote the pseudo-code of the Qknn algorithm to compute $(\hat{B}_{N,D}^Q)$ in Algorithm 1.

We discuss in Remark 4.4 the reasons why we can apply Theorem 4.1.

Remark 4.4. *When the number of jumps of the LOB $N \geq 1$ is fixed, the set of all the states that can take the controlled order book by jumping less than N times, denoted by \mathcal{K} in the sequel, is finite. Hence, the reward function r , defined in (3.5), is bounded and Lipschitz on \mathcal{K} .*

The following proposition states that $\hat{V}_n^{Q,(N,D)}$, built from the combination of time-discretization, k -nearest neighbors and optimal quantization methods, is a consistent estimator of the value function at time T_n , for $n = 0, \dots, N - 1$. It provides a rate of convergence for the Qknn-estimations of the value functions.

Algorithm 1 Generic Qknn Algorithm

Inputs:

- N : number of time steps
- z : state in E at time $T_0 = 0$
- $\Gamma^\varepsilon = \{e_1, \dots, e_L\}$ and $(p_\ell)_{\ell=1}^L$: the grid and the weights for the optimal quantization of $(\varepsilon_n)_{n=1}^N$.
- Γ_n and Γ_n^E the grids for the projection of respectively the time and the state components at time n , for $n = 0, \dots, N$.

- 1: **for** $i = N - 1, \dots, 0$ **do**
- 2: Compute the approximated $Qknn$ -value at time n :

$$\begin{aligned} \hat{Q}_n(z, a) &= r(T_n, z, a) \\ &\quad + \sum_{\ell=1}^L p_\ell \hat{V}_{n+1}^Q(\text{Proj}(G(z, e_\ell, a), \Gamma_{n+1}^T), \text{Proj}(F(z, e_\ell, a), \Gamma_{n+1}^E)), \\ &\hspace{15em} \text{for } (z, a) \in \Gamma_n \times A_z; \end{aligned}$$

- 3: Compute the optimal control at time n

$$\hat{A}_n(z) \in \underset{a \in A_z}{\text{argmin}} \hat{Q}_n(z, a), \quad \text{for } z \in \Gamma_n,$$

where the argmin is easy to compute since A_z is finite for all $z \in E$;

- 4: Estimate analytically by quantization the value function:

$$\hat{V}_n^Q(z) = \hat{Q}_n(z, \hat{A}_n(z)), \quad \forall z \in \Gamma_n;$$

- 5: **end for**

Output:

- (\hat{V}_0^Q) : Estimate of $V(0, z)$;
-

Proposition 4.1. *The estimators of the value functions provided by Qknn algorithm are consistent. Moreover, it holds as $M \rightarrow +\infty$:*

$$\left\| \widehat{V}_n^{Q,(N,M)}(\widehat{T}_n, \widehat{Z}_n) - V_n(T_n, Z_n) \right\|_{M,2} = \mathcal{O}\left(\alpha^N + \frac{1}{M^{2/d}}\right), \quad \text{for } n = 0, \dots, N-1,$$

where we denote by $\|\cdot\|_{M,2}$ the $\mathbb{L}^2(\mu)$ norm conditioned by the training sets that have been used to build the estimator $\widehat{V}_{n+1}^{Q,(N,M)}$.

Proof. Splitting the error of time cutting and quantization, we get:

$$\begin{aligned} \|V_n(T_n, Z_n) - \widehat{V}_n^{(N,M)}(\widehat{T}_n, \widehat{Z}_n)\|_{M,2} &\leq \|V_n(T_n, Z_n) - V_n^{(N)}(T_n, Z_n)\|_{M,2} \\ &\quad + \|V_n^{(N)}(T_n, Z_n) - \widehat{V}_n^{(N,M)}(\widehat{T}_n, \widehat{Z}_n)\|_{M,2}. \end{aligned} \quad (4.8)$$

Step 1: Applying Lemma 3.2, we get the following bound on the first term in the r.h.s. of (4.8):

$$\|V_n(T_n, Z_n) - V_n^{(N)}(T_n, Z_n)\|_{M,2} \leq \frac{\alpha^N}{1-\alpha} \|b\|_\infty, \quad (4.9)$$

where $\|b\|_\infty$ stands for the supremum of b over $[0, T] \times E$.

Step 2: Note that the assumptions of Theorem 4.1 are met as noticed in Remark 4.4, so that the latter provides the following bound for the second term in the r.h.s. of (4.8):

$$\left\| V_n^{(N)}(T_n, Z_n) - \widehat{V}_n^{Q,(N,M)}(\widehat{T}_n, \widehat{Z}_n) \right\|_{M,2} \stackrel{M \rightarrow \infty}{=} \mathcal{O}\left(\frac{1}{M^{2/d}}\right). \quad (4.10)$$

It remains to plug (4.9) and (4.10) into (4.8) to complete the proof of Proposition 4.1. \square

We provide a diagram in figure 3 to summarize the two main steps in the estimation of the value function of the market-making problem defined in (3.1).

4.4 Numerical results

In this section, we propose several settings to test the efficiency of Qknn on simulated order books. We take no running reward, i.e. $f = 0$, and take the wealth of the market maker as terminal reward, i.e. $g(z) = x$. The intensities are taken constant in some tests, and state dependent on other tests. The values of the intensities are similar to the ones in [HLR15]. Although the intensities are assumed uncontrolled in section 3 for predictability reasons, the latter are controlled processes in this section, i.e. the intensities of the order arrivals depends on the orders in the order book from all the participant plus the ones of the market maker. The optimal trading strategies have been computed among two different classes of strategies: in section 4.4.1, we tested the algorithm to approximate the optimal strategy among those where the market maker is only allowed to place orders only at the best bid and the best ask. The dynamics of the controlled order book for such a class of controls are available in Section B in the Appendix. In Section 4.4.2, we computed the optimal trading strategy among the class of the strategies where the market maker allows

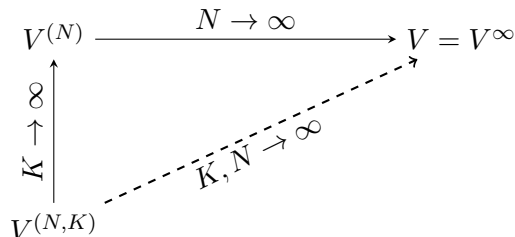


Figure 3 – Numerical resolution of the algorithmic control problem. We first bound the number of events and then quantize the state space.

herself to place orders on the two best limits on each side of the order book. Note that the second class of controls is more general than the first one.

The search of the k nearest neighbors, that arises when estimating the conditional expectations using the Qknn algorithm, is very time-consuming; especially in the considered market-making problem which is of dimension more than 10. The efficiency of Qknn then highly depends on the algorithm used to find the k nearest neighbors in high-dimension. Qknn algorithm has been implemented using the Fast Library for Approximate Nearest Neighbors algorithm (FLANN), introduced in [ML09] and already available in libraries in C++, Python, Julia and many other languages. This algorithm is based on tree methods. Note that recent algorithms based on graph also proved to perform well, and can also be used.

4.4.1 Case 1: The market maker only place orders at the best ask and best bid.

Denote by A1lim the class of controls where the placements of orders in allowed on the best ask and best bid exclusively. We implement the Qknn algorithm to compute the optimal strategy among those in A1lim. We then compared the optimal strategy with a naive strategy which consists in always placing one order at the best bid and one order at the best ask. The naive strategy is called 11 in the plots, and can be seen as a benchmark. The naive strategy is a good benchmark when the model for the intensities of order arrivals is symmetrical, i.e. the intensities for the bid and the ask sides are the same. Indeed, in this case, the market maker can expect to earn the spread in average.

Numerical results:

In Figure 4, we take constant intensities to model the limit and market orders arrivals, and linear intensity to model the cancel orders. In this setting, as we can see in the figure, the strategy computed using Qknn algorithm performs as well as the naive strategy. Note that, obviously, the market maker has to take enough points for the state quantization in order for Qknn algorithm to perform well. In figure 5, we plotted the P&L of the market maker when the latter compute the optimal strategy using only 6000

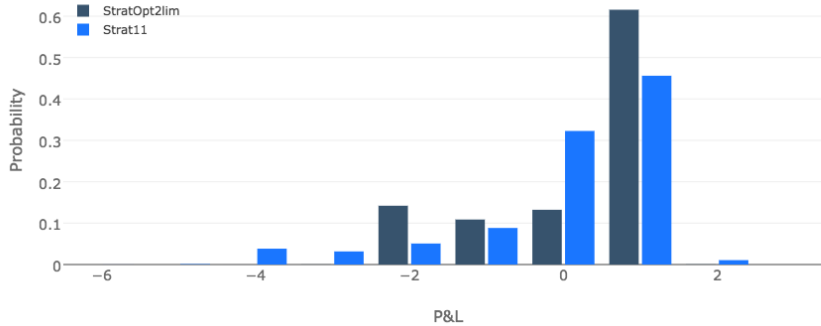


Figure 4 – Symmetrical intensities, Size of the grids: 100000. Short terminal time: $T=1$. Notice that the Qknn strategy reduces the variance of the P&L, but the expected wealth when following Qknn strategy (StratOpt2lim) is the same as the one following the naive strategy (Strat11).

points for the state space discretization, and for such a low number of points for the grid, Qknn algorithm performs poorly.

In Figure 6, we plotted the empirical histogram of the P&L of the market maker using the Qknn-estimated optimal strategy, computed with grids of size $N = 1000, 10000, 100000, 1000000$ for the state space discretization; and the empirical histogram of the P&L of the market maker using the naive strategy. One can see that the larger the size of the grids are, the better the Qknn-estimation of the optimal strategy is.

We plot in Figure 7 the results of simulations run taking a short terminal time $T=1$, and intensities that depend on the size of the queues. In this setting, notice that the naive strategy does not perform well anymore, but the Qknn algorithm still does well, when the market maker takes enough points for state space discretization.

In figure 7, we plot the P&L of the market maker following the Qknn strategy and the naive strategy, and we took the same parameters as in figure 6 to run the simulations expect from the terminal time that we set as $T=10$. As expected³, the expected wealth of the market maker is larger when terminal time is larger and when the latter follows the Qknn-estimated optimal strategy. Note that the expectation of the latter remains the same when she follows the naive strategy.

4.4.2 Case 2: the market maker place orders on the first two limits of the Orders Book

We extend the class of admissible controls to the ones where the market maker places order on the first two limits on the bid and ask sides of the order book. Denote by $A2lim$ the latter. We run simulations to test the Qknn algorithm on A2lim. In figure 8 and figure 9, we plot the empirical distributions of the P&L when the market maker follows the three different strategies:

³The value function for the market-making problem is by definition a non-decreasing function w.r.t. the time component

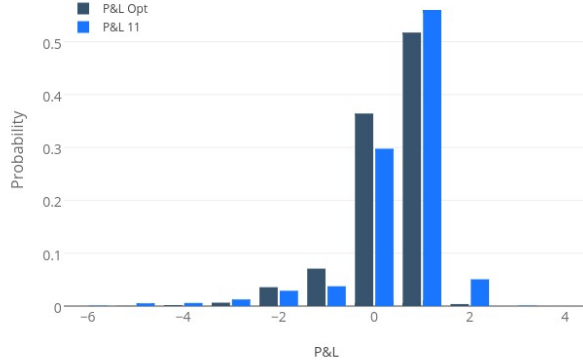


Figure 5 – Symmetrical and constant intensities. Size of the grids: 6000. The computed optimal strategy is less efficient than the naive strategy.

