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# ORDINAL REAL NUMBERS 1. The ordinal characteristic.

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## Abstract

In this paper are introduced the ordinal integers ,the ordinal rational numbers ,the ordinal real numbers ,the ordinal p-adic numbers ,the ordinal complex numbers and the ordinal quaternion numbers .It is also introduced the ordinal characteristic of linearly ordered fields. The final result of this series of papers shall be that the three different techniques of surreal numbers, of transfinite real numbers ,of ordinal real numbers give by inductive limit or union the same class of numbers known already as the class No and that would deserve the name the "infinitary totally ordered Newton-Leibniz realm of numbers ".

**Key words** :Hessenberg natural operations, principal ordinal,linearly ordered commutative fields,Archimedean complete fields, abstract additive valuation,formal power series fields,transfinite real numbers

**Subject Classification of AMS** 03,04,08,13,46

**§0 Introduction.** This is the third paper of a series of five papers that have as goal the definition of topological complete linearly ordered fields (continuous numbers) that include the real numbers and are obtained from the ordinal numbers in a method analogous to the way that Cauchy derived the real numbers from the natural numbers. We may call them linearly ordered Newton-Leibniz numbers. The author initiated and completed this research in the island of Samos in Greece during 1990-1992 .

The present papers define the topological and algebraic structure of the ordinal real numbers and does not refer at all to their stochastic interpretation. Nevertheless as in practical applications of pre-emptive Goal Programming in Operations Research and operating systems of computers, the non-Archimedean or lexicographic order is usually called **pre-emptive prioritization order** .the ordinal real numbers could as well be called (for the sake of practical applications) **Linearly ordered pre-emptive real numbers** .

In a communication (1992) that the author had with N.L. Alling and his group of researchers on analysis on surreal numbers, suggested the term ordinal real numbers instead of surreal numbers. Some years later and before the present work appears for publication, it appeared in the bibliography conferences about *real ordinal numbers* .

In these last three papers is studied a special Hierarchy of transcendental over the real numbers, linearly ordered fields that are characterized by the property that they are fundamentally (Cauchy ) complete. It shall turn out that they are isomorphic to the transfinite real numbers (see [Glazal A. (1937)]).The author was not familiar with the 5 pages paper of [ Glazal A. (1937)] ,and his original term was "transfinite real numbers". When one year later (1991) he discovered the paper by A. Glazal ,he changed the term to the next closest : "Ordinal Real Numbers" .One more year

later he proved that the transfinite real numbers, the surreal numbers and the ordinal real numbers were three different techniques leading to isomorphic field of numbers. He then suggested (1992) to researchers of surreal numbers, like N.L.Alling to use the more casual term “ordinal or transfinite real numbers “ for the surreal numbers. In the present work it is introduced a new, better, classifying and more natural technique in order to define them. This technique I call "*free operations-fundamental completion*". It

is actually the same ideas that lead to the process of construction of the real numbers from the natural numbers through fundamental (Cauchy) sequences. In the modern conceptual context of the theory of categories this may demand at least three adjunctions (see [MacLane S 1971]). It is developed their elementary theory which belongs to algebra. Their definition uses the Hessenberg operations of the ordinal numbers. It may be considered as making use of an infinite dimensional K-theory which is mainly not created yet. In this first paper it is also introduced the ordinal characteristic of any linearly ordered field. It is a principal ordinal number, that is of type  $\omega^{\alpha}$ . These numbers, as defined with the present technique of the "free operations-fundamental completion" and prior to the proof that the resulting linearly ordered fields are isomorphic to the transfinite real numbers (as in [Glazal A. (1937)]), we shall call Ordinal real numbers. The relevancy with the surreal numbers and the non-standard (hyper) real numbers, shall be studied in a later paper. In detail, the next Hierarchies are defined:

1) The **Ordinal natural numbers**, denoted by  $N_\alpha$ . 2) The **Ordinal integral numbers**, denoted by  $Z_\alpha$  3) The **Ordinal rational numbers**, denoted by  $Q_\alpha$  4) The **Ordinal p-adic numbers**, denoted by  $Q_{\alpha,p}$  5) The **Ordinal real numbers**, denoted by  $R_\alpha$  6) The **Ordinal complex numbers**, denoted by  $C_\alpha$  7) The **Ordinal quaternion numbers**, denoted by  $H_\alpha$ , of characteristic  $\alpha$ . The fields  $Q_{\alpha,p}$ ,  $R_\alpha$ ,  $C_\alpha$ ,  $H_\alpha$  are fundamentally (Cauchy) complete topological fields.

The field  $R_\alpha$  is also the unique maximal field of characteristic  $\alpha$  (that is, it is Hilbert complete), and the unique fundamentally (Cauchy) complete field of characteristic  $\alpha$ . It is also a real closed field, according to the theory of Artin-Schreier. These will be proved in the next paper on ordinal real numbers.

As it is known there are three more techniques and Hierarchies of transcendental over the real numbers, linearly ordered fields. Namely (in the historical order): The transfinite real numbers (see [Glazal A. 1937]), and the surreal numbers (see [Conway J.H (1976)]).

