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We consider a way of defining quantum Hamiltonians involving particle creation and annihilation based on an *interior-boundary condition* (IBC) on the wave function, where the wave function is the particle-position representation of a vector in Fock space, and the IBC relates (essentially) the values of the wave function at any two configurations that differ only by the creation of a particle. Here we prove, for a model of particle creation at one or more point sources using the Laplace operator as the free Hamiltonian, that a Hamiltonian can indeed be rigorously defined in this way without the need for any ultraviolet regularization, and that it is self-adjoint. We prove further that introducing an ultraviolet cut-off (thus smearing out particles over a positive radius) and applying a certain known renormalization procedure (taking the limit of removing the cut-off while subtracting a constant that tends to infinity) yields, up to addition of a finite constant, the Hamiltonian defined by the IBC.

MSC: 81T10, 81Q10, 47F05. Key words: Ultraviolet divergence problem; renormalization in quantum field theory; self-adjoint Hamiltonian; self-adjoint extensions of the Laplace operator; point interactions; particle-position representation; ultraviolet cut-off.

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1 Introduction

Interior-boundary conditions (IBCs) provide a method of defining Hamiltonian operators with particle creation and annihilation that has received little attention so far. These Hamiltonians are related to extensions of differential operators to singular functions, somewhat like Hamiltonians describing point interactions. At least for some models, the IBC approach provides an alternative solution to the problem of ultraviolet (UV) divergences. In this paper, we consider a specific non-relativistic model of quantum field theory with point-shaped sources for creation and annihilation of bosonic particles, for which the UV problem has been solved by renormalization. For this model we prove that the IBC Hamiltonian H_{IBC} is a well-defined self-adjoint operator and agrees, up to addition of a constant, with the renormalized Hamiltonian H_∞.

    The UV problem, in the form relevant to us, is the following. In the Fock space formulation of quantum field theories, the Hamiltonian involves annihilation and

creation operators $a(\chi)$ and $a^*(\chi)$ that annihilate or create particles with wave function χ. For square-integrable functions χ these operators are densely defined operators on Fock space. However, in most physically relevant field theories the particles are created and annihilated at points in space, and χ is a distribution that is not square-integrable. For our model, χ will be the Dirac δ-distribution. While $a(\delta)$ can still be given mathematical sense as a densely defined operator, this is no longer possible for $a^*(\delta)$. Renormalization then amounts to making sense of the limit $\chi \to \delta$.

The IBC approach allows for the direct definition of Hamiltonians H_{IBC}, without a renormalization procedure, also in cases where χ is not square-integrable. It starts out from the particle-position representation of a vector in Fock space as a wave function on a configuration space of a variable number of particles. In this representation, the absorption of particle 1 by particle 2 corresponds to a jump from a configuration with 1 at the same location as 2 to the configuration without 1, while the emission of a particle corresponds to the opposite jump. These processes are therefore related to the flux of probability into (or out of) the set $\mathcal{C}$ of collision configurations in configuration space (i.e., the configurations with two particles at the same location). As we will show, a non-trivial such flux is possible for wave functions satisfying a suitable boundary condition, with $\mathcal{C}$ regarded as the boundary of configuration space; the relevant boundary condition is a relation between the values of the wave function at the two configurations connected by the jump just mentioned; since it relates a boundary point to an interior point of another sector, we call this condition an *interior-boundary condition* (IBC). Since wave functions in the domain of the Hamiltonian satisfy the IBC, the domain is not the same as that of a free field Hamiltonian. As we will show, the only common element of these domains is the zero vector. As a consequence, IBC Hamiltonians cannot be obtained as perturbations of free field Hamiltonians in any simple way.

While we discuss more general situations in [TeTu15, LaSch18, Lam18], we focus in our present rigorous study on the simple model of a single non-relativistic bosonic scalar field whose quanta are created or annihilated at one or more point sources at fixed locations. For a single source at the origin, the formal expression for the Hamiltonian reads

$$H_\chi := \mathrm{d}\Gamma(h) + g\big(a(\chi) + a^*(\chi)\big) \tag{1.1}$$

with $\chi = \delta$. The free Hamiltonian $\mathrm{d}\Gamma(h)$ is the second quantization of the non-relativistic 1-particle Hamiltonian $h = -\Delta + E_0$, E_0 is a real constant called the rest energy, and g is a real coupling constant. Operators of the form (1.1) belong to the class of *van Hove Hamiltonians* [vH52, Schw61, Der03]. Our model can be regarded as a non-relativistic variant of the Lee model [Lee54] or Schweber's scalar field model [Schw61, Sec. 12a].

A rigorous definition of van Hove Hamiltonians is discussed by Dereziński [Der03] for general h and χ. For our case of $h = -\Delta + E_0$ and $\chi = \delta$, the Hamiltonian H_∞ of [Der03] can be obtained through the following renormalization procedure. Consider a sequence of square-integrable functions χ_n approaching the δ distribution, $\chi_n \to \delta$. Then the sequence H_{χ_n} of Hamiltonians defined by (1.1) converges, after subtraction of a suitable divergent sequence of constants E_n, to H_∞. As described in more detail in Section 3, for $E_0 > 0$ there exists a unitary Weyl operator W_∞ such that $H_\infty = W_\infty^* \, \mathrm{d}\Gamma(h) \, W_\infty$ with domain $D(H_\infty) = W_\infty^* D(\mathrm{d}\Gamma(h))$. For $E_0 \in \mathbb{R}$, [Der03] provides an explicit formula for $\mathrm{e}^{-\mathrm{i}H_\infty t}$ and defines H_∞ as its generator. For a broader discussion of the UV problem, see, e.g., [vH52, Lee54, Schw61, GlJa85, GlJa87] and also Section 3.

Here we show instead that the IBC Hamiltonian H_{IBC} corresponding to the formal expression

$$H_\delta := \mathrm{d}\Gamma(h) + g\big(a(\delta) + a^*(\delta)\big) \tag{1.2}$$

is rigorously defined, self-adjoint, and (if $E_0 \geq 0$) bounded from below. The domain of H_{IBC} is explicitly characterized in terms of interior-boundary conditions. The action of H_{IBC} involves extensions of the Laplacian to functions singular on the set $\mathcal{C}$ where one particle hits the origin. Moreover, we show that H_{IBC} is equal, up to addition of a finite constant, to the Hamiltonian H_∞ obtained through renormalization. This yields a new explicit characterization of the domain of H_∞ and its action thereon that is not easily available otherwise. Thus, one conclusion from our results is that quantum field Hamiltonians obtained through renormalization can have a simple and explicit form when expressed in the particle-position representation, albeit not in terms of creation and annihilation operators but in terms of IBCs. And, as mentioned already, they are no longer defined on the domain of the free operator $\mathrm{d}\Gamma(h)$, but $D(\mathrm{d}\Gamma(h)) \cap D(H_\infty) = \{0\}$.

As a mathematical problem we have to study an infinite system of inhomogeneous boundary value problems. Here, the configuration space is the disjoint union of n-particle configuration spaces called sectors, and the boundary on each sector is a union of codimension-three planes. A particular difficulty arises from the fact that, in sectors with more than one particle, these planes intersect. This makes the regularity issues more complicated, and general approaches to elliptic problems with boundaries of higher codimension (e.g., [Ma91]) cannot be applied directly. The intersections of these planes play an important role in the theory of point interactions involving more than two particles, see [Min11, CDFMT12, CDFMT15, MiOt17, MoSe17]. See also Remark 5.7 at the end of Section 5 for the relation of our results to the theory of abstract boundary value problems (e.g., [BM14]). In our case, some of the technical difficulties associated with the boundary value problem could be circumvented if we contented ourselves with proving merely *essential* self-adjointness, as we do for the generalized models of Section 4. However, in that

case we do not obtain an explicit characterization of the domain of self-adjointness. Moreover, the enhanced understanding of these boundary value problems provided by our direct approach proves useful when dealing with further variants of the IBC approach and point interactions. In particular, in [LaSch18] the IBC approach is applied to a large class of models with dynamical sources, including the Nelson model [Nel64] and the Fröhlich polaron. In [Lam18] the model of the present paper is generalized to dynamical sources, a case that is of particular interest as no rigorous definition of the corresponding Hamiltonian was known before.

The plan of the paper is as follows: In Section 2 we motivate and define the IBC Hamiltonian $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ and state the main theorem about its self-adjointness for a single point source at the origin. In Section 3 we discuss the relation of the IBC Hamiltonian to a Hamiltonian obtained from a standard renormalization procedure. In Section 4 we explain that our results also apply to the situation of *several* (finitely many) point sources that can emit and absorb particles, located at fixed points in $\mathbb{R}^3$. Furthermore, we also provide in Section 4 a discussion of a 4-parameter family of IBCs and possibilities for further generalizations. In Sections 5–7 and the Appendix, we provide the proofs: In Section 5 we prove symmetry of H_{IBC} based on the regularity results provided in the Appendix. In Section 6 (essential) self-adjointness is proved by combining the symmetry result with the explicit knowledge of a core of the renormalized Hamiltonian H_{∞}. In Section 7 we prove the statements on generalized IBC Hamiltonians from Section 4.

Let us end the introduction with remarks on related literature. IBCs have been considered in the past, in some form or another, in [LaPei30, Mo51a, Mo51b, Mo51c, Tho84, MoLo91, Yaf92, TuGe04]. Recent and upcoming works exploring various aspects of IBCs include [TeTu15, TeTu16, KeSi16, Gal16, DGTTZ18, LaSch18, Lam18]. Introductory presentations of the kind of models considered here can be found in [TeTu15, TeTu16], and the physical motivation is discussed in [TeTu15]. Landau and Peierls [LaPei30] obtained conditions similar to IBCs when trying to formulate quantum electrodynamics in the particle-position representation, although their Hamiltonian was still ultraviolet divergent (and thus mathematically ill defined). Moshinsky [Mo51a, Sec. III] considered (as an effective description of nuclear reactions) a model with IBCs that is essentially equivalent to ours (including the 4-parameter family of IBCs discussed in Section 4), except that he considered only the sectors with $n = 0$ and $n = 1$ particles; he did not provide rigorous results about the Hamiltonian. Yafaev [Yaf92] independently considered the same model (again only the sectors with $n = 0$ and $n = 1$ particles), proved that the Hamiltonian is well defined and self-adjoint, and showed that the 4-parameter family mentioned above exhausts all possible IBC Hamiltonians in this case. Thomas [Tho84] considered a model analogous to ours with moving sources, but only (what corresponds to) the sectors with $n = 2$ and $n = 1$ particles [Tho84, Sec. III], respectively [Tho84, Sec. II] with $n = 1$ and $n = 0$ particles, proving self-adjointness of the corresponding Hamil-

tonian. Moshinsky and Lopez [MoLo91] proposed a non-local kind of IBC for the Dirac and Klein-Gordon equations. Tumulka and Georgii [TuGe04, Sec. 6] considered IBCs for boundaries of codimension 1 (whereas the boundary relevant here has codimension 3) and did not provide rigorous results. Keppeler and Sieber [KeSi16] described a physical reasoning leading to IBCs and discussed IBCs in 1 space dimension (though not rigorously). Galvan [Gal16] suggested another approach towards a well defined Hamiltonian that has strong parallels to the IBC approach.

The mathematical study of Hamiltonians with IBCs is closely related to that of point interactions, a field that has recently received renewed attention. Hamiltonians for N-particle systems with point interactions were constructed rigorously using quadratic forms by Correggi, Dell'Antonio, Finco, Michelangeli, Teta [CDFMT12, CDFMT15] and by Moser, Seiringer [MoSe17]. The problem was approached from the point of view of self-adjoint extensions by Minlos [Min11] and more recently by Michelangeli and Ottolini [MiOt17] (see also references therein for a more complete bibliography). Note that most of the literature on point interactions concerns fermionic systems. We expect that the IBC approach can also be applied to creation and annihilation of fermions.

2 The IBC Hamiltonian

We model the emission and absorption of non-relativistic particles at a point in $\mathbb{R}^3$, which we choose to be the origin. We thus call the origin the "source" and may think of it as a different kind of particle (which however remains at a fixed location).

Let $\mathfrak{H} := L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathbb{C})$ be the one-particle Hilbert space, $\mathfrak{H}^n := \mathrm{Sym}\,\mathfrak{H}^{\otimes n}$ its n-fold symmetric tensor product, and $\mathfrak{F} := \Gamma(\mathfrak{H}) = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{H}^n$ with $\mathfrak{H}^0 := \mathbb{C}$ the symmetric Fock space over $\mathfrak{H}$. An element ψ of $\mathfrak{F}$ has the form $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots)$ with

$$\psi^{(n)} = \psi^{(n)}(x_1, \ldots, x_n) \in L^2(\mathbb{R}^{3n}) \tag{2.1}$$

symmetric under permutations of its arguments and $\sum_{n=0}^{\infty} \|\psi^{(n)}\|_{\mathfrak{H}^n}^2 < \infty$. For a bounded operator T on $\mathfrak{H}$, an operator $\Gamma(T)$ on $\mathfrak{F}$ is defined by $(\Gamma(T)\psi)^{(n)} = T^{\otimes n}\psi^{(n)}$, and for a self-adjoint operator h (possibly unbounded), we define $\mathrm{d}\Gamma(h)$ as the generator of $\Gamma(\mathrm{e}^{-ith})$. Its action is given by

$$(\mathrm{d}\Gamma(h)\psi)^{(n)} = \sum_{j=1}^{n} h_j \psi^{(n)}, \tag{2.2}$$

where $h_j = \mathbf{1} \otimes \ldots \otimes h \otimes \ldots \otimes \mathbf{1}$ is h acting on the jth factor. From now on we reserve the symbol h for the free one-particle Hamiltonian,

$$(h, D(h)) = (-\Delta + E_0, \mathrm{H}^2(\mathbb{R}^3)). \tag{2.3}$$

As a little digression, we point out how to set up a Hamiltonian with ultraviolet cut-off. We write $\overline{z}$ for the complex conjugate of $z \in \mathbb{C}$. For $\chi \in \mathfrak{H}$, the annihilation operator

$$(a(\chi)\psi)^{(n)}(x_1, \ldots, x_n) := \sqrt{n+1} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\chi(x)} \, \psi^{(n+1)}(x, x_1, \ldots, x_n) \qquad (2.4)$$

and its adjoint, the creation operator

$$(a^*(\chi)\psi)^{(n)}(x_1, \ldots, x_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \chi(x_j) \, \psi^{(n-1)}(x_1, \ldots, \hat{x}_j, \ldots, x_n) \qquad (2.5)$$

(where ˆ denotes omission) are densely defined, closed operators on $\mathfrak{F}$ that are in-finitesimally $\mathrm{d}\Gamma(h)$-bounded when $E_0 > 0$. Thus, for $E_0 > 0$ and any coupling constant $g \in \mathbb{R}$, the total Hamiltonian H_χ defined in (1.1) is self-adjoint on the domain of $\mathrm{d}\Gamma(h)$ by the Kato-Rellich theorem.

We now explain how to construct explicitly an operator H_{IBC} that captures, as we believe, the physical meaning of "H_δ" and agrees, as we will show, with the renormalized Hamiltonian up to addition of a finite constant. Recall that with the free Schrödinger evolution generated by the Laplacian on $\mathrm{L}^2(\mathbb{R}^3)$ there is associated a probability current

$$j^\psi(x) = 2 \, \mathrm{Im} \, \overline{\psi(x)} \, \nabla \psi(x) \,. \qquad (2.6)$$

In order to allow for annihilation or creation of particles at the origin, a non-vanishing probability current into or out of the origin must be possible. Using spherical coordinates $r = |x|$ and $\omega = \frac{x}{|x|} \in S^2 = \{v \in \mathbb{R}^3 : |v| = 1\}$, this current is

$$
\begin{aligned}
j_0^\psi &:= 2 \lim_{r \to 0} \int_{S^2} \mathrm{d}\omega \, r^2 \, \omega \cdot \mathrm{Im} \, \overline{\psi(r\omega)} \, \nabla \, \psi(r\omega) \\
&= 2 \lim_{r \to 0} \int_{S^2} \mathrm{d}\omega \, r^2 \, \mathrm{Im} \, \overline{\psi(r\omega)} \, \partial_r \, \psi(r\omega) \,.
\end{aligned}
\qquad (2.7)
$$

However, for j_0^ψ to be non-vanishing, ψ or $\partial_r \psi$ must be sufficiently singular at the origin. Since such singular functions are not in the standard domain $\mathrm{H}^2(\mathbb{R}^3)$ of the Laplacian, we need to consider the one-particle Laplace operator on a domain that includes singular functions that allow for non-vanishing currents into and out of the origin. Of course, such operators cannot be self-adjoint, since they cannot generate unitary groups.[1] In order to obtain a self-adjoint Hamiltonian and a unitary

---

[1] Note that operators with δ-like potentials are defined in a similar way by enlarging the domain of the Laplacian, cf. [DFT08]. However, in order to obtain a self-adjoint operator, an additional condition of the form $\lim_{r \to 0} (\partial_r r \psi(r\omega) - \alpha \, r \, \psi(r\omega)) = 0$ with $\alpha \in \mathbb{R}$ is imposed, precisely to ensure $j_0^\psi = 0$.

evolution on Fock space one thus needs to compensate the loss of probability in one sector by a corresponding gain in another sector. This is achieved by connecting different sectors with boundary conditions. Here, the configuration space is $\cup_{n=0}^{\infty} \mathbb{R}^{3n}$, and the "boundary" of its n-particle sector is the set

$$\mathcal{C}^n := \Big\{ x \in \mathbb{R}^{3n} \ \Big| \ \prod_{j=1}^{n} |x_j| = 0 \Big\} \tag{2.8}$$

of those n-particle configurations with at least one particle at the origin. (This is the relevant set of collision configurations here; at these configurations, one of the moving particles collides with the source.) The "interior-boundary condition" connects the wave function $\psi^{(n)}$ on $\mathcal{C}^n$ with the wave function $\psi^{(n-1)}$ one sector below.