- Qknn-estimated optimal strategy among those in A2lim (PLOpt2lim).
- Qknn-estimated optimal strategy among those in A1lim (PLOpt1lim).
- naive strategy, i.e. always place orders on the best bid and best ask queues (PL11).

Note that the P&L of the market maker is always better when the class of admissible controls is extended, see figure 8, but in some models of order books, the extended set of controls does not improve the P&L, i.e. $\sup_{\alpha \in A2lim} V^\alpha = \sup_{\alpha \in A1lim} V^\alpha$.

5 Model extension to Hawkes Processes

We consider in this section a market maker who aims at maximizing a function of her terminal wealth, penalizing her inventory at terminal time T in the case where the orders arrivals are driven by Hawkes processes.

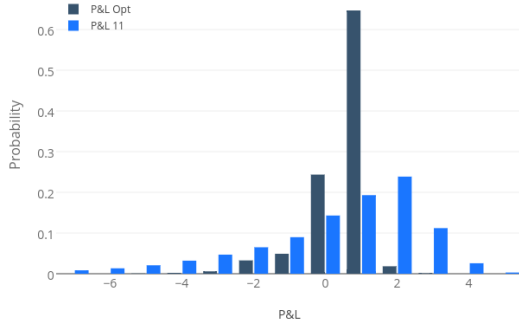
Let us first present the model with Hawkes processes for the LOB.

Model for the LOB:

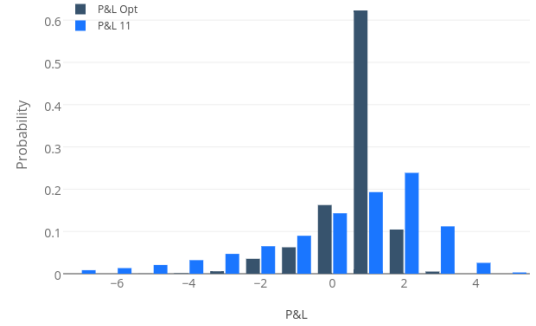
We assume that the order book receives limit, cancel, and market orders. We denote by L^+ (resp. L^-) the limit order arrivals process the ask (resp. bid) side; by C^+ (resp. C^-) the cancel order on the ask (resp. bid) side; and by M^+ (resp. M^-) the buy (resp. sell) market order arrivals processes. In this section, the limit orders arrivals are assumed to follow Hawkes processes dynamics, and moreover we assume the kernel to be exponential. The order arrivals are then modeled by a $(4K+2)$ -variate Hawkes process (N_t) with a vector of exogenous intensities λ_0 and exponential kernel ϕ , i.e. $\phi^{ij}(t) = \alpha^{ij} \beta_{ij} e^{\beta_{ij} t} \mathbf{1}_{t \geq 0}$. Note that in the presented model, the following holds:

(H λ) λ is assumed to be independent of the control.

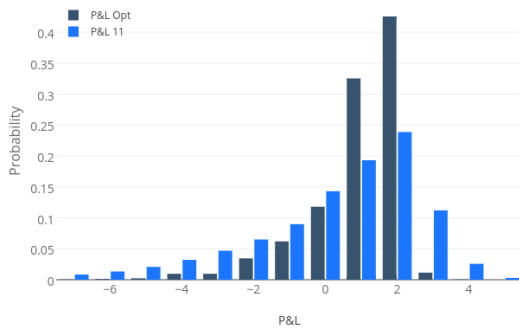
Denoting by $D = 4K + 2$ the dimension of (N_t) , the m^{th} component of the intensity λ of N_t writes, under



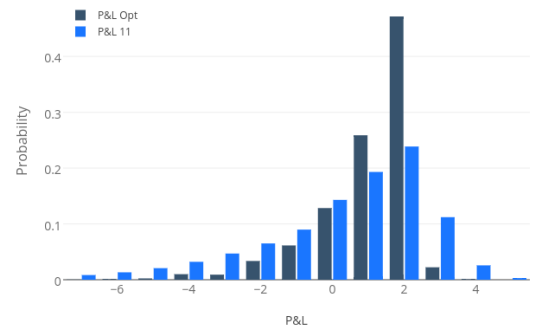
(a)



(b)



(c)



(d)

Figure 6 – P&L when the intensities λ^M , λ_i^L and λ_i^C depend on the state of the order book. Figure 6a shows the P&L of the market maker when following the Qknn-estimated optimal strategy computed with 1000 points for the state space discretization. Figure 6b shows the P&L when following the Qknn-estimated optimal strategy computed with 9000 points for the state space discretization. Figure 6c shows the P&L when following the Qknn-estimated optimal strategy computed with 100000 points for the state space discretization. Figure 6d shows the P&L when following the Qknn-estimated optimal strategy computed with 1000000 points for the state space discretization.

The reader can see that the market maker increases her expected terminal wealth by taking more and more points for the state space discretization. Also, the naive strategy is beaten when the intensities are state dependent.

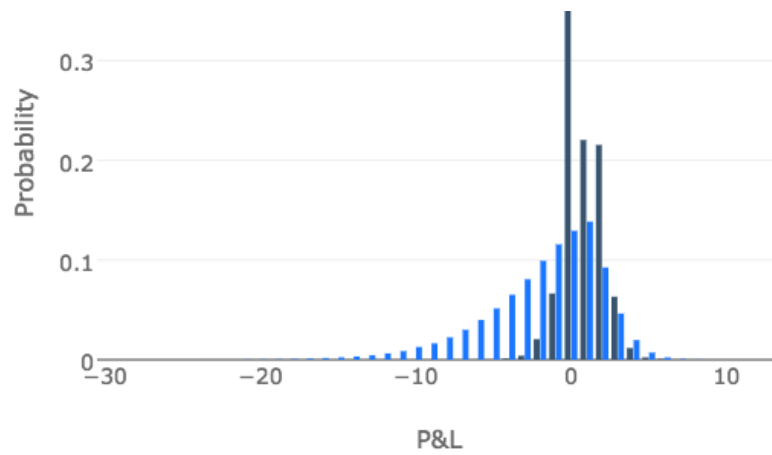


Figure 7 – P&L of the market maker following the optimal strategy and following the naive strategy 11. Symmetrical state dependent intensities. Long Terminal Time: $T=10$. Notice that the Qknn strategy does better than the naive strategy when the intensities are state dependent.

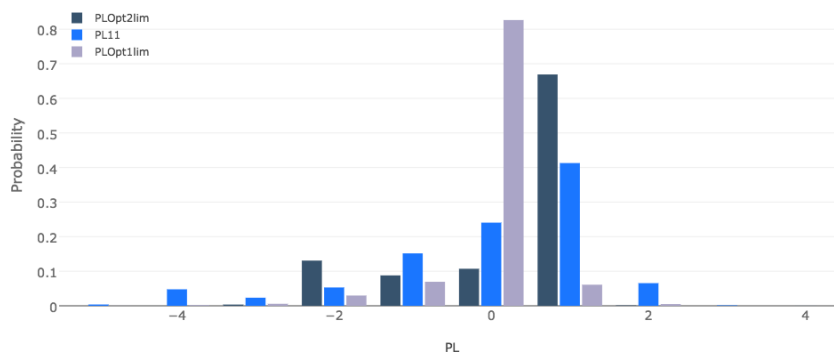


Figure 8 – P&L of the market maker who follows optimal strategies and the naive strategy (PL11). Short Terminal Time. asymmetrical intensities for the market order arrivals: the intensity for the buying market order process is taken higher than the one for the selling market order process. The wealth of the market maker is greater when she places orders on the two first limits of each sides of the order book, rather than when she places orders only on the best limits at the bid and ask sides.

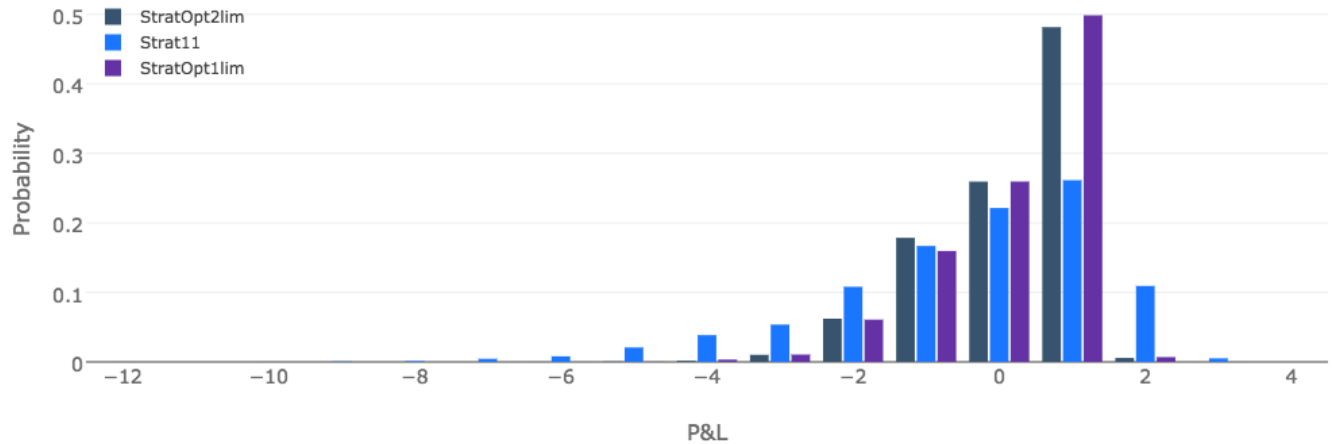


Figure 9 – P&L when following the optimal strategy or the naive strategy (PL11). Long Terminal Time. Symmetrical intensities for the arrival of market orders. 400000 points for the quantization. Notice that the Qknn strategy computed on the extended class of controls, i.e. order placements on the two first limits (StratOpt2lim), performs as well as the one computed on the original class of controls, i.e. order placements on the best-bid and best-ask (StratOpt1lim).

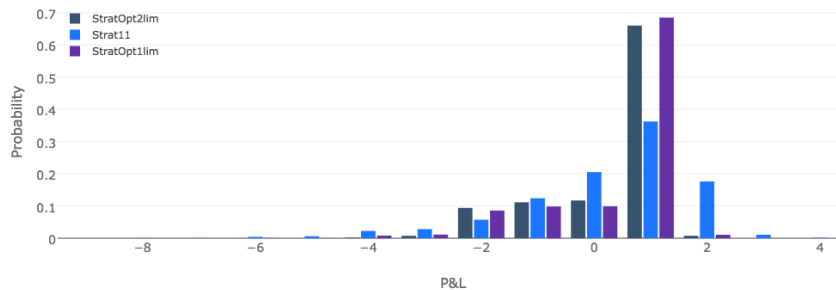


Figure 10 – P&L of the market maker who follows the optimal strategy and following the naive strategy 11. Long Terminal Time. Constant and symmetrical intensities for the arrivals of orders. Notice that the strategies computed by Qknn algorithm when taking A2lim performs as well as the one computed on the two best limits of the order book exclusively. Then, in this setting, placing orders only at the best-ask and best-bid seems to be the the optimal strategy.

