



# RANDOM WALKS ON BRATTELI DIAGRAMS

Jean N Renault

► To cite this version:

| Jean N Renault. RANDOM WALKS ON BRATTELI DIAGRAMS. 2017. hal-01514270

**HAL Id: hal-01514270**

**<https://hal.science/hal-01514270>**

Preprint submitted on 25 Apr 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# RANDOM WALKS ON BRATTELI DIAGRAMS

JEAN RENAULT

ABSTRACT. In a 1989 article, A. Connes and E. J. Woods made a connection between hyperfinite von Neumann algebras and Poisson boundaries of time dependent random walks. I will explain this connection and will present two theorems given there: the description of an almost periodic state on a hyperfinite von Neumann algebra (due to A. Connes) and the ergodic decomposition of a Markov measure via harmonic functions (a classical result in J. Neveu 64). The crux of the first theorem is a model for conditional expectations on finite dimensional  $C^*$ -algebras. Our proof of the second theorem hinges on the notion of cotransition probability.

## 1. INTRODUCTION.

This article is based on

A. Connes and E. J. Woods, Hyperfinite von Neumann algebras and Poisson boundaries of time dependent random walks, *Pacific J. Math.* 137 (1989), no 2, 225-243.

It contains the statement of the two theorems which I am going to describe:

- (i) the description of an arbitrary state on a hyperfinite von Neumann algebra (due to A. Connes);
- (ii) the ergodic decomposition of a Markov measure via harmonic functions (a classical result in J. Neveu 64).

## 2. CONDITIONAL EXPECTATIONS ON FINITE DIMENSIONAL $C^*$ -ALGEBRAS

Although the material of this section is not new, I have not found a reference for Theorem 2.8 below which gives a complete invariant for a faithful conditional expectation on a finite dimensional  $C^*$ -algebra. The description of an inclusion of finite dimensional  $C^*$ -algebras in terms of matrix units, which is a part of the theorem, and its graphical description, have been given by O. Bratteli in his fundamental paper [3] (see Proposition 1.7 and section 1.8). The graphical description of a conditional expectation on a finite dimensional  $C^*$ -algebra appears in section 3 (iii) of [8] but without much detail. The only originality may be our systematic use of Cartan subalgebras. The corresponding groupoid models are also known as path models or tail equivalence relations.

We first give the ingredients and the recipe to construct a faithful conditional expectation  $Q$  of a finite dimensional  $C^*$ -algebra  $\underline{M}$  onto a sub- $C^*$ -algebra  $M$ . Then, we show that every faithful conditional expectation is obtained by this construction.

We first recall that we can associate to an equivalence relation  $R$  on a finite set  $X$  a finite dimensional  $C^*$ -algebra  $M = C^*(R)$ : its elements are the functions  $f : R \rightarrow \mathbb{C}$

---

1991 *Mathematics Subject Classification.* Primary 22A22; Secondary 54H20, 43A65, 46L55.

*Key words and phrases.* Bratteli diagrams, hyperfinite von Neumann algebras.

(here  $R \subset X \times X$  is the graph of the equivalence relation), the product is the matrix multiplication and the involution is the usual complex conjugate of a matrix. It has a canonical matrix unit  $(e(x, y))_{(x, y) \in R}$  indexed by  $R$ .

We consider now finite sets  $X, V, E, \underline{V}$  equipped with surjections  $r : X \rightarrow V$ ,  $s : E \rightarrow V$ ,  $r : E \rightarrow \underline{V}$ . We define

$$\underline{X} = \{(x, a) \in X \times E : r(x) = s(a)\}$$

and the equivalence relations:

$$R = \{(x, y) \in X \times X : r(x) = r(y)\}$$

$$\underline{R} = \{(xa, yb) \in \underline{X} \times \underline{X} : r(a) = r(b)\}.$$

We construct the  $C^*$ -algebras  $C^*(R)$  and  $C^*(\underline{R})$ . The map  $j : C^*(R) \rightarrow C^*(\underline{R})$  given by

$$j(f)(xa, yb) = \begin{cases} f(x, y) & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

identifies  $C^*(R)$  to a subalgebra of  $C^*(\underline{R})$ . We shall make this identification and view the elements of  $C^*(R)$  as functions on  $\underline{R}$ . Then  $(V, E, \underline{V})$  is the graph of the inclusion. We leave as an exercise to the reader the proof of the following lemma:

**Lemma 2.1.** *Let  $R'$  be the following equivalence relation on  $E$ :*

$$R' = \{(a, b) \in E \times E : s(a) = s(b), r(a) = r(b)\}$$

*Then the map  $k : C^*(R') \rightarrow C^*(\underline{R})$  given by*

$$k(g)(xa, yb) = \begin{cases} g(a, b) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

*identifies  $C^*(R')$  to the commutant of  $C^*(R)$  in  $C^*(\underline{R})$ .*

**Definition 2.1.** A transition probability on the graph  $E$  is a function  $p : E \rightarrow \mathbf{R}_+^*$  such that for all  $v \in V$ ,  $\sum_{s(c)=v} p(c) = 1$ .

**Proposition 2.2.** *Let  $X, V, E, \underline{V}$  be as above and let  $p$  be a transition probability on  $E$ . The map  $Q : C^*(\underline{R}) \rightarrow C^*(R)$  defined by*

$$Q(f)(x, y) = \sum_c p(c) f(xc, yc),$$

*where the sum is over all edges  $c \in E$  originating from the common range of  $x$  and  $y$ , is a faithful conditional expectation onto  $C^*(R)$ .*

*Proof.* This is a straightforward verification. □

We are going to prove a converse to the proposition: namely all faithful conditional expectations  $Q : \underline{M} \rightarrow M$ , where  $M$  is a sub- $C^*$ -algebra of a finite dimensional  $C^*$ -algebra  $\underline{M}$ , are of that form. We first recall the notion of Cartan subalgebra which will be our main tool. It is an algebraic characterization of the canonical abelian subalgebra  $C(X)$  of the  $C^*$ -algebra  $C^*(R)$  of an equivalence relation  $R$  on  $X$  as above.

**Definition 2.2.** An abelian subalgebra  $A$  of a von Neumann algebra  $M$  is called a Cartan subalgebra if it is maximal self-adjoint, regular and there exists a faithful normal conditional expectation  $P : M \rightarrow A$ . We then say that  $(M, A)$  is a Cartan pair.

Regularity means that the normalizer of  $A$  in  $M$ , which is defined here as

$$N_M(A) = \{v \text{ partial isometry of } M : vAv^* \subset A, v^*Av \subset A\},$$

generates  $M$  as a von Neumann algebra. We recall the fact that the conditional expectation  $P$  is unique.

The main result of [24] is that every Cartan pair  $(M, A)$  (if one assumes that  $M$  acts on a separable Hilbert space) is of the form  $(W^*(R, \tau), L^\infty(X, \mu))$  where  $R$  is a countable Borel equivalence relation on a standard measured space  $(X, \mu)$ , where  $\mu$  is a quasi-invariant measure and  $\tau \in Z^2(R, \mathbf{T})$  is a Borel twist. When  $M$  is finite dimensional, the result of [24] is elementary: we let  $X$  be the spectrum of  $A$  and  $V$  be the spectrum of the centre  $Z(M)$  of  $M$ . The inclusion  $Z(M) \subset A$  gives a surjective map  $r : X \rightarrow V$ . We let  $R$  be the equivalence relation admitting  $r$  as quotient map. Each  $x \in X$  corresponds to a minimal projection  $e(x)$  in  $A$ ;  $e(x)$  and  $e(y)$  are equivalent if and only if  $(x, y) \in R$ . We choose a matrix unit  $(e(x, y))_{(x, y) \in R}$  such that for all  $x \in X$ ,  $e(x, x) = e(x)$ . This matrix unit defines an isomorphism  $M \rightarrow C^*(R)$  sending  $A$  to  $C(X)$ . Thus, when  $M$  is finite dimensional, the twist is trivial. However, it does not admit a canonical trivialization. Note also that in a finite dimensional  $C^*$ -algebra, the notions of Cartan subalgebra and of maximal abelian self-adjoint subalgebra agree. We shall need an easy lemma about extension of matrix units.