In this series of papers, it is proved (among other results) that all the previous three different techniques and Hierarchies give by inductive limit, or by union, the same class of numbers (already known as the class **No**).

## § 1. The ordinal characteristic of linearly ordered fields.

Corrective Remark (2022) : The ordinal characteristic is essentially a measurement of the size of a linearly ordered commutative field with a semi-

ring of Ordinal natural numbers (Hessenberg natural commutative operations in the ordinal numbers, as developed in the two previous papers-sections). We embed systems of ordinal natural numbers in a linearly ordered field, so that not “gaps” exist. There is always a minimal such system the natural numbers themselves. The definition of the ordinal characteristic of such ordinal natural numbers is always the supremum of the ordinals which are contained in it, and it is a principal ordinal numbers as we have described in the previous paper-section. Then we embed with monomorphisms and with 1-1 functions, such semi-rings of ordinal natural numbers in a linearly ordered commutative field so that the 0 and 1 of the ordinal real numbers goes to the 0, 1 of the linearly ordered field and there are no “gaps”, in other words the image is the minimal such possible set in the linearly ordered field. All such possible monomorphic with no gaps in a linearly ordered field, which is a set, give a set of corresponding ordinal characteristics of such semi-rings of ordinal natural numbers which is upper bounded, because of the cardinal and corresponding ordinal of the set and linearly ordered field. Thus as such ordinal are a subset of a well ordered set of ordinals it holds the supremum property, and there is such a supremum ordinal. Since also such a maximal embedding is also a semi-ring of ordinal natural numbers, this supremum is also a principal ordinal number which exist and its unique, it measures the size of the linearly ordered field and we call the it its ordinal characteristic. Having defined the ordinal characteristic as above the next definitions follow with minor corrections.

**Definition 0.** We remind the reader that a linearly (totally) ordered, double abelian semigroup (semiring)  $M$  is a set with two operations denoted by  $+, \cdot$ , such that with each one of them it is an abelian semigroup. Furthermore the distribution law holds for multiplication over addition. A linear ordering is supposed defined in  $M$  that satisfies the following compatibility conditions with the two operations 1) if  $x > y$ ,  $x' > y'$   $x, x', y, y' \in M$  then  $x + x' > y + y'$  and  $xx' + yy' > xy' + yx'$  (The symbol  $<$  is used for  $\leq$  and not equal) if  $M$  is also a monoid relative to the two operations, and zero is absorbent unit for  $M$ ,  $M$  is called *ordered double abelian monoid*. (semiring)

(e.g. The set of natural numbers, denoted by  $N$ ).

In the next we shall consider linearly (totally) ordered fields. (For a definition see [Lang S.] ch xi §1 pp 391).

Also in the next we shall use ordinal numbers. (For a reference to standard symbolism and definitions see [Kuratowski K.-Mostowski A. 1968] ch vii, [Cohn P.M. 1965] pp 1-36)

In the following paragraphs we will not avoid the use of larger totalities than the sets of the Zermelo-Frankel set theory, namely classes.

We may suppose that we work in the Zermelo-Frankel set theory, augmented with axioms for classes also, as is presented for instance in bibliography [Cohn P.M. 1965] p.1-36 with axioms A1-A11. We denote by  $\Omega_1$  the class of the ordinal numbers. (The last capital letter of the Greek alphabet with subscript 1). The axioms A1-A11 allow

for larger entities than sets, to define algebraic fields or integral domains or semi-groups. Hence we will also study classes that have two algebraic operations (Their Cartesian square treated as classes of sets of the form  $\{\{x, y\}, \{x\}\}$ , that is of ordered pairs) that satisfy the axioms of an algebraic field and have a subclass called the class of positive elements, with properties 1. 2., that they define a compatible ordering in the field (again as a class of ordered pairs) such classes that are ordered fields we will call again ordered fields and if we want to discriminate them from set-fields, especially when they are classes that are not sets, we will write for them that they are c-fields similarly we write c-integral domains or c-semigroups. We must not confuse the term "c-field" with the term "class-field" of the ordinary set-fields of "class-fields theory" (see [ Van der Waerden B.L 1970], [Artin E.-Tate J. 1967]). A subset (or subclass) denoted by  $X \subseteq F$  of a linearly ordered field  $F$ , is said to be cofinal with  $F$ , if for every  $a \in F$  there is a  $b \in X$  with  $a \leq b$ .