We now prepare for the precise definition of H_{IBC}. Define the operator Δ_n to be the Laplacian with domain $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^n) \subset \mathrm{L}^2(\mathbb{R}^{3n})$, which is defined as the closure of $C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^n)$ in the H^2-norm. We then set

$$\big(\Delta_n^*, D(\Delta_n^*)\big) \ \text{is the adjoint of } \big(\Delta_n, \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^n)\big). \tag{2.9}$$

Since Δ_n is densely defined, closed and symmetric, the adjoint Δ_n^* extends Δ_n and its domain is given by (cf. [ReSi75, Sect. X.1])

$$D(\Delta_n^*) = D(\Delta_n) \oplus \ker(\Delta_n^* - \mathrm{i}) \oplus \ker(\Delta_n^* + \mathrm{i}) \,. \tag{2.10}$$

We will always regard $D(\Delta_n^*)$ as a Banach space with the graph norm of Δ_n^*. Combining the Δ_n^* yields an operator $\Delta_{\mathfrak{F}}^*$ on Fock space, whose action is given by

$$(\Delta_{\mathfrak{F}}^* \psi)^{(n)} := \Delta_n^* \psi^{(n)} \,, \tag{2.11}$$

for those $\psi \in \mathfrak{F}$ such that $\psi^{(n)} \in D(\Delta_n^*)$.

The role of the annihilation operator $a(\delta)$ will be played by an operator A on Fock space that we define sector-wise by[2]

$$(A\psi)^{(n)}(x_1, \ldots, x_n) := \frac{\sqrt{n+1}}{4\pi} \lim_{r \to 0} \partial_r \, r \int_{S^2} \mathrm{d}\omega \, \psi^{(n+1)}(r\omega, x_1, \ldots, x_n) \,. \tag{2.12}$$

Its dense domain will be specified later. It is not difficult to see that for $\psi^{(n+1)} \in \mathrm{H}^2(\mathbb{R}^{3(n+1)}) \cap \mathfrak{H}^{n+1}$,

$$(A\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \, \psi^{(n+1)}(0, x_1, \ldots, x_n) = (a(\delta)\psi)^{(n)}(x_1, \ldots, x_n) \,. \tag{2.13}$$

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[2] Here and throughout the paper, we follow the convention, in order to write fewer brackets, that a derivative operator acts on all factors to the right of it, not just the one immediately to the right, unless otherwise indicated by brackets. Thus, in (2.12), ∂_r acts also on ψ.

However, some $\psi^{(n+1)}(r\omega,\dots)$ in the domain of H_{IBC} diverge like $1/r$ as $r \to 0$, and A is an extension of $a(\delta)$ to such functions.

The boundary conditions are formulated in terms of an operator B on Fock space that can again be defined sector-wise by

$$(B\psi)^{(n)}(x_1,\dots,x_n) := -4\pi\sqrt{n+1}\,\lim_{r\to 0}\,r\,\psi^{(n+1)}(r\omega, x_1,\dots,x_n)\,. \tag{2.14}$$

We will define B on a dense domain where, in particular, the right hand side does not depend on ω. Again it is easy to see that for $\psi^{(n+1)} \in \mathrm{H}^2(\mathbb{R}^{3(n+1)}) \cap \mathfrak{H}^{n+1}$ we have $(B\psi)^{(n)} = 0$.

In the one-particle sector, $n = 1$, the domain $D(\Delta_1^*)$ is explicitly known and it is straightforward to prove that A and B are well defined functionals on $D(\Delta_1^*)$. For $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$ define the function

$$f_\gamma(x) := -\frac{1}{4\pi}\frac{\mathrm{e}^{-\gamma|x|}}{|x|}\,. \tag{2.15}$$

Clearly, $f_\gamma \in \mathrm{L}^2(\mathbb{R}^3)$ but $f_\gamma \notin \mathrm{H}^2(\mathbb{R}^3)$. Moreover, $\Delta_1^* f_\gamma = \gamma^2 f_\gamma$ and f_γ is the unique L^2-solution to this equation. Consequently, with (2.10) it follows that

$$D(\Delta_1^*) = D(\Delta_1) \oplus V \qquad V = \mathrm{span}\Big\{ f_\gamma \Big| \gamma \in \{(1 \pm \mathrm{i})/\sqrt{2}\}\Big\}\,. \tag{2.16}$$

Then, writing $\psi \in D(\Delta_1^*)$ as $\psi_0 + \phi$ with $\psi_0 \in D(\Delta_1)$ and $\phi \in V$ and integrating by parts in spherical coordinates, one finds that that the degree of asymmetry of Δ_1^* can be expressed by A and B, that is

$$\langle \varphi, \Delta_1^* \psi \rangle_{\mathfrak{H}} - \langle \Delta_1^* \varphi, \psi \rangle_{\mathfrak{H}} = \langle B\varphi, A\psi \rangle_{\mathbb{C}} - \langle A\varphi, B\psi \rangle_{\mathbb{C}}\,. \tag{2.17}$$

We will give a rigorous proof of this equation and generalize it to the case $n \geq 2$ in Propositions 5.1 and 5.4 in Section 5. We remark that this implies that $-\Delta_1$ has a one-parameter family of self-adjoint extensions, known as point interactions (cf. [AGHH88]). Their domains correspond to one-dimensional subspaces of V on which the right hand side of Equation (2.17) vanishes.

To illustrate the importance of Equation (2.17), we define the simplest possible IBC Hamiltonian on the truncated Fock space $\mathfrak{F}^{(1)} := \mathbb{C} \oplus \mathrm{L}^2(\mathbb{R}^3)$ by

$$H_{\text{IBC}}^{(1)} := \begin{pmatrix} 0 & gA \\ 0 & -\Delta_1^* + E_0 \end{pmatrix} \tag{2.18}$$

on the domain

$$D_{\text{IBC}}^{(1)} := \Big\{ (\psi^{(0)}, \psi^{(1)}) \in \mathfrak{F}^{(1)} \,\Big|\, \psi^{(1)} \in D(\Delta_1^*),\, B\psi^{(1)} = g\psi^{(0)} \Big\}\,. \tag{2.19}$$

Here $B\psi^{(1)} = g\psi^{(0)}$ is the interior-boundary condition (IBC). Equation (2.17) now implies that, contrary to what it may seem like, $H_{\text{IBC}}^{(1)}$ is symmetric: for $\varphi, \psi \in D_{\text{IBC}}^{(1)}$

$$
\begin{aligned}
\langle \varphi, H_{\text{IBC}}^{(1)} \psi \rangle_{\mathfrak{F}^{(1)}} &- \langle H_{\text{IBC}}^{(1)} \varphi, \psi \rangle_{\mathfrak{F}^{(1)}} = \\
&= -\langle \varphi^{(1)}, \Delta_1^* \psi^{(1)} \rangle_{\mathfrak{H}} + \langle \Delta_1^* \varphi^{(1)}, \psi^{(1)} \rangle_{\mathfrak{H}} + \langle \varphi^{(0)}, gA\psi^1 \rangle_{\mathbb{C}} - \langle gA\varphi^{(1)}, \psi^{(0)} \rangle_{\mathbb{C}} \\
&\overset{(2.17)}{=} \langle A\varphi^{(1)}, B\psi^{(1)} \rangle_{\mathbb{C}} - \langle B\varphi^{(1)}, A\psi^{(1)} \rangle_{\mathbb{C}} + g\langle \varphi^{(0)}, A\psi^1 \rangle_{\mathbb{C}} - g\langle A\varphi^{(1)}, \psi^{(0)} \rangle_{\mathbb{C}} \\
&\overset{\text{IBC}}{=} g\langle A\varphi^{(1)}, \psi^{(0)} \rangle_{\mathbb{C}} - g\langle \varphi^{(0)}, A\psi^1 \rangle_{\mathbb{C}} + g\langle \varphi^{(0)}, A\psi^1 \rangle_{\mathbb{C}} - g\langle A\varphi^{(1)}, \psi^{(0)} \rangle_{\mathbb{C}} \\
&= 0.
\end{aligned}
\tag{2.20}
$$

It is not difficult to see (and was also shown in [Yaf92]) that $H_{\text{IBC}}^{(1)}$ is even self-adjoint.

Our main result states that also the natural extension of $H_{\text{IBC}}^{(1)}$ to the whole Fock space is (essentially) self-adjoint.

Theorem 2.1. *For every $g, E_0 \in \mathbb{R}$ the operator*

$$
H_{\text{IBC}} := -\Delta_{\mathfrak{F}}^* + d\Gamma(E_0) + gA
\tag{2.21}
$$

is essentially self-adjoint on the domain

$$
D_{\text{IBC}} := \left\{ \psi \in \mathfrak{F} \;\middle|\; \begin{array}{l} \psi^{(n)} \in D(\Delta_n^*) \cap \mathfrak{H}^n \text{ for all } n \in \mathbb{N}, \\ H\psi \in \mathfrak{F}, \; A\psi \in \mathfrak{F}, \text{ and } B\psi = g\psi \end{array} \right\}.
\tag{2.22}
$$

If $g \neq 0$ then $D_{\text{IBC}} \cap D(d\Gamma(-\Delta)^{1/2}) = \{0\}$. For $E_0 \geq 0$, the Hamiltonian H_{IBC} is bounded from below.

If moreover $E_0 > 0$, the domain of self-adjointness $D(\overline{H}_{\text{IBC}})$ equals D_{IBC} and $D(\overline{H}_{\text{IBC}}) \cap D(d\Gamma(h)^{1/2}) = \{0\}$. In this case, the spectrum of H_{IBC} is given by $\{E_{min}\} \cup [E_{min} + E_0, \infty)$, and $E_{min} = g^2 \sqrt{E_0}/4\pi$ is a simple eigenvalue.

Note that the first two conditions in (2.22) just ensure that H maps the domain D_{IBC} back into Fock space. The third condition, $A\psi \in \mathfrak{F}$, might be redundant and follow from the second one, but we cannot show that. The last condition,

$$
B\psi = g\psi,
\tag{2.23}
$$

is the interior-boundary condition, which connects the limiting behavior of $\psi^{(n)}$ at the boundary of the n-particle sector (where one particle reaches the origin) with the wave function $\psi^{(n-1)}$ one sector below. In particular, the IBC (2.23) immediately yields that if $\psi^{(n)} \neq 0$, then $\psi^{(k)} \neq 0$ for all $k > n$, and hence the Fock vacuum does not belong to D_{IBC}.

Formally, an analogous computation to the one for $H_{\text{IBC}}^{(1)}$ shows that H_{IBC} is symmetric (see the proof of Corollary 5.5). However, in order to establish Equation (2.17) for $n \geq 2$, we need to first investigate the regularity of functions in the

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adjoint domain $D(\Delta_n^*)$. This will be carried out in Section 5, with the main result given by Proposition 5.4. The proof of (essential) self-adjointness in Section 6 uses the symmetry established in Section 5, a dense domain of coherent states contained in (2.22) and a Weyl operator to be discussed in Section 6.2.

3 The connection to renormalization

As mentioned already, the formal expression H_δ as in (1.2) can be regularized by means of an ultraviolet cut-off, then the cut-off can be removed (while constants E_n tending to $\pm\infty$ get subtracted) in order to obtain a renormalized Hamiltonian H_∞. Our main result in this section, Theorem 3.1, asserts that H_{IBC} agrees with H_∞ (up to addition of a finite constant relative to the standard choice of E_n). We state Theorem 3.1 in Section 3.1 and then put it into perspective in Section 3.2 by connecting it to known facts, techniques, and hitherto open questions about H_∞.

3.1 Definition of H_∞ and relation to H_{IBC}

We approximate the formal Hamiltonian H_δ with regularized (cut-off) Hamiltonians

$$H_n = \mathrm{d}\Gamma(h) + g\Big(a(\chi_n) + a^*(\chi_n)\Big) = H_0 + H_{I_n} \tag{3.1}$$

with any choice of $\chi_n \in \mathrm{L}^2(\mathbb{R}^3)$ such that $\chi_n \to \delta$ as $n \to \infty$ in the sense that $\hat{\chi}_n \to \hat{\chi}_\infty := \hat{\delta} = (2\pi)^{-3/2}$ pointwise with $\|\hat{\chi}_n\|_\infty$ uniformly bounded. Here $\mathcal{F}\chi = \hat{\chi}$ denotes the Fourier transform of $\chi \in \mathrm{L}^2(\mathbb{R}^d)$. It is easy to see using standard arguments (and will be explained below) that if $E_0 > 0$ then $H_n - E_n$ converges in the strong resolvent sense for

$$E_n := -g^2 \langle \chi_n, h^{-1}\chi_n \rangle_{\mathrm{L}^2} \,. \tag{3.2}$$

Note that for $E_0 > 0$ the free one-particle operator $h = -\Delta + E_0 \geq E_0 > 0$ is invertible. The limit is called the renormalized Hamiltonian,

$$H_\infty := \lim_{n \to \infty} (H_n - E_n) \,. \tag{3.3}$$

For $E_0 \leq 0$, a modification of the same procedure (or alternatively a formula for the unitary group $\mathrm{e}^{-iH_\infty t}$) allows to define a Hamiltonian H_∞ as well [Der03]. However, we will compare H_∞ to H_{IBC} only for $E_0 > 0$.

Theorem 3.1. *For $E_0 > 0$, the renormalized operator $(H_\infty, D(H_\infty))$ agrees with $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ up to an additive constant:*

$$D_{\mathrm{IBC}} = D(H_\infty) \quad and \quad H_{\mathrm{IBC}} = H_\infty + \frac{g^2\sqrt{E_0}}{4\pi}\,\mathbf{1}_{\mathfrak{F}} \,. \tag{3.4}$$

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Theorem 3.1 is established in Section 6. Together with Theorem 2.1, it provides a new characterization of $D(H_\infty) = D_{\mathrm{IBC}}$, and of the action of H_∞ thereon (2.21), and shows that $D(H_\infty) \cap D(H_0^{1/2}) = \{0\}$.

3.2 Remarks on the renormalization procedure

The above described renormalization scheme is a particularly simple case of a somewhat more general renormalization procedure that can be applied to a wider class of UV divergent Hamiltonians with the following common structure. There is a self-adjoint operator $(H_0, D(H_0))$ and a sequence of operators H_{I_n} that are small perturbations of H_0 in the sense that

$$H_n := H_0 + H_{I_n} \tag{3.5}$$

is self-adjoint on $D(H_0)$. If the interaction operator H_{I_n} converged as $n \to \infty$ to an operator that is relatively (form-)bounded by H_0 with relative bound smaller than one, then no renormalization would be necessary. In a typical manifestation of the UV problem, however, H_{I_n} does not converge. But in the cases of interest, there is a sequence of numbers $E_n \to \pm\infty$ such that $H_\infty = \lim_{n\to\infty}(H_n - E_n)$ exists in the strong resolvent sense.

In the examples we have in mind, the essential steps in finding this sequence E_n and proving the convergence of $H_n - E_n$ are, first, to construct a certain sequence of unitary operators W_n on Fock space, called dressing transformations, such that $W_n H_n W_n^*$ assumes a manageable form; second, to split $W_n H_n W_n^*$ into

$$W_n H_n W_n^* = H_n' + E_n \tag{3.6}$$

such that H_n' converges in the strong resolvent sense to a well defined operator H_∞'. Third, one shows that W_n has a strong limit W_∞ (which is automatically unitary). Then it follows that

$$H_n - E_n = W_n^* H_n' W_n \xrightarrow{n\to\infty} W_\infty^* H_\infty' W_\infty = H_\infty \tag{3.7}$$

in the strong resolvent sense.