**Lemma 2.3.** *Let  $(M, A)$  be a finite dimensional Cartan pair and let  $(X, R)$  be the corresponding equivalence relation. Then every partial matrix unit  $(e(x, y))_{(x, y) \in S}$  in  $M$ , where  $S$  is a subequivalence relation of  $R$ , can be extended to a full matrix unit  $(e(x, y))_{(x, y) \in R}$ .*

*Proof.* We fix an arbitrary full matrix unit  $(\underline{e}(x, y), (x, y) \in R)$ . There exists a function  $c : S \rightarrow \mathbf{T}$ , where  $\mathbf{T}$  is the group of complex numbers of module 1, such that  $e(x, y) = c(x, y)\underline{e}(x, y)$  for all  $(x, y) \in S$ . It is a cocycle. Every cocycle on  $S$  is trivial: there exists  $b : X \rightarrow \mathbf{T}$  such that  $c(x, y) = b(x)\overline{b(y)}$  for all  $(x, y) \in S$ . Then, we define  $e(x, y) = b(x)\underline{e}(x, y)\overline{b(y)}$  for all  $(x, y) \in R$ .  $\square$

The following lemma is a complement to Lemma III.1.14 of [40].

**Lemma 2.4.** *Given an inclusion  $M \subset \underline{M}$  of finite dimensional  $C^*$ -algebras, a faithful conditional expectation  $Q : \underline{M} \rightarrow M$  and a Cartan subalgebra  $A$  of  $M$ , there exists a Cartan subalgebra  $\underline{A}$  of  $\underline{M}$  such that*

- (i)  $A \subset \underline{A}$ ,
- (ii)  $N_M(A) \subset N_{\underline{M}}(\underline{A})$ , and
- (iii)  $Q \circ \underline{P} = P \circ Q_0$ , where  $P$  is the conditional expectation from  $M$  onto  $A$ ,  $\underline{P}$  is the conditional expectation from  $\underline{M}$  to  $\underline{A}$  and  $Q_0$  is the restriction of  $Q$  to  $\underline{A}$ .

*Proof.* We let  $(X, R)$  be the equivalence relation defined by the pair  $(M, A)$ :  $X$  is the spectrum of  $A$ ,  $V$  is the spectrum of the centre  $Z(M)$  of  $M$  and  $r : X \rightarrow V$  is the quotient map. We choose a matrix unit  $(e(x, y))_{(x, y) \in R}$  of  $M$  with  $e(x, x) = e(x)$  minimal projection corresponding to  $x$ . We choose a section  $\sigma$  for the map  $r : X \rightarrow V$ . For each  $v \in V$ , we set  $\underline{M}_v = e(\sigma(v))\underline{M}e(\sigma(v))$ . There exists a unique state  $\underline{\varphi}_v$  of the algebra  $\underline{M}_v$  such that  $Q(\underline{f}) = \underline{\varphi}_v(\underline{f})e(\sigma(v))$  for all  $\underline{f} \in \underline{M}_v$ . It is faithful because  $Q$  is faithful. Since

self-adjoint matrices are diagonalizable, there exists a Cartan subalgebra  $\underline{A}_v$  of  $\underline{M}_v$  such that  $\varphi_v = \varphi_v \circ \underline{P}_v$ , where  $\underline{P}_v$  is the conditional expectation onto  $\underline{A}_v$ . For  $x \in X$ , we define

$$\underline{A}_x = e(x, \sigma(r(x))) \underline{A}_{r(x)} e(\sigma(r(x)), x)$$

Then  $\underline{A} = \oplus_{x \in X} \underline{A}_x$  is a Cartan subalgebra of  $\underline{M}$ . It contains  $A$  because for all  $x \in X$ ,  $\underline{A}_x$  contains  $e(x)$  as its unit element. By construction  $e(x, y)$  belongs to the normalizer of  $\underline{A}$  in  $\underline{M}$ , hence  $N_M(A) \subset N_{\underline{M}}(\underline{A})$ . Let  $v \in V$  and  $\underline{a} \in \underline{M}_v$ . Then

$$P \circ Q(\underline{a}) = P(\varphi_v(\underline{a})e(\sigma(v))) = \varphi_v(\underline{a})e(\sigma(v)).$$

On the other hand, since  $\underline{P}_v$  is the restriction of  $\underline{P}$  to  $\underline{M}_v$ ,

$$Q \circ \underline{P}(\underline{a}) = \varphi_v(\underline{P}_v(\underline{a}))e(\sigma(v)) = \varphi_v(\underline{a})e(\sigma(v)).$$

Thus,  $P \circ Q$  and  $Q \circ \underline{P}$  agree on  $\underline{M}_v$ . Suppose now that  $\underline{a}$  belongs to  $e(x)\underline{M}e(y)$ , where  $(x, y) \in R$ . We write  $\underline{a} = e(x, v)\underline{a}_v e(v, y)$  with  $\underline{a}_v \in \underline{M}_v$ . Since  $e(x, v)$  and  $e(v, x)$  belong to  $M$ ,  $Q(\underline{a}) = e(x, v)Q(\underline{a}_v)e(v, y)$  and since  $e(x, v)$  and  $e(v, x)$  belong to  $N_M(A)$ ,

$$P \circ Q(\underline{a}) = e(x, v)P \circ Q(\underline{a}_v)e(v, y).$$

On the other hand, since  $e(x, v)$  and  $e(v, x)$  belong also to  $N_{\underline{M}}(\underline{A})$ ,  $\underline{P}(\underline{a}) = e(x, v)\underline{P}(\underline{a}_v)e(v, y)$ . Hence

$$Q \circ \underline{P}(\underline{a}) = e(x, v)Q \circ \underline{P}(\underline{a}_v)e(v, y).$$

Therefore,  $P \circ Q$  and  $Q \circ \underline{P}$  agree on  $e(x)\underline{M}e(y)$ . We deduce that they agree on  $\underline{M}$ . This implies that  $Q(\underline{A}) = A$  and that we have the equality  $Q \circ \underline{P} = P \circ Q_0$  where  $Q_0$  is the restriction of  $Q$  to  $\underline{A}$ . □

**Definition 2.3.** Let  $Q : \underline{M} \rightarrow M$  be a conditional expectation and let  $A, \underline{A}$  be Cartan subalgebras of  $M, \underline{M}$  respectively. We say that  $(M, A) \subset (\underline{M}, \underline{A})$

- (i) is a Cartan pairs inclusion if it satisfies the conditions (i) and (ii) of the lemma;
- (ii) is compatible with  $Q$  if moreover the condition (iii) of the lemma is also satisfied.