(H λ):

$$\lambda_t^m = \lambda_0^m + \sum_{j=1}^D \alpha_{mj} \int_0^t e^{-\beta_{mj}(t-s)} dN_s^j, \quad \text{for } m = 1, \dots, D,$$

or equivalently:

$$d\lambda_t^m = \sum_{j=1}^D \alpha_{mj} \left[-\beta_{mj}(\lambda_t^m - \lambda_0^m) dt + \alpha_{mj} dN_t^j \right], \quad \text{for } m = 1, \dots, D,$$

with given initial conditions: $\lambda_0^m \in \mathbb{R}_+^*$ for $m = 1, \dots, D$. It is well-known that for this choice of intensity, the couple $(N_t, \lambda_t)_{t \geq 0}$ becomes Markovian. See e.g. Lemma 6 in [Mas98] for a proof of this result.

We can now rewrite the control problem (3.1) in the particular case where the order book is driven by Hawkes processes, there is no running reward, i.e. $f = 0$, and where the terminal reward G stands for the terminal wealth of the market maker penalized by her inventory. We then consider the following problem in this section:

$$V(t, \lambda, z) := \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t,z,\lambda}^{\alpha} [G(Z_T)], \quad (5.1)$$

where $G(z)$ denotes the wealth of the market maker when the controlled order book is at state z , plus a term of penalization of her inventory; and where \mathbb{A} is the set of the admissible controls, i.e. the predictable decisions taken by the market maker until a terminal time $T > 0$.

We now present the main result of this section.

Theorem 5.1. *V is characterized as the unique solution of the following HJB equation:*

$$\left\{ \begin{array}{l} f(T, z, \lambda) = G(z), \text{ for } z \in E \\ 0 = \frac{\partial f}{\partial t}(t, z, \lambda) - \sum_{m=1}^D \left[\sum_{j=1}^D \beta_{mj}(\lambda^m - \lambda_0^m) \frac{\partial f}{\partial \lambda^m}(t, z, \lambda) \right. \\ \quad \left. + \lambda^m \sup_{a \in A_z} [f(t, e_m^a(z), \lambda + \alpha_m) - f(t, z, \lambda)] \right], \\ \text{for } 0 \leq t < T, \text{ and } (t, z, \lambda) \in \mathbb{R}_+ \times E \times \mathbb{R}_+^*. \end{array} \right. \quad (5.2)$$

Moreover, V admits the following representation

$$V(t, z, \lambda) = \sup_{\alpha \in \mathbb{A}} \sum_{n=0}^{\infty} \mathbb{E}_{t,z,\lambda}^{\alpha} \left[1_{T_n \leq T} G(Z_{T_n}^{\alpha}) \exp \left\{ -|\lambda_0|(T - T_n) \right. \right. \\ \left. \left. + \sum_{m=1}^D \frac{\lambda_{T_n}^m - \lambda_0^m}{\sum_{j=1}^D \beta_{mj}} \left(e^{-\sum_{j=1}^D \beta_{mj}(T - T_n)} - 1 \right) \right\} \right], \quad (5.3)$$

where, for $n \geq 0$, T_n stands for the n^{th} jump time of Z after time t , and $(Z_{T_n}^{\alpha})_{n=0}^{\infty}$ is seen as a MDP controlled by $\alpha \in \mathbb{A}$; and where $\mathbb{E}_{t,z,\lambda}^{\alpha}[\cdot]$ stands for the expectation conditioned by $Z_t = z, \lambda_t = \lambda$ when the control α is followed.

Remark 5.1. V is characterized in (5.3) as the value function associated to an MDP with infinite horizon, where the instantaneous reward reads:

$$r(t, z, \lambda) = \mathbf{1}_{t \leq T} G(z) \exp \left\{ -|\lambda_0|_1(T-t) + \sum_{m=1}^D \frac{\lambda^m - \lambda_0^m}{\sum_{j=1}^D \beta_{mj}} \left(e^{-\sum_{j=1}^D \beta_{mj}(T-t)} - 1 \right) \right\},$$

where $|\cdot|_1$ denotes the $\mathbb{L}^1(\mathbb{R}^D)$ norm.

Proof: (of Theorem 5.1)

Step 1: Let us check that (5.3) holds, where V is defined as solution of (5.1).

We want to show that (5.10) is the expression of the maximal reward operator associated to the PDMDP (3.9) that we will define later. First notice that $(\lambda_t, Z_t)_t$ is a PDMDP, since $(\lambda_t, Z_t)_t$ is deterministic between two jumping times. We then aim at rewriting the expression of the value function defined in (5.1) as the value function associated to a infinite horizon control problem of the PDMDP $(\lambda_t, Z_t)_t$. To do so, we first notice that by conditioning on the time jumps we get:

$$\begin{aligned} V(t, z, \lambda) &= \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t, z, \lambda}^\alpha \left[G(Z_T^\alpha) \right] \\ &= \sup_{\alpha \in \mathbb{A}} \mathbb{E}_{t, z, \lambda}^\alpha \left[\sum_{n=0}^{\infty} \mathbf{1}_{T_n \leq T < T_{n+1}} G(Z_{T_n}^\alpha) \right] \\ &= \sup_{\alpha \in \mathbb{A}} \sum_{n=0}^{\infty} \mathbb{E}_{t, z, \lambda}^\alpha \left[\mathbf{1}_{T_n \leq T} G(Z_{T_n}^\alpha) \mathbb{P}(T - T_n \leq T_{n+1} - T_n | T_n) \right], \end{aligned} \quad (5.4)$$

where $(T_n)_n$ is the sequence of jump times of N . This process is a jump process with intensity $\mu_s = \sum_{m=1}^D \lambda_s^m$. Since it holds, conditioned to \mathcal{F}_{T_n} :

$$\mu_s = \sum_{m=1}^D (\lambda_{T_n}^m - \lambda_0^m) e^{-\sum_{j=1}^D \beta_{mj}(s-t)}, \quad \text{for } s \in [T_n, T_{n+1}),$$

then, we have:

$$\begin{aligned} \mathbb{P}(T_{n+1} - T_n \geq T - T_n | T_n) &= \int_{T-T_n}^{\infty} \mu_s e^{-\int_0^s \mu_u du} ds \\ &= \exp \left\{ -|\lambda_0|(T - T_n) + \sum_{m=1}^D \frac{\lambda_{T_n}^m - \lambda_0^m}{\sum_{j=1}^D \beta_{mj}} \left(e^{-\sum_{j=1}^D \beta_{mj}(T-T_n)} - 1 \right) \right\}. \end{aligned} \quad (5.5)$$

Plugging (5.5) into (5.4), the value function rewrites:

$$\begin{aligned} V(t, z, \lambda) &= \sup_{\alpha \in \mathbb{A}} \sum_{n=0}^{\infty} \mathbb{E}_{t, z, \lambda}^\alpha \left[\mathbf{1}_{T_n \leq T} G(Z_{T_n}^\alpha) \exp \left\{ -|\lambda_0|(T - T_n) \right. \right. \\ &\quad \left. \left. + \sum_{m=1}^D \frac{\lambda_{T_n}^m - \lambda_0^m}{\sum_{j=1}^D \beta_{mj}} \left(e^{-\sum_{j=1}^D \beta_{mj}(T-T_n)} - 1 \right) \right\} \right], \end{aligned} \quad (5.6)$$

which completes the step 1. The r.h.s of (5.6) can be seen as the value function of an infinite horizon control problem associated to the PDMDP.

Step 2: Let us show that V is the unique solution to (5.2).

Notice first that the solutions to the following HJB equation

$$\begin{cases} G(z) = f(T, z, \lambda) \\ 0 = \frac{\partial f}{\partial t} - \sum_{m=1}^D \sum_{j=1}^D \beta_{mj} (\lambda^m - \lambda_0^m) \frac{\partial f}{\partial \lambda^m} + \lambda^m \sup_{a \in A_z} [f(t, e_m^a(z), \lambda + \alpha_m) - f(t, z, \lambda)], \end{cases} \text{ for } 0 \leq t < T.$$

are the fixed points of the operator $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2$ where \mathcal{T}_1 and \mathcal{T}_2 are defined as follows:

$$\mathcal{T}_1 : F \mapsto f \text{ solution of } \begin{cases} \frac{\partial f}{\partial t} - \sum_{m=1}^D \sum_{j=1}^D \beta_{mj} (\lambda^m - \lambda_0^m) \frac{\partial f}{\partial \lambda^m} = F(t, z, \lambda) \\ f(T, z, \lambda) = G(z), \end{cases}$$

and:

$$\mathcal{T}_2 : f \mapsto - \sum_{m=1}^D \lambda^m \sup_{a \in A_z} [f(t, e_m^a(z), \lambda + \alpha_m) - f(t, z, \lambda)].$$

We now use the characteristic method to rewrite the image of \mathcal{T}_1 .

Let us take function F , and define $f = \mathcal{T}_1(F)$. Let us fix $t \in [0, T]$ and $\lambda \in (\mathbb{R}_+)^D$, and denote by g the function $g(s, z) = f(s, z, \lambda_s^1, \dots, \lambda_s^D)$ where, for $m = 1, \dots, D$, $s \mapsto \lambda_s^m$ is a differentiable function defined on $[t, T]$ as solution to the following ODE:

$$\begin{cases} \frac{d\lambda_s^m}{ds} = - \sum_{j=1}^D \beta_{mj} (\lambda_s^m - \lambda_0^m), & \text{for all } t < s \leq T, \\ \lambda_t^m = \lambda^m. \end{cases} \quad (5.7)$$

For $m = 1, \dots, D$, basic theory on ODE provides existence and uniqueness of a solution to (5.7), which is given by:

$$\lambda_s^m = \lambda_0^m + (\lambda^m - \lambda_0^m) e^{-\sum_{j=1}^D \beta_{mj} (s-t)}, \quad \text{for } s \in [t, T], \text{ and } m = 1, \dots, D.$$

Since $\frac{\partial g}{\partial s} = \frac{\partial f}{\partial s} + \sum_{m=1}^D \frac{d\lambda_s^m}{ds} \frac{\partial f}{\partial \lambda^m}$, then $g(t, z) = G(z) - \int_t^T F(s, z, \lambda_s) ds$, which finally leads to the following expression of $\mathcal{T}_1(F)$:

$$\mathcal{T}_1(F) = f(t, z, \lambda) = G(z) - \int_t^T F(s, z, \lambda_s) ds. \quad (5.8)$$

Replacing F by $\mathcal{T}_2(f)$ in (5.8), we get that f is fixed point of $\mathcal{T}_1 \circ \mathcal{T}_2$ if and only if:

$$f(t, \lambda, z) + \sum_{m=1}^D \int_t^T \lambda_s^m f(s, z, \lambda_s) ds = G(z) - \sum_{m=1}^D \int_t^T \lambda_s^m \sup_{a \in A_z} f(s, e_m^a(z), \lambda_s + \alpha_m) ds.$$