**Definition 1.** We say that an ordinal number  $\alpha'$  is contained in a linearly ordered field (or integral domain or double abelian monoid) denoted by  $F$ , if there is an ordinal  $\alpha$ ,  $\alpha' < \alpha$  and  $\Omega = \omega^{\omega^x}$ , (omega in the power of omega in the power of  $x$ ) where  $x$  is an ordinal, and a subset  $A$  of  $F^+ \cup \{0\}$  and a function  $h: W(\alpha) \rightarrow A$  which is an order isomorphism (similarity) of  $W(\alpha)$  and  $A$  and such that  $h(0)=0$  and if  $\beta$  is an ordinal number with  $\beta < \alpha$  then  $h(s(\beta))=h(\beta)+1$  in the field operations and furthermore the set  $A$  is closed to sum and product in the field (integral domain or double monoid) operations and isomorphic by  $h$  to the  $W(\alpha)$  relative to the Hessenberg natural operations, furthermore the closure in the order topology of the field of the set  $A$  (range of  $h$ ) is the **minimal** such set with the previous properties (so we ensure that there are no gaps at the limits that are included).

**Remark 2.** If an ordinal number  $\alpha'$  is contained in the field  $F$ , then also the sequent of  $\alpha'$ ,  $S(\alpha')$  is contained in  $F$ . This holds since the sequent of  $\alpha'$  is again in  $W(\alpha)$  where  $\alpha$  as in the definition above.

**Remark 3.** If the ordinal number  $\alpha$  is contained in the field  $F$ , then obviously every ordinal number less than  $\alpha$ , is also contained in the field  $F$ . *In the next, we will suppose (for simplification of symbolism) that if the ordinal  $\alpha$  is contained in  $F$ , the set  $\alpha$  is a subset of  $F$ , and also  $\alpha$  is the element  $h(\alpha)$  of the field  $F$ .* We fix a mapping  $h$  for each ordinal that is contained in  $F$ . So we can talk about the set of ordinals contained in  $F$  as if it is a subset of  $F$ . The set of ordinal numbers that are contained in a linearly ordered set-field, is obviously a non-empty set. (Because as  $F$  is linearly ordered,  $\text{char}F = \infty$  hence for every natural number  $n$ , we have that it is a (finite) ordinal contained in the field  $F$ ).

But even more by the remarks 2, 3, we have that the set of ordinals contained in a linearly ordered field, which of course by the non-Neuman definition of ordinals is itself an ordinal, is either of the form  $W(x)$  or  $W(x) \cup \{x\} = W(S(x))$  for some ordinal number  $x$  (in other words either it shall be a limit ordinal or it shall have a immediately previous ordinal). The last case is directly excluded (by remark 2) hence it is of the form  $W(x) = x$ , that is this set is itself a limit ordinal number. In case the linearly ordered field  $F$ , is a c-field then all the ordinals contained in  $F$  is again a set which is limit ordinal number, or the class  $\Omega_1$  of all ordinal numbers.

**Definition 4.** Let a linearly ordered set-field (or integral domain or double abelian monoid) denoted by  $F$ . Let  $\alpha$  be the set of ordinals contained in  $F$  (which is itself a limit ordinal number). We say that the field (or integral domain or double abelian monoid)  $F$  is of characteristic  $\alpha$  and we shall write  $\text{char}F = \alpha$ .

If  $F$  is a c-field we include the case of characteristic  $\Omega_1$  and we write  $\text{char}F = \Omega_1$  if all ordinals contained in  $F$  is the class  $\Omega_1$  and also it is a cofinal subclass with  $F$ .

**Remark .** In the case of a set-field  $F$  with  $\alpha = \text{char}F$ , we do not need to suppose that the subset of elements of  $F$  corresponding to the ordinal in  $\alpha$  by the definition 1 (it always exists ,by making use of the definition by transfinite induction and its version that uses only a set of functions sufficient for an inductive rule), see appendix A), is cofinal with  $F$ , as this is a consequence of the definition. For, if there is an element  $X_0 \in F$  with  $\beta < X_0$  for every ordinal number  $\beta$  with  $\beta \leq \alpha$ , then the set  $\alpha \cup \{X_0\}$  can be extended , with the field operations ,to its closure in the natural Hessenberg operations (a semiring) (see [Kyritsis C. Alt] ) and it becomes similar to an initial segment of a principal ordinal number Thus  $\alpha+1$  is an ordinal contained in  $F$ , contradiction with the definition of  $\alpha$  .

By the previous definitions we realize that every linearly ordered set-field has characteristic which is a limit ordinal number.

The fact that the linearly ordered field  $F$  has characteristic  $\omega$  (the least infinite ordinal) is equivalent with the statement that the field  $F$  is Archimedean.

In the followings when we will work on a linearly ordered field denoted by  $F$  of ordinal characteristic  $\alpha$ ,  $\alpha = \text{char}F$  (or  $\Omega_1 = \text{char}F$ ) we will supposed that is fixed an embedding of the ordinal numbers of the initial segment  $w(\alpha)$  in the set  $F$  (or of  $\Omega_1$  in  $F$ ).

If the characteristic is  $\omega$ , the embedding is obviously unique as it can be proved by finite induction.