**Lemma 2.5.** *Let  $(M, A) \subset (\underline{M}, \underline{A})$  be an inclusion of finite dimensional Cartan pairs. Then the spectrum  $\underline{X}$  of  $\underline{A}$  is canonically identified to the fibered product  $X \times_V E$ , where  $X$  is the spectrum of  $A$ ,  $E$  is the spectrum of  $M' \cap \underline{A}$ ,  $V$  is the spectrum of  $Z(M)$  and the fibered product is relative to the maps  $r : X \rightarrow V$  and  $s : E \rightarrow V$  given by the inclusions  $Z(M) \subset M$  and  $Z(M) \subset M' \cap \underline{A}$ .*

*Proof.* We let  $\alpha$  be the action of  $N_M(A)$  on  $X$  and  $\underline{\alpha}$  be the action of  $N_{\underline{M}}(\underline{A})$  on  $\underline{X}$ . We let  $\pi : \underline{X} \rightarrow X$  and  $\pi : \underline{X} \rightarrow E$  be the surjections corresponding to the inclusions  $A \subset \underline{A}$  and  $M' \cap \underline{A} \subset \underline{A}$ . They satisfy  $r \circ \pi = s \circ q$ . Hence  $(\pi, q)$  maps  $\underline{X}$  into the fibered product  $X \times_V E$ . This map is injective: let  $\underline{x}, \underline{y} \in \underline{X}$  such that  $\pi(\underline{x}) = \pi(\underline{y})$  and  $q(\underline{x}) = q(\underline{y})$ . The elements of  $M' \cap \underline{A}$  are exactly the functions on  $\underline{X}$  which are constant under the action  $\underline{\alpha}$  of  $N_{\underline{M}}(\underline{A})$  on  $\underline{X}$ . Therefore, the relation  $q(\underline{x}) = q(\underline{y})$  implies the existence of  $u \in N_{\underline{M}}(\underline{A})$  such that  $\underline{y} = \underline{\alpha}_u(\underline{x})$ . This implies that  $\pi(\underline{x}) = \pi(\underline{y}) = \alpha_u(\pi(\underline{x}))$ , hence  $ue(x) = e(x)$  and  $\underline{y} = \underline{x}$ . The map is surjective. Let  $(x, c) \in X \times E$  such that  $r(x) = s(c)$ . Pick  $\underline{y} \in \underline{X}$  such that  $q(\underline{y}) = c$ . Since  $r(\pi(\underline{y})) = r(x)$ , there exists  $u \in N_M(A)$  such that  $x = \alpha_u(\pi(\underline{y}))$ . Then  $\underline{x} = \underline{\alpha}_u(\underline{y})$  does the job. □

An equivalent statement of the lemma is that  $\underline{A}$  is canonically identified to  $A \otimes_{Z(M)} (M' \cap \underline{A})$ .

**Lemma 2.6.** *Let  $(M, A) \subset (\underline{M}, \underline{A})$  be an inclusion of finite dimensional Cartan pairs. The commutant of  $M$  in  $\underline{M}$  is denoted by  $M'$ .*

- (i) *If  $\underline{a}$  belongs to  $M'$ , then  $\underline{e}(xa)\underline{a}\underline{e}(yb) = 0$  if  $x \neq y$ .*
- (ii)  *$M' \cap \underline{A}$  is a Cartan subalgebra of  $M'$ .*
- (iii) *the equivalence relation induced on the spectrum  $E$  of  $M' \cap \underline{A}$  by the normalizer is*

$$R' = \{(a, b) \in E \times E : s(a) = s(b), r(a) = r(b)\}$$

*Proof.* i) Assume that  $\underline{f}$  commutes with  $M$ . If  $x \neq y$ ,

$$\underline{e}(xa)\underline{f}\underline{e}(yb) = \underline{e}(xa)e(x)\underline{f}\underline{e}(yb) = \underline{e}(xa)\underline{f}e(x)\underline{e}(yb) = 0.$$

ii) For  $c \in E$ , we denote by  $\epsilon(c)$  the corresponding projection in  $M' \cap \underline{A}$ . According to (i),  $\epsilon(c) = \sum_{r(x)=s(c)} \underline{e}(xc)$ . Suppose that  $\underline{f} \in M'$  commutes with the elements of  $M' \cap \underline{A}$ . Consider  $xa$  and  $xb$  with  $a \neq b$ . Then

$$\underline{e}(xa)\underline{f}\underline{e}(xb) = \underline{e}(xa)\epsilon(a)\underline{f}\underline{e}(yb) = \underline{e}(xa)\underline{f}\epsilon(a)\underline{e}(yb) = 0.$$

Thus  $\underline{e}(xa)\underline{f}\underline{e}(yb) = 0$  if  $xa \neq yb$ , therefore  $\underline{f}$  belongs to  $\underline{A}$ .

iii) Assume that  $\epsilon(a)M'\epsilon(b) \neq 0$ . Then according to (i), there exists  $x \in X$  such that  $(xa, xb) \in \underline{R}$ . This implies that  $(a, b) \in R'$ . Conversely, if  $(a, b) \in R'$ , we pick  $x \in X$  such that  $r(x) = s(a) = s(b)$ . We choose a partial isometry  $\underline{u} \in \underline{M}$  such that  $\underline{u}\underline{u}^* = \underline{e}(xa)$ ,  $\underline{u}^*\underline{u} = \underline{e}(xb)$ . Then  $\sum e(y, x)\underline{u}e(x, y)$ , where  $(e(x, y))_{(x, y) \in R}$  is a matrix unit for  $R$  and the sum is over the  $y$ 's such that  $r(y) = r(x)$  is a partial isometry in  $M'$  with domain  $\epsilon(b)$  and range  $\epsilon(a)$ . □

**Lemma 2.7.** *Let  $(M, A) \subset (\underline{M}, \underline{A})$  be an inclusion of finite dimensional Cartan pairs and let  $Q : \underline{M} \rightarrow M$  be a faithful conditional expectation which satisfies the condition (iii) of Lemma 2.4. Then, with above notations, there exists a transition probability  $p$  on the graph  $E$  such that for all  $c \in E$ ,*

$$Q(\epsilon(c)) = p(c)e(s(c))$$

where  $\epsilon(c)$  is the minimal projection in  $M' \cap \underline{A}$  corresponding to  $c \in E$  and  $e(v)$  is the minimal projection in  $Z(M)$  corresponding to  $v \in V$ .

*Proof.* As earlier, we denote by  $Q_0$  the restriction of  $Q$  to  $\underline{A}$ . We first check that  $Q_0(\epsilon(c))$  belongs to  $Z(M)$ : for  $a \in M$ ,

$$aQ_0(\epsilon(c)) = Q_0(a\epsilon(c)) = Q_0(\epsilon(c)a) = Q_0(\epsilon(c))a$$

Then, we observe that  $e(x)\epsilon(c) = 0$  if  $r(x) \neq s(c)$ . Therefore  $e(v)Q_0(\epsilon(c)) = 0$  for all  $v \in V$  distinct from  $s(c)$ :  $Q_0(\epsilon(c))$  is proportional to  $e(s(c))$ . The constant of proportionality is non zero because  $Q$  is supposed to be faithful. The equality  $e(v) = \sum_{s(c)=v} \epsilon(c)$  gives the equality  $\sum_{s(c)=v} p(c) = 1$ . □

**Theorem 2.8.** *Let  $Q : \underline{M} \rightarrow M$  be a faithful conditional expectation on a finite dimensional  $C^*$ -algebra and let  $A$  be Cartan subalgebra of  $M$ . We let  $(X, R)$  be the associated equivalence relation. Then,*

- (i) *there exists  $\underline{A}$  Cartan subalgebra of  $\underline{M}$  such that  $(M, A) \subset (\underline{M}, \underline{A})$  is a Cartan pairs inclusion compatible with  $Q$ .*
- (ii) *any isomorphism  $\Phi : M \rightarrow C^*(R)$  carrying  $A$  onto  $C_0(X)$  can be extended to an isomorphism  $\underline{\Phi} : \underline{M} \rightarrow C^*(\underline{R})$  carrying  $Q$  into the model expectation  $Q_p : C^*(\underline{R}) \rightarrow C^*(R)$  constructed from the graph  $(V, E, \underline{V})$  of the inclusion, the spectrum  $X$  of  $A$  and the transition probability  $p$  of Lemma 2.7.*