Notice

$$\frac{\partial f(s, \lambda_s, z) e^{-\sum_{j=1}^D \int_t^s \lambda_u^j du}}{\partial s} = - \sum_{m=1}^D \lambda_s^m e^{-\sum_{j=1}^D \int_t^s \lambda_u^j du} \sup_{a \in A_z} f(s, e_m^a(z), \lambda_s + \alpha_m),$$

so that:

$$\begin{aligned}
f(t, \lambda, z) &= G(z) e^{-\sum_{m=1}^D \int_t^T \lambda_s^m ds} + \sum_{m=1}^N \int_t^T \lambda_s^m e^{-\int_t^s \lambda_u du} \sup_{a \in A_z} f(s, e_m^a(z), \lambda_s + \alpha_m) ds \\
&= G(z) e^{-\sum_{m=1}^D \int_t^T \lambda_s^m ds} + \sup_{a \in A_z} \mathbb{E}_{t, \lambda, z}^a [f(T_1, Z_1, \lambda_{T_1} + \alpha_m)],
\end{aligned} \tag{5.9}$$

where T_1 is the first jump time of N larger than t , we denote $Z_1 = Z_{T_1}$. Equation (5.9) shows that the fixed point of $\mathcal{T}_1 \circ \mathcal{T}_2$ is characterized as the fixed point of the operator \mathcal{T} defined for any smooth enough function f by:

$$\mathcal{T}(f) = G(z) e^{-\sum_{m=1}^D \int_t^T \lambda_s^m ds} + \sup_{a \in A_z} \mathbb{E}_{t, \lambda, z}^a [f(T_1, Z_1, \lambda_{T_1} + \alpha_m)], \tag{5.10}$$

where $\mathbb{E}_{t, \lambda, z}^a[\cdot]$ stands for the expectation conditioned by the events $\lambda_t = \lambda$ and $Z_t = z$, when decision a is taken at time t . We recognize here the maximal reward operator of the value function defined in (5.6). Basic theory on PDMDP shows that the maximal reward operator \mathcal{T} admits V as unique fixed point, which completes step 2. \square

Appendix A From uncontrolled to controlled intensity

Remind that the results state in Section 3 hold when assuming that the intensities of the orders arrivals are uncontrolled. In particular, we assumed in this section that the market maker has no influence on the next exogenous event that will occur. This can be seen as a weak assumption if the market maker is a small player, but never holds in the case where the latter is a large player.

In this section, we show how to alleviate Assumption (**Harrivals2**) by rewriting the initial control problems (3.1) with controlled intensities as a control problems with uncontrolled intensities under a new (controlled) probability measure. The results and proofs in this section are inspired from [Bré81].

Consider a LOB which can receive at any time limit, cancel, and market orders. Denote by L^+ (res. L^-) the limit sell (resp. buy) order arrival process, received on the ask (resp. buy) side. Denote by C^+ (resp. C^-) the cancel order on the ask (resp. bid) side. Denote by M^+ (resp. M^-) the buy (resp. sell) market order process. The orders arrivals process is then a $(4K + 2)$ dimensional process. Recall that E is the state space of the order book. The order book is modeled by a jump process $Z : [0, T] \rightarrow E$ such that the order arrivals processes have uncontrolled stochastic intensities $\lambda^i(\underline{a}, \underline{b})$, for $i = 1, \dots, 4K + 2$, that only depend on the bid and ask sides, i.e. $(\underline{a}, \underline{b})$ of the order book under \mathbb{P} . We underline that, by assumption, the intensities are uncontrolled under \mathbb{P} .

Let us fix $(\alpha_t)_{0 \leq t \leq T} \in \mathbb{A}$ an admissible control, i.e. a predictive process w.r.t. the natural filtration $(\mathcal{F}_t)_{t > 0}$ generated by the uncontrolled orders arrivals processes under \mathbb{P} .

(HarrivalsL): We assume in this section that the intensities are Lipschitz and bounded, i.e. there exist $[\lambda]_L > 0$ and $\|\lambda\|_\infty > 0$ such that

$$|\lambda^i(\underline{a}, \underline{b}) - \lambda^i(\underline{a}', \underline{b}')| \leq [\lambda]_L (|\underline{a} - \underline{a}'| + |\underline{b} - \underline{b}'|),$$

and

$$\lambda^i(\underline{a}, \underline{b}) \leq \|\lambda\|_\infty, \quad \text{for } i = 1, \dots, 4K + 2,$$

for $\underline{a}, \underline{a}' \in \mathbb{N}^K$ and $\underline{b}, \underline{b}' \in (-\mathbb{N})^K$.

We want to define the probability \mathbb{P}_α as the absolutely continuous probability w.r.t. \mathbb{P} , which Radon-Nikodym derivative writes:

$$L_t^\alpha = \frac{d\mathbb{P}_\alpha}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \prod_{i=1}^{4K+2} \prod_{n=1}^{\infty} \mu_{\alpha, T_n^i}^i \mathbf{1}_{T_n^i \leq t} \exp \left\{ \int_0^t (1 - \mu_s^i) \lambda_s^i ds \right\}, \quad \text{for } 0 \leq t \leq T, \quad (\text{A.1})$$

where for $i = 1, \dots, 4K + 2$, we denote by $\mu_{\alpha, T_n^i}^i$ the quotient of the controlled intensity at time T_n^i of the n^{th} jump of the i^{th} process and the uncontrolled intensity, i.e. denoting by \underline{a}^α and \underline{b}^α the ask and bid where the market order's orders are counted, we define:

$$\mu_\alpha^i = \frac{\lambda(\underline{a}^\alpha, \underline{b}^\alpha)}{\lambda(\underline{a}, \underline{b})}.$$

Remark A.1. Under **(HarrivalsL)**, it holds:

$$|\lambda(\underline{a}^\alpha, \underline{b}^\alpha) - \lambda(\underline{a}, \underline{b})| \leq [\lambda]_L M, \quad \text{for } \underline{a}, \underline{a}' \in \mathbb{N}^K, \quad \text{and } \underline{b}, \underline{b}' \in (-\mathbb{N})^K, \quad (\text{A.2})$$

where we remind that M stands for the limit number of orders that can be hold by the market maker at the same time in the LOB.

Remark A.2. From Remark A.1, it is straightforward to see that μ_α^i is bounded under **(HarrivalsL)**, and moreover:

$$\mu_\alpha^i \leq 1 + \frac{[\lambda]_L M}{\lambda_{\min}}, \quad \text{for } i = 1, \dots, 4K + 2,$$

where we denote $\lambda_{\min} = \inf_{i=1, \dots, 4K+2} \inf_{z \in E} \lambda^i(z)$, and assume the latter to be strictly positive. Note that the bound is uniform w.r.t. the control and the state variables.

Proposition A.1. For every $\alpha \in \mathbb{A}$, it holds under **(HarrivalsL)**:

$$\mathbb{E}[L_T^\alpha] = 1, \quad (\text{A.3})$$

which implies in particular that \mathbb{P}_α is well-defined.

Moreover, the orders arrivals admit the controlled intensities $\lambda(\underline{a}^\alpha, \underline{b}^\alpha)$, for $i = 1, \dots, 4K + 2$, under \mathbb{P}_α , where we remind that \underline{a}^α and \underline{b}^α stand for the vector of orders on the ask and the bid sides, where the market maker's orders are counted.

Proof. We divided the proof of Proposition A.1 into two steps.

Step 1: We show (A.3).

Let us fix $\alpha \in \mathbb{A}$ and write the integral representation of $(L_t^\alpha)_{0 \leq t \leq T}$: for $t \in [0, T]$,

$$L_t^\alpha = 1 + \sum_{i=1}^{4K+2} \int_0^t L_{s-}^\alpha (\mu_{\alpha, s}^i - 1) d\tilde{M}_s^i, \quad \text{for } i = 1, \dots, 4K + 2, \quad (\text{A.4})$$

where \tilde{M} stands for the local martingale which dynamic writes: $d\tilde{M}_s^i = dN_s^i - \lambda^i(\underline{a}_s, \underline{b}_s) ds$. It is then sufficient to show that

$$\mathbb{E} \left[\int_0^T L_{s-}^\alpha (\mu_{\alpha,s}^i - 1) \lambda_s^i ds \right] < +\infty, \quad \text{for } i = 1, \dots, 4K + 2, \quad (\text{A.5})$$

to get that the $\left(\int_0^t L_{s-}^\alpha (\mu_{\alpha,s}^i - 1) d\tilde{M}_s^i \right)_{0 \leq t \leq T}$ are martingales for $i = 1, \dots, 4K + 2$ (as proved e.g. in [Br 81]), and complete the proof of Step 1, using (A.4).

Plugging (A.2) into (A.1), we get:

$$L_s \leq \|\mu\|_\infty^{A_s} e^{[\lambda]_L M T}, \quad \text{for } 0 \leq s \leq T, \quad (\text{A.6})$$

where we denote $\|\mu\|_\infty := 1 + \frac{[\lambda]_L M}{\lambda_{\min}}$, and where $(A_t)_{t \in [0, T]}$ stands for the sum of all the order arrivals process up to time t , for $t \in [0, T]$.

Moreover, as stated in Remark A.1, we have for all $i = 1, \dots, 4K + 2$:

$$\begin{aligned} |(\mu_s^i - 1) \lambda_s^i(z)| &= |\lambda(\underline{a}^\alpha, \underline{b}^\alpha) - \lambda(\underline{a}, \underline{b})| \\ &\leq [\lambda]_L M. \end{aligned} \quad (\text{A.7})$$

Plugging (A.7) and (A.6) into the l.h.s. of (A.5), we get:

$$\mathbb{E} \left[\int_0^T L_{s-}^\alpha (\mu_{\alpha,s}^i - 1) \lambda_s^i ds \right] \leq \int_0^T \mathbb{E} [\|\mu\|_\infty^{A_s}] e^{[\lambda]_L M T} [\lambda]_L M ds \quad (\text{A.8})$$

Notice that the intensity of A is bounded by $\|\lambda\|_\infty$, under **(HarrivalsL)**, so that:

$$\begin{aligned} \mathbb{E} [\|\mu\|_\infty^{A_s}] &\leq e^{-\|\lambda\|_\infty s} \sum_{n=0}^{+\infty} \frac{\|\mu\|_\infty^n (\|\lambda\|_\infty s)^n}{n!} \\ &\leq \exp \{ \|\lambda\|_\infty T (\|\mu\|_\infty - 1) \}, \quad \text{for } s \in [0, T], \end{aligned} \quad (\text{A.9})$$

Combining (A.8) and (A.9), we can prove that (A.5) holds, which completes the proof of Step 1.

Step 2: We refer to the T3 Theorem in Chapter VI of [Br 81] for a proof of the second assertion in Proposition A.1. \square

Appendix B Dynamics of the controlled order book (simplified version)

In this section, we give the expressions for the dynamics of the controlled order book process (Z_t) . The market maker control has been simplified to a couple (la_t, lb_t) , where $la = 1$ (resp. 0) if the market maker holds (does not hold) a sell order at the best ask limit, and $lb = 1$ (resp. 0) if the market maker holds (does not hold) a buying order at the best bid limit. So to speak, the market maker considers to place orders at the best ask limit or at the best bid limit exclusively. In the numerical simulations that we run, we also had to calculate the dynamics of (Z_t) for the set of generalized controls in which the market

maker is allowed to post orders on the two first limits at the bid and at the ask side. The expression of the dynamics for the generalized controls are very similar to the ones for the simplified controls.