Depending on the concrete model, the determination of the limiting Hamiltonian $H_\infty' = \lim_{n\to\infty} H_n'$ can be more or less tricky and, as a consequence, its domain can be more or less explicit. In all examples discussed in the following, W_n leaves invariant the domain $D(H_0)$, but this is no longer true for W_∞.

In his seminal paper [Nel64], Nelson showed that the model nowadays named after him can be renormalized according to the general scheme just sketched. He used the so-called Gross transformation for W_n and was able to characterize $(H_\infty', D(H_\infty'))$ as a form perturbation of H_0. Hence, he could not explicitly determine $D(H_\infty')$ but merely conclude that $D(H_\infty') \subset D(H_0^{1/2})$.

Whenever H'_∞ is an operator-bounded perturbation of H_0, one has $D(H'_\infty) = D(H_0)$ and $D(H_\infty) = W^*_\infty D(H_0)$ can be determined through the mapping properties of W^*_∞. Recently, Griesemer and Wünsch [GrWü16] proved that the Fröhlich Hamiltonian, which describes polarons, is of that type. In this case, one can define H_∞ also directly via its quadratic form without the detour via the dressing transformation. However, then the domain of H_∞ remains unknown, while the result of [GrWü16] provides an explicit characterization of it. In our model (1.2), the situation is even simpler, since it turns out that $H'_n = H'_\infty = H_0$.

After the existence of a self-adjoint renormalized Hamiltonian H_∞ is established, two questions remain in general open. First, is there a direct characterization of the domain $D(H_\infty) = W^*_\infty D(H'_\infty)$? And second, how does H_∞ act explicitly? As Nelson [Nel64] put it:

It would be interesting to have a direct description of the operator H_∞. Is $D(H_\infty) \cap D(H_0^{1/2}) = 0$?

The answer to the second question has been given by Griesemer and Wünsch for the Fröhlich Hamiltonian in [GrWü16] and for the Nelson model in [GrWü17] by studying the mapping properties of W^*_∞. A direct description in terms of IBCs, and thus a complete answer to both questions, is provided for our model in Theorem 3.1, and for the Fröhlich and Nelson Hamiltonians in [LaSch18].

Here is what the dressing transformation W_n looks like for our model (1.2). Since $h^{-1}\chi_n \in \mathrm{L}^2(\mathbb{R}^3)$ for $n \leq \infty$, the field operator

$$\Phi(h^{-1}\chi_n) := a(h^{-1}\chi_n) + a^*(h^{-1}\chi_n) \tag{3.8}$$

is self-adjoint. Therefore,

$$W_n := \mathrm{e}^{-\mathrm{i}\Phi(igh^{-1}\chi_n)} \tag{3.9}$$

is unitary for all $n \leq \infty$. It is straightforward to show that (3.6) now holds with E_n as in (3.2) and $H'_n := \mathrm{d}\Gamma(h)$. The proof can be found in Section 6.3, or, for example, also in [Deck04, Der03]. Then $\lim_{n\to\infty} E_n = -\infty$, and $H'_\infty = \lim_{n\to\infty} H'_n = \mathrm{d}\Gamma(h)$ clearly exists. As a consequence,

$$H_\infty = W^*_\infty \, \mathrm{d}\Gamma(h) \, W_\infty \qquad \text{on} \quad D(H_\infty) = W^*_\infty D(\mathrm{d}\Gamma(h)) \,. \tag{3.10}$$

4 Variants of the IBC Hamiltonian

4.1 General interior-boundary conditions

The IBC $B\psi = g\psi$ discussed in the previous sections is not the only possibility of implementing interior-boundary conditions for the Laplacian. In this section we

present a four-parameter family of different interior-boundary conditions that all lead to a self-adjoint Hamiltonian on Fock space. In a certain sense, this family covers all possible types of IBCs.

The wider class of IBCs involves, instead of the values of the wave function on the boundary (like a Dirichlet boundary condition), a linear combination of the values and the derivative of the wave function on the boundary (like a Robin boundary condition); such IBCs were formulated in [TeTu15, TeTu16] for boundaries of codimension 1 (and are also considered in [SchTu18] for particle creation, where the boundary has codimension 3). Specifically, in this wider class, we replace

$$B \to e^{i\theta}(\alpha B + \beta A), \quad A \to e^{i\theta}(\gamma B + \delta A), \tag{4.1}$$

where $\theta \in [0, 2\pi)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are such that

$$\alpha\delta - \beta\gamma = 1, \tag{4.2}$$

so that four of the five parameters can be chosen independently. We absorb the coupling constant g into the constants $\alpha, \beta, \gamma, \delta$. That is, we replace the IBC $B\psi = g\psi$ by

$$e^{i\theta}(\alpha B + \beta A)\psi = \psi \tag{4.3}$$

and the Hamiltonian $H_{\text{IBC}} = -\Delta_{\mathfrak{F}}^* + d\Gamma(E_0) + gA$ by

$$\tilde{H}_{\text{IBC}} = -\Delta_{\mathfrak{F}}^* + d\Gamma(E_0) + e^{i\theta}(\gamma B + \delta A). \tag{4.4}$$

The previous IBC (2.23) and Hamiltonian (2.21) are obviously contained in this scheme by chosing $\theta = 0 = \beta = \gamma$ and $\alpha^{-1} = g = \delta$. As discussed in detail in [SchTu18], the phase θ can be removed by means of the gauge transformation $\psi^{(n)} \to e^{-i\theta n}\psi^{(n)}$ if there is a single source, but not if there are several sources with different θ's, a situation that we consider in the next section. We refrain from stating and proving the analogue to Theorem 2.1 also for $\tilde{H}_{\text{IBC}}$, although it could be proved along the same lines as for H_{IBC}. Instead, Theorem 4.1 below implies already a statement that is merely slightly weaker, namely that, for $E_0 > 0$, $\tilde{H}_{\text{IBC}}$ is essentially self-adjoint on a dense domain satisfying the IBC (4.3).

To which extent does the family $\tilde{H}_{\text{IBC}}$ cover all possible Hamiltonians with IBCs? Yafaev [Yaf92] showed that for the model on the truncated Fock space $\mathbb{C} \oplus L^2(\mathbb{R}^3)$ with either zero or one particle all possible extensions of the (not densely defined) operator

$$H^\circ = (0, -\Delta) \quad \text{on} \quad D(H^\circ) = \{0\} \oplus C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \tag{4.5}$$

are of the above type. On Fock space, however, one has in principle much more freedom. We could connect different sectors by different IBCs, i.e., make $\theta, \alpha, \beta, \gamma, \delta$ all depend on n, or even let them depend on the configuration of the other particles. But if we exclude such a dependence, then Yafaev's result shows that the family $\tilde{H}_{\text{IBC}}$ is complete.

4.2 IBCs for multiple sources

We now consider a finite number N of sources fixed at (pairwise distinct) locations $\xi_1, \ldots, \xi_N \in \mathbb{R}^3$. To keep things simple, we assume $E_0 > 0$ for the remainder of this section. For each source ξ_i, $1 \leq i \leq N$, we choose parameters

$$v_i := (\theta_i, \alpha_i, \beta_i, \gamma_i, \delta_i) \in [0, 2\pi) \times \mathbb{R}^4 \tag{4.6}$$

which fullfill separately

$$\alpha_i \delta_i - \beta_i \gamma_i = 1 \qquad 1 \leq i \leq N \,. \tag{4.7}$$

We write v for $(v_1, \ldots, v_N)$. For suitable $\psi \in \mathfrak{H}$, define

$$A_i \psi := \lim_{x \to \xi_i} \partial_{r_i} (r_i \psi(x)) \,, \qquad B_i \psi := -4\pi \lim_{x \to \xi_i} (r_i \psi(x)) \,, \qquad \text{where } r_i := |x - \xi_i| \,, \tag{4.8}$$

and

$$X_i := \mathrm{e}^{\mathrm{i}\theta_i} (\alpha_i B_i + \beta_i A_i) \,, \qquad Y_i := \mathrm{e}^{\mathrm{i}\theta_i} (\gamma_i B_i + \delta_i A_i) \,, \qquad 1 \leq i \leq N \,. \tag{4.9}$$

The corresponding Fock space operators

$$X_i^{\mathfrak{F}}\big|_{\mathfrak{H}^{n+1}} := \sqrt{n+1}\, X_i \otimes \mathbf{1}_{\mathfrak{H}^n} \,, \qquad Y_i^{\mathfrak{F}}\big|_{\mathfrak{H}^{n+1}} := \sqrt{n+1}\, Y_i \otimes \mathbf{1}_{\mathfrak{H}^n} \tag{4.10}$$

are densely defined in $\mathfrak{F}$. Then $(\Delta_1^*, D(\Delta_1^*)) := (\Delta_1, C_0^\infty(\mathbb{R}^3 \setminus \{\xi_1, \xi_2, \ldots, \xi_N\}))^*$ is a closed but non-symmetric operator on $\mathfrak{H}$. Nevertheless, we will use the symbol $\mathrm{d}\Gamma(-\Delta_1^*)$ to denote the operator which acts as $-\sum_{j=1}^n \mathbf{1}_{1,\ldots,j-1} \otimes \Delta_1^* \otimes \mathbf{1}_{j+1,\ldots,n}$ on the n-th sector of Fock space. It is well known [AGHH88, DFT08] that

$$\tilde{h} := -\Delta_1^* + E_0 \quad \text{on} \quad U(v) := \bigcap_{i=1}^N \ker X_i \subset D(\Delta_1^*) \tag{4.11}$$

is a self-adjoint operator that is bounded from below. It is called the N-center point interaction with energy offset E_0 and parameters $a_i := \frac{\alpha_i}{\beta_i}$, where $\beta_i = 0$ corresponds to $a_i = +\infty$.

Theorem 4.1. *Let $E_0 > 0$ and v be any set of parameters obeying the condition (4.7) given above. There exists a dense subspace $\tilde{D}_{\mathrm{IBC}} \subset \mathfrak{F}$ such that for $\psi \in \tilde{D}_{\mathrm{IBC}}$ the IBCs*

$$X_i^{\mathfrak{F}} \psi = \psi \qquad \forall\, 1 \leq i \leq N \tag{4.12}$$

hold and such that

$$\tilde{H}_{\mathrm{IBC}} := \mathrm{d}\Gamma(-\Delta_1^* + E_0) + \sum_{i=1}^{N} Y_i^{\mathfrak{F}} \tag{4.13}$$

is essentially self-adjoint on $\tilde{D}_{\mathrm{IBC}}$. If $\tilde{h}$ is strictly positive[3], then $\tilde{H}_{\mathrm{IBC}}$ is bounded from below and possesses a unique ground state.

Remark 4.2. Suppose that $\beta_i = 0$ for all $1 \leq i \leq N$. Then

$$\tilde{h} = h = (-\Delta_1^* + E_0, \mathrm{H}^2(\mathbb{R}^3)) \tag{4.14}$$

is the free one-particle operator, which is strictly positive. In this case $\tilde{H}_{\mathrm{IBC}}$ is bounded from below for any choice of distinct points $\xi_1, \ldots, \xi_N$.

Remark 4.3. Let $N = 1$. In this case, for all values of $a_1 = \frac{\alpha_1}{\beta_1} \in (-\infty, \infty]$, the essential spectrum of the point-interaction operator is $\sigma_{ess}(\tilde{h}) = [E_0, \infty)$, cf. [AGHH88]. If $a_1 \geq 0$, then $\tilde{h}$ has no point spectrum. If $a_1 < 0$, then there is exactly one eigenvalue λ_0 of $\tilde{h}$. It is explicitly given as $\lambda_0 = E_0 - 16\pi^2 a_1^2$. Therefore $\tilde{H}_{\mathrm{IBC}}$ is bounded from below if $a_1 > 0$ or if $a_1 \leq 0$ but still $a_1 > -\frac{\sqrt{E_0}}{4\pi}$.

Under certain assumptions on v and E_0, we are able to further characterize $\tilde{H}_{\mathrm{IBC}}$. In order to state the theorem, we have to introduce some abbreviations: For any $\lambda > 0$ let

$$w_i^\lambda(x) := f_{\sqrt{\lambda}}(x - \xi_i) = -\frac{\mathrm{e}^{-\sqrt{\lambda}|x-\xi_i|}}{4\pi|x-\xi_i|} \in \mathrm{L}^2(\mathbb{R}^3) , \tag{4.15}$$

and define the matrices

$$G_{ij}^\lambda := w_i^\lambda(\xi_j) = w_j^\lambda(\xi_i) , \tag{4.16}$$

and

$$S_{ij}(\lambda) := \delta_{ij}\mathrm{e}^{\mathrm{i}\theta_i}\left(\alpha_i + \frac{\sqrt{\lambda}}{4\pi}\beta_i\right) + (1 - \delta_{ij})\mathrm{e}^{\mathrm{i}\theta_i}\beta_i G_{ij}^\lambda , \tag{4.17}$$

where δ_{ij} denotes the Kronecker symbol. Note that S depends on all of $\lambda, \xi_1, \ldots, \xi_N$, $v_1, \ldots, v_N$.

Theorem 4.4. *Let $(\tilde{H}_{\mathrm{IBC}}, \tilde{D}_{\mathrm{IBC}})$ also denote the unique self-adjoint extension that has been constructed in Theorem 4.1. If the vector $(1, 1, \ldots, 1)^T$ lies in the range of $S(E_0)$, then there exists $\phi \in D(\Delta_1^*) \subset \mathfrak{H}$ such that we have the equality*

$$\mathrm{e}^{\mathrm{i}\Phi(\mathrm{i}\phi)} \tilde{H}_{\mathrm{IBC}} \, \mathrm{e}^{-\mathrm{i}\Phi(\mathrm{i}\phi)} = \mathrm{d}\Gamma(\tilde{h}) + C(\phi)\mathbf{1}_{\mathfrak{F}} \tag{4.18}$$

as self-adjoint operators on Fock space $\mathfrak{F}$. Here $C(\phi) \in \mathbb{R}$ is a constant, Φ has been defined in (3.8) and $\mathrm{d}\Gamma(\tilde{h})$ denotes the second quantization of $\tilde{h} = (-\Delta_1^* + E_0, U)$.

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[3]i.e., there is a positive constant c such that $\tilde{h} \geq c$.

The definition of $\tilde{D}_{\mathrm{IBC}}$ in terms of coherent states obtained from vectors in $D(\Delta_1^*)$, as well as the proof of Theorems 4.1 and 4.4 and the explicit form of the ground state, of ϕ and of $C(\phi)$ are given in Section 7. As discussed in detail in [SchTu18], $\tilde{H}_{\mathrm{IBC}}$ is time reversal invariant if and only if all θ_i coincide up to addition of an integer multiple of π.

5 Symmetry of H_{IBC}

In this section we prove symmetry of $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$. The main ingredient is Equation (2.17), which will be proved in Proposition 5.1 below, and its generalization to $n \geq 2$.

Proposition 5.1. *For $n = 1$ the maps A and B are well-defined continuous linear functionals on $D(\Delta_1^*)$ and for any $\varphi, \psi \in D(\Delta_1^*)$ we have*

$$\langle \varphi, \Delta_1^* \psi \rangle_{\mathfrak{H}} - \langle \Delta_1^* \varphi, \psi \rangle_{\mathfrak{H}} = \langle B\varphi, A\psi \rangle_{\mathbb{C}} - \langle A\varphi, B\psi \rangle_{\mathbb{C}} . \tag{5.1}$$

Proof. Recall that $D(\Delta_1^*) = D(\Delta_1) \oplus V$ with $V = \operatorname{span} \left\{ f_\gamma \,\middle|\, \gamma \in \{(1 \pm \mathrm{i})/\sqrt{2}\} \right\}$. On the functions f_γ one easily evaluates

$$A f_\gamma = \frac{\gamma}{4\pi} \quad \text{and} \quad B f_\gamma = 1 . \tag{5.2}$$

On $D(\Delta_1)$ we have $A = 0$, since for $\psi \in C^1(\mathbb{R}^3)$

$$A\psi = \frac{1}{4\pi} \lim_{r \to 0} \int_{S^2} \left(\psi(r\omega) + r\omega \cdot \nabla \psi(r\omega) \right) \mathrm{d}\omega = \psi(0) , \tag{5.3}$$

and the point evaluation is continuous on $D(\Delta_1) = \mathrm{H}_0^2(\mathbb{R}^3 \setminus \{0\})$. Clearly also $B = 0$ on $D(\Delta_1)$. Now since H_0^2 is a closed subspace of $D(\Delta_1^*)$, the projection $p : D(\Delta_1^*) \to D(\Delta_1^*)/\mathrm{H}_0^2 \cong V$ is continuous. Thus $A, B : D(\Delta_1^*) \to \mathbb{C}$ are continuous as they can be written as the composition of p with a linear functional on a finite dimensional space.