*Proof.* The first assertion is Lemma 2.4. We fix a Cartan subalgebra  $\underline{A}$  satisfying (i). We recall that the spectrum  $\underline{X}$  of  $\underline{A}$  can be identified with the fibered product  $X \times_V E$  and that the spectrum of  $M' \cap \underline{A}$  can be identified with  $E$ . We also recall from Lemma 2.6 that  $(M', M' \cap \underline{A})$  is a Cartan pair defining the equivalence relation  $R'$  on  $E$ . We pick a matrix unit  $(\epsilon(a, b))_{(a, b) \in R'}$  for the Cartan pair  $(M', M' \cap \underline{A})$ . Let  $\Phi : M \rightarrow C^*(R)$  be an isomorphism carrying  $A$  onto  $C_0(X)$ . There exists a unique matrix unit  $(e(x, y))_{(x, y) \in R}$  for the Cartan pair  $(M, A)$  which is sent by  $\Phi$  onto the canonical matrix unit of  $C^*(R)$ . We define  $\underline{e}(xa, yb) = e(x, y)\epsilon(a, b)$  if  $r(x) = r(y) = s(a) = s(b)$  and  $r(a) = r(b)$ . This defines a partial matrix unit on a subequivalence relation of  $\underline{R}$ . According to Lemma 2.3, it can be completed into a full matrix unit  $(\underline{e}(xa, yb))_{(xa, yb) \in \underline{R}}$ . The isomorphism  $\underline{\Phi} : \underline{M} \rightarrow C^*(\underline{R})$  defined by this matrix unit extends  $\Phi$  and satisfies  $\underline{\Phi} \circ Q = Q_p \circ \underline{\Phi}$ .  $\square$

In particular, the theorem shows that our path model of a conditional expectation gives every faithful conditional expectation. We can recover from this theorem the main result of [11], namely every conditional expectation on a finite dimensional  $C^*$ -algebra can be written as a pinching followed by slicing and averaging: one introduces an intermediate level  $V_1$  in the inclusion graph  $(V, E, \underline{V})$ , whose vertices label the edges (thus  $V_1 = E$ ). The graph  $(V, E, \underline{V})$  is then written as the concatenation of two graphs  $(V, E_1, V_1)$  and  $(V_1, E_2, \underline{V})$ . In the first graph, the vertices of  $V_1$  receive a single edge. In the second graph, the vertices of  $V_1$  emit a single edge. With the ingredient  $X_1 = \underline{X}$  and  $(V_1, E_2, \underline{V})$ , our recipe gives the inclusion  $C^*(R_1) \subset C^*(\underline{R})$  where  $(xa, yb) \in R_1$  if and only if  $a = b$ . The transition probability  $p_2 \equiv 1$  gives the restriction map  $Q_2 : C^*(\underline{R}) \rightarrow C^*(R_1)$  as its associated conditional expectation. It is a pinching: in other words, it is of the form

$$Q_2(\underline{f}) = \sum_{c \in E} \epsilon(c) \underline{f} \epsilon(c).$$

The conditional expectation  $Q_1 : C^*(R_1) \rightarrow C^*(R)$  is an averaging: for every  $v \in V$

$$Q_1(\underline{f})(x, y) = \sum_{s(c)=v} p(c) \underline{f}(xc, yc) \quad \text{for } r(x) = r(y) = v.$$

In [7], A. Connes uses a similar decomposition of an inclusion of type I von Neumann algebras to construct inclusions of Cartan pairs.

### 3. RANDOM WALKS ON DISCRETE BRATTELI DIAGRAMS.

We first recall the classical definition of a Bratteli diagram.

**Definition 3.1.** A Bratteli diagram is a directed graph  $(V, E)$  where the set of vertices  $V = \coprod_{n=0}^{\infty} V(n)$  and the set of edges  $E = \coprod_{n=1}^{\infty} E(n)$  are graded. For each  $n \geq 1$ ,  $s(E(n)) = V(n-1)$  and  $r(E(n)) = V(n)$ , where  $s(e)$  and  $r(e)$  are respectively the source and the range of the edge  $e$ .

We assume that each level of vertices  $V(n)$  is at most countable; we also assume that each vertex emits finitely many but at least one edge and that each vertex of a level  $n \geq 1$  receives finitely many but at least one edge.

**Definition 3.2.** Let  $(V, E)$  be a Bratteli diagram.

- A transition probability is a map  $p$  assigning to each vertex  $v \in V$  a probability measure  $p(v)$  on the set of edges  $E_v = s^{-1}(v)$  emanating from  $v$ . We shall view  $p$  as a map  $p : E \rightarrow \mathbf{R}$  such that for all  $v \in V$ ,  $\sum_{s(e)=v} p(e) = 1$ . We shall denote by  $p_n$  its restriction to  $E(n)$ .
- An initial probability measure is a probability measure  $\nu_0$  on the set of initial vertices  $V(0)$ .
- A random walk is a pair  $(p, \nu_0)$ , where  $p$  is a transition probability and  $\nu_0$  is an initial probability measure.

We shall always assume that  $p$  and  $\nu_0$  have full support, in the sense that  $p(e) > 0$  for all  $e \in E$  and  $\nu_0(v) > 0$  for all  $v \in V(0)$ .

A Bratteli diagram  $(V, E)$  defines an étale equivalence relation  $(X, R)$  called the *tail equivalence relation* of the diagram:  $X$  is the set of infinite paths  $x = e_1 e_2 \dots$  where  $e_n \in E(n)$  and  $r(e_n) = s(e_{n+1})$ . It is a locally compact Hausdorff totally disconnected space admitting the cylinders

$$Z(a) = \{ae_{n+1}e_{n+2}\dots\}$$

where  $a = a_1 a_2 \dots a_n$  is a finite path (we assume implicitly that  $a_i \in E(i)$ ), as a base of compact open subsets. Two infinite paths  $x = e_1 e_2 \dots$  and  $y = f_1 f_2 \dots$  are tail equivalent if there exists  $n$  such that  $e_i = f_i$  for  $i > n$ . Its graph  $R$  is a locally compact Hausdorff totally disconnected space admitting the cylinders

$$Z(a, b) = \{(ae_{n+1}e_{n+2}\dots, be_{n+1}e_{n+2}\dots)\}$$

where  $(a, b)$  is a pair of equivalent finite paths: this means that they have same length  $n$  and same range  $r(a) = r(b)$ , where we define  $r(a_1 a_2 \dots a_n) = r(a_n) \in V(n)$ .

A random walk on a Bratteli diagram defines a measure on the path space  $X$ ; it is a particular case of the well-known construction of Markov measures.

**Proposition 3.1.** *Given a random walk  $(p, \nu_0)$  on a Bratteli diagram  $(V, E)$ , there is a unique probability measure  $\mu$  on  $X$ , called the Markov measure of the random walk whose values on cylinder sets is given by*

$$\mu(Z(a)) = \nu_0(s(a))p(a)$$

where, for the finite path  $a = a_1 a_2 \dots a_n$ ,

$$p(a) = p_1(a_1)p_2(a_2)\dots p_n(a_n) \quad \text{and} \quad s(a) = s(a_1).$$



As observed in [41, Section 3.2], Markov measures are quasi-invariant under the tail equivalence relation. Let us recall that a measure  $\mu$  on  $X$  is quasi-invariant under the equivalence relation  $R$  if the measures  $r^*\mu$  and  $s^*\mu$  on  $R$  are equivalent, where  $\int f d(r^*\mu) = \int \sum_y f(x, y) d\mu(x)$  for  $f \in C_c(R)$  and  $s^*\mu$  is similarly defined. Then its Radon-Nikodym derivative  $D_\mu = d(r^*\mu)/d(s^*\mu)$  is a cocycle, i.e. it satisfies  $D_\mu(x, y)D_\mu(y, z) = D_\mu(x, z)$  for a.e.  $(x, y, z) \in R^{(2)}$ . Here is a way to construct cocycles on the tail equivalence relation  $R$  of a Bratteli diagram  $(V, E)$ .