To understand the dynamics of the rank of the orders of the market maker, we need a model for the cancellation of orders. Suppose for example that the market maker holds an order whose rank is na in the queue, with $na < a_{A^{-1}(0)}$. Suppose that the cancel process $L_{A^{-1}(0)}^{C+}$ jumps. Then two scenarios can occur:

- If the rank of the canceled order is greater than the one of the market maker, then na_t stays constant.
- If the rank of the canceled order is smaller than the one of the market maker, then $na_t = na_{t-} + 1$.

Model:

We consider a Bernoulli variable X^a with parameter:
$$\underbrace{\frac{na - 1}{a_{A^{-1}(0)}}}_{\alpha} \delta_1 + \underbrace{\frac{a_{A^{-1}(0)} + 1 - na}{a_{A^{-1}(0)}}}_{\beta} \delta_0.$$

We assume that the canceled order is in front of the market maker's order in the queue if $X^a = 1$, and behind it if $X^a = 0$.

We proceed for the bid side as we just did for the ask side. We consider a random variable X^b following a Bernoulli law with parameter:
$$\underbrace{\frac{nb - 1}{|b_{B^{-1}(0)}|}}_{\alpha} \delta_1 + \underbrace{\frac{|b_{B^{-1}(0)}| + 1 - nb}{|b_{B^{-1}(0)}|}}_{\beta} \delta_0.$$

B.1 Dynamics of X_t et Y_t

The dynamic of the amount hold by the market maker on a no-interest-bearing account $(X_t)_{t \in \mathbb{R}_+}$ is as follows:

$$dX_t = la_t p a_{t-} \mathbf{1}_{\{na_{t-}=1\}} dM_t^+ - lb_t p b_{t-} \mathbf{1}_{\{nb_{t-}=1\}} dM_t^-$$

The market maker's inventory (Y_t) follows the dynamic:

$$dY_t = -la_t \mathbf{1}_{\{na_{t-}=1\}} dM_t^+ + \mathbf{1}_{\{nb_{t-}=1\}} lb_t dM_t^-$$

where:

- $\hat{a} = \sup\{a_i : \sum_{j=1}^{i-1} a_j = 0\}$ et $\hat{b} = \sup\{b_i : \sum_{j=1}^{i-1} b_j = 0\}$
- M_t^\pm are Cox processes with intensities λ^{M^\pm}

B.2 Dynamics of the a_t et b_t

We remind that a_i is the number of orders located i ticks away from the best buy order.

We denote by \mathcal{J} the *shift* operator that re-index a side of the book when an event occurred on the opposite side.

$\mathcal{J}^{L_i^-}, i \in \{1, \dots, B^{-1}(0)\}$ is the shift operator that shifts the bid side due to the jump of a L_i^+ for $i \in \{0, K\}$. We get:

$$\mathcal{J}^{L_i^-}(\underline{a}) = \left(a_{i+1}, \dots, a_K, \underbrace{a_\infty, \dots, a_\infty}_{i \text{ times}} \right)$$

Dynamics of a_i :

$$\begin{aligned} da_i &= (1 - lb_t) dL_i^+ + lb_t dL_{i-(A^{-1}(0)-rb_{t-})}^+ + \left[(1 - lb_t) + lb_t \mathbb{1}_{\{nb_{t-} > 1\}} \right] \left(\mathcal{J}^{M^-}(a_i) - a_i \right) dM^-(t) \\ &\quad - (1 - lb_t) dC_i^+ - lb_t dC_{i-(A^{-1}(0)-rb_{t-})}^+ \\ &\quad + (1 - la_t) \left[- \mathbb{1}_{\{i=A^{-1}(0)\}} dM_t^+ + (\mathcal{J}^{C^-}(a_i) - a_i) dC_{A^{-1}(0)}^- \right. \\ &\quad \quad \left. + (1 - lb_t) \sum_{j=1}^{A^{-1}(0)-1} (\mathcal{J}_{0,0}^{L_j^-}(a_i) - a_i) dL_j^-(t) \right. \\ &\quad \quad \left. + lb_t \sum_{j=1}^{A^{-1}(0)-1} (\mathcal{J}_{0,1}^{L_j^-}(a_i) - a_i) dL_j^-(t) \right] \\ &\quad + la_t \left[- \mathbb{1}_{\{na_{t-} > 1\}} \mathbb{1}_{\{i=A^{-1}(0)\}} dM_t^+ + (\mathcal{J}^{C^-}(a_i) - a_i) dC_{ra_{t-}}^- \right. \\ &\quad \quad \left. + lb_t \sum_{j=1}^{ra_{t-}-1} (\mathcal{J}_{1,1}^{L_j^-}(a_i) - a_i) dL_j^-(t) \right. \\ &\quad \quad \left. + (1 - lb_t) \sum_{j=1}^{ra_{t-}-1} (\mathcal{J}_{1,0}^{L_j^-}(a_i) - a_i) dL_j^-(t) \right] \end{aligned}$$

with \mathcal{J} such that:

$$\mathcal{J}^{C^-}(a_i) = \begin{cases} a_\infty & \text{si } i > B^{-1}(1) - B^{-1}(0) + K \\ a_{i-(B^{-1}(1)-B^{-1}(0))} & \text{si } i > (B^{-1}(1) - B^{-1}(0)) \\ 0 & \text{si } i \leq B^{-1}(1) - B^{-1}(0) \end{cases}$$

$$\mathcal{J}^{M^-}(a_i) = \begin{cases} a_{i-(B^{-1}(1)-B^{-1}(0))} & \text{si } i > (B^{-1}(1) - B^{-1}(0)) \\ 0 & \text{si } i \leq B^{-1}(1) - B^{-1}(0) \end{cases}$$

$$\mathcal{J}_{0,0}^{L_j^-}(a_i) = \begin{cases} a_{i+j} & \text{si } i+j \leq K \\ 0 & \text{si } i+j > K \end{cases}$$

$$\mathcal{J}_{0,1}^{L_j^-}(a_i) = \begin{cases} a_{i+rb_{t-}-j} & \text{si } i+rb_{t-}-j \leq K \\ a_\infty & \text{si } i+rb_{t-}-j > K \end{cases}$$

$$\mathcal{J}_{1,0}^{L_j^-}(a_i) = \begin{cases} a_{i+ra_{t-}-j} & \text{si } i+ra_{t-}-j \leq K \\ a_\infty & \text{si } i+ra_{t-}-j > K \end{cases}$$

$$\mathcal{J}_{1,1}^{L_j^-}(a_i) = \begin{cases} 0 & \text{si } i + rb_{t-} - j < 0 \\ a_{i+rb_{t-}-j} & \text{si } i + rb_{t-} - j \leq K \\ a_\infty & \text{si } i + rb_{t-} - j > K \end{cases}$$

We remind that b_i is the number of buy order located i ticks away from the best sell order.

Dynamics of b_i :

$$\begin{aligned} db_i = & -(1 - la_t)dL_i^- - la_t dL_{i-(A^{-1}(0)-ra_{t-})}^- + \left[(1 - la_t) + la_t \mathbb{1}_{\{na_{t-} > 1\}} \right] \left(\mathcal{J}^{M^+}(b_i) - b_i \right) dM^+(t) \\ & + (1 - la_t) dC_i^- + la_t dC_{i-(A^{-1}(0)-ra_{t-})}^- \\ & + (1 - lb_t) \left[\mathbb{1}_{\{i=A^{-1}(0)\}} dM_t^- + (\mathcal{J}^{C^+}(b_i) - b_i) dC_{A^{-1}(0)}^+ \right. \\ & \quad + (1 - la_t) \sum_{j=1}^{B^{-1}(0)-1} (\mathcal{J}_{0,0}^{L_j^+}(b_i) - b_i) dL_j^+(t) \\ & \quad \left. + la_t \sum_{j=1}^{B^{-1}(0)-1} (\mathcal{J}_{1,0}^{L_j^+}(b_i) - b_i) dL_j^+(t) \right] \\ & + lb_t \left[\mathbb{1}_{\{nb_{t-} > 1\}} \mathbb{1}_{\{i=A^{-1}(0)\}} dM_t^- + (\mathcal{J}^{C^+}(b_i) - b_i) dC_{rb_{t-}}^+ \right. \\ & \quad + la_t \sum_{j=1}^{rb_{t-}-1} (\mathcal{J}_{1,1}^{L_j^+}(b_i) - b_i) dL_j^+(t) \\ & \quad \left. + (1 - la_t) \sum_{j=1}^{rb_{t-}-1} (\mathcal{J}_{0,1}^{L_j^+}(b_i) - b_i) dL_j^+(t) \right] \end{aligned}$$

with \mathcal{J} the shift operators:

$$\begin{aligned} \mathcal{J}^{C^+}(b_i) &= \begin{cases} b_\infty & \text{si } i + A^{-1}(1) - A^{-1}(0) > K \\ b_{i-(A^{-1}(1)-A^{-1}(0))} & \text{si } i > (A^{-1}(1) - A^{-1}(0)) \\ 0 & \text{si } i \leq A^{-1}(1) - A^{-1}(0) \end{cases} \\ \mathcal{J}^{M^+}(b_i) &= \begin{cases} b_\infty & \text{si } i + A^{-1}(1) - A^{-1}(0) > K \\ b_{i-(A^{-1}(1)-A^{-1}(0))} & \text{si } i > (A^{-1}(1) - A^{-1}(0)) \\ 0 & \text{si } i \leq A^{-1}(1) - A^{-1}(0) \end{cases} \\ \mathcal{J}_{0,0}^{L_j^+}(b_i) &= \begin{cases} b_{i+j} & \text{si } i + j \leq K \\ 0 & \text{si } i + j > K \end{cases} \\ \mathcal{J}_{1,0}^{L_j^+}(b_i) &= \begin{cases} b_{i-j+A^{-1}(0)} & \text{si } i - j + A^{-1}(0) \leq K \\ b_\infty & \text{si } i - j + A^{-1}(0) > K \end{cases} \end{aligned}$$

$$\mathcal{J}_{0,1}^{L_j^+}(b_i) = \begin{cases} b_{i+rb_{t-}-j} & \text{si } i + rb_{t-} - j \leq K \\ b_\infty & \text{si } i + rb_{t-} - j > K \end{cases}$$

$$\mathcal{J}_{1,1}^{L_j^+}(b_i) = \begin{cases} 0 & \text{si } i + rb_{t-} - j < 0 \\ b_{i+rb_{t-}-j} & \text{si } i + rb_{t-} - j \leq K \\ b_\infty & \text{si } i + rb_{t-} - j > K \end{cases}$$

B.3 Dynamics of na_t and nb_t

Dynamics of na_t :

X^a has been introduced in part B. It models whether the canceled order is behind or in front of the market maker's order in the queue.