**Remark.5** Let a linearly ordered field denoted by  $F$  .Obviously there is an extension which is a real field .Let us denote by  $R(F)$  the real closure of  $F$  .(For results of the theory of Artin-Schreier on real and real closed fields see e.g.[ Lang S. 1984] ch xi .or [Artin E.-Shreier O. 1927]) Since  $R(F)$  can be obtained by adjunction of the square roots of the positive elements of  $F$  and Zorn's Lemma on algebraic extensions see[Lang S. 1984] ch i proposition 2.10 theorem 2.11 pp 397), it is direct that the characteristic of the real closure  $R(F)$  is the same with that of  $F$ .

For the definitions of the terms infinite, finite, infinitesimal elements in an extension of such fields, see e.g.[ Lang S] ch xi paragraph 1 pp 391, the definitions can be given relative to extensions of any linearly ordered field to an other linearly ordered field ,and not only extensions of the real numbers.

## §2 The ordinal natural numbers $N$ . The ordinal- integers $Z$ .

Let  $w(\alpha)$  a principal initial segment of ordinal numbers. Let us denote by  $+$  and  $\cdot$  the Hessenberg's natural sum and product in  $w(\alpha)$ . They satisfy properties 0.1.2.3.4.5.6. after lemma 1 in §1 in [ Kyritsis C.1991 Alter]

**Definition 6.** The set  $w(\alpha) = \alpha$  where  $\alpha = \omega^x$  for some ordinal  $x$ , is an abelian double monoid relative to sum and product and furthermore it satisfies the cancellation law (see [ Kyritsis C. 1991 Alter] lemma 1 ).This set I call the (double abelian) monoid of ordinal natural numbers of characteristic  $\alpha$  and I denote it by  $N_\alpha$ . Thus  $N_\alpha = \alpha$ .

**Remark 7.** It is obvious that the (double abelian, well ordered ) monoid  $N_\alpha$ , is the minimal such monoid of characteristic  $\alpha$  and the embedding of the ordinal numbers

of  $W(\alpha)$  in it is unique. Furthermore it can be proved by transfinite induction that it is a unique factorization monoid (called simply factorial monoid also).

The additive cancellation law in  $\alpha$  has as a consequence that  $\alpha$  is monomorphically embedded in its Grothendieck group denoted by  $k(\alpha)$  (see [Lang S. 1984] Ch.1 §9 p. 44). Furthermore the Grothendieck group  $k(N_\alpha)$  can be ordered by defining the set of positive elements  $k(\alpha)^+ = \{v/v = (x,y) \text{ with } x,y \in w(\alpha) \text{ and } x > y\}$ . We remind the reader that if we denote by  $F_{ab}(\alpha)$  the free abelian group generated by  $\alpha$ , and by  $((x+y)-x-y)$  the normal subgroup of  $F_{ab}(\alpha)$  generated by elements of the form  $(x+y)-x-y$ ,

$$k(\alpha) \cong \frac{F_{ab}(\alpha)}{((x+y)-x-y)}$$

then

By  $(x,y)$  we denote the equivalence class that is defined in  $F_{ab}(\alpha)$  in the process of taking the quotient group  $F_{ab}(\alpha)/((x+y)-x-y)$  by the representative  $x+(-y)$ .

The first part of property 6. (lemma 1 in [Kyritsis C.1991 Alter]) guarantees that this ordering in  $k(\alpha)$  restricted on  $\alpha$  coincides with the usual ordering of ordinal numbers.

**Definition 8.** *The ordered Grothendieck group  $k(\alpha)$  of an initial segment of ordinals relative to natural sum, we call transfinite cyclic group of exponent  $\alpha$  and we denote it by  $\Gamma_\alpha$ . (by [Kuratowski K. Mostowski A. 1968] ch vii §7 pp 252-253 exercises 1.2.3.the ordinal  $\alpha$  has to be of the type  $\omega^x$ . If the ordinal  $\alpha$  is principal then I denote it also by  $Z_\alpha$ ).*

Every element of the group  $Z_\alpha$  is represented as a difference  $x-y$  with  $x,y \in w(\alpha)$ . Then we define multiplication in  $Z_\alpha$  by the rule

$$(*) (x-y).(x'-y')=(x.x'+y.y')-(xy'+x'y)$$

where sum and product are the natural sum and product in  $w(\alpha)$ . This makes  $Z_\alpha$  a commutative ring with unit (the element 1).

If  $(x-y)(x'-y') = 0$  and both  $(x-y)$ ,  $(x'-y')$  are not zero, we get by property 6 in lemma 1 in [Kyritsis C. 1991Alter] that  $xx'+yy' \neq xy'+yx'$  or  $(x-y)(x'-y') \neq 0$ , contradiction. Then one of

$(x-y)$ ,  $(x'-y')$  is zero that is the ring  $Z_\alpha$  has no divisors of zero and it is an integral domain. Remembering that  $Z_\alpha^+ = \{v|v \in Z_\alpha \text{ and } v = (x,+y) \text{ with } x,y \in w(\alpha) \text{ } x > y\}$ , by property 6 lemma 1 in [Kyritsis C. 1991 Alter], we get that the sum and product of elements of  $Z_\alpha^+$  are again elements of  $Z_\alpha^+$ . From all these we get:

**Lemma 9.** *The ring  $Z_\alpha$  is a linearly ordered integral domain of characteristic the principal ordinal  $\alpha$  (see § 1 Def.1).The set  $Z_\alpha^+$  is a linearly ordered double abelian monoid and  $Z_\alpha^+ \neq N_\alpha$*

**Definition 10 .** *The integral domain  $Z_\alpha$  I call ordinal integers of characteristic  $\alpha$  .*

The integral domain  $Z_\alpha$  of characteristic  $\alpha$  has minimality relative to its property of being an integral domain of characteristic  $\alpha$ , in the following sense: Every integral domain of characteristic  $\alpha$  contains a monomorphic image of  $Z_\alpha$ .