**Definition 3.3.** Let  $G$  be a group. A map  $D : R \rightarrow G$  is called a quasi-product cocycle if there exists a map  $q : E \rightarrow G$ , called a potential, such that for all pairs of equivalent finite paths  $(a, b)$  and all  $(az, bz) \in Z(a, b)$ ,  $D(az, bz) = q(a)q(b)^{-1}$  and where, as before,  $q(a_1a_2 \dots a_n) = q(a_1)q(a_2) \dots q(a_n)$ .

Since a quasi-product cocycle is locally constant, it takes at most countably many values and it is continuous. The following result, which is a simple observation, is essential here.

**Proposition 3.2.** [41, Proposition 3.3] *Let  $(p, \nu_0)$  be a random walk on a Bratteli diagram  $(V, E)$ .*

- (i) *The associated Markov measure  $\mu$  is quasi-invariant under the tail equivalence relation  $R$*
- (ii) *Its Radon-Nikodym derivative  $D_\mu$  is the quasi-product cocycle given by the potential  $q = (q_n)$  defined by the relation*

$$\nu_{n-1}(s(e))p_n(e) = q_n(e)\nu_n(r(e)) \quad \text{for } e \in E(n).$$

*where  $\nu_n$  is the distribution of the random walk on  $V(n)$ , defined inductively by  $\nu_n(w) = \sum_{r(e)=w} p_n(e)\nu_{n-1}(s(e))$  for  $w \in V(n)$ .*

Note that the Radon-Nikodym derivative depends only on the potential  $q$ . This potential  $q$  has a simple probabilistic interpretation: it is the *cotransition probability* of the random walk: let  $e$  be an edge in  $E(n)$  with range  $r(e) = w$ , then  $q(e)$  is the probability that the random walk passes through  $e$  given that it is at  $w$  at time  $n$ . As shown by the next example, different random walks may share the same cotransition probability. In his recent papers [44, 45], A. Vershik also emphasizes the importance of cotransition probabilities in the asymptotic study of random walks on Bratteli diagrams.

**Example 3.1.** Random walks on the Pascal triangle. It is the time development of the simple random walk on  $\mathbf{Z}$ . Here, the Bratteli diagram is  $(V, E)$  where

$$V(n) = \{(n, k) : k = 0, 1, \dots, n\}; \quad E(n) = \{(n-1, k, \epsilon) : k = 0, 1, \dots, n-1; \epsilon \in \{0, 1\}\}.$$

We consider the random walk defined by the transition probability

$$p_n(n-1, k) = (1-t)\delta_{(n-1, k, 0)} + t\delta_{(n-1, k, 1)}$$

where  $0 < t < 1$ . Since  $V(0)$  has a single vertex, the initial measure  $\nu_0$  is the point mass at this vertex. The infinite path can be written as the infinite product  $X = \prod_{n=1}^{\infty} \{0, 1\}$ . Then, the Markov measure is the product measure  $\mu_t = \prod_{n=1}^{\infty} ((1-t)\delta_0 + t\delta_1)$ . An elementary computation gives the cotransition probability

$$q_n(n, k) = \left(1 - \frac{k}{n}\right) \delta_{(n-1, k, 0)} + \frac{k}{n} \delta_{(n-1, k-1, 1)}$$

It does not depend on  $t$ . For a finite path  $\epsilon_1\epsilon_2\ldots\epsilon_n$  ending at  $(n, k = \epsilon_1 + \ldots + \epsilon_n)$ , one has

$$q(\epsilon_1\epsilon_2\ldots\epsilon_n) = \binom{n}{k}^{-1}.$$

One deduces that the Radon-Nikodym of  $\mu_t$  is  $D \equiv 1$ . In other words, the measures  $\mu_t$  are invariant under the tail equivalence relation on  $(V, E)$ . It is a well-known result. It is also well-known (see for example [42, Example 4.2]) that these are the extremal invariant probability measures (one has to add  $\mu_0$  and  $\mu_1$  which we have excluded from our discussion).

We use now the construction given by Feldman and Moore in [24]: since  $(X, R, \mu)$  is a countable standard measured equivalence relation, one can construct its von Neumann algebra  $\mathcal{M} = W^*(X, R, \mu)$  and its state  $\varphi = \mu \circ P$ , where  $P$  is the expectation of  $\mathcal{M}$  onto  $\mathcal{A} = L^\infty(X, \mu)$ , which is normal and faithful. By construction,  $\mathcal{M}$  acts on the Hilbert space  $L^2(R, s^*\mu)$ . This representation is standard. It is known that the modular operator  $\Delta$  of  $\varphi$  is the operator of multiplication by  $D_\mu$  and that the modular automorphism  $\sigma_t^\varphi$  is implemented by the operator of multiplication by  $D_\mu^{it}$ . A. Connes has given the following characterization of the pairs  $(\mathcal{M}, \varphi)$  arising from this construction.

**Theorem 3.3.** [7, Theorem 1] *Let  $\varphi$  be a faithful normal state on a von Neumann algebra  $\mathcal{M}$ . Then the following conditions are equivalent:*

- (i) *there exists an increasing sequence  $(M_n)$  of finite dimensional subalgebras stable under the automorphism group  $\sigma$  of  $\varphi$  whose union is weakly dense in  $\mathcal{M}$ ;*
- (ii) *there exists a Bratteli diagram  $(V, E)$  and a random walk  $(p, \nu_0)$  on it such that the pair  $(\mathcal{M}, \varphi)$  is isomorphic to  $(W^*(X, R, \mu), \mu \circ P)$ , where  $R$  is the tail equivalence relation on the infinite path space  $X$  of the diagram and  $\mu$  is the Markov measure of the random walk.*

The original theorem of Connes is in terms of Krieger's factors. It is an intermediate step to show that all hyperfinite type  $\text{III}_0$  factors are Krieger's factors. We consider here von Neumann algebras arising from hyperfinite measured equivalence relations rather than Krieger's factors. This makes the statement easier to prove.

*Proof.* (ii) $\Rightarrow$ (i). We assume that  $\mathcal{M} = W^*(X, R, \mu)$  as above. We let  $M_n$  be the subalgebra of  $\mathcal{M}$  generated by the characteristic functions  $\mathbf{1}_{Z(a,b)}$ , where  $(a, b)$  is a pair of joining paths of length  $n$ . Since the Radon-Nikodym derivative  $D_\mu$  is constant on the cylinder sets  $Z(a, b)$ ,  $M_n$  is stable under the automorphism group of  $\varphi_\mu$ . Since  $Z(a, b)$  is the disjoint union of  $Z(ae, be)$ 's where  $e \in E(n+1)$  and  $s(e) = r(a) = r(b)$ , we have the inclusion  $M_n \subset M_{n+1}$ . The elements of the union  $M_\infty$  of the  $M_n$ 's are the locally constant functions with compact support. Since  $M_\infty$  is dense in  $C_c(R)$  with respect to the inductive limit topology, it is dense in the weak topology. Since  $C_c(R)$  is weakly dense in  $\mathcal{M}$ , so is  $M_\infty$ .