The difference on the left hand side of (2.17) vanishes if either φ or ψ are elements of $\mathrm{H}_0^2(\mathbb{R}^3 \setminus \{0\})$, and so does the right hand side by the considerations above. Thus, it is sufficient to verify the claim for $\varphi = f_{\gamma_1}$, $\psi = f_{\gamma_2}$. As noted before we have $\Delta_1^* f_\gamma = \gamma^2 f_\gamma$ and

$$\langle f_{\gamma_1}, f_{\gamma_2} \rangle = \frac{1}{4\pi} \int_0^\infty \mathrm{d}r \, \mathrm{e}^{-(\overline{\gamma}_1 + \gamma_2)r} = \frac{1}{4\pi(\overline{\gamma}_1 + \gamma_2)} . \tag{5.4}$$

Thus

$$\begin{aligned}
\langle f_{\gamma_1}, \Delta_1^* f_{\gamma_2} \rangle - \langle \Delta_1^* f_{\gamma_1}, f_{\gamma_2} \rangle &= \frac{\gamma_2^2 - \overline{\gamma}_1^2}{4\pi(\overline{\gamma}_1 + \gamma_2)} = \frac{\gamma_2 - \overline{\gamma}_1}{4\pi} \\
&= \overline{B f_{\gamma_1}} A f_{\gamma_2} - \overline{A f_{\gamma_1}} B f_{\gamma_2} .
\end{aligned} \tag{5.5}$$

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$\square$

Proposition 5.1 can be understood as a generalized integration-by-parts formula for the singular functions in $D(\Delta_1^*)$. Its generalization to the case $n \geq 2$, given in Proposition 5.4 below, requires knowledge of the regularity properties of functions in $D(\Delta_n^*)$. These are rather subtle, as the following example shows:

Let $f \in \mathrm{H}^{-1/2}(\mathbb{R}^3)$, and set

$$\psi(x, y) = -\frac{\mathrm{e}^{T|x|}}{4\pi|x|} f(y), \tag{5.6}$$

where $\mathrm{e}^{T|x|}$ denotes the contraction semi-group with generator $T = -\sqrt{-\Delta_y + 1}$, $D(T) = \mathrm{H}^1(\mathbb{R}^3)$, acting on $\mathrm{L}^2(\mathbb{R}_y^3)$. One easily checks that $\psi \in \mathrm{L}^2(\mathbb{R}^6)$ with norm proportional to $\|f\|_{\mathrm{H}^{-1/2}}$. By the smoothing properties of the semi-group, ψ is a smooth function on $\mathbb{R}^6 \setminus \{x = 0\} \supset \mathbb{R}^6 \setminus \mathcal{C}^2$. The action of Δ_2^* on ψ is thus given by differentiating on $\mathbb{R}^6 \setminus \mathcal{C}^2$ and yields

$$\Delta_2^* \psi = \psi, \tag{5.7}$$

so $\psi \in D(\Delta_2^*)$ is an eigenfunction of Δ_2^* with eigenvalue one. However, applying only the differential expression Δ_x gives $\Delta_x \psi = T^2 \psi$, which is not an element of $\psi \in \mathrm{L}^2(\mathbb{R}^6)$ unless $f \in \mathrm{H}^{3/2}(\mathbb{R}^3)$. Thus we have $\psi \in D(\Delta_2^*)$, but applying the Laplacian in only one of the variables does not give a square-integrable function, i.e. $\psi \notin D(\Delta_1^* \otimes 1)$. Furthermore, the formula for ψ suggests that $B\psi = \sqrt{2}f \in \mathrm{H}^{-1/2}(\mathbb{R}^3)$ is a distribution, so the "boundary values" of ψ on the collision configurations $\mathcal{C}^2$ will be of low regularity.

We now state our results concerning the definition of the operators A and B on $D(\Delta_n^*)$, which we prove in Appendix A. To allow for a lighter notation, we will use the symbol Ω_n to denote the configuration space of n particles, that is $\Omega_n := \mathbb{R}^{3n} \setminus \mathcal{C}^n$.

Lemma 5.2. *For any $n \in \mathbb{N}$, every $\varphi \in D(\Delta_n^*)$ has a representative for which the limits*

$$(A^{(n)}\varphi)(x_1, \dots, x_{n-1}) := \frac{\sqrt{n}}{4\pi} \lim_{r \to 0} \partial_r \int_{S^2} r\varphi(r\omega, x_1, \dots, x_{n-1}) \, \mathrm{d}\omega \tag{5.8}$$

and

$$(B^{(n)}\varphi)(x_1, \dots, x_{n-1}) := -4\pi\sqrt{n} \lim_{r \to 0} r\varphi(r\omega, x_1, \dots, x_{n-1}) \tag{5.9}$$

exist in $\mathrm{H}^{-2}(\Omega_{n-1})$ and this defines continuous linear maps

$$A^{(n)}, B^{(n)} : D(\Delta_n^*) \to \mathrm{H}^{-2}(\Omega_{n-1}). \tag{5.10}$$

Furthermore, $B^{(n)}$ vanishes on $\mathrm{H}^1(\mathbb{R}^{3n}) \cap D(\Delta_n^*)$ and the restriction of $A^{(n)}$ to $\mathrm{H}^2(\mathbb{R}^{3n})$ is given by the Sobolev-trace on $\{x_1 = 0\}$.

In the following we will drop the superscript from $A^{(n)}$ and $B^{(n)}$ for better readability. Let

$$D_n^* := \left\{ \psi \in D(\Delta_n^*) \cap \mathfrak{H}^n \;\middle|\; A\psi \in \mathrm{L}^2(\mathbb{R}^{3n-3}), \; B\psi \in \mathrm{L}^2(\mathbb{R}^{3n-3}) \right\} \subset \mathfrak{H}^n. \quad (5.11)$$

and equip this space with the norm $\|\psi\|_{\mathfrak{H}^n} + \|\Delta_n^* \psi\|_{\mathfrak{H}^n} + \|A\psi\|_{\mathfrak{H}^{n-1}} + \|B\psi\|_{\mathfrak{H}^{n-1}}$. The following Proposition characterizes $\mathrm{H}^2 \subset D_n^*$ in terms of boundary values.

Proposition 5.3. *Let $\varphi \in D_n^*$. Then $B\varphi = 0$ if and only if $\varphi \in \mathrm{H}^2(\mathbb{R}^{3n})$.*

With this a priori information on the functions in D_n^* we can now characterize the asymmetry of Δ_n^* in terms on the operators A and B.

Proposition 5.4. *For all $\psi, \varphi \in D_n^*$ we have that*

$$\langle \Delta_n^* \psi, \varphi \rangle_{\mathfrak{H}^n} - \langle \psi, \Delta_n^* \varphi \rangle_{\mathfrak{H}^n} = \langle A\psi, B\varphi \rangle_{\mathfrak{H}^{n-1}} - \langle B\psi, A\varphi \rangle_{\mathfrak{H}^{n-1}}. \quad (5.12)$$

Proof. By definition of the norm on D_n^*, the maps $A, B : D_n^* \to \mathfrak{H}^{n-1}$ are continuous, and so is the map

$$\mathfrak{B} : D_n^* \to \mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}, \qquad \psi \mapsto (B\psi, A\psi). \quad (5.13)$$

The skew-hermitean sesquilinear form

$$\beta(\psi, \varphi) := \langle \Delta_n^* \psi, \varphi \rangle - \langle \psi, \Delta_n^* \varphi \rangle \quad (5.14)$$

is also continuous on D_n^*. Suppose for the moment that there exists a continuous, skew-hermitean sesquilinear form α on $\operatorname{ran} \mathfrak{B} \subset \mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}$ such that $\beta = \alpha \circ \mathfrak{B}$. Any continuous sesquilinear form on $\operatorname{ran} \mathfrak{B}$ is already determined by its values on any subspace of $\operatorname{ran} \mathfrak{B}$ which is dense in the $\| \cdot \|_{n-1} + \| \cdot \|_{n-1}$-norm. Therefore, β is already determined by its values on a subspace D_0 whose image $\mathfrak{B}(D_0)$ is dense in $\mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}$. That is, it suffices to verify (5.12) on D_0. Such a subspace is given by

$$D_0 := \{\psi \in D_n^* | \psi = \psi_A + \psi_B, \; \psi_A \in D_A^n, \; \psi_B \in D_B^n\} \qquad D_{A/B} := \ker A/B \subset D(\Delta_1^*). \quad (5.15)$$

Here D_A^n and D_B^n are the spans of symmetric n-fold tensor products of elements of $\ker A$ and $\ker B$ on $D(\Delta_1^*)$. These kernels are the domains of self-adjoint extensions of Δ_1; in fact $\ker B = \mathrm{H}^2(\mathbb{R}^3)$, and $\ker A$ is the domain of a point source with infinite scattering length. We have, by Proposition 5.1,

$$\mathfrak{B}(D_0) = (B(D_A^n), A(D_B^n)) = \left(D_A^{n-1}, D_B^{n-1} \right) \subset \mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}, \quad (5.16)$$

so $\mathfrak{B}(D_0)$ is in fact dense.

Now let $\psi = \psi_A + \psi_B, \varphi = \varphi_A + \varphi_B \in D_0$. The action of Δ_n^* on ψ, φ is given by the action of Δ_1^* on every factor. Because (Δ_1^*, D_A) and (Δ_1^*, D_B) are symmetric operators and β is skew-Hermitean, $\beta(\psi_A, \varphi_A) = \beta(\psi_B, \varphi_B) = 0$ and we only need to compute one cross-term $\beta(\psi_A, \varphi_B)$. Applying Proposition 5.1 yields

$$
\begin{aligned}
\beta(\psi_A, \varphi_B) &= \sum_{i=1}^n \langle (\Delta_1^*)_{x_i} \psi_A, \varphi_B \rangle_{\mathfrak{H}^n} - \langle \psi_A, (\Delta_1^*)_{x_i} \varphi_B \rangle_{\mathfrak{H}^n} \\
&= n \left(\langle (\Delta_1^*)_{x_1} \psi_A, \varphi_B \rangle_{\mathfrak{H}^n} - \langle \psi_A, (\Delta_1^*)_{x_1} \varphi_B \rangle_{\mathfrak{H}^n} \right) \\
&= \langle A\psi_A, B\varphi_B \rangle_{\mathfrak{H}^{n-1}} - \langle B\psi_A, A\varphi_B \rangle_{\mathfrak{H}^{n-1}} \\
&= -\langle B\psi_A, A\varphi_B \rangle_{\mathfrak{H}^{n-1}} .
\end{aligned}
\tag{5.17}
$$

We still have to construct an α with $\beta = \alpha \circ \mathfrak{B}$. Here Proposition 5.3 enters as the key ingredient: we have that

$$
\ker \mathfrak{B} = \ker B \cap \ker A = \{\psi \in \mathrm{H}^2(\mathbb{R}^{3n}) \cap \mathfrak{H}^n | A\psi = \psi|_{\mathcal{C}^n} = 0\} = \mathrm{H}_0^2(\Omega_n). \tag{5.18}
$$

As a consequence $\beta(\psi, \varphi) = 0$ for all $\varphi \in D_n^*$ if $\psi \in \ker \mathfrak{B}$. Thus we can define on the quotient the sesquilinear form

$$
\tilde{\alpha} : D_n^*/\ker \mathfrak{B} \times D_n^*/\ker \mathfrak{B} \to \mathbb{C}, \quad ([\psi], [\varphi]) \mapsto \beta(\psi, \varphi) \tag{5.19}
$$

and (5.18) guarantees that this is well defined. Let π denote the quotient map. Then $\beta = \tilde{\alpha} \circ \pi$, which means that $\tilde{\alpha}$ is continuous in the quotient topology. There exists a unique continuous isomorphism $\mathfrak{B}' : D_n^*/\ker \mathfrak{B} \to \operatorname{ran} \mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{B}' \circ \pi$. Inserting the identity we get

$$
\beta = \tilde{\alpha} \circ \pi = \tilde{\alpha} \circ (\mathfrak{B}')^{-1} \circ \mathfrak{B}' \circ \pi = \tilde{\alpha} \circ (\mathfrak{B}')^{-1} \circ \mathfrak{B} . \tag{5.20}
$$

If we define $\alpha := \tilde{\alpha} \circ (\mathfrak{B}')^{-1}$, it is obviously continuous. This proves the claim. $\quad\square$

Corollary 5.5. $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ *is symmetric for all $E_0 \in \mathbb{R}$.*

Proof. Recall the definition of the domain

$$
D_{\mathrm{IBC}} := \left\{ \psi \in \mathfrak{F} \left| \begin{array}{l} \psi^{(n)} \in D(\Delta_n^*) \cap \mathfrak{H}^n \text{ for all } n \in \mathbb{N}, \\ H\psi \in \mathfrak{F}, \ A\psi \in \mathfrak{F}, \text{ and } B\psi = g\psi \end{array} \right. \right\} . \tag{5.21}
$$

Now $H\psi \in \mathfrak{F}$ together with $A\psi \in \mathfrak{F}$ clearly implies $(-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\psi \in \mathfrak{F}$, so we

may split the operator and compute with the help of Proposition 5.4:

$$
\begin{aligned}
\langle \varphi, H\psi \rangle_{\mathfrak{F}} &= \langle \varphi, (-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&= \sum_{n \in \mathbb{N}} \langle \varphi^{(n)}, -\Delta_n^* \psi^{(n)} \rangle_n + \langle \varphi, \mathrm{d}\Gamma(E_0)\psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&\stackrel{(5.12)}{=} \sum_{n \in \mathbb{N}} \langle -\Delta_n^* \varphi^{(n)}, \psi^{(n)} \rangle_n + \langle A\varphi^{(n)}, B\psi^{(n)} \rangle_{n-1} - \langle B\varphi^{(n)}, A\psi^{(n)} \rangle_{n-1} \\
&\qquad + \langle \varphi, \mathrm{d}\Gamma(E_0)\psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&\stackrel{\mathrm{IBC}}{=} \langle (-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\varphi, \psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&\qquad + \sum_{n \in \mathbb{N}} \langle A\varphi^{(n)}, g\psi^{(n-1)} \rangle_{n-1} - \langle g\varphi^{(n-1)}, A\psi^{(n)} \rangle_{n-1} \\
&= \langle (-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\varphi, \psi \rangle_{\mathfrak{F}} + \langle gA\varphi, \psi \rangle_{\mathfrak{F}} = \langle H\varphi, \psi \rangle_{\mathfrak{F}} \,. \qquad (5.22)
\end{aligned}
$$

$\square$

Another simple corollary of our results in this section is the fact that, if $g \neq 0$, the intersection of D_{IBC} and the form-domain of $\mathrm{d}\Gamma(-\Delta)$ contains only the zero vector. For $E_0 \geq 0$, the form-domain of the free operator $\mathrm{d}\Gamma(h)$ is of course contained in that of $\mathrm{d}\Gamma(-\Delta)$.

Corollary 5.6. *If $g \neq 0$ we have for any $E_0 \in \mathbb{R}$*

$$
D_{\mathrm{IBC}} \cap D\left(\mathrm{d}\Gamma(-\Delta)^{1/2}\right) = \{0\} \,. \tag{5.23}
$$

Proof. Take $\psi \neq 0 \in D_{\mathrm{IBC}}$. Then $\psi^{(n)} \neq 0$ for some $n \in \mathbb{N}$. This implies that $B\psi^{(n+1)} = g\psi^{(n)} \neq 0$. But $D(\mathrm{d}\Gamma(-\Delta)^{1/2})|_{\mathfrak{H}^{n+1}} = \mathrm{H}^1(\mathbb{R}^{3(n+1)}) \cap \mathfrak{H}^{n+1}$, and, by Lemma 5.2, B vanishes on this set. Hence $\psi \notin D(\mathrm{d}\Gamma(-\Delta)^{1/2})$. $\square$

Remark 5.7. Propositions 5.3 and 5.4 prove that $(\mathfrak{H}^{n-1}, B, A)$ is a quasi boundary triple (in the sense of [BM14]) for the operator $(-\Delta_n^*, D_n^*)$. This allows for a complete characterization of the adjoint domain $D(\Delta_n^*)$ and the self-adjoint extensions of Δ_n (restricted to symmetric functions $\mathfrak{H}^n$). The following statements are consequences of the general theory [BM14, Prop. 2.9, 2.10], but can also be concluded directly in our setting from Propositions 5.3 and 5.4.