**Theorem 11 (Minimality).**

*Every integral domain  $Z_\alpha$  is minimal integral domain of characteristic  $\alpha$ . That is every integral domain of characteristic  $\alpha$ , contains a monomorphic image of  $Z_\alpha$ .*

*Proof.* Put  $R_\alpha$  an integral domain of characteristic  $\alpha$ , where  $\alpha$  is a principal ordinal number ( $\alpha = \omega^{\omega^x}$ ).

Then the initial segment  $w(\alpha)$  is contained in  $R_\alpha$  (more precisely an order preserving image of  $w(\alpha)$ ). The principal initial segment is closed to the integral domain operations and by theorem 13,14 of [Kyritsis C. 1991 Alter], they coincide with the natural sum and product of Hessenberg. Then, applying the construction of this paragraph for the integral-domain  $Z_\alpha$ , we remain inside the integral-domain  $R_\alpha$ , that is  $Z_\alpha \subseteq R_\alpha$ . This proves the minimality.

**Remark 12.** The ordinal integers are semigroup-rings of quotient monoids of semigroups that are used to define as semigroup-rings the hierarchy of integral domains of the transfinite integers (see [Gleyzal A. 1937] pp 586). I use the term hierarchy not only as a well ordered sequence but also as a net (thus partially ordered). The transfinite real numbers are thus an hierarchy.

The transfinite integers over the order-type  $\lambda$  symbolised by  $Z(\lambda)$ , is the semigroup-ring

(also module  $Z$ -algebra and integral domain) of the linearly ordered monoid  $\sum_{\lambda} \mathbb{N}$ ,

where  $\sum_{\lambda} \mathbb{N}$  is the coproduct, or direct sum denoted also by  $\coprod_{\lambda} \mathbb{N}$ , of a family of

isomorphic copies of  $\mathbb{N}$  with set of indices the order-type  $\lambda$ . Thus  $Z(\lambda) = Z[\sum_{\lambda} \mathbb{N}]$ . Thus any

ring of polynomials of a linearly ordered set of variables with integer coefficients is an

integral domain of transfinite integers and conversely. It can be proved

with the axiom of choice and transfinite induction, as in the case of

finite set of variables, that  $Z(\lambda)$  is a unique factorization domain. On the other hand the

Cantor normal form in the Hessenberg operations of the ordinal numbers (see lemma 6 in

[Kyritsis C. 1991 Alter]) gives that any element  $x$  of  $Z_\alpha$  is of the

form  $x = \omega^{x_1} y_1 + \dots + \omega^{x_n} y_n, y_i \in \mathbb{Z}, i = 1, \dots, n, n \in \mathbb{N}$   $x_i$  are ordinals with  $x_1 > \dots > x_n$ . The ordinal

powers of  $\omega$  in  $Z_\alpha$  is an abelian well ordered monoid (see e.g. [Neumann B.H. 1949] §2 pp

204-205) of ordinal characteristic  $\beta = \omega^\alpha$ , if  $\alpha = \omega^{\log(\alpha)}$ . Let us denote it by  $M_\beta$ . Actually  $M_\beta = \beta$ .

Let us denote by  $\omega^{\log(\alpha)}$ , or simply by  $\lambda_\alpha$  the order type of the Archimedean equivalent

classes of  $M_\beta$ . Then we get by the Cantor normal form that  $Z_\alpha = Z[M_\beta]$  (The semigroup ring

of  $M_\beta$ ). The monoid  $M_\beta$  can be obtained as quotient monoid of the free abelian

multiplicative monoid of  $\lambda_\alpha$  variables, which is the monoid  $\prod_{\lambda_\alpha} \mathbb{N}$ .

But  $Z[\prod_{\lambda_\alpha} \mathbb{N}] = Z(\lambda_\alpha)$ , which was the assertion to be proved.

**Remark 13** The equation  $Z_{\omega^{\omega^\alpha}} = Z[M_{\omega^\alpha}]$  gives an alternative, simpler definition of

the ordinal integers without the use of the Hessenberg multiplication, since the ordinal

powers of  $\omega$  coincide in the abelian Hessenberg operations and the usual ordinal operations

(see [Kyritsis C.1991 Alter] Remark 7.5) and without the use of the Grothentick group. The

monoid  $M_x$  is defined as the initial segment  $W(\omega^x)$  (or simply as the ordinal  $\omega^x$ ) in the

Hessenberg addition.