(i) $\Rightarrow$ (ii). Let  $(M_n)_{n \in \mathbf{N}}$  be as in (i). Without loss of generality, we may assume that  $M_0 = \mathbf{C}1$ . Since for all  $n$ ,  $M_n$  is stable under  $\sigma$ , the modular automorphism of the restriction  $\varphi_n$  to  $M_n$  is the restriction  $\sigma_n$  of  $\sigma$  to  $M_n$ . Since for all  $n \geq 1$ ,  $M_{n-1}$  is

invariant under  $\sigma_n$ , there exists a faithful expectation  $Q_{n-1,n} : M_n \rightarrow M_{n-1}$  such that  $\varphi_n = \varphi_{n-1} \circ Q_{n-1,n}$ . We use inductively Theorem 2.8, to construct an increasing sequence  $(A_n)$  of abelian subalgebras such that for all  $n \geq 1$ ,  $(M_{n-1}, A_{n-1}) \subset (M_n, A_n)$  is a Cartan pair inclusion compatible with the conditional expectation  $Q_{n-1,n}$ . The construction is initialized by the only possible choice  $A_0 = M_0$ . Thus we obtain for each  $n \in \mathbf{N}$  the spectrum  $X_n$  of  $A_n$  and for each  $n \geq 1$  the graph  $(V(n-1), E(n), V(n))$  of the inclusion  $M_{n-1} \subset M_n$  and the transition probability  $p_n : E(n) \rightarrow \mathbf{R}_+^*$ . From the same theorem, we obtain for all  $n$  an isomorphism  $\Phi_n : M_n \rightarrow C^*(R_n)$  sending  $A_n$  to  $C(X_n)$ , where  $(X_n, R_n)$  is the equivalence relation defined by  $(M_n, A_n)$ , such that  $\Phi_n$  extends  $\Phi_{n-1}$  and carrying the conditional expectation  $Q_{n-1,n} : M_n \rightarrow M_{n-1}$  into the model conditional expectation  $F_{p_n} : C^*(R_n) \rightarrow C^*(R_{n-1})$ . Again, the construction is initialized by the only possible isomorphism  $\Phi_0 : M_0 \rightarrow C(X_0)$ . Since the conditional expectations  $P_n : M_n \rightarrow A_n$  satisfy  $Q_{n-1,n} \circ P_n = P_{n-1} \circ Q_{n-1,n}$ , we have  $\varphi_n = \nu_n \circ P_n$ , where  $\nu_n$  is the restriction of  $\varphi_n$  to  $A_n$ . The Bratteli diagram of the increasing sequence  $(M_n)$  of finite dimensional algebras is  $(V = \coprod_{n=0}^{\infty} V(n), E = \coprod_{n=1}^{\infty} E(n))$ . The transition probabilities  $p_n : E(n) \rightarrow \mathbf{R}_+^*$  define a transition probability on  $E$ . The initial measure  $\nu_0$  is the point mass at the unique point of  $X_0$ . This defines the random walk. We let  $X$  be its infinite path space,  $R$  be the tail equivalence and  $\mu$  be the Markov measure of the random walk. The isomorphisms  $\Phi_n : M_n \rightarrow C^*(R_n)$  extend to an isomorphism  $\Phi_{\infty} : M_{\infty} \rightarrow C_{00}(R)$ , where  $M_{\infty}$  is the union of the  $M_n$ 's and  $C_{00}(R)$  is the  $*$ -algebra of locally constant functions with compact support. This isomorphism carries the restriction of the state  $\varphi$  to the restriction of the state  $\mu \circ P$ , where  $P$  is the expectation of  $W^*(R)$  onto  $L^{\infty}(X, \mu)$ . Since both von Neumann algebras  $\mathcal{M}$  and  $W^*(X, R, \mu)$  can be obtained from the GNS representation of these states,  $\Phi_{\infty}$  extends to a normal  $*$ -isomorphism  $\Phi : \mathcal{M} \rightarrow W^*(X, R, \mu)$  which sends the weak closure  $\mathcal{A}$  of the union of the  $A_n$ 's to  $L^{\infty}(X, \mu)$  and  $\varphi$  to  $\mu \circ P$ . □

*Remark 3.1.* States on AF-algebras constructed from a random walk on a Bratteli diagram are called quasi-product states in [19]. Do we have a characterization (besides condition (i) of the theorem) of the normal faithful states on a hyperfinite von Neumann algebra which can be described as quasi-product states? Necessarily, these states are almost periodic (their modular operators are diagonalizable) and their centralizers contain a Cartan subalgebra. In part II of [7], A. Connes shows that any faithful semifinite normal weight on a hyperfinite factor of type  $III_0$  whose modular operator  $\Delta$  is diagonalizable and such that 1 is isolated in its spectrum and its point spectrum contained in  $\mathbf{Q}$  satisfies condition (i) of the theorem (with  $M_n$  type  $I_{\infty}$  rather than finite-dimensional).

#### 4. MARKOV CHAINS AND BRATTELI DIAGRAMS

In order to recover the general theory of time-dependent Markov chains (with discrete time), it is necessary to generalize the notion of Bratteli diagram. Indeed, the original definition is limited to Markov chains with at most countably many states. Generalized Bratteli diagrams have been considered before, mostly in the topological setting, and are part of the theory of topological graphs. Since we are considering objects of measure-theoretical nature, we choose the Borel setting. We assume implicitly that the Borel spaces are analytic.

**Definition 4.1.** We say that a directed graph  $(V, E)$  is a *Borel graph* if the sets of edges  $E$  and the set of vertices  $V$  are endowed with a Borel structure and the source and range maps are Borel. A *Borel Bratteli diagram* is a Bratteli diagram which is a Borel graph.

Before extending to Borel Bratteli diagrams Definition 3.2, we need to make precise our assumptions.

**Definition 4.2.** Let  $E, V$  be Borel spaces and let  $s : E \rightarrow V$  be a Borel surjection. An  $s$ -system of probability measures  $p$  is a map assigning to each  $v \in V$  a probability measure  $p_v$  on  $s^{-1}(v)$ . We say that the  $s$ -system  $p$  is

- (i) Borel if for all bounded Borel function  $f$  on  $E$ , the map  $v \rightarrow \int f dp_v$  is Borel;
- (ii)  $\nu$ -measurable, where  $\nu$  is a probability measure on  $V$ , if for all bounded Borel function  $f$  on  $E$ , the map  $v \rightarrow \int f dp_v$  is  $\nu$ -measurable.

If the  $s$ -system is  $\nu$ -measurable, we can form the probability measure  $\mu = \nu p$  on  $E$ , defined by  $\int f d(\nu p) = \int (\int f dp_v) d\nu(v)$  for  $f$  bounded Borel function on  $E$ . In fact it suffices to have  $p_v$  defined for a.e.  $v$  to define  $\nu p$ . The measure  $\nu$  is the image  $s_*(\nu p)$  of  $\nu p$  and  $\mu = \nu p$  is the disintegration of  $\mu$  along  $s$ . Conversely, given a probability measure  $\mu$  on  $E$ , we can disintegrate  $\mu$  along  $s$ : there exists a  $\nu$ -measurable  $s$ -system of probability measures  $p$ , where  $\nu = s_*(\mu)$ , such that  $\mu = \nu p$ . It is unique in the sense that if  $\nu p = \nu p'$ , then  $p_v = p'_v$  for  $\nu$  a.e.  $v$ .