For any $\lambda > 0$ we have that

$$
D(\Delta_n^*) \cap \mathfrak{H}^n = \mathrm{H}^2(\mathbb{R}^{3n}) \cap \mathfrak{H}^n \oplus K_\lambda \,, \tag{5.24}
$$

with $K_\lambda = \ker(-\Delta_n^* + \lambda) \cap \mathfrak{H}^n$. The map

$$
B : K_\lambda \to \left(\mathrm{H}^{1/2}(\mathbb{R}^{3(n-1)}) \cap \mathfrak{H}^{n-1}\right)' \subset \mathrm{H}^{-1/2}(\mathbb{R}^{3(n-1)}) \tag{5.25}
$$

is continuous, as can easily be seen from the proof of Lemma 5.2. By Proposition 5.3 it is one-to-one. It is also surjective, with inverse given, as in (5.6), by

$$
f \mapsto \mathrm{Sym}_n \left(\frac{\mathrm{e}^{-\sqrt{-\Delta+1}|x_n|}}{4\pi|x_n|} f(x_1, \ldots, x_{n-1}) \right)
$$
$$
= \mathrm{Sym}_n \left((-\Delta+1)^{-1} f(x_1, \ldots, x_{n-1})\delta(x_n) \right) . \tag{5.26}
$$

Such formulas for functions in $D(\Delta_n^*)$ have been widely used in the literature on point interactions, see e.g. [Min11]. An alternative proof that, for a similar problem with $n = 2$, the whole adjoint domain can be obtained in this way has been indicated recently in [MiOt17, Prop. 4].

6 Essential Self-Adjointness of H_{IBC}

6.1 Coherent Vectors and Denseness

The aim of this subsection is to introduce a set of coherent vectors in the domain D_{IBC} on which we can perform many computations explicitly. A standard choice of a dense set in Fock space is the space $\mathfrak{F}_0$ containing the vectors with a bounded number of particles, i.e., $\psi \in \mathfrak{F}_0$ iff there exists $N \in \mathbb{N}$ such that $\psi^{(n)} = 0$ for $n > N$. However, $\mathfrak{F}_0 \cap D_{\mathrm{IBC}} = \{0\}$ since the IBC $B\psi = g\psi$ immediately yields that if $\psi^{(n)} \neq 0$, then $\psi^{(k)} \neq 0$ for all $k > n$.

For $u \in \mathfrak{H}$ the associated *coherent vector* $\varepsilon(u) \in \mathfrak{F}$ is defined by

$$
\varepsilon(u)^{(n)} := \frac{u^{\otimes n}}{\sqrt{n!}} . \tag{6.1}
$$

It holds that $\langle \varepsilon(v), \varepsilon(u) \rangle_{\mathfrak{F}} = \exp(\langle v, u \rangle_{\mathfrak{H}})$; thus, the nonlinear map $\varepsilon : \mathfrak{H} \to \mathfrak{F}$, $u \mapsto \varepsilon(u)$, is continuous,

$$
\begin{aligned}
\|\varepsilon(v) - \varepsilon(u)\|^2 &= \langle \varepsilon(v), \varepsilon(v) \rangle_{\mathfrak{F}} + \langle \varepsilon(u), \varepsilon(u) \rangle_{\mathfrak{F}} - 2\mathrm{Re}\left(\langle \varepsilon(v), \varepsilon(u) \rangle_{\mathfrak{F}} \right) \\
&= \mathrm{e}^{\|v\|_{\mathfrak{H}}^2} + \mathrm{e}^{\|u\|_{\mathfrak{H}}^2} - 2\mathrm{Re}\ \mathrm{e}^{\langle v, u \rangle_{\mathfrak{H}}} \xrightarrow{v \to u} 0 .
\end{aligned} \tag{6.2}
$$

For a subset $D \subseteq \mathfrak{H}$, consider the subspace spanned by coherent vectors of elements of D, that is

$$
E(D) := \mathrm{span}\{\varepsilon(u)|u \in D\} \subset \mathfrak{F} . \tag{6.3}
$$

We will refer to this subspace as the *coherent domain over D*. When working with coherent vectors, we will need the following generalized polarization identity.

Proposition 6.1. *Let V be a complex vector space and $v_1, \ldots, v_n \in V$. Then there exist vectors $u_1, \ldots, u_m \in V$ and coefficients $d_1, \ldots, d_m \in \mathbb{C}$ such that*

$$\operatorname{Sym}(v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^{m} d_k \, u_k^{\otimes n} \, . \tag{6.4}$$

See Appendix B for the proof, including an explicit formula for u_k and d_k. For a densely defined, non-self-adjoint operator (T, D), we use the expression $\mathrm{d}\Gamma(T)$ to denote the operator which acts as $\sum_{j=1}^{n} \mathbf{1}_{1,\ldots,j-1} \otimes T \otimes \mathbf{1}_{j+1,\ldots,n}$ on the n-th sector of Fock space. This expression obviously has meaning on $E(D)$.

Proposition 6.2. *If $D \subset \mathfrak{H}$ is dense, then $E(D)$ is a dense subspace of $\mathfrak{F}$. Moreover, let (T, D) be a densely defined operator on $\mathfrak{H}$. Then for $f \in \mathfrak{H}$ we have*

$$a(f)\,\varepsilon(u) \;=\; \langle f, u \rangle_{\mathfrak{H}}\,\varepsilon(u) \qquad \text{for all } u \in \mathfrak{H}, \tag{6.5}$$

$$a^*(f)\,\varepsilon(u) \;=\; \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varepsilon(u + tf) \qquad \text{for all } u \in \mathfrak{H}, \tag{6.6}$$

$$\mathrm{d}\Gamma(T)\,\varepsilon(u) \;=\; a^*(Tu)\,\varepsilon(u) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varepsilon(u + tTu) \qquad \text{for all } u \in D. \tag{6.7}$$

Proof. For $u \in \mathfrak{H}$ the map $\mathbb{R} \to \mathfrak{F}$, $t \mapsto \varepsilon(tu)$, has derivatives of any order at $t = 0$ with

$$\left(\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} \varepsilon(tu) \right)^{(m)} = \begin{cases} 0 & m \neq n \\ \sqrt{n!}\, u^{\otimes n} & m = n \, . \end{cases} \tag{6.8}$$

Thus, $E(\mathfrak{H})$ is dense in the span of all vectors of the form $(0, \ldots, u^{\otimes n}, 0 \ldots)$. Then, by the generalized polarization identity (Proposition 6.1) and standard approximation arguments, $E(\mathfrak{H})$ is also dense in $\mathfrak{F}$. The continuity of the map $u \mapsto \varepsilon(u)$ finally implies that $E(D)$ is dense in $E(\mathfrak{H})$ whenever D is dense in $\mathfrak{H}$. The formulas (6.5)–(6.7) follow directly from the definitions of the corresponding operators. $\qquad \square$

The natural candidate for the set D is of course $D(\Delta_1^*)$. However, we still need to make sure that the coherent vectors generated by D satisfy the boundary condition. Let

$$D_g^\gamma := \left\{ \varphi \in \mathfrak{H} \,\middle|\, \varphi = g f_\gamma + \phi, \ \phi \in \mathrm{H}^2(\mathbb{R}^3) \right\} \tag{6.9}$$

for some γ with $\operatorname{Re}\gamma > 0$. The affine subspace D_g^γ is dense in $\mathfrak{H}$ because $\mathrm{H}^2(\mathbb{R}^3)$ is dense. Then, according to Proposition 6.2, the coherent domain $E(D_g^\gamma)$ over D_g^γ is a dense subspace of $\mathfrak{F}$; in fact, it is included in D_{IBC}:

Corollary 6.3. *We have that $E(D_g^\gamma) \subset D_{\mathrm{IBC}}$ for the value of g used in D_{IBC} and any $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. As a consequence, D_{IBC} is dense in $\mathfrak{F}$.*

Proof. Let $\varphi \in D_g^\gamma \subset D(\Delta_1^*)$. Then obviously $\varepsilon(\varphi)^{(n)} \in D_n^*$ as in (5.11), and

$$(B\varepsilon(\varphi))^{(n)} = \sqrt{n+1}(B\varphi)\frac{\varphi^{\otimes n}}{\sqrt{(n+1)!}} = g\frac{\varphi^{\otimes n}}{\sqrt{(n)!}} = g\varepsilon(\varphi)^{(n)}, \qquad (6.10)$$

so $\varepsilon(\varphi)$ satisfies the interior-boundary condition. Additionally,

$$(A\varepsilon(\varphi))^{(n)} = \sqrt{n+1}(A\varphi)\frac{\varphi^{\otimes n}}{\sqrt{(n+1)!}} = (A\varphi)\varepsilon(\varphi)^{(n)}, \qquad (6.11)$$

which defines an element of $\mathfrak{F}$ since A is bounded on $D(\Delta_1^*)$ by Proposition 5.1. Observe that $(\Delta_1^*)_{x_j}\varepsilon(\varphi)^{(n)} \in \mathrm{L}^2(\mathbb{R}_{x_j}^3, \mathrm{L}^2(\mathbb{R}^{3n-3}))$. Therefore the action of Δ_n^* coincides on $E(D_g^\gamma)$ with that of $\sum_{j=1}^n (-\Delta_1^*)_{x_j}$. It is also straightforward to check that $\Delta_\mathfrak{F}^*\varepsilon(\varphi) \in \mathfrak{F}$, and this completes the proof. $\qquad \square$

6.2 Unitary Equivalence

To avoid unnecessary technicalities, we define the dressing transformation $\mathrm{e}^{-\mathrm{i}\Phi}$ directly for coherent states and not in terms of its generator $\Phi = a + a^*$. That is, we write $W(\varphi)$ for $\mathrm{e}^{-\mathrm{i}\Phi(\mathrm{i}\varphi)}$ and construct $W(\varphi)$ as follows. For $\varphi, u \in \mathfrak{H}$, let

$$W(\varphi)\,\varepsilon(u) := \mathrm{e}^{-\langle\varphi,u\rangle_\mathfrak{H} - \frac{\|\varphi\|_\mathfrak{H}^2}{2}}\,\varepsilon(u + \varphi). \qquad (6.12)$$

Lemma 6.4. *For every $\varphi \in \mathfrak{H}$, the map $W(\varphi)$ can be extended uniquely to a unitary transformation on Fock space; its inverse is given by $W(-\varphi)$.*

See, e.g., Section IV.1.9 in [Mey93] for the rather elementary proof.

Proposition 6.5. *Let (T, D) be a self-adjoint operator on $\mathfrak{H}$. Then its second quantization $\mathrm{d}\Gamma(T)$ is essentially self-adjoint on the coherent domain $E(D)$.*

Proof. The coherent domain $E(D)$ is a subspace of $D(\mathrm{d}\Gamma(T))$ and the associated unitary group of $\mathrm{d}\Gamma(T)$ is given by $\Gamma(\mathrm{e}^{-\mathrm{i}Tt})$. Since its action on coherent vectors is extremely simple, $\Gamma(\mathrm{e}^{-\mathrm{i}Tt})\varepsilon(u) = \varepsilon(\mathrm{e}^{-\mathrm{i}Tt}u)$, the coherent domain over D is *invariant* under $\Gamma(\mathrm{e}^{-\mathrm{i}Tt})$ because D is. Now the statement follows from Nelson's invariant domain theorem [ReSi80, Thm. VIII.11]. $\qquad \square$

Lemma 6.6. *Let (T, D) be a densely defined operator on $\mathfrak{H}$. Suppose that $\varphi, u \in D$, and let $W(\varphi)$ be the corresponding unitary dressing transformation defined by (6.12). Then*

$$W(-\varphi)\mathrm{d}\Gamma(T)W(\varphi)\big|_{E(D)} = \mathrm{d}\Gamma(T) + a^*(T\varphi) + a(T\varphi) + G(T,\varphi)\,\big|_{E(D)}, \quad (6.13)$$

where $G(T, \varphi)$ is an operator on $E(D)$ whose action is given by

$$G(T,\varphi)\varepsilon(u) = \left(\langle\varphi, Tu\rangle_\mathfrak{H} - \langle T\varphi, u\rangle_\mathfrak{H} + \langle\varphi, T\varphi\rangle_\mathfrak{H}\right)\varepsilon(u). \qquad (6.14)$$

Proof. This is a consequence of Proposition 6.2 and the following straightforward computation:

$$W(-\varphi)\mathrm{d}\Gamma(T)W(\varphi)\varepsilon(u) \tag{6.15}$$

$$\overset{(6.7)}{=} W(-\varphi)\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varepsilon(u+\varphi+tT(u+\varphi))\mathrm{e}^{-\langle\varphi,u\rangle-\frac{\|\varphi\|}{2}}$$

$$\overset{(6.12)}{=} \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varepsilon(u+tT(u+\varphi))\mathrm{e}^{t\langle\varphi,T(u+\varphi)\rangle}$$

$$\overset{(6.6)}{=} \left(a^*(T(u+\varphi))+\langle\varphi,Tu\rangle_{\mathfrak{H}}+\langle\varphi,T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u)$$

$$\overset{(6.7)}{=} \left(\mathrm{d}\Gamma(T)+a^*(T\varphi)+\langle\varphi,Tu\rangle_{\mathfrak{H}}+\langle\varphi,T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u)$$

$$\overset{(6.5)}{=} \left(\mathrm{d}\Gamma(T)+a^*(T\varphi)+a(T\varphi)+\langle\varphi,Tu\rangle_{\mathfrak{H}}-\langle T\varphi,u\rangle_{\mathfrak{H}}+\langle\varphi,T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u). \qquad \square$$

Corollary 6.7. *Let (T,D) be a self-adjoint operator on $\mathfrak{H}$ which is invertible, i.e. $0\in\rho(T)$. Then for $\psi\in\mathfrak{H}$ and $u\in D$ it holds that*

$$W(-T^{-1}\psi)\mathrm{d}\Gamma(T)W(T^{-1}\psi)\big|_{E(D)} = \mathrm{d}\Gamma(T)+a^*(\psi)+a(\psi)+\langle\psi,T^{-1}\psi\rangle_{\mathfrak{H}}\mathbf{1}_{\mathfrak{F}}\big|_{E(D)} \tag{6.16}$$

Proof. Apply Lemma 6.6 with $\varphi=T^{-1}\psi$ and observe that, because T is symmetric, it holds that $\langle\varphi,Tu\rangle_{\mathfrak{H}}-\langle T\varphi,u\rangle_{\mathfrak{H}}=0$. So the operator $G(T,\varphi)$ reduces to multiplication with the constant $\langle T^{-1}\psi,\psi\rangle_{\mathfrak{H}}=\langle\psi,T^{-1}\psi\rangle_{\mathfrak{H}}$. $\qquad \square$

Corollary 6.8. *Let $E_0\in\mathbb{R}$, $\gamma>0$, f_γ be given by (2.15) and let $h=-\Delta+E_0$ with domain $\mathrm{H}^2(\mathbb{R}^3)$. Then on the coherent domain $E(\mathrm{H}^2(\mathbb{R}^3))$ we have*

$$W(-gf_\gamma)\,H_{\mathrm{IBC}}\,W(gf_\gamma)\big|_{E(\mathrm{H}^2(\mathbb{R}^3))}$$
$$= \mathrm{d}\Gamma(h)+(-\gamma^2+E_0)\,(a^*(gf_\gamma)+a(gf_\gamma))+C(g,\gamma,E_0)\mathbf{1}_{\mathfrak{F}}\big|_{E(\mathrm{H}^2(\mathbb{R}^3))} \tag{6.17}$$

where the constant reads

$$C(g,\gamma,E_0)=(-\gamma^2+E_0)\|gf_\gamma\|_{\mathfrak{H}}^2+g^2\frac{\gamma}{4\pi}. \tag{6.18}$$

Proof. We start by noting that (6.11) gives for $u\in\mathrm{H}^2(\mathbb{R}^3)$

$$gAW(gf_\gamma)\varepsilon(u)=g(A(gf_\gamma+u))W(gf_\gamma)\varepsilon(u)=\left(\frac{g^2\gamma}{4\pi}+gu(0)\right)W(gf_\gamma)\varepsilon(u). \tag{6.19}$$

Now set $(T,D)=(-\Delta_1^*+E_0,D(\Delta_1^*))$ and $\varphi=gf_\gamma$ in Lemma 6.6. Then

$$W(-gf_\gamma)\,H_{\mathrm{IBC}}\,W(gf_\gamma)\varepsilon(u) \tag{6.20}$$

$$= W(-gf_\gamma)\mathrm{d}\Gamma(-\Delta_1^*+E_0)W(gf_\gamma)\varepsilon(u)+\left(\frac{g^2\gamma}{4\pi}+gu(0)\right)\varepsilon(u)$$

$$= \left(\mathrm{d}\Gamma(h)+(E_0-\gamma^2)\,(a^*(gf_\gamma)+a(gf_\gamma))+G(T,\varphi)+\left(\frac{g^2\gamma}{4\pi}+gu(0)\right)\right)\varepsilon(u).$$

25

It remains to show that for $u \in \mathrm{H}^2(\mathbb{R}^3)$

$$\left(G(T, \varphi) + \frac{g^2 \gamma}{4\pi} + gu(0)\right) \varepsilon(u) = C(g, \gamma, E_0)\varepsilon(u) \,. \tag{6.21}$$

It follows from Proposition 5.1 that

$$
\begin{aligned}
G(T, \varphi) + gu(0) &= g\langle f_\gamma, Tu \rangle - g\langle T(f_\gamma), u \rangle + g^2 \langle f_\gamma, Tf_\gamma \rangle + gAu \\
&= g\langle f_\gamma, -\Delta_1^* u \rangle - g\langle -\Delta_1^* f_\gamma, u \rangle + gAu + (-\gamma^2 + E_0)\|gf_\gamma\|_{\mathfrak{H}}^2 \\
&= gAf_\gamma Bu - gBf_\gamma Au + gAu + (-\gamma^2 + E_0)\|gf_\gamma\|_{\mathfrak{H}}^2 \\
&= (-\gamma^2 + E_0)\|gf_\gamma\|_{\mathfrak{H}}^2 \,,
\end{aligned} \tag{6.22}
$$

since $Bu = 0$ and $Bf_\gamma = 1$. $\qquad\square$

Proposition 6.9. *For all $E_0 \in \mathbb{R}$ the operator $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ is essentially self-adjoint and for any $\gamma > 0$ the space $W(gf_\gamma)E(\mathrm{H}^2(\mathbb{R}^3)) \subset D_{\mathrm{IBC}}$ is a core. If $E_0 \geq 0$, then the Hamiltonian H_{IBC} is bounded from below.*

Proof. According to Corollary 6.8 and by symmetry of $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ it suffices to show that

$$\mathrm{d}\Gamma(h) + (-\gamma^2 + E_0)\left(a^*(gf_\gamma) + a(gf_\gamma)\right) \tag{6.23}$$

is essentially self-adjoint on $E(\mathrm{H}^2(\mathbb{R}^3))$. By Proposition 6.5, the operator $(\mathrm{d}\Gamma(h), E(\mathrm{H}^2(\mathbb{R}^3)))$ is essentially self-adjoint.