### §3 The definition of the fields $Q_\alpha, R_\alpha, C_\alpha, H_\alpha$ .



In this paragraph, I shall introduce the hierarchies of fields of ordinal rational ,real, complex ,quaternion numbers. These hierarchies give the unification of the other three techniques and hierarchies, namely of the transfinite real numbers, of the surreal numbers. Furthermore we introduced the hierarchies of transfinite complex and transfinite quaternion numbers.

**Definition 14.** *The localization (field of quotients) of the integral domain  $Z_\alpha$ , I will denote by  $Q_\alpha$  and I will call ordinal rational numbers (of characteristic  $\alpha$ ) (see [Lang S. 1984] ChII §3).*

**Remark.** Since we have that cancellation law holds, we do not have to use the Malcev-Neuman theorem (see [Cohn P.M. 1965] Ch VII §3. Theorem 3.8). We define as set of

positive element of  $Q_\alpha$  the set 
$$Q_\alpha^+ = \left\{ \frac{m}{1} \mid m \in Z_\alpha \text{ and } m \in Z_\alpha^+ \right\}$$
. It is elementary in algebra that if the integral domain is linearly ordered then also its field of quotients (localization) with the previous definition for its set of positive elements, is a linearly ordered field with the restriction of its ordering on the integral domain to coincide with the ordering of the integral domain. Obviously the ordinals of the initial segment of  $w(\alpha)$  are contained in  $Z_\alpha$  and also in  $Q_\alpha$ . By a direct argument, holds also that the characteristic of  $Q_\alpha$  is  $\alpha$ : Char  $Q_\alpha = \alpha$ .

**Remark** From the construction of  $Q_\alpha$  we infer easily that  $\aleph(Q_\alpha) = \aleph(\alpha)$  and if  $\alpha < \beta$  where  $\alpha, \beta$  are two principal ordinals then  $Q_\alpha \subseteq Q_\beta$ . The converse obviously holds.

**Lemma 15.** Every element  $x$  of the field  $Q_\alpha$  is of the form 
$$x = \frac{\omega^{\alpha_1} \cdot a_1 + \dots + \omega^{\alpha_n} \cdot a_n}{\omega^{\beta_1} \cdot b_1 + \dots + \omega^{\beta_m} \cdot b_m}$$
 where  $\alpha_i, \beta_j \in w(\alpha)$  and  $\alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0$ ,  $\beta_1 > \beta_2 > \dots > \beta_m \geq 0$  and  $a_i, b_j$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, m$  are finite integers.

*Proof.* Direct from the definition of localization and lemma 6 in [Kyritsis C. 1991Alter].

**Theorem 17. (Minimality)**

*The field  $Q_\alpha$  is a minimal field of characteristic  $\alpha$ , in the sense that every field of characteristic  $\alpha$ , contains the field  $Q_\alpha$  (more precisely an order preserving monomorphic image of  $Q_\alpha$ ).*

**Remark.** This property is already obvious for the field of rational numbers, that in the statement of Theorem 17 is denoted by  $Q_\omega$ .

*Proof.* Let a field of characteristic  $\alpha$ , that we denote by  $F_\alpha$ . Then the principal initial segment  $w(\alpha)$  of ordinals is contained in  $F_\alpha$  and the field-inherited operations coincide with the natural sum and product of Hessenberg (see theorem 14 in [Kyritsis C. 1991 Alter]). Then constructing first the integral domain  $Z_\alpha$  and afterwards its localization  $Q_\alpha$  we always remain in the field  $F_\alpha$ .

Thus  $Q_\alpha \subseteq F_\alpha$  (or more precisely  $h(Q_\alpha) \subseteq F_\alpha$  where  $h$  is a order-preserving monomorphism of  $Q_\alpha$  in to  $F_\alpha$ ) Q.E.D.

**Definition 18.** *The (strong) Cauchy completion of the topological field  $Q_\alpha$  we denote by  $R_\alpha$  and I call ordinal real numbers of characteristic  $\alpha$ .*

**Remark.** The process of extensions ,beginning with a principal initial ordinal  $\alpha = N_\alpha$  which is the minimal double, abelian monoid of

characteristic  $\alpha$ , and ending with the field  $R_\alpha$  which is the maximal field of characteristic  $\alpha$ , we call K-fundamental densification .

**Lemma 19.** *The characteristic of the (strong) Cauchy completion of a linearly ordered field  $F$ , is the same with that of the field  $F$ .*

*Proof.* If the characteristic of the field is  $\alpha$ , let us denote it by  $F_\alpha$ , and its completion by  $\hat{F}_\alpha$ . Obviously the characteristic of  $\hat{F}_\alpha$  is not less than  $\alpha$ .