We get:

$$\begin{aligned} dna_t = la_t & \left[-X^a \left((1-lb_t) dC_{A^{-1}(0)}^+(t) + lb_t dC_{rb_{t-}}^+ \right) + \left(-\mathbb{1}_{\{na_{t-}>1\}} + (a_{A^{-1}(0)} + 1 - na_{t-}) \mathbb{1}_{\{na_{t-}=1\}} \right) dM_t^+ \right. \\ & \left. + (2 - na_{t-}) \left\{ (1-lb_t) \sum_{i=1}^{ra_{t-}(t)-1} dL_i^+ + lb_t \sum_{i=1}^{ra_{t-}(t)-(A^{-1}(0)-rb_{t-})-1} dL_i^+ \right\} \right] \\ & + (1-la_t) \left[\left(a_{A^{-1}(0)} \mathbb{1}_{\{a_{A^{-1}(0)}>1\}} + (a_{A^{-1}(1)} + 1) \mathbb{1}_{\{a_{A^{-1}(0)}=1\}} - na_{t-} \right) dM_t^+ \right. \\ & \left. + lb_t \left[(2 - na_{t-}) \sum_{j=1}^{rb_{t-}-1} dL_j^+ + (a_{A^{-1}(0)} + 2 - na_{t-}) dL_{rb_{t-}}^+ \right. \right. \\ & \left. \left. + (a_{A^{-1}(0)} + 1 - na_{t-}) \sum_{j=rb_{t-}+1}^K (dL_j^+ + dC_j^+) \right. \right. \\ & \left. \left. + \left(a_{A^{-1}(0)} \mathbb{1}_{\{a_{A^{-1}(0)}>1\}} + (a_{A^{-1}(1)} + 1) \mathbb{1}_{\{a_{A^{-1}(0)}=1\}} - na_{t-} \right) dC_{rb_{t-}}^+ \right] \right. \\ & + (1-lb_t) \left[(2 - na_{t-}) \sum_{j=1}^{B^{-1}(0)-1} dL_j^+ + (a_{A^{-1}(0)} + 2 - na_{t-}) dL_{A^{-1}(0)}^+ \right. \\ & \left. + (a_{A^{-1}(0)} + 1 - na_{t-}) \sum_{j=B^{-1}(0)+1}^K (dL_j^+ + dC_j^+) \right. \\ & \left. + \left(a_{A^{-1}(0)} \mathbb{1}_{\{a_{A^{-1}(0)}>1\}} + (a_{A^{-1}(1)} + 1) \mathbb{1}_{\{a_{A^{-1}(0)}=1\}} - na_{t-} \right) dC_{A^{-1}(0)}^+ \right] \\ & \left. + (a_{A^{-1}(0)} + 1 - na_{t-}) \left[dM_t^- + \sum_{j=1}^K (dL_j^- + dC_j^-) \right] \right] \end{aligned}$$

$$\begin{aligned}
dna_t &= (la_t == 0)(-1 - na_{t-}) \left[dM_t^+ + dM_t^- \right. \\
&\quad + (lb_{t!} = 1) \sum_{i=1}^K (dL_i^+ + dL_i^- + dC_i^+ + dC_i^-) \\
&\quad \left. + (lb_t == 1) \left[\sum_{i=0}^{K-(A^{-1}(0)-rb_{t-})} (dL_i^+ + dC_i^+) + \sum_{i=1}^K (dL_i^- + dC_i^-) \right] \right] \\
&\quad + \mathbf{1}_{la_t=1} \left\{ \left(\mathbf{1}_{na_{t-}=-1} + \mathbf{1}_{na_{t-}!=-1} \mathbf{1}_{ra_{t-} > A^{-1}(0)} \right) \left[(a_{A^{-1}(0)} - na_{t-}) \left[dM_t^+ + \mathbf{1}_{lb_{t!}=1} dC_{rb_{t-}}^+ + \mathbf{1}_{lb_{t!}=1} dC_{A^{-1}(0)}^+ \right] \right. \right. \\
&\quad \left. \left. + (a_{A^{-1}(0)} + 1 - na_{t-}) \left[\mathbf{1}_{lb_{t!}=1} \sum_{i=1}^K (dL_i^+ + dC_i^+) + \mathbf{1}_{lb_t=1} \sum_{i=1}^{K-(A^{-1}(0)-rb_{t-})} (dL_i^+ + dC_i^+) \right] \right] \right\}
\end{aligned}$$

Dynamics of nb_t :

$$\begin{aligned}
dnb_t &= lb_t \left[-X^b \left((1 - la_t) dC_{B^{-1}(0)}^-(t) + la_t dC_{ra_{t-}}^- \right) + \left(-\mathbf{1}_{\{nb_{t-} > 1\}} + (|b_{B^{-1}(0)}| + 1 - nb_{t-}) \mathbf{1}_{\{nb_{t-}=1\}} \right) dM_t^- \right. \\
&\quad \left. + (2 - nb_{t-}) \left\{ (1 - la_t) \sum_{i=1}^{rb_{t-}(t)-1} dL_i^- + la_t \sum_{i=1}^{rb_{t-}(t)-(B^{-1}(0)-ra_{t-})-1} dL_i^- \right\} \right] \\
&\quad + (1 - lb_t) \left[\left(|b_{A^{-1}(0)}| \mathbf{1}_{\{|b_{B^{-1}(0)}| > 1\}} + (|b_{B^{-1}(1)}| + 1) \mathbf{1}_{\{|b_{B^{-1}(0)}|=1\}} - nb_{t-} \right) dM_t^- \right. \\
&\quad \left. + la_t \left[(2 - nb_{t-}) \sum_{j=1}^{ra_{t-}-1} dL_j^- + (|b_{A^{-1}(0)}| + 2 - nb_{t-}) dL_{ra_{t-}}^- \right. \right. \\
&\quad \left. \left. + (|b_{A^{-1}(0)}| + 1 - nb_{t-}) \sum_{j=ra_{t-}+1}^K (dL_j^- + dC_j^-) \right. \right. \\
&\quad \left. \left. + \left(|b_{A^{-1}(0)}| \mathbf{1}_{\{|b_{B^{-1}(0)}| > 1\}} + (|b_{B^{-1}(1)}| + 1) \mathbf{1}_{\{|b_{B^{-1}(0)}|=1\}} - nb_{t-} \right) dC_{ra_{t-}}^- \right] \\
&\quad + (1 - la_t) \left[(2 - nb_{t-}) \sum_{j=1}^{B^{-1}(0)-1} dL_j^- + (|b_{B^{-1}(0)}| + 2 - nb_{t-}) dL_{A^{-1}(0)}^- \right. \\
&\quad \left. + (|b_{B^{-1}(0)}| + 1 - nb_{t-}) \sum_{j=B^{-1}(0)+1}^K (dL_j^- + dC_j^-) \right. \\
&\quad \left. + \left(|b_{A^{-1}(0)}| \mathbf{1}_{\{|b_{B^{-1}(0)}| > 1\}} + (|b_{B^{-1}(1)}| + 1) \mathbf{1}_{\{|b_{B^{-1}(0)}|=1\}} - nb_{t-} \right) dC_{A^{-1}(0)}^- \right] \\
&\quad + (|b_{A^{-1}(0)}| + 1 - nb_{t-}) \left[dM_t^+ + \sum_{j=1}^K (dL_j^+ + dC_j^+) \right]
\end{aligned}$$

B.4 Dynamics of pa and pb

Dynamics of $(pa_t)_t$:

Denoting by δ the tick, we have:

$$\begin{aligned}
dP_t^A &= \delta(1 - la_t) \left[\left((A^{-1}(1) - ra_{t-}) dM^+(t) \right. \right. \\
&\quad + lb_t \left[- \sum_{i=1}^{rb_{t-}-1} \left[rb_{t-} - (A^{-1}(0) - ra_{t-}) - j \right] dL_i^+(t) + (A^{-1}(0) - ra_{t-}) \sum_{j=rb_{t-}}^K dL_j^+ \right. \\
&\quad \left. \left. + (A^{-1}(1) - ra_{t-}) dC_{rb_{t-}}^+ + \sum_{j=rb_{t-}+1}^K (A^{-1}(0) - ra_{t-}) dC_j^+ \right] \right. \\
&\quad + (1 - lb_t) \left[- \sum_{i=1}^{A^{-1}(0)-1} (ra_{t-} - j) dL_i^+(t) + (A^{-1}(0) - ra_{t-}) \sum_{j=A^{-1}(0)}^K dL_j^+ \right. \\
&\quad \left. \left. + (A^{-1}(1) - ra_{t-}) dC_{A^{-1}(0)}^+ + \sum_{j=A^{-1}(0)+1}^K (A^{-1}(0) - ra_{t-}) dC_j^+ \right] \right. \\
&\quad \left. + \mathbb{1}_{\{ra_{t-} \neq A^{-1}(0)\}} (A^{-1}(0) - ra_{t-}) \left(dM_t^- + \sum_{j=1}^K dL_t^- + \sum_{j=1}^K dC_t^- \right) \right] \\
&+ \delta la_t \left[(A^{-1}(0) - r_t^A) dM^+(t) \right. \\
&\quad - lb_t \sum_{i=1}^{rb_{t-} - (A^{-1}(0) - ra_{t-}) - 1} \left(ra_{t-} - (j + A^{-1}(0) - rb_{t-}) \right) dL_i^+(t) \\
&\quad \left. - (1 - lb_t) \sum_{i=1}^{ra_{t-} - 1} (ra_{t-} - j) dL_i^+(t) \right]
\end{aligned}$$

Dynamics of $(pb_t)_t$:

$$\begin{aligned}
dP_t^B = & -\delta(1-lb_t) \left[\left((B^{-1}(1) - rb_{t-}) dM^-(t) \right. \right. \\
& + la_t \left[- \sum_{i=1}^{ra_{t-}-1} \left[ra_{t-} - (B^{-1}(0) - rb_{t-}) - j \right] dL_i^-(t) + (B^{-1}(0) - rb_{t-}) \sum_{j=ra_{t-}}^K dL_i^- \right. \\
& \left. \left. + (B^{-1}(1) - rb_{t-}) dC_{rb_{t-}}^- + \sum_{j=ra_{t-}+1}^K (B^{-1}(0) - rb_{t-}) dC_j^- \right] \right. \\
& + (1-la_t) \left[- \sum_{i=1}^{B^{-1}(0)-1} (rb_{t-} - j) dL_i^-(t) + (B^{-1}(0) - rb_{t-}) \sum_{j=B^{-1}(0)}^K dL_i^- \right. \\
& \left. \left. + (B^{-1}(1) - rb_{t-}) dC_{A^{-1}(0)}^- + \sum_{j=A^{-1}(0)+1}^K (B^{-1}(0) - rb_{t-}) dC_j^- \right] \right. \\
& \left. \left. + \mathbb{1}_{\{rb_{t-} \neq A^{-1}(0)\}} (A^{-1}(0) - rb_{t-}) \left(dM_t^+ + \sum_{j=1}^K dL_j^+ + \sum_{j=1}^K dC_j^+ \right) \right] \right. \\
& - \delta lb_t \left[(B^{-1}(0) - r_t^B) dM^-(t) \right. \\
& - la_t \sum_{i=1}^{ra_{t-} - (B^{-1}(0) - rb_{t-}) - 1} \left(rb_{t-} - (j + B^{-1}(0) - ra_{t-}) \right) dL_i^-(t) \\
& \left. - (1-la_t) \sum_{i=1}^{rb_{t-}-1} (rb_{t-} - j) dL_i^-(t) \right]
\end{aligned}$$

B.5 Dynamics of ra and rb

We remind that ra_t denotes the number of ticks between the market maker's order and the best buy order in the order book. We assumed in this simplified control problem that the market maker is allowed to place no more than one order on the best ask and best bid limits. So ra and rb are vectors of size 1 here.