For $E_0 \geq 0$ the perturbation $a^*(gf_\gamma) + a(gf_\gamma)$ is infinitesimally bounded with respect to $\mathrm{d}\Gamma(h)$ (see Proposition 3.8 in [Der03]) and thus, by Kato-Rellich, essential self-adjointness of (6.23) on $E(\mathrm{H}^2(\mathbb{R}^3))$ holds. Here one uses the fact that

$$\hat{f}_\gamma(k) = -(2\pi)^{-\frac{3}{2}}(|k|^2 + \gamma^2)^{-1} = -\hat{\delta}(k) \cdot (|k|^2 + \gamma^2)^{-1} \qquad \mathrm{Re}(\gamma) > 0 \,, \tag{6.24}$$

and therefore $\langle \hat{f}_\gamma, \hat{h}^{-1}\hat{f}_\gamma \rangle < \infty$ even for $E_0 = 0$.

If $E_0 < 0$, essential self-adjointness of (6.23) is shown using Nelson's Commutator Theorem (Theorem X.36 in [ReSi75]) with comparison operator $N = \mathbf{1}_{\mathfrak{F}} + \mathrm{d}\Gamma(h - E_0 + 1)$, cf. Proposition 3.11 in [Der03]. $\qquad\square$

Proposition 6.10. *If $E_0 > 0$, then the operator $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ is self-adjoint and*

$$H_{\mathrm{IBC}} = W(gf_{\sqrt{E_0}})\left[\mathrm{d}\Gamma(h) + \frac{g^2\sqrt{E_0}}{4\pi}\right]W(-gf_{\sqrt{E_0}}) \,. \tag{6.25}$$

Proof. As $E_0 > 0$, we may choose $\gamma = \sqrt{E_0}$ in Corollary 6.8 and set $\phi := gf_{\gamma = \sqrt{E_0}}$. The constant $C(g, \sqrt{E_0}, E_0)$ then reduces to $\frac{g^2\sqrt{E_0}}{4\pi}$ and the equality (6.25) holds on

the common core $W(\phi)E(\mathrm{H}^2(\mathbb{R}^3))$. This extends to the common domain of self-adjointness $W(\phi)D(\mathrm{d}\Gamma(h))$.

The inclusion $D_{\mathrm{IBC}} \subseteq W(\phi)D(\mathrm{d}\Gamma(h))$ follows from the symmetry of $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$, Proposition 5.5. To show that also $W(\phi)D(\mathrm{d}\Gamma(h)) \subseteq D_{\mathrm{IBC}}$, we use that $W(\phi)D(\mathrm{d}\Gamma(h))$ is the closure of $W(\phi)E(\mathrm{H}^2(\mathbb{R}^3))$ in the graph norm of $W(\phi)\mathrm{d}\Gamma(h)W(-\phi)$. We need to show that for $\psi \in W(\phi)D(\mathrm{d}\Gamma(h))$ we have $\psi^{(n)} \in D(\Delta_n^*)$ and $A\psi \in \mathfrak{F}$. Let $u \in \mathrm{H}^2(\mathbb{R}^3)$, then we have the estimate

$$
\begin{aligned}
\|u(0)W(\phi)\varepsilon(u)\|_{\mathfrak{F}}^2 &= \sum_{n \geq 0} \frac{1}{n!} \|u(0)u^{\otimes n}\|_{\mathfrak{H}^n}^2 \\
&\leq \sum_{n \geq 0} \frac{C}{(n+1)!}(n+1)\|(-\Delta_{x_{n+1}} + E_0)u^{\otimes(n+1)}\|_{\mathrm{L}^2(\mathbb{R}^{3(n+1)})}^2 \\
&\leq C\|\mathrm{d}\Gamma(h)\varepsilon(u)\|_{\mathfrak{F}}^2 ,
\end{aligned}
\tag{6.26}
$$

where we have used that $|u(0)| \leq C\|u\|_{\mathrm{H}^2}$ and that $\langle \Delta_{x_j}u^{\otimes(n+1)}, \Delta_{x_i}u^{\otimes(n+1)}\rangle \geq 0$. In view of Equation (6.19) this implies that

$$
\|AW(\phi)\varepsilon(u)\|_{\mathfrak{F}} \leq C\|\mathrm{d}\Gamma(h)\varepsilon(u)\|_{\mathfrak{F}}
\tag{6.27}
$$

for some constant $C > 0$. This clearly implies that for any $n \in \mathbb{N}$

$$
\|(-\Delta_n^* + nE_0)(W(\phi)\varepsilon(u))^{(n)}\|_{\mathfrak{H}^n} \leq \|(H - gA)W(\phi)\varepsilon(u)\|_{\mathfrak{F}} \leq C\|\mathrm{d}\Gamma(h)\varepsilon(u)\|_{\mathfrak{F}} .
\tag{6.28}
$$

As Δ_n^* is closed, it follows that $W(\phi)D(\mathrm{d}\Gamma(h))|_{\mathfrak{H}^n} \subset D(\Delta_n^*)$.

Consequently by Lemma 5.2 the expressions for A and B are well defined (as distributions) and continuous on each sector of $W(\phi)D(\mathrm{d}\Gamma(h))$. Now (6.27) implies that A maps $W(\phi)D(\mathrm{d}\Gamma(h))$ to $\mathfrak{F}$, so in particular $A\psi^{(n)} \in \mathrm{L}^2(\mathbb{R}^{3n-3})$. Since $B\psi = g\psi$ on the dense set $W(\phi)E(\mathrm{H}^2)$, this also holds on $W(\phi)D(\mathrm{d}\Gamma(h))$ by continuity, and we have proved $W(\phi)D(\mathrm{d}\Gamma(h)) \subset D_{\mathrm{IBC}}$. $\qquad\square$

We remark that the expressions A and B defined on some natural domain $D \subset \bigoplus_n D(\Delta_n^*)$ are not necessarily closable, e.g., B vanishes on the dense (in $\mathfrak{F}$) subspace $D(\mathrm{d}\Gamma(h))$, so we cannot directly conclude from an estimate such as (6.27) that these expressions are well defined on the closure of $W(\phi)E(\mathrm{H}^2)$.

By virtue of the unitary equivalence, we can compute the ground state of H_{IBC} explicitly, provided $E_0 > 0$. The unique ground state of the free field $\mathrm{d}\Gamma(h)$ is the vector $\Omega_0 := (1, 0, 0, \dots) \in \mathfrak{F}$, which is called the Fock vacuum. With $\phi = gf_{\gamma = \sqrt{E_0}}$ we conclude that $\psi_{min} := W(\phi)\Omega_0$ is the unique ground state of H_{IBC} with ground state energy $\frac{g^2\sqrt{E_0}}{4\pi}$, i.e.

$$
H_{\mathrm{IBC}}\,\psi_{min} = \frac{g^2\sqrt{E_0}}{4\pi}\,\psi_{min} .
\tag{6.29}
$$

27

Note that because of $\Omega_0 = \varepsilon(0)$ we can calculate ψ_{min} explicitly by using (6.12),

$$\psi_{min} = W(\phi)\Omega_0 = W(\phi)\varepsilon(0) = \mathrm{e}^{-\frac{\|\phi\|^2}{2}} \, \varepsilon(\phi) \,. \tag{6.30}$$

Taken together, Corollary 5.6 and Propositions 6.9, 6.10 prove Theorem 2.1.

6.3 Renormalization: Proof of Theorem 3.1

Let $h = (-\Delta + E_0, \mathrm{H}^2(\mathbb{R}^3))$, where we now assume that $E_0 > 0$. This operator is self-adjoint and invertible. In Section 3 we defined $W_n := W(gh^{-1}\chi_n)$ where χ_n is any sequence of elements of $\mathrm{L}^2(\mathbb{R}^3)$ such that $\chi_n \to \delta$ as $n \to \infty$ in the sense that $\hat{\chi}_n \to \hat{\chi}_\infty := \hat{\delta} = (2\pi)^{-3/2}$ pointwise with $\|\hat{\chi}_n\|_\infty$ uniformly bounded.

We first use Corollary 6.7 with $\psi = g\chi_n$ and $T = h$ to establish that, in the notation of Section 3,

$$W_n H_n W_n^* = W_n \left(\mathrm{d}\Gamma(h) + a^*(g\chi_n) + a(g\chi_n) \right) W_n^* = \mathrm{d}\Gamma(h) - g^2\langle \chi_n, h^{-1}\chi_n \rangle_{\mathfrak{H}}$$
$$= \mathrm{d}\Gamma(h) + E_n \,. \tag{6.31}$$

The assumptions we made on the sequence χ_n imply that $\mathcal{F}(gh^{-1}\chi_n)$ converges in L^2 to the function $g(2\pi)^{-3/2}\hat{h}^{-1}$. Therefore, according to (6.24), $gh^{-1}\chi_n$ converges to $-gf_{\sqrt{E_0}}$. We have defined the family of unitary operators $W(\varphi)$ in (6.12) via coherent vectors. From this definition it follows that the mapping $\varphi \mapsto W(\varphi)\psi$ is continuous because the mapping $\varphi \mapsto \varepsilon(\varphi)$ is. As a consequence, the W_n converge strongly, and the limiting operator is

$$W_\infty = \lim_{n\to\infty} W_n = \lim_{n\to\infty} W(gh^{-1}\chi_n) = W(\lim_{n\to\infty} gh^{-1}\chi_n) = W(-gf_{\sqrt{E_0}}) \,. \tag{6.32}$$

Moreover, for any $z \in \mathbb{C} \setminus \mathbb{R}$ also

$$\lim_{n\to\infty} (H_n - E_n - z)^{-1} = \lim_{n\to\infty} W_n^*(\mathrm{d}\Gamma(h) - z)^{-1} W_n = W_\infty^*(\mathrm{d}\Gamma(h) - z)^{-1} W_\infty$$
$$= (W_\infty^* \mathrm{d}\Gamma(h) W_\infty - z)^{-1} \tag{6.33}$$

converges strongly because $\sup_n \|W_n^*\| = 1$. Recalling the definition (3.3) of H_∞, we find that

$$H_\infty := \lim_{n\to\infty} (H_n - E_n) = W_\infty^* \mathrm{d}\Gamma(h) W_\infty = W(gf_{\sqrt{E_0}}) \, \mathrm{d}\Gamma(h) \, W(-gf_{\sqrt{E_0}})$$
$$\overset{(6.25)}{=} H_{\mathrm{IBC}} - \frac{g^2\sqrt{E_0}}{4\pi} \tag{6.34}$$

on $W(gf_{\sqrt{E_0}})D(\mathrm{d}\Gamma(h)) = D_{\mathrm{IBC}}$. We have proven Theorem 3.1.

7 Variants of the Model

Throughout this section, let $E_0 > 0$ and $N \in \mathbb{N}$ be fixed. We will use the notation that has been introduced in Section 4 and in particular assume the condition (4.7). Here we will properly define $\tilde{D}_{\mathrm{IBC}}$ and prove Theorems 4.1 and 4.4.

Observe that $w_i^\lambda \in D(\Delta_1^*)$ and that $\Delta_1^* w_i^\lambda = \lambda w_i^\lambda$ for $1 \le i \le N$, cf. [AGHH88]. It is known that that the maps $\psi \mapsto A_i \psi$ and $\psi \mapsto B_i \psi$ define continuous linear functionals on $D(\Delta_1^*)$. Furthermore, using a partition of unity, the degree of non-symmetry of Δ_1^* may be expressed with their help:

$$\langle \varphi, -\Delta_1^* \psi \rangle_{\mathfrak{H}} - \langle -\Delta_1^* \varphi, \psi \rangle_{\mathfrak{H}} = \sum_{i=1}^N \langle B_i \varphi, A_i \psi \rangle_{\mathbb{C}} - \langle A_i \varphi, B_i \psi \rangle_{\mathbb{C}} . \tag{7.1}$$

Note the following: The set $U(v) := \bigcap_{i=1}^N \ker X_i$ is a subspace of $D(\Delta_1^*)$, which is L^2-dense. By further inspection $X_i(\psi) = 0$ for all $1 \le i \le N$ is identified with the conditions that specify the domain of point interactions centered in $\xi_1, \dots, \xi_N$ with parameters $a_i = \frac{\alpha_i}{\beta_i}$, where $\beta_i = 0$ formally corresponds to $a_i = \infty$, see [DFT08].

The matrix $S(\lambda)$ is invertible if and only if $-\lambda$ is *not* an eigenvalue of the point-interaction operator $(-\Delta_1^*, U(v))$, see Theorem II.1.1.4 in [AGHH88]. The number of eigenvalues of this operator is finite, and all its eigenvalues are negative and situated below the essential spectrum, which covers the non-negative real axis. That implies, in particular, that for all $E_0 > 0$ and for all admissible choices of v there exists $\lambda > 0$ such that $S(\lambda)$ is invertible.

Lemma 7.1. *Let v obey the condition (4.7) and let $(1, 1, \dots, 1)^T \in \operatorname{ran} S(\lambda)$. Then there exists $\phi = \phi(\lambda) \in D(\Delta_1^*)$ with the properties*

$$\Delta_1^* \phi = \lambda \phi \tag{7.2}$$

$$X_k(\phi) = 1 \qquad 1 \le k \le N . \tag{7.3}$$

Proof. For every choice of $c_1, \dots, c_N \in \mathbb{C}$ the sum $\sum_{l=1}^N c_l w_l^\lambda$ is an eigenvector of Δ_1^* with eigenvalue λ. To obtain (7.3), we first compute

$$X_k \left(\sum_{l=1}^N c_l w_l^\lambda \right) = \sum_{l=1}^N c_l X_k(w_l^\lambda) = \sum_{l=1}^N c_l \alpha_k \mathrm{e}^{\mathrm{i}\theta_k} B_k(w_l^\lambda) + c_l \beta_k \mathrm{e}^{\mathrm{i}\theta_k} A_k(w_l^\lambda)$$

$$= \sum_{l=1}^N c_l \alpha_k \mathrm{e}^{\mathrm{i}\theta_k} \delta_{kl} + c_l \beta_k \mathrm{e}^{\mathrm{i}\theta_k} (\delta_{kl} \frac{\sqrt{\lambda}}{4\pi} + (1 - \delta_{kl}) G_{kl}^\lambda) = \sum_{l=1}^N S_{kl} c_l . \tag{7.4}$$

Since $(1, 1, \dots, 1)^T \in \operatorname{ran} S(\lambda)$, there are numbers $c_l \in \mathbb{C}$ such that $\sum_{l=1}^N S_{kl} c_l = 1$ for all $1 \le k \le N$. Then we set $\phi := \sum_{l=1}^N c_l w_l$. $\qquad \square$

Lemma 7.2. *Let v obey the condition (4.7). Then the degree of non-symmetry of Δ_1^* can be expressed using X_i and Y_i: for $\varphi, \psi \in D(\Delta_1^*)$,*

$$\langle \varphi, -\Delta_1^* \psi \rangle_{\mathfrak{H}} - \langle -\Delta_1^* \varphi, \psi \rangle_{\mathfrak{H}} = \sum_{i=1}^{N} \langle X_i \varphi, Y_i \psi \rangle_{\mathbb{C}} - \langle Y_i \varphi, X_i \psi \rangle_{\mathbb{C}}. \tag{7.5}$$

Lemma 7.3. *Let $\psi \in U(v) = \bigcap_{i=1}^{N} \ker X_i$ and let $\phi(\lambda) \in D(\Delta_1^*)$ with the properties (7.2) and (7.3). Then*

(a) $\displaystyle\sum_{i=1}^{N} Y_i(\psi) = \langle \phi, (-\Delta_1^* + E_0)\psi \rangle_{\mathfrak{H}} - \langle (-\Delta_1^* + E_0)\phi, \psi \rangle_{\mathfrak{H}}$

(b) $\displaystyle\sum_{i=1}^{N} Y_i(\phi) \in \mathbb{R}$.