Suppose that there is an ordinal  $\beta$  with  $\alpha < \beta$  which is contained in  $\hat{F}_\alpha$  (see Definition 1). Then there is a Cauchy net  $\{x_i | i \in I\}$  of elements of  $F_\alpha$  that converges to  $\beta$ . Let  $\varepsilon \in F_\alpha$   $0 < \varepsilon < 1$ , then there is  $i_0 \in I$  such that for every  $i \in I$   $i \geq i_0$   $x_i \in (b-\varepsilon, b+\varepsilon)$ . But this gives an element of  $F_\alpha$  greater than  $\alpha$ , hence than every element of  $F_\alpha$ , which is a contradiction. Thus  $\text{Char } R_\alpha = \alpha$ . Q.E.D.

**Corollary 20.** *The characteristic of  $R_\alpha$  is  $\alpha$ .*

From the definition of  $R_\alpha$  we infer that  $\aleph(R_\alpha) \leq 2^{\aleph(\alpha)}$  and that  $\alpha < \beta \Leftrightarrow R_\alpha \subsetneq R_\beta$  for two principal ordinals denoted by  $\alpha, \beta$ .

**Remark.21** We denote by  $R(\lambda)$  the transfinite real numbers of order-base  $\lambda$ . It holds by definition that  $R(\lambda) = R((LR^\lambda))$ , where  $LR^\lambda$  is the lexicographic product of a family of isomorphic copies of the real numbers  $R$ , with set of indices the order-type  $\lambda$ .

**Remark .** It is said that a field  $F$  has formal power series representation, if there is a formal power series ring  $R((G))$  and a ideal  $I$  of it such that  $F$  has a monomorphic image in  $R((G))/I$ . From the universal embedding property of the hierarchy of transfinite real numbers we get that every linearly ordered field has formal power series representation. Thus:

**Corollary 22.** *The fields of ordinal real numbers  $R$ , have formal power series representation, with real coefficients.*

**Definition 23** *The field  $C_\alpha = R_\alpha[i]$  I call ordinal complex numbers of characteristic  $\alpha$ .*

**Definition 24.** *The field  $C(\lambda) = R(\lambda)[i]$  we call transfinite complex numbers of base-order  $\lambda$ . Actually it is the field  $C(\lambda) = C((LR^\lambda))$ .*

**Definition 25.** *The quaternion extension field of the field  $R_\alpha$  (or of  $C_\alpha$ ) by the units  $i, j, k$  with  $i^2 = j^2 = k^2 = ijk = -1$ , I call the ordinal quaternion numbers of characteristic  $\alpha$  and I denote them by  $H_\alpha$ . They are non-commutative fields (following the terminology e.g. of A.Weil in [ Weil A. 1967]) that are transcendental extension of the non-commutative field  $H$  of quaternion numbers.*

**Definition 26.** *The formal power series fields  $H(\lambda) = H((LR^\lambda))$  we call transfinite quaternion numbers of base-order  $\lambda$ .*

For a proof that  $H((LR^\lambda))$  is a (non commutative ) field see [Neumann B.H.1949] part I.

#### §4 The ordinal p-adic numbers $Q_{\alpha,p}$ .

As it is known if  $F$  is a linearly ordered field, and  $K$  a linearly ordered subfield of the real numbers and  $F|K$  is an extension respecting the ordering, then this extension defines the order-valuation (see [N.L.Alling 1987] ch 6 § 6.00 pp 207). Actually every extension of any two linearly ordered fields  $F, K, K \subseteq F$ , respecting

the ordering, defines a place, thus a valuation  $v$ . (I use the place and valuation as are defined e.g. by O.Zariski in [Zariski O.-. Samuel P.1958] vol ii ch vi §2, §8.and not as are defined by A.Weil in [Weil A. 1967] ch iii or by v.der Waerden in [Van der Waerden B.L. 1970] vol ii ch 18 .The definition of Zariski is equivalent with the definition of v.der Waerden only for the non Archimedean valuations of the latter).

The place-ring is the  $F_v = \{x/x \in F \text{ and there are } a, b \in K \text{ with } a < x < b \}$ . The maximal ideal of the place (or valuation  $v$ ) is the ideal of infinitesimals of  $K$  relative to  $F$ .

This valuation we call extension - valuation (and the corresponding place extension - place) It has as special case the order valuation .The rank of the extension- place (see [Zariski O.-. Samuel P.1958] vol.II §3 pp 9) we call the rank of the extension .If  $\text{char}(F) > \text{char}(K)$  then the extension is transcendental ,and has transcendental degree and basis ;the latter is to be found in the ideal of infinitesimals or in the set of infinite elements .

**Definition 27 .**

*Let  $F$  a field of ordinal characteristic. Let  $R$  a subring of  $F$  that has  $F$  as its field of quotients. Let  $p$  a prime ideal of  $R$ , such that the triple  $(pR_p, R_p, F)$  where  $R_p$  is the localization of  $R$  at  $p$ , defines a place of  $F$ . Such a place (or valuation denoted by  $v_p$ ) I call  $p$ -adic of the field  $F$ . In the valuation topology of the valuation  $v_p$ , that has a local base of zero the ideals of  $R$  ) the field  $F$  is a topological field and the (strong) Cauchy completion I denote by  $F_p$ , it is a (topological field ) and I call  $p$ -adic extension field of  $F$ .*

**Definition 28.** For  $F=Q_\alpha$  and  $R=Z_\alpha$  in the previous definition the field  $Q_{\alpha,p}$  I call ordinal  $p$ -adic numbers of characteristic  $\alpha$ .