*Notation.* Consider the  $n$ -th floor  $V(n-1) \xleftarrow{s} E(n) \xrightarrow{r} V(n)$  of a Borel Bratteli diagram. A probability measure  $\mu_n$  on  $E(n)$  admits a disintegration  $\mu_n = \nu_{n-1} p_n$  along  $s$  and a disintegration  $\mu_n = \nu_n q_n$  along  $r$ , where  $\nu_{n-1} = s_* \mu_n$  and  $\nu_n = r_* \mu_n$ . This establishes a bijection between pairs  $(\nu_{n-1}, p_n)$ , where  $\nu_{n-1}$  is a probability measure on  $V(n-1)$  and  $p_n$  is a  $\nu_{n-1}$ -measurable system of probability measures along  $s$  and pairs  $(\nu_n, q_n)$ , where  $\nu_n$  is a probability measure on  $V(n)$  and  $q_n$  is a  $\nu_n$ -measurable system of probability measures along  $r$ , given by the equation  $\nu_{n-1} p_n = \nu_n q_n$ .

**Definition 4.3.** Let  $(V, E)$  be a Borel Bratteli diagram.

- A *random walk* on  $(V, E)$  is a sequence of probability measures  $\mu_n$  on  $E(n)$  which are compatible in the sense that for all  $n \geq 1$ ,  $r_* \mu_n = s_* \mu_{n+1}$ .
- The measures  $\nu_n = r_* \mu_n$  are called the *one-dimensional distributions* of the random walk.
- The measure  $\nu_0 = s_* \mu_1$  is called the *initial distribution* of the random walk.
- The sequence  $p = (p_n)$  of  $s$ -systems, where  $\mu_n = \nu_{n-1} p_n$  as above, is called the *transition probability* of the random walk.
- The sequence  $q = (q_n)$  of  $r$ -systems, where  $\mu_n = \nu_n q_n$  as above, is called the *cotransition probability* of the random walk.

*Remark 4.1.* Note that by construction, the sequence  $(\nu_n)$  of one-dimensional distributions satisfies the relations  $\nu_{n-1} = s_*(\nu_n q_n)$  and  $\nu_n = r_*(\nu_{n-1} p_n)$  for all  $n \geq 1$ . We say that it is  $q$ -compatible and  $p$ -compatible respectively.

**Proposition 4.1.** We have two constructions of a random walk on a Borel Bratteli diagram  $(V, E)$ .

- (i) Let  $\nu_0$  be a probability measure on  $V(0)$  and let  $p = (p_n)$  be a sequence of  $\nu_{n-1}$ -measurable systems of probability measures for  $s : E(n) \rightarrow V(n-1)$ , where the measures  $\mu_n = \nu_{n-1}p_n$  and  $\nu_n = r_*(\nu_{n-1}p_n)$  are constructed inductively. Then there exists a unique random on  $(V, E)$  admitting  $\nu_0$  as initial distribution and  $p$  as transition probability.
- (ii) Let  $(\nu_n)$  be a sequence of probability measures  $\nu_n$  on  $V(n)$  and let  $q = (q_n)$  be a sequence of  $\nu_n$ -measurable systems of probability measures for  $r : E(n) \rightarrow V(n)$  which is  $q$ -compatible in the sense that  $\nu_{n-1} = s_*(\nu_n q_n)$  for all  $n \geq 1$ . Then there exists a unique random on  $(V, E)$  admitting  $(\nu_n)$  as one-dimensional distributions and  $q$  as cotransition probability.

*Proof.* This is clear. In the first case, we define inductively  $\mu_n = \nu_{n-1}p_n$ . In the second case, we define  $\mu_n = \nu_n q_n$ .  $\square$

We recall the construction of the Markov measure of a random walk (see [39, V-1]). As earlier, we introduce the infinite path space  $X$ . We let  $X^n$  denote the space of paths  $e_1 \dots e_n$  of length  $n$  endowed with the product Borel structure. Then  $X = \varprojlim X^n$  is the projective limit with respect to the canonical projection  $X^n \leftarrow X^{n+1}$ . Given a random walk  $(p, \nu_0)$ , one first constructs by induction a probability measure  $\mu^n$  on  $X^n$  such that

$$\int f d\mu^1 = \int f(e_1) dp_{\nu_0}(e_1) d\nu_0(v), \quad \int f d\mu^n = \int f(e_1 \dots e_n) dp_{r(e_{n-1})}(e_n) d\mu^{n-1}(e_1 \dots e_{n-1})$$

The sequence of measures  $(\mu^n)$  is consistent. Therefore, there exists a unique probability measure  $\mu$  on  $X$  whose image in  $X^n$  is  $\mu^n$ . Note that the one-dimensional distribution  $\nu_n$  on  $V(n)$  is the image of  $\mu^n$  by the range map  $r : X^n \rightarrow V(n)$ . It is also the image of  $\mu$  by the map  $r_n : X \rightarrow V(n)$  such that  $r_n(e_1 e_2 \dots) = r(e_n)$ .

It remains to characterize the Markov measure  $\mu$  on  $X$  in terms of the cotransition probability  $q$ . We have seen that in the framework of the previous section, the Markov measure  $\mu$  is quasi-invariant under the tail equivalence relation and its Radon-Nikodym derivative  $D$  is the quasi-product cocycle defined by  $q$ . We then say that  $\mu$  is a  $D$ -measure. The notion of quasi-product cocycle does not admit a straightforward generalization in the general framework. However, there exists (see [42, Proposition 3.7]) an equivalent definition of a  $D$ -measure (known in statistical mechanics as the Dobrushin-Lanford-Ruelle condition for Gibbs states) which can be easily extended. We let  $X_{|n}$  be the space of infinite paths  $e_{n+1}e_{n+2}\dots$  starting at level  $n$ . The sequence of quotient maps

$$X \xrightarrow{\pi_1} X_{|1} \xrightarrow{\pi_2} X_{|2} \xrightarrow{\pi_3} \dots \xrightarrow{\pi_n} X_{|n} \xrightarrow{\pi_{n+1}} \dots$$

defines the tail equivalence relation  $R$  on  $X$ : two infinite paths  $x$  and  $y$  are tail equivalent if and only if there exist  $n$  such that  $\pi_n \circ \dots \circ \pi_2 \circ \pi_1(x) = \pi_n \circ \dots \circ \pi_2 \circ \pi_1(y)$ . The cotransition probability  $q$  defines an inductive system of expectations

$$L^\infty(X, \mu) \xrightarrow{\tilde{q}_1} L^\infty(X_1, \mu_{|1}) \xrightarrow{\tilde{q}_2} \dots \xrightarrow{\tilde{q}_n} L^\infty(X_n, \mu_{|n}) \xrightarrow{\tilde{q}_{n+1}} \\ \tilde{q}_n(f)(e_{n+1}e_{n+2}\dots) = \int f(e_n e_{n+1} e_{n+2} \dots) dq^{s(e_{n+1})}(e_n).$$

**Definition 4.4.** Let  $q$  be a cotransition probability on the Borel Bratteli diagram  $(V, E)$ . A  $q$ -measure is a measure on the infinite path space  $X$  which factors through all expectations  $\tilde{q}_n \dots \tilde{q}_2 \tilde{q}_1$ .