The proofs can be found in the Appendix B.

As mentioned above, the operator $\tilde{h} = (-\Delta_1^* + E_0, U)$ is self-adjoint and is called the N-center point-interaction with energy offset $E_0 > 0$. The coherent domain $E(U)$ is a core of $d\Gamma(\tilde{h})$, see Proposition 6.5. Next we turn to another subset of $D(\Delta_1^*)$, which is an affine subspace. If $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(\lambda)$, define

$$M = M(\lambda) := \{\varphi \in D(\Delta_1^*) | \varphi = \phi(\lambda) + \psi, \psi \in U(v)\}. \tag{7.6}$$

Since $U(v)$ is L^2-dense, so is $M(\lambda)$ and therefore the coherent domain over $E(M)$ is a dense subspace of the symmetric Fock space $\mathfrak{F}$. Set $\tilde{D}_{\mathrm{IBC}} := E(M)$. Then on $\tilde{D}_{\mathrm{IBC}}$ we find

$$Y_i^{\mathfrak{F}}(\varepsilon(\varphi)) = Y_i(\phi + \psi)\varepsilon(\varphi) = (Y_i(\phi) + Y_i(\psi))\varepsilon(\varphi) \tag{7.7}$$

and

$$X_i^{\mathfrak{F}}(\varepsilon(\varphi)) = X_i(\phi + \psi)\varepsilon(\varphi) = X_i(\phi)\varepsilon(\varphi) = \varepsilon(\varphi). \tag{7.8}$$

We are now in a position to define the operator $(\tilde{H}_{\mathrm{IBC}}, \tilde{D}_{\mathrm{IBC}})$ which depends on the set of parameters (v, E_0) where v obeys the relation (4.7):

$$\tilde{H}_{\mathrm{IBC}} := d\Gamma(-\Delta_1^* + E_0) + \sum_{i=1}^{N} Y_i^{\mathfrak{F}} \quad \text{on} \quad \tilde{D}_{\mathrm{IBC}} := E(M). \tag{7.9}$$

Proof of Theorem 4.1 and Theorem 4.4. Let $\psi \in U$. Choose $\lambda > 0$ such that $S(\lambda)$ is invertible and use $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(\lambda)$ to construct $\phi(\lambda)$ with the properties (7.2) and (7.3). Due to property (7.2) of $\phi = \phi(\lambda)$, using Lemma 6.6 we

get

$$W(-\phi)d\Gamma(-\Delta_1^* + E_0)W(\phi)\varepsilon(\psi) + W(-\phi)\left[\sum_{i=1}^N Y_i^{\mathfrak{F}}\right]W(\phi)\varepsilon(\psi)$$

$$= \left(d\Gamma(\tilde{h}) + (-\lambda + E_0)\left(a(\phi)^* + a(\phi)\right)\right)\varepsilon(\psi)$$

$$+ \left(\langle\phi, (-\Delta_1^* + E_0)\psi\rangle_{\mathfrak{H}} - \langle(-\Delta_1^* + E_0)\phi, \psi\rangle_{\mathfrak{H}}\right)\varepsilon(\psi)$$

$$+ \langle\phi, (-\Delta_1^* + E_0)\phi\rangle_{\mathfrak{H}}\varepsilon(\psi) + \left[\sum_{i=1}^N Y_i(\phi) + Y_i(\psi)\right]\varepsilon(\psi)$$

$$= \left(d\Gamma(\tilde{h}) + (-\lambda + E_0)\left(a(\phi)^* + a(\phi)\right)\right)\varepsilon(\psi)$$

$$+ \left[(-\lambda + E_0)\,\|\phi\|_{\mathfrak{H}} + \sum_{i=1}^N Y_i(\phi)\right]\varepsilon(\psi). \tag{7.10}$$

We have used statement (a) of Lemma 7.3. Due to statement (b) of this lemma, the constant in brackets is real. Because $\tilde{h}$ is bounded from below, we can use Nelson's Commutator Theorem to show essential self-adjointness of the operator on $E(U)$, cf. Proposition 6.9 and [Der03]. Now essential self-adjointness of $\tilde{H}_{\text{IBC}}$ on $W(\phi(\lambda))E(U) = E(M) = \tilde{D}_{\text{IBC}}$ follows.

If $(1, 1, \dots, 1)^T \in \operatorname{ran} S(E_0)$, set $\lambda = E_0$ to get (4.18). We have proven Theorem 4.4. In this case $\tilde{H}_{\text{IBC}}$ may be unbounded from below.

If $\tilde{h}$ is strictly positive, then $-E_0$ is not an eigenvalue of $(-\Delta_1^*, U)$ and $S(E_0)$ is invertible. From the explicit form (4.18) we see that, because $d\Gamma(\tilde{h})$ is strictly positive as well, Ω_0 is the unique ground state of $d\Gamma(\tilde{h})$. As a consequence $\tilde{H}_{\text{IBC}}$ is bounded from below by

$$C(\phi(E_0)) = \sum_{i=1}^N Y_i(\phi(E_0)) \tag{7.11}$$

and

$$\psi_{min} = \mathrm{e}^{-\frac{\|\phi(E_0)\|^2}{2}}\varepsilon(\phi(E_0)) \tag{7.12}$$

is the unique ground state of $\tilde{H}_{\text{IBC}}$. $\qquad\square$

A Regularity

Here, we give the details on the regularity questions regarding $D(\Delta_n^*)$, $A^{(n)}$, and $B^{(n)}$. We will need to work with Hilbert-space-valued distributions. Keep in mind for the following that for defining distributions the removal of a point $\{0\}$ from $\mathbb{R}^3$ or the sets $\mathcal{C}^n$ from $\mathbb{R}^{3n}$ matters, while $\mathrm{L}^2(\mathbb{R}^3 \setminus \{0\}, X) = \mathrm{L}^2(\mathbb{R}^3, X)$ and $\mathrm{L}^2(\mathbb{R}^{3n} \setminus \mathcal{C}^n, X) = \mathrm{L}^2(\Omega_n, X) = \mathrm{L}^2(\mathbb{R}^{3n}, X)$.

Lemma A.1. *Let $\varphi \in D(\Delta_n^*)$ and equip this space with the graph norm. Then for $j = 1, \ldots, n$*

$$\Delta_{x_j}\varphi \in \mathrm{L}^2\left(\mathbb{R}_{x_j}^3, \mathrm{H}^{-2}(\Omega_{n-1})\right), \tag{A.1}$$

where φ is regarded as a vector valued distribution on $\mathbb{R}_{x_j}^3 \setminus \{0\}$ and Δ_{x_j} is the Laplacian of distributions on that domain taking values in $\mathrm{H}^{-2}(\Omega_{n-1})$. Moreover,

$$\|\Delta_{x_j}\varphi\|_{\mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2})} \leq \sqrt{2}\|\varphi\|_{D(\Delta_n^*)}. \tag{A.2}$$

Proof. We will show the case $j = 1$. Recall that $D(\Delta_n) = \mathrm{H}_0^2(\Omega_n)$. For any $\varphi \in D(\Delta_n^*)$, the map

$$\Delta_{x_1}\varphi : \mathrm{H}_0^2(\Omega_n) \to \mathbb{C}, \quad \psi \mapsto \langle \varphi, \Delta_{x_1}\psi \rangle = \langle \Delta_n^*\varphi, \psi \rangle - \sum_{i=2}^{N} \langle \varphi, \Delta_{x_i}\psi \rangle \tag{A.3}$$

extends by density to a bounded linear functional on the Bochner space $\mathrm{L}^2\left(\mathbb{R}_{x_1}^3, \mathrm{H}_0^2(\Omega_{n-1})\right)$, i.e.,

$$\Delta_{x_1}\varphi \in \mathrm{L}^2\left(\mathbb{R}_{x_1}^3, \mathrm{H}_0^2(\Omega_{n-1})\right)'. \tag{A.4}$$

Since $\mathrm{H}^{-2}(\Omega_{n-1}) := \mathrm{H}_0^2(\Omega_{n-1})'$ and this space is reflexive, we obtain that $\Delta_{x_1}\varphi \in \mathrm{L}^2\left(\mathbb{R}_{x_1}^3, \mathrm{H}^{-2}(\Omega_{n-1})\right)$. It remains to show that this $\Delta_{x_1}\varphi$ is in fact also the Laplacian of φ in the sense of $\mathrm{H}^{-2}(\Omega_{n-1})$-valued distributions, i.e. that for all $\phi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ and $\xi \in \mathrm{H}_0^2(\Omega_{n-1})$ we have

$$(\Delta_{x_1}\varphi)(\phi\xi) = \int_{\mathbb{R}^3} \langle \varphi(x_1, \ldots, x_n)), \xi(x_2, \ldots, x_n) \rangle_{(\mathrm{H}^{-2}, \mathrm{H}_0^2)} \Delta\phi(x_1) \, \mathrm{d}x_1. \tag{A.5}$$

The left hand side is, by its definition (A.3),

$$(\Delta_{x_1}\varphi)(\phi\xi) = \langle \varphi, (\Delta\phi)\xi \rangle_{\mathrm{L}^2(\mathbb{R}^{3n})}, \tag{A.6}$$

and, since $\varphi \in \mathrm{L}^2(\mathbb{R}_{x_1}^3, \mathrm{L}^2(\mathbb{R}^{3n-3}))$, the right hand side equals

$$\int_{\mathbb{R}^3} \int_{\Omega_{n-1}} \overline{\varphi}(x_1, \ldots, x_n)\xi(x_2, \ldots, x_n)\Delta\phi(x_1) \, \mathrm{d}x = \langle \varphi, (\Delta_1\phi)\xi \rangle_{\mathrm{L}^2(\mathbb{R}^{3n})}. \tag{A.7}$$

Finally, the bound on the norm of $\Delta_{x_1}\varphi$ follows from Equation (A.3) by

$$\|\Delta_{x_1}\varphi\|_{\mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2}(\Omega_{n-1}))} \leq \sup_{\|\psi\|_{\mathrm{L}^2\left(\mathbb{R}^3, \mathrm{H}_0^2\right)}=1} \left(\|\Delta_n^*\varphi\|_{\mathrm{L}^2}\|\psi\|_{\mathrm{L}^2} + \|\varphi\|_{\mathrm{L}^2}\|\psi\|_{\mathrm{L}^2(\mathbb{R}^3, \mathrm{H}_0^2)} \right)$$

$$\leq \|\Delta_n^*\varphi\|_{\mathrm{L}^2(\mathbb{R}^{3n})} + \|\varphi\|_{\mathrm{L}^2(\mathbb{R}^{3n})} \leq \sqrt{2}\|\varphi\|_{D(\Delta_n^*)}. \qquad \square$$

Proof of Lemma 5.2. For clarity, we use the notation $A^{(n)}$ and $B^{(n)}$ in this proof for the operators on $D(\Delta_n^*) \subset L^2(\mathbb{R}^{3n})$. The case $n = 1$ has been proved in Proposition 5.1 and we will use it here to show continuity of $A^{(n)}$ and $B^{(n)}$ for $n \geq 2$. Our proof basically follows ideas for the construction of distribution-valued trace maps on Sobolev spaces, as presented, e.g, in [LiMa72].

Define the space

$$D_{H^{-2}}^* := \left\{ \varphi \in L^2(\mathbb{R}^3, H^{-2}(\Omega_{n-1})) | \Delta_x \varphi \in L^2(\mathbb{R}^3, H^{-2}(\Omega_{n-1})) \right\}, \tag{A.8}$$

where Δ_x denotes the Laplacian on vector-valued distributions on $\mathbb{R}^3 \setminus \{0\}$, and

$$\|\varphi\|_{D_{H^{-2}}^*}^2 := \|\varphi\|_{L^2(\mathbb{R}^3, H^{-2})}^2 + \|\Delta_x \varphi\|_{L^2(\mathbb{R}^3, H^{-2})}^2. \tag{A.9}$$

Then, by Lemma A.1, we have the continuous injection

$$D(\Delta_n^*) \hookrightarrow D_{H^{-2}}^*. \tag{A.10}$$

We will show that $A^{(n)}$ is continuous on $D_{H^{-2}}^*$, which of course implies continuity on $D(\Delta_n^*)$. To do so, we approximate any $\varphi \in D_{H^{-2}}^*$ by a sequence φ_N in the following way: Let $(\eta_k)_{k \in \mathbb{N}}$ be a complete orthonormal set in $H^{-2}(\Omega_{n-1})$ and set $\varphi_k(x) := \langle \eta_k, \varphi(x, \cdot) \rangle_{H^{-2}}$. Because $\varphi \in L^2(\mathbb{R}^3, H^{-2}(\Omega_{n-1}))$, it holds that

$$\sum_{k=1}^N \varphi_k(x) \eta_k := \varphi_N(x) \overset{N \to \infty}{\to} \varphi(x) \tag{A.11}$$

pointwise in $H^{-2}(\Omega_{n-1})$ and by dominated convergence in $L^2(\mathbb{R}^3, H^{-2}(\Omega_{n-1}))$. Now let $\psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ and observe that, because $\langle \eta_k, \cdot \rangle_{H^{-2}}$ is continuous on $H^{-2}(\Omega_{n-1})$ and $\varphi(x, \cdot) \Delta \psi$ is integrable, we have that

$$\int_{\mathbb{R}^3} \varphi_k \Delta \psi \, \mathrm{d}x = \int_{\mathbb{R}^3} \langle \eta_k, \varphi(x, \cdot) \Delta \psi \rangle_{H^{-2}} \, \mathrm{d}x = \left\langle \eta_k, \int_{\mathbb{R}^3} \varphi(x, \cdot) \Delta \psi \, \mathrm{d}x \right\rangle_{H^{-2}}$$

$$= \left\langle \eta_k, \int_{\mathbb{R}^3} \psi \Delta_x \varphi(x, \cdot) \, \mathrm{d}x \right\rangle_{H^{-2}} = \int_{\mathbb{R}^3} \langle \eta_k, \Delta_x \varphi(x, \cdot) \rangle_{H^{-2}} \psi(x) \, \mathrm{d}x. \tag{A.12}$$

Since $\varphi \in D_{H^{-2}}^*$, $\langle \eta_k, \Delta_x \varphi(x, \cdot) \rangle_{H^{-2}} \in L^2(\mathbb{R}^3)$ and thus $\varphi_k \in D(\Delta_1^*)$ with $\Delta_1^* \varphi_k = \langle \eta_k, \Delta_x \varphi(x, \cdot) \rangle_{H^{-2}}$.