**Final remark** .Using inductive limit ,or union of the elements of the hierarchies of the previous ordinal and transfinite number systems, we get corresponding classes of numbers .The classes of ordinal natural, integer, rational, real, complex, quaternion numbers denoted respectively by  $\Omega_1$ , (or  $\Omega_n$ ),  $\Omega_1Z$ ,  $\Omega_1Q$ ,  $\Omega_1R$ ,  $\Omega_1C$ ,  $\Omega_1H$ .

And the classes of transfinite integer, rational, real, complex, quaternion numbers denoted respectively by:

CZ, CQ, CR, CC, CH.

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#### List of special symbols

$\omega$  : Small Greek letter omega, the first infinite number.

$\alpha, \beta$  : Small Greek letter alfa, an ordinal.

$\alpha = \omega^{\omega^x}$  Ordinal alpha  $\alpha$  equal to omega in the power of omega in the power of x

$\Omega^0$  : Capital Greek letter omega with the superscript zero.

$F^a$  : Capital letter with superscript a. The of algebraic elements of a field F.

char F : The characteristic of a field denoted by F.

$\cong$  : Equivalence relation of Commensurateness.

$\sim$  : Equivalence relation of comparability.

tr.d.(x) : The transcendence degree initial of words tr.(ascendance) and d.(egree).

$N(x)$  : Aleph of x, the cardinality of the set X. N: the first capital letter of the Hebrew alphabet.

cf(X)=cf(Y) : The sets x and Y are cofinal.

$W(\alpha)$  : Initial segment of ordinal numbers defined by the ordinal number a.

$\oplus, \circ$  : Natural sum and product of G. Hessenberg plus and point in parenthesis.

$N_\alpha, Z_\alpha, Q_\alpha, R_\alpha,$ : Double-lined capital letters with subscript small Greek letters

$C_\alpha, H_\alpha$  namely transfinite positive integers, intergals, rationals reats, complex and quatenion numbers.

$Z_{\alpha_1}^* \omega$  : The dual lually compact abelian groups of the transfinite integers  $Z_\alpha$ . The capital letter Z double-lined wiuth subscripts two Greek let- $\alpha$  (alpha) and  $\omega$  (omega) and superscript a star

$T_\alpha$  : Transfinite circle groups: Capital letter T with subscript a small Greek letter.

$^*X, ^*R$  et.c : A non-standard enlergement structure capital letter X with left superscript a star.

$\xi\text{No}$  : A surreal number field of characteristic  $\xi$ . A small Greek letter followed by the symbol No.

$C, RC^*R, \text{No}$  : The c-structures (classes) previous symbols following the capital

$CN, CZ, CQ, \dots$  latin letter C

$CC, CH$

$\hat{X}$  : Strong Cauchy completion of a topological space capital letter with cap.

$\Sigma$  : Capital Greek letter sigma symbol for summation.

$\overset{\circ}{D}_\alpha$  : The open full-binary tree of height  $\alpha$ . Capital latin D with subscript  $\alpha$  a small Greek letter and in upper place a small zero.

The ordinal real numbers 1. The ordinal characteristic.

## APPENTIX A.

### A MORE EFFECTIVE FORM OF DEFINITION BY TRANSFINITE INDUCTION.

1. Given a set  $Z$  and an ordinal  $\alpha$ , let  $\Phi$  be a set of  $\xi$ -sequences with the properties:

a) If  $f$  belongs to  $\Phi$  then  $f/w(\xi)$  belongs to  $\Phi$  for every  $\xi \leq \text{domain of } f$ .

b) For every  $\xi < \alpha$  there is at least one  $f$  belonging to  $\Phi$  with  $\xi = w(\xi) = \text{domain}(f)$  and values belonging to  $Z$ .

c) If  $f_\xi$  is an  $\alpha$ -sequence of  $\xi$ -sequences of  $\Phi$  such that whenever  $\gamma < \xi_1$

,  $\xi_2 < \alpha$ ,  $f_{\xi_1}/w(\gamma) = f_{\xi_2}/w(\gamma)$ ; then the  $\alpha$ -sequence  $c_\alpha(\xi) = f_\xi(\xi)$ , belongs to  $\Phi$  also.

For each function  $h$  in  $Z^\Phi$ , there is one and only one transfinite sequence  $f$  defined on  $\xi \leq \alpha$ ,  $f$  in  $\Phi$  and such that  $f(\xi) = h[f/w(\xi)]$  for every  $\xi \leq \alpha$ .

The function  $h$  is called **a recursive rule for  $\Phi$** . The set  $\Phi$  with the properties a). b), c), is called **sufficient** for recursive rules.

*Proof:* Not much different than the ordinary form of definition by transfinite induction.