Then we have the easy generalisation of Proposition 3.2:

**Theorem 4.2.** *Let  $(V, E)$  be a Borel Bratteli diagram. Then the Markov measure of a random walk  $(p, \nu_0)$  is a  $q$ -measure, where  $q$  is the cotransition probability of the random walk.*

*Acknowledgements.*

## REFERENCES

- [1] C. Anantharaman-Delaroche and J. Renault: *Amenable groupoids*, Monographie de l'Enseignement Mathématique No **36**, Genève, 2000.
- [2] N. Bourbaki: *Topologie générale, chapitres 1 à 4*, Eléments de Mathématique, Diffusion C.C.L.S., Paris, 1971.
- [3] O. Bratteli: *Inductive limits of finite dimensional  $C^*$ -algebras*, Trans. Amer. Math. Soc., **171** (1971), 195–234.
- [4] T. Bousch: *La condition de Walters*, Janvier 2000
- [5] G. Brown and A. H. Dooley: *Odometer actions on  $G$ -measures*, Ergodic Theory Dyn. System, **11** (1991), 279–307.
- [6] D. Capocaccia: *A definition of Gibbs state for a compact set with  $\mathbf{Z}^\nu$  action*, Commun. Math. Phys., **48** (1976), 85–88.
- [7] A.-Connes: *On hyperfinite factors of type  $III_0$  and Krieger's factors*, J. Funct. Anal., **18** (1975), 318–327.
- [8] A. Connes and E. J. Woods: *Hyperfinite von Neumann algebras and Poisson boundaries of time dependent random walks*, Pacific J. Math. **137** (1989), no 2, 225–243.
- [9] J. Cuntz: *A class of  $C^*$ -algebras and topological Markov chains II: reducible chains and the Ext-functor for  $C^*$ -algebras*, Inventiones Math., **63** (1981), 25–40.
- [10] J. Cuntz and W. Krieger: *A class of  $C^*$ -algebras and topological Markov chains*, Inventiones Math., **56** (1980), 251–268.
- [11] C. Davis: *Various averaging operations on subalgebras*, Illinois J. Math., **3** (1959), 538–553.
- [12] V. Deaconu: *Groupoids associated with endomorphisms*, Trans. Amer. Math. Soc., **347** (1995), 1779–1786.
- [13] E. Dynkin: *The exit space of a Markov process*, Russian Math. Surveys **24**, No. 4, (1969), 89–152.
- [14] E. Effros: *Dimensions and  $C^*$ -algebras*, CBMS Regional Conf. Series in Math., **46**, Amer. Math. Soc., 1981.
- [15] G. Elliott: *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra, **38** (1976), 29–44.
- [16] M. Enomoto, M. Fujii and Y. Watatani: *KMS states for gauge action on  $O_A$* , Math. Japon., **29** (1984), 607–619.
- [17] D. E. Evans: *On  $O_n$* , Publ. RIMS Kyoto Univ. **16** (1980), 915–927.
- [18] D. E. Evans: *The  $C^*$ -algebras of topological Markov chains*, Tokyo Metropolitan University Lecture Notes (1983).
- [19] D. E. Evans: *Quasi-product states on  $C^*$ -algebras*, Lecture Notes in Mathematics, No. 1132, Springer-Verlag, Berlin-New York, 1985, 129–151.
- [20] R. Exel: *KMS states for generalized gauge actions on Cuntz-Krieger algebras*, preprint, (2001).
- [21] R. Exel and M. Laca: *Cuntz-Krieger algebras for infinite matrices*, J. reine angew. Mth. (Crelle) **512** (1999), 119–172.

- [22] R. Exel and M. Laca: *Partial Dynamical Systems and the KMS Condition*, preprint, (2000).
- [23] T. Fack and O. Maréchal: *Sur la classification des symétries des  $C^*$ -algèbres UHF*, Canadian Journal of Math.
- [24] J. Feldman and C. Moore: *Ergodic equivalence relations, cohomologies, von Neumann algebras, I and II*, Trans. Amer. Math. Soc., **234** (1977), 289–359.
- [25] W. Feller: *An introduction to probability theory and its applications*, vol. II, John Wiley, N.Y.
- [26] A. H. Fan: *A proof of the Ruelle theorem*, Reviews Math. Phys. **7**, no. 8 (1995), 1241–1247.
- [27] A. H. Fan and Y. P. Jiang: *On Ruelle-Perron-Frobenius Operators. I. Ruelle theorem*, Commun. Math. Phys. **223** (2001), 125–141.
- [28] H.-O. Georgii: *Gibbs measures and phase transitions*, de Gruyter Studies in Mathematics, 1988.
- [29] T. Giordano, I. Putnam and C. Skau: *Affable equivalence relations and orbit structure of Cantor dynamical systems*, preprint 2002.
- [30] K. R. Goodearl: *Partially ordered abelian groups with interpolation*, Mathematical Surveys and Monographs, **20**, Amer. Math. Soc., 1986.
- [31] G. Keller: *Equilibrium States in Ergodic Theory*, London Mathematical Society Student Texts **42**, Cambridge University Press (1998).
- [32] S. Kerov: *The boundary of Young lattice and random Young tableaux*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **24**, Amer. Math. Soc., (1996), 133–158.
- [33] D. Kerr and C. Pinzari: *Noncommutative pressure and the variational principle in Cuntz-Krieger type  $C^*$ -algebras*, J. Funct. Anal., **188** (2002), 156–215.
- [34] W. Krieger: *On dimension functions and topological Markov chains*, Inventiones Math., **56** (1980), 239–250.
- [35] A. Kumjian: *On localizations and simple  $C^*$ -algebras*, Pacific J. Math. **112** (1984), 141–192.
- [36] A. Kumjian and J. Renault: *KMS states on  $C^*$ -algebras associated to expansive maps*, in preparation.
- [37] D. Lind and B. Marcus: *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.
- [38] P. Muhly, D. Williams: *Continuous trace groupoid  $C^*$ -algebras*, Math. Scand. **66** (1990), 231–241.
- [39] J. Neveu: *Bases mathématiques du calcul des probabilités*, Masson & Cie, Paris, 1964.
- [40] J. Renault: *A groupoid approach to  $C^*$ -algebras*, Lecture Notes in Mathematics, Vol. **793** Springer-Verlag Berlin, Heidelberg, New York (1980).
- [41] J. Renault: *AF equivalence relations and their cocycles*, Operator Algebras and Mathematical Physics, Conference Proceedings, Constanza 2001, the Theta Foundation (2003), 365–377.
- [42] J. Renault: *The Radon-Nikodym problem for approximately proper equivalence relations*, Ergodic Theory Dyn. System, **25** (2005), 1643–1672.
- [43] A. Török: *AF-algebras with unique trace*, Acta. Sci. Math. (Szeged) **55**, No 1-2 (1991), 129–139.
- [44] A. Vershik: *Equipped graded graphs, projective limits of simplices, and their boundaries*, Zapiski Nauchn. Semin. POMI **432**, (2015), 83–104, English translation to appear in J. Math. Sci.
- [45] A. Vershik: *Asymptotic theory of path spaces of graded graphs and its applications*, Japanese J. Math. **11**, No. 2, (2016), 151–218.
- [46] A. Vershik and S. Kerov: *Asymptotic character theory of the symmetric group*, Funct. Analysis and its Applications **15**, (1981), 25–36.
- [47] P. Walters: *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc., **236**, (1978), 121–153.
- [48] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, **79**, Springer-Verlag, New York-Berlin, 1982.
- [49] P. Walters: *Convergence of the Ruelle operator for a function satisfying Bowen’s condition*, Trans. Amer. Math. Soc., **353**, No 1 (2000), 327–347.
- [50] A. Wassermann: *Automorphic actions of compact groups on operator algebras*, Ph.D thesis, U. of Pennsylvania (1981).
- [51] M. Zinsmeister: *Formalisme thermodynamique et systèmes dynamiques holomorphes*, Panoramas et Synthèses, **4**, Société Mathématique de France, 1996.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'ORLÉANS, 45067 ORLÉANS, FRANCE  
*E-mail address:* `Jean.Renault@univ-orleans.fr`