Dynamics of ra :

$$\begin{aligned}
d ra_t = & la_t \left[\mathbf{1}_{\{na=1\}} \left(A^{-1}(0) - r_t^A \right) dM_t^+ \right. \\
& + (1 - lb_t) \sum_{i=1}^{ra_{t-}} \left(i - ra_{t-} \right) dL_i^+ + lb_t \sum_{i=1}^{rb_{t-} - (B^{-1}(0) - ra_{t-}) - 1} \left(i + B^{-1}(0) - rb_{t-} - ra_{t-} \right) dL_i^+ \\
& + \sum_{i=1}^{ra_{t-} - 1} \left(i - ra_{t-} \right) dL_i^- + \left(B^{-1}(1) - B^{-1}(0) \right) dC_{ra_{t-}}^- \\
& \left. \left(\left((1 - lb_t) + lb_t \mathbf{1}_{\{nb_{t-} > 1\}} \right) \left[B^{-1}(1) - B^{-1}(0) \right] \right) dM_t^- \right] \\
& + (l - la_t) \left[lb_t \left[\sum_{j=1}^{rb_{t-} - 1} \left(j + B^{-1}(0) - rb_{t-} - ra_{t-} \right) dL_j^+ + \left(A^{-1}(0) - ra_{t-} \right) \sum_{j=rb_{t-}}^K dL_j^+ \right. \right. \\
& \quad \left. \left(A^{-1}(1) - ra_{t-} \right) dC_{rb_{t-}}^+ + \left(A^{-1}(0) - ra_{t-} \right) \sum_{j=rb_{t-} + 1}^K dC_j^+ \right] \\
& + (1 - lb_t) \left[\sum_{j=1}^{B^{-1}(0) - 1} \left(j - ra_{t-} \right) dL_j^+ + \left(A^{-1}(0) - ra_{t-} \right) \sum_{j=B^{-1}(0)}^K \left(dC_j^+ + dC_j^- \right) \right. \\
& \quad \left. + \left(A^{-1}(1) - ra_{t-} \right) dC_{A^{-1}(0)}^+ + \sum_{j=A^{-1}(0) + 1}^K \left(A^{-1}(0) - ra_{t-} \right) dC_j^+ \right. \\
& \quad \left. + \sum_{j=1}^{A^{-1}(0) - 1} \left(j - ra_{t-} \right) dL_j^- + \sum_{j=A^{-1}(0)}^K \left(A^{-1}(0) - ra_{t-} \right) dL_j^- \right. \\
& \quad \left. + \left(B^{-1}(1) - ra_t \right) dC_{B^{-1}(0)}^- + \left(A^{-1}(0) - ra_{t-} \right) \sum_{j=A^{-1}(0) + 1}^K dC_j^- \right. \\
& \quad \left. + \left[\left((1 - lb_t) + lb_t \mathbf{1}_{\{nb_{t-} > 1\}} \right) \left(B^{-1}(1) - ra_{t-} \right) + lb_t \mathbf{1}_{\{nb_{t-} = 1\}} \left(B^{-1}(0) - ra_{t-} \right) \right] dM_t^- \right. \\
& \quad \left. + \left(A^{-1}(1) - ra_{t-} \right) dM_t^+ \right]
\end{aligned}$$

We remind that rb_t is the number of ticks between the market maker's order and the best sell order in the order book.

Dynamics of rb :

$$\begin{aligned}
d rb_t = & lb_t \left[\mathbf{1}_{\{nb=1\}} \left(B^{-1}(0) - rb \right) dM_t^- \right. \\
& + (1 - la_t) \sum_{i=1}^{rb_{t-}-1} \left(i - rb_{t-} \right) dL_i^- + la_t \sum_{i=1}^{ra_{t-} - (A^{-1}(0) - rb_{t-}) - 1} \left(i + A^{-1}(0) - ra_{t-} - rb_{t-} \right) dL_i^- \\
& + \sum_{i=1}^{rb_{t-}-1} \left(i - rb_{t-} \right) dL_i^+ + \left(A^{-1}(1) - A^{-1}(0) \right) dC_{rb_{t-}}^+ \\
& \left. \left(\left((1 - la_t) + la_t \mathbf{1}_{\{na_{t-} > 1\}} \right) \left[A^{-1}(1) - A^{-1}(0) \right] \right) dM_t^+ \right] \\
& + (l - lb_t) \left[la_t \left[\sum_{j=1}^{ra_{t-}-1} \left(j + A^{-1}(0) - ra_{t-} - rb_{t-} \right) dL_j^- + \sum_{j=ra_{t-}}^K \left(B^{-1}(0) - rb_{t-} \right) dL_j^- \right. \right. \\
& \quad \left. \left(B^{-1}(1) - rb_{t-} \right) dC_{ra_{t-}}^- + \sum_{j=ra_{t-}+1}^K \left(B^{-1}(0) - rb_{t-} \right) dC_j^- \right] \\
& + (1 - la_t) \left[\sum_{j=1}^{A^{-1}(0)-1} \left(j - rb_{t-} \right) dL_j^- + \sum_{j=A^{-1}(0)}^K \left(B^{-1}(0) - rb_{t-} \right) dL_j^- \right. \\
& \quad \left. + \left(B^{-1}(1) - rb_{t-} \right) dC_{B^{-1}(0)}^- + \sum_{j=B^{-1}(0)+1}^K \left(B^{-1}(0) - rb_{t-} \right) dC_j^- \right. \\
& + \sum_{j=1}^{B^{-1}(0)-1} \left(j - rb_{t-} \right) dL_j^+ + \sum_{j=B^{-1}(0)}^K \left(B^{-1}(0) - rb_{t-} \right) dL_j^+ \\
& + \left(A^{-1}(1) - rb_{t-} \right) dC_{A^{-1}(0)}^+ + \left(B^{-1}(0) - rb_{t-} \right) \sum_{j=B^{-1}(0)+1}^K dC_j^+ \\
& \left. + \left[\left((1 - la_t) + la_t \mathbf{1}_{\{na_{t-} > 1\}} \right) \left(A^{-1}(1) - rb_{t-} \right) + la_t \mathbf{1}_{\{na_{t-} = 1\}} \left(A^{-1}(0) - rb_{t-} \right) \right] dM_t^+ \right. \\
& \left. + \left(B^{-1}(1) - rb_{t-} \right) dM_t^- \right]
\end{aligned}$$

Appendix C Proof of Theorem 4.1 and Corollary 4.1

We divided the proofs of Theorem 4.1 and Corollary 4.1 into several Lemmas that we state and prove now.

Lemma C.1 aims at bounding the projection error. It relies on [GKKW02], see p.93, as well as Zador's theorem, stated in Section D for the sake of completeness.

Lemma C.1. Assume $d \geq 3$, and take $K = M^{d+2}$ points for the optimal quantization of ε_n , then it holds under $(\mathbf{H}\mu)$ and (\mathbf{HF}) , as $M \rightarrow +\infty$,

$$\varepsilon_n^{proj} = \mathcal{O}\left(\frac{1}{M^{1/d}}\right), \quad (\text{C.1})$$

where we remind that $\varepsilon_n^{proj} := \sup_{a \in A} \|\text{Proj}_{n+1}(F(X_n, a, \hat{\varepsilon}_n)) - F(X_n, a, \hat{\varepsilon}_n)\|_2$ stands for the average projection error.

Proof. Let us take $\eta > 0$, and observe that

$$\begin{aligned} \mathbb{P}\left(\left|\text{Proj}_{n+1}[F(X_n, a, \hat{\varepsilon}_{n+1})] - F(X_n, a, \hat{\varepsilon}_{n+1})\right|^2 > \eta\right) &= \mathbb{E}\left[\prod_{m=1}^M \mathbb{E}\left[\mathbf{1}_{|X_{n+1}^{i,(m)} - F(X_n, a, \hat{\varepsilon}_{n+1})| > \sqrt{\eta}} \middle| X_n, \hat{\varepsilon}_{n+1}\right]\right] \\ &= \mathbb{E}\left[\left(1 - \mu[B(F(X_n, a, \hat{\varepsilon}_{n+1}), \sqrt{\eta})]\right)^M\right], \end{aligned}$$

where for all $x \in E$ and $\eta > 0$, $B(x, \eta)$ denote the ball of center x and radius η . Since $x \mapsto (1 - x)^M$ is M -Lipschitz, we get by application of Zador's theorem:

$$\begin{aligned} \mathbb{P}\left(\left|\text{Proj}_{n+1}[F(X_n, a, \hat{\varepsilon}_{n+1})] - F(X_n, a, \hat{\varepsilon}_{n+1})\right|^2 > \eta\right) &\leq M[F]_L[\mu]_L \|\hat{\varepsilon}_{n+1} - \varepsilon_{n+1}\|_2 + \mathbb{E}\left[\left(1 - \mu(B(F(X_n, a, \varepsilon_{n+1}), \sqrt{\eta}))\right)^M\right] \\ &= \frac{M[F]_L[\mu]_L}{K^{1/d}} + \mathbb{E}\left[\left(1 - \mu(B(F(X_n, a, \varepsilon_{n+1}), \sqrt{\eta}))\right)^M\right] + \mathcal{O}\left(\frac{M}{K^{1/d}}\right), \end{aligned}$$

as the number of points for the quantization of the exogenous noise K goes to $+\infty$, and where M stands for the size of the grids Γ_n .

Let us introduce $A_1, \dots, A_{N(\eta)}$, a cubic partition of $\text{Supp}(\mu)$, which is bounded under $(\mathbf{H}\mu)$, such that for all $j = 1, \dots, N(\eta)$, A_j has diameter η . Also, Notice that there exists $c > 0$, which only depends on $\text{Supp}(\mu)$, such as

$$N(\eta) \leq \frac{c}{\eta^d}. \quad (\text{C.2})$$

If $x \in A_j$, then $A_j \subset B(x, \eta)$, therefore:

$$\begin{aligned} \mathbb{E}\left[\left(1 - \mu(B(X_n, \eta))\right)^M\right] &= \sum_{j=1}^{N(\eta)} \int_{A_j} \left(1 - \mu(B(x, \eta))\right)^M \mu(dx) \\ &\leq \sum_{j=1}^{N(\eta)} \int_{A_j} \left(1 - \mu(A_j)\right)^M \mu(dx). \end{aligned} \quad (\text{C.3})$$

Also notice that:

$$\sum_{j=1}^{N(\eta)} \mu(A_j) \left(1 - \mu(A_j)\right)^M \leq \sum_{j=1}^{N(\eta)} \max_z z(1-z)^M \leq \frac{e^{-1}N(\eta)}{M}. \quad (\text{C.4})$$

Combining (C.3) and (C.4) leads to

$$\mathbb{E} \left[(1 - \mu(B(X_n, \eta)))^M \right] \leq \frac{e^{-1}N(\eta)}{M}. \quad (\text{C.5})$$

Let $L = 2\|\mu\|_\infty$ stands for the diameter of the support of μ . We then get, as $M \rightarrow +\infty$,

$$\begin{aligned} & \mathbb{E} \left[\left| \text{Proj}_{n+1}[F(X_n, a, \hat{\epsilon}_{n+1})] - F(X_n, a, \hat{\epsilon}_{n+1}) \right|^2 \right] \\ &= \int_0^\infty \mathbb{P} \left(\left| \text{Proj}_{n+1}[F(X_n, a, \hat{\epsilon}_{n+1})] - F(X_n, a, \hat{\epsilon}_{n+1}) \right|^2 > \eta \right) d\eta \\ &\leq \int_0^{L^2} \frac{M[F]_L[\mu]_L}{K^{2/d}} + \mathbb{P} \left(\left| \text{Proj}_{n+1}[F(X_n, a, \hat{\epsilon}_{n+1})] - F(X_n, a, \epsilon_{n+1}) \right| > \sqrt{\eta} \right) d\eta \\ &= \int_0^{L^2} \min \left(1, \frac{e^{-1}N(\sqrt{\eta})}{M} \right) d\eta + \mathcal{O} \left(\frac{M}{K^{1/d}} \right) \\ &= \int_0^{L^2} \min \left(1, \frac{c\eta^{-d/2}}{eM} \right) d\eta + \mathcal{O} \left(\frac{M}{K^{1/d}} \right) \\ &= \int_0^{(c/(eM))^{(2/d)}} 1 d\eta + \int_{(c/(eM))^{(2/d)}}^{L^2} \frac{c\eta^{-d/2}}{eM} d\eta + \mathcal{O} \left(\frac{M}{K^{1/d}} \right) \\ &= \frac{\tilde{c}^2}{M^{2/d}} + \mathcal{O} \left(\frac{M}{K^{1/d}} \right), \end{aligned} \quad (\text{C.6})$$

where \tilde{c} is defined as $\tilde{c} := \sqrt{\frac{d}{d-2}} \left(\frac{c}{e}\right)^{1/d}$, and where we used (C.5) and (C.2) to go from the second to the third line. It remains to take $K = M^{d+1}$ points for the optimal quantization of the exogenous noise, and then take square root of equality (C.6), in order to derive (C.1). \square

Lemma C.2. *Assume $d \geq 3$, take $K = M^{d+2}$ points for the optimal quantization of ϵ_n , and let $x \in E$. Then it holds under (H μ) and (HF), as $M \rightarrow +\infty$:*

$$\varepsilon_n^{proj}(x) = \mathcal{O} \left(\frac{1}{M^{1/d}} \right),$$

where $\varepsilon_n^{proj}(x)$, defined as $\varepsilon_n^{proj}(x) := \sup_{a \in A} \|\text{Proj}_{n+1}(F(x, a, \hat{\epsilon}_n)) - F(x, a, \hat{\epsilon}_n)\|_2$, stands for the later-projection error at state x .

Proof. Following the same steps as those used to prove Lemma C.1, we show that:

$$\begin{aligned} & \mathbb{P} \left(\left| \text{Proj}_{n+1}[F(x, a, \hat{\epsilon}_{n+1})] - F(x, a, \hat{\epsilon}_{n+1}) \right|^2 > \eta \right) \\ &= \frac{M[F]_L[\mu]_L}{K^{1/d}} + \mathbb{E} \left[\left(1 - \mu(B(F(x, a, \epsilon_{n+1}), \sqrt{\eta})) \right)^M \right] + \mathcal{O} \left(\frac{M}{K^{1/d}} \right), \end{aligned}$$

as $K \rightarrow +\infty$, and moreover,

$$\mathbb{E} \left[\left(1 - \mu(B(F(x, a, \varepsilon_{n+1}), \sqrt{\eta})) \right)^M \right] \leq \frac{e^{-1}N(\eta)}{M},$$

holds, which is enough to complete the proof of Lemma C.2. \square

Lemma C.3. *Under (HF), for $n = 0, \dots, N$ there exists constant $[\hat{V}_n^Q]_L > 0$ such that for $x, x' \in E$, it holds as $M \rightarrow \infty$:*

$$|\hat{V}_n^Q(x) - \hat{V}_n^Q(x')| \leq [\hat{V}_n^Q]_L |x - x'| + \mathcal{O}\left(\frac{1}{M^{1/d}}\right). \quad (\text{C.7})$$

Moreover, following bounds holds on $[\hat{V}_n^Q]_L$, for $n = 0, \dots, N$:

$$\begin{cases} [\hat{V}_N^Q]_L & \leq [g]_L \\ [\hat{V}_n^Q]_L & \leq [f]_L + [F]_L [\hat{V}_{n+1}^Q]_L, \end{cases} \quad \text{for } n = 0, \dots, N-1. \quad (\text{C.8})$$

Proof. Let us show that by induction that \hat{V}_N^Q is Lipschitz. First, notice that (C.7) holds at terminal time $n = N$, if one define $[\hat{V}_N^Q]_L$ as $[\hat{V}_N^Q]_L = [g]_L$. Let us take $x, x' \in E$. Assume $|\hat{V}_{n+1}^Q(x) - \hat{V}_{n+1}^Q(x')| \leq [\hat{V}_{n+1}^Q]_L |x - x'| + \mathcal{O}\left(\frac{1}{M^{1/d}}\right)$ holds for some $n = 0, \dots, N-1$. Let us show that

$$|\hat{V}_n^Q(x) - \hat{V}_n^Q(x')| \leq [\hat{V}_n^Q]_L |x - x'| + \mathcal{O}\left(\frac{1}{M^{1/d}}\right),$$

where $[\hat{V}_n^Q]_L$ is defined in (C.8). Notice that, by the dynamic programming principle and the triangular inequality, it holds:

$$\begin{aligned} |\hat{V}_n^Q(x) - \hat{V}_n^Q(x')| & \leq [f]_L |x - x'| \\ & \quad + \sup_a \mathbb{E}_n^a \left[\left| \hat{V}_{n+1}^Q(\text{Proj}_{n+1}(F(x, a, \hat{\varepsilon}_{n+1}))) - \hat{V}_{n+1}^Q(\text{Proj}_{n+1}(F(x', a, \hat{\varepsilon}_{n+1}))) \right| \right] \\ & \leq [f]_L |x - x'| + [\hat{V}_{n+1}^Q]_L \sup_a \mathbb{E} \left[|\text{Proj}_{n+1}(F(x, a, \hat{\varepsilon}_{n+1})) - F(x, a, \hat{\varepsilon}_{n+1})| \right] \\ & \quad + \mathcal{O}\left(\frac{1}{M^{1/d}}\right) \\ & \leq ([f]_L + [\hat{V}_{n+1}^Q]_L [F]_L) |x - x'| + \mathcal{O}\left(\frac{1}{M^{1/d}}\right) \\ & \leq [\hat{V}_n^Q]_L |x - x'| + \mathcal{O}\left(\frac{1}{M^{1/d}}\right), \end{aligned}$$

which completes the proof of (C.7). \square

We now proceed to the proof of Theorem 4.1.

Proof. (of Theorem 4.1) Combining inequality $|u_1 + u_2 + u_3|^2 \leq 3(|u_1|^2 + |u_2|^2 + |u_3|^2)$ that holds for all $u_1, u_2, u_3 \in \mathbb{R}$ with inequality $\left| \sup_{i \in I} a_i - \sup_{i \in I} b_i \right| \leq \sup_{i \in I} |a_i - b_i|$ that holds for all families $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ of reals, and all subset I of \mathbb{R} , we have:

$$\begin{aligned} \|\hat{V}_n^Q(X_n) - V_n(X_n)\|_2^2 &\leq 3 \mathbb{E} \left[\sup_{a \in A} \mathbb{E}_{n, X_n} \left| \hat{V}_{n+1}^Q(\text{Proj}_{n+1}(F(X_n, a, \hat{\varepsilon}_{n+1}))) - \hat{V}_{n+1}^Q(F(X_n, a, \hat{\varepsilon}_{n+1})) \right|^2 \right. \\ &\quad + \sup_{a \in A} \mathbb{E}_{n, X_n} \left| \hat{V}_{n+1}^Q(F(X_n, a, \hat{\varepsilon}_{n+1})) - \hat{V}_{n+1}^Q(F(X_n, a, \varepsilon_{n+1})) \right|^2 \\ &\quad \left. + \sup_{a \in A} \mathbb{E}_{n, X_n} \left| \hat{V}_{n+1}^Q(F(X_n, a, \varepsilon_{n+1})) - V_{n+1}(F(X_n, a, \varepsilon_{n+1})) \right|^2 \right] \end{aligned}$$

where \mathbb{E}_{n, X_n} stands for the expectation conditioned by the state X_n at time n . It holds as $M \rightarrow +\infty$, using Lemma C.3:

$$\begin{aligned} \|\hat{V}_n^Q(X_n) - V_n(X_n)\|_2^2 &\leq 3 \left[\hat{V}_n^Q \right]_L \mathbb{E} \left[\sup_a \mathbb{E}_{n, X_n} [|\text{Proj}_{n+1}(F(X_n, a, \hat{\varepsilon}_{n+1})) - F(X_n, a, \hat{\varepsilon}_{n+1})|^2] \right. \\ &\quad \left. + \sup_a \mathbb{E}_{n, X_n} [|F(X_n, a, \hat{\varepsilon}_{n+1}) - F(X_n, a, \varepsilon_{n+1})|^2] \right] \\ &\quad + 3 \|r\|_\infty \mathbb{E} [|\hat{V}_{n+1}^Q(X_{n+1}) - V_{n+1}(X_{n+1})|^2] + \left(\frac{1}{M^{1/d}} \right) \end{aligned} \quad (\text{C.9})$$

Under **(HF)**, (C.9) can then be rewritten as:

$$\begin{aligned} \|\hat{V}_n^Q(X_n) - V_n(X_n)\|_2^2 &\leq 3 \left[\hat{V}_n^Q \right]_L \left([F]_L^2 (\epsilon_n^Q)^2 + (\epsilon_n^{proj})^2 \right) \\ &\quad + 3 \|r\|_\infty \|\hat{V}_{n+1}^Q(X_{n+1}) - V_{n+1}(X_{n+1})\|_2^2 + \left(\frac{1}{M^{1/d}} \right). \end{aligned}$$

(4.5) then follows by induction, which completes the proof of Theorem 4.1. \square

Proof. (of Corollary 4.1)

Corollary 4.1 is straightforward by plugging the bound for the projection error provided by Lemma C.1 and the one of the quantization error provided by the Zador's Theorem into (4.5). \square

Appendix D Zador's Theorem

Theorem D.1 (Zador's theorem). *Let us take $n = 0, \dots, N$, and denote by K the number of points for the quantization of the exogenous noise ε_n .*

Assume that $\mathbb{E} [|\varepsilon_n|^{2+\eta}] < +\infty$ for some $\eta > 0$. Then, there exists a universal constant $C > 0$ such that:

$$\lim_{M \rightarrow +\infty} \left(M^{\frac{1}{d}} \|\hat{\varepsilon}_n - \varepsilon_n\|_2 \right) = C$$

Proof. We refer to [GL00] for a proof of Theorem D.1. \square

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