To prove that the limit in the expression for $A^{(n)}$ exists, let

$$\tilde{\varphi}_k(r) := \frac{1}{4\pi} \int_{S^2} r \varphi_k(r\omega) \mathrm{d}\omega. \tag{A.13}$$

One easily sees that $\|\tilde{\varphi}_k\|_{H^2((0,\infty))} = \|\varphi_k\|_{D(\Delta_1^*)}$, and thus $\tilde{\varphi}_k$ has a representative in $C^{1,\frac{1}{4}}([0, \infty))$ and there exists a constant K such that for $R, r \in [0, 1]$

$$|\tilde{\varphi}_k'(R) - \tilde{\varphi}_k'(r)| \leq |R - r|^{1/4} \|\tilde{\varphi}_k\|_{C^{1,\frac{1}{4}}[0,1]} \leq K |R - r|^{1/4} \|\varphi_k\|_{D(\Delta_1^*)}. \tag{A.14}$$

where $\tilde{\varphi}'_k$ denotes the derivative of $\tilde{\varphi}_k$. Then we also have that

$$
\begin{aligned}
\left\| \sum_{k=1}^{\infty} (\tilde{\varphi}'_k(R) - \tilde{\varphi}'_k(r))\eta_k \right\|_{\mathrm{H}^{-2}(\Omega_{n-1})}^2 &= \sum_{k=1}^{\infty} |\tilde{\varphi}'_k(R) - \tilde{\varphi}'_k(r)|^2 \\
&\leq K^2 |R-r|^{1/2} \sum_{k=1}^{\infty} \|\varphi_k\|_{D(\Delta_1^*)}^2 \\
&= K^2 |R-r|^{1/2} \sum_{k=1}^{\infty} \left(\|\varphi_k\|_{\mathrm{L}^2(\mathbb{R}^3)}^2 + \|\Delta_1^* \varphi_k\|_{\mathrm{L}^2(\mathbb{R}^3)}^2 \right) \\
&= K^2 |R-r|^{1/2})^2 \left(\sum_{k=1}^{\infty} \|\varphi_k \eta_k\|_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2})}^2 + \sum_{k=1}^{\infty} \|\Delta_x \varphi_k \eta_k\|_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2})}^2 \right) \\
&= K^2 |R-r|^{1/2} \|\varphi\|_{D_{\mathrm{H}^{-2}}^*}^2 .
\end{aligned}
\tag{A.15}
$$

It follows that the limit $\lim_{r\to 0} \sum_{k=0}^{\infty} \tilde{\varphi}'_k(r)\eta_k$ exists for this representative of φ and yields the value of $A^{(n)}/\sqrt{n}$. In addition, we have that

$$
\begin{aligned}
\|A^{(n)} \varphi_N\|_{\mathrm{H}^{-2}}^2 = \left\| A^{(n)} \sum_{k=1}^{N} \varphi_k \eta_k \right\|_{\mathrm{H}^{-2}}^2 &= n \left\| \sum_{k=1}^{N} (A^{(1)}\varphi_k)\eta_k \right\|_{\mathrm{H}^{-2}}^2 \\
&\leq n\|A^{(1)}\|_{D(\Delta_1^*)'}^2 \sum_{k=1}^{N} \|\varphi_k\|_{D(\Delta_1^*)}^2 \\
&= n\|A^{(1)}\|_{D(\Delta_1^*)'}^2 \|\varphi_N\|_{D_{\mathrm{H}^{-2}}^*}^2 .
\end{aligned}
\tag{A.16}
$$

Thus, $A^{(n)}$ defines a bounded linear map. The proof for $B^{(n)}$ follows the same steps.

This proof shows that the action of $A^{(n)}$, $B^{(n)}$ is determined by the action of $A^{(1)}$, $B^{(1)}$ on the φ_k. If φ is an element of $\mathrm{H}^2(\mathbb{R}^{3n})$ or $\mathrm{H}^1(\mathbb{R}^{3n})$, then the φ_k are in the corresponding space over $\mathbb{R}^3$. In case $\varphi \in \mathrm{H}^1(\mathbb{R}^{3n})$ we thus have that $B^{(n)}\varphi = 0$ since $B^{(1)} = 0$ on $D(\Delta_1^*) \cap \mathrm{H}^1(\mathbb{R}^3) = \mathrm{H}^2(\mathbb{R}^3)$ because $f_\gamma \notin \mathrm{H}^1(\mathbb{R}^3)$. If $\varphi \in \mathrm{H}^2(\mathbb{R}^{(3n)})$, $A^{(n)}$ acts as the Sobolev-trace, because $A^{(1)}\varphi_k = \varphi_k(0)$. $\square$

In order to establish regularity of the functions $\varphi \in D(\Delta_n^*)$ with $B^{(n)}\varphi = 0$, we use a theorem of Hörmander, which is formulated using the following spaces:

$$
H_{(2,s)} := \mathrm{L}^2([0,\infty), \mathrm{H}^{2+s}(\mathbb{R}^d)) \cap \mathrm{H}^2((0,\infty), \mathrm{H}^s(\mathbb{R}^d)) .
\tag{A.17}
$$

Theorem A.2. *Let $\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ and $\Delta_{\mathbb{R}^d}$ denote the distributional Laplacians on $(0,\infty)$ and $\mathbb{R}^d$, respectively. The map*

$$
\begin{aligned}
H_{(2,s)} &\to \mathrm{L}^2([0,\infty), \mathrm{H}^s) \oplus \mathrm{H}^{s+\frac{3}{2}}(\mathbb{R}^d) , \\
\eta &\mapsto \left(\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \Delta_{\mathbb{R}^d} - 1 \right)\eta,\ \eta(0) \right)
\end{aligned}
\tag{A.18}
$$

is an isomorphism of topological vector spaces.

This theorem is a direct consequence of [Hoer64, Corollary 10.4.1]. It gives rise to the following regularity lemma, where we denote by $P : \mathrm{L}^2(\mathbb{R}^3) \to \mathrm{L}^2(\mathbb{R}^3)$ the projection to the space of radial functions; for $j \in \{1, \ldots, n\}$, P_j is the projection P acting on the j-th factor of $\mathrm{L}^2(\mathbb{R}^{3n}) = \mathrm{L}^2(\mathbb{R}^3)^{\otimes n}$; and $Q_j = 1 - P_j$.

Lemma A.3. *Let $\varphi \in D(\Delta_n^*)$ with $B\varphi = 0$ and $\chi_\varepsilon \in C_{\mathrm{b}}^\infty(\mathbb{R}^{3n-3})$ such that, for some $\varepsilon > 0$,*

$$\mathrm{supp}\chi_\varepsilon \subset \mathcal{U}_\varepsilon(\mathcal{C}^{n-1}) := \left\{ (x_2, \ldots, x_n) \in \mathbb{R}^{3n-3} \,\middle|\, |x_i| > \varepsilon \text{ for all } i \right\}. \qquad (A.19)$$

Then $\chi_\varepsilon P_1 \varphi \in \mathrm{H}^2(\mathbb{R}^{3n})$.

Proof. We assume without loss of generality that φ is radial in the first argument, i.e., $\varphi = P_1 \varphi$. Let $\tilde{\varphi}(r, y) := r \, \chi_\varepsilon(y) \varphi(r, y) \in \mathrm{L}^2([0, \infty), \mathrm{L}^2(\mathbb{R}^{3n-3}))$. First note that

$$\begin{aligned} \Delta \tilde{\varphi} &= \chi_\varepsilon(\tfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \Delta_y) r\varphi + (\Delta_y \chi_\varepsilon) r\varphi + 2r \nabla_y \chi_\varepsilon \cdot \nabla_y \varphi \\ &= \chi_\varepsilon \underbrace{r\Delta_n^* \varphi}_{\in \mathrm{L}^2} + (\Delta_y \chi_\varepsilon) \underbrace{r\varphi}_{\in \mathrm{L}^2} + 2\nabla_y \chi_\varepsilon \cdot \underbrace{\nabla_y r\varphi}_{\in \mathrm{L}^2([0,\infty), \mathrm{H}^{-1})} \end{aligned} \qquad (A.20)$$

and that $B\varphi = 0$ implies $\tilde{\varphi}(0) = 0 \in \mathrm{H}^{-2}$. This of course means that $\tilde{\varphi}(0) \in \mathrm{H}^{s+\frac{3}{2}}$ for any $s \in \mathbb{R}$. Thus, Theorem A.2 implies that

$$\tilde{\varphi} \in H_{(2,-1)} \subset \mathrm{L}^2([0, \infty), \mathrm{H}^1(\mathbb{R}^{3n-3})). \qquad (A.21)$$

Plugging this information into Equation (A.20), we conclude that $\Delta \tilde{\varphi} \in \mathrm{L}^2([0, \infty), \mathrm{L}^2)$. Another use of Theorem A.2 then yields $\tilde{\varphi} \in H_{(2,0)}$ with $\tilde{\varphi}(0) = 0$. Hence

$$\frac{\tilde{\varphi}}{r} = \chi_\varepsilon P_1 \varphi \in \mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^2(\mathbb{R}^{3n-3})) \cap \mathrm{H}^2(\mathbb{R}^3, \mathrm{L}^2(\mathbb{R}^{3n-3})) = \mathrm{H}^2(\mathbb{R}^{3n}). \qquad (A.22)$$

$\square$

For $I \subset \{1, 2, \ldots, n\}$ define the following sets:

$$\mathcal{C}^I := \left\{ x \in \mathbb{R}^{3n} \,\middle|\, \prod_{j \in I} |x_j| = 0 \right\}. \qquad (A.23)$$

Then we have $\mathcal{C}^I \subset \mathcal{C}^n = \mathcal{C}^{\{1,2,\ldots,n\}}$. We will also use the abbreviation $\mathcal{C}^k := \mathcal{C}^{\{n-k+1, n-k+2, \ldots, n\}}$.

Proof of Proposition 5.3. We will prove that $\varphi \in D(\Delta_n^*) \cap \mathfrak{H}^n$ together with $B\varphi = 0$ implies $\varphi \in \mathrm{H}^2(\mathbb{R}^{3n})$. This will prove the statement when combined with Lemma 5.2.

In this proof we write $D^*(X)$ for the adjoint domain of the Laplacian defined on $X \subset \mathrm{H}^2(\mathbb{R}^{3n})$. For $I \subset \{1, \dots, n\}$ let $P_I := \prod_{i \in I} P_i$ and $Q_I := \prod_{i \in I}(1 - P_i)$. Then for $f \in \mathrm{L}^2(\mathbb{R}^{3n})$

$$f = \prod_{i=1}^n (P_i + Q_i) f = \sum_{I \subset \{1, \dots, n\}} P_I Q_{I^c} f. \tag{A.24}$$

Now let $\psi \in \mathrm{H}^2(\mathbb{R}^{3n})$. Then

$$\langle \varphi, \Delta\psi \rangle = \sum_{I \subset \{1, \dots, n\}} \langle P_I Q_{I^c} \varphi, \Delta\psi \rangle = \sum_{I \subset \{1, \dots, n\}} \langle P_I \varphi, \Delta Q_{I^c} \psi \rangle. \tag{A.25}$$

Since $Q_j \psi|_{x_j=0} = 0$, we have that $Q_{I^c}\psi \in \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{I^c})$ (cf. [Sve81]), and so it is sufficient to show that

$$P_I \varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{I^c})) \tag{A.26}$$

in order to conclude $\varphi \in D^*(\mathrm{H}^2(\mathbb{R}^{3n})) = \mathrm{H}^2(\mathbb{R}^{3n})$. By symmetry it suffices to consider the sets $I = \{1, \dots, k\}$ for $k \leq n$, which will be done by induction over k.

For $k = 1$, $I = \{1\}$, Equation (A.26) follows from Lemma A.3 in the following way: Let $\psi \in \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{n-1})$ and let ψ_ε be a sequence in $C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^{n-1})$ with $\mathrm{supp}\,\psi_\varepsilon \subset \mathcal{U}_{2\varepsilon}$ converging to ψ in H^2. Then Lemma A.3 implies

$$\langle P_1 \varphi, \Delta_{x_1} \psi_\varepsilon \rangle = \langle \chi_\varepsilon(x_2, \dots, x_n) P_1 \varphi, \Delta_{x_1} \psi_\varepsilon \rangle = \langle \chi_\varepsilon \Delta_{x_1} P_1 \varphi, \psi_\varepsilon \rangle = \langle \Delta_{x_1} P_1 \varphi, \psi_\varepsilon \rangle, \tag{A.27}$$

where we have used a cutoff χ_ε with $\chi_\varepsilon \equiv 1$ on $\mathcal{U}_{2\varepsilon}$. Since $\psi_\varepsilon \in \mathrm{L}^2\left(\mathbb{R}^3_{x_1}, \mathrm{H}_0^2(\Omega_{n-1})\right)$, we find that

$$\langle P_1\varphi, \Delta\psi \rangle = \lim_{\varepsilon \to 0} \langle P_1\varphi, \Delta\psi_\varepsilon \rangle \overset{\text{(A.4)}}{=} \lim_{\varepsilon \to 0} \left(\langle \Delta_{x_1} P_1\varphi, \psi_\varepsilon \rangle + \langle P_1\varphi, \sum_{j=2}^n \Delta_{x_j} \psi_\varepsilon \rangle \right)$$

$$\overset{\text{(A.3)}}{=} \lim_{\varepsilon \to 0} \langle \Delta_n^* P_1\varphi, \psi_\varepsilon \rangle = \langle \Delta_n^* P_1\varphi, \psi \rangle. \tag{A.28}$$

Hence, $P_1\varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{n-1}))$.

Now assume the induction hypothesis

$$P_{\{1, \dots, k\}} \varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+1, \dots, n\}})). \tag{A.29}$$

By symmetry, the argument for $k = 1$ independently gives also

$$P_{\{k+1\}} \varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{1, \dots, k, k+2, \dots, n\}})). \tag{A.30}$$

Thus, $P_{\{1,\ldots,k+1\}}\varphi$ is in the intersection of these two domains (A.29) and (A.30). Clearly, for two dense domains D_1, D_2 it holds that $D^*(D_1) \cap D^*(D_2) \subset D^*(D_1 + D_2)$. We thus need to show that

$$\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+1,\ldots,n\}}) + \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{1,\ldots,k,k+2,\ldots,n\}}) \tag{A.31}$$

is dense in $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+2,\ldots,n\}})$, as this implies that the adjoint domains are equal. The functions in this sum vanish on

$$\widetilde{\mathcal{C}} := \left(\mathcal{C}^{\{k+1\}} \cap \mathcal{C}^{\{1,\ldots,k\}} \right) \cup \mathcal{C}^{\{k+2,\ldots,n\}} . \tag{A.32}$$

Conversely, any function $f \in C_0^\infty(\mathbb{R}^{3n} \setminus \widetilde{\mathcal{C}})$ can be written as a sum $f = f_1 + f_2$ with $f_1 \in C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+1\}})$ and $f_2 \in C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{1,\ldots,k\}})$. Thus the sum (A.31) is dense in $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \widetilde{\mathcal{C}})$, but the latter space is equal to $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+2,\ldots,n\}})$, as $\mathcal{C}^{\{k+1\}} \cap \mathcal{C}^{\{1,\ldots,k\}}$ has codimension six, see [Sve81]. $\qquad\square$

B Algebraic identities

Proof of Proposition 6.1*.* We will prove the following formula:

$$\mathrm{Sym}(v_1 \otimes \cdots \otimes v_n) = \sum_{\mathbf{j} \in J} d_{\mathbf{j}} u_{\mathbf{j}}^{\otimes n} , \tag{B.1}$$

where $J = \{0,1\}^n$ and

$$u_{\mathbf{j}} = \sum_{k=1}^n (-1)^{j_k} v_k , \qquad d_{\mathbf{j}} = \frac{(-1)^{j_1 + \ldots + j_n}}{2^n n!} . \tag{B.2}$$

Note that we may rewrite $u_{\mathbf{j}}^{\otimes n}$ as a sum:

$$
\begin{aligned}
u_{\mathbf{j}}^{\otimes n} = \left(\sum_{k=1}^n (-1)^{j_k} v_k \right)^{\otimes n} &= \sum_{\mathbf{k} \in \{1,\ldots,n\}^n} (-1)^{j_{k_1} + \ldots + j_{k_n}} \, v_{k_1} \otimes \cdots \otimes v_{k_n} \\
&= \sum_{\mathbf{k} \in P} (-1)^{j_{k_1} + \ldots + j_{k_n}} \, v_{k_1} \otimes \cdots \otimes v_{k_n} \\
&\quad + \sum_{\mathbf{k} \in \{1,\ldots,n\}^n \setminus P} (-1)^{j_{k_1} + \ldots + j_{k_n}} \, v_{k_1} \otimes \cdots \otimes v_{k_n} \\
&=: (u_{\mathbf{j}})_P + (u_{\mathbf{j}})_{P^C} .
\end{aligned}
\tag{B.3}
$$

Here we have introduced a set P of multi-indices:

$$P := \left\{ x \in \mathbb{N}^n \,\middle|\, \exists \sigma \in S_n : x = \sigma(1,2,\ldots,n) \right\} \subset \{1,2,\ldots,n\}^n . \tag{B.4}$$

We will focus on $(u_{\mathbf{j}})_P$ first and insert it into our ansatz (B.1):

$$\sum_{\mathbf{j}\in J} d_{\mathbf{j}}(v_{\mathbf{j}})_P = \frac{1}{2^n n!}\sum_{\mathbf{j}\in J}\sum_{\mathbf{k}\in P}(-1)^{j_1+\dots+j_n}(-1)^{j_{k_1}+\dots+j_{k_n}}\, v_{k_1}\otimes\cdots\otimes v_{k_n}$$

$$= \frac{1}{2^n n!}\sum_{\mathbf{j}\in J}\sum_{\sigma\in S_n}(-1)^{j_1+\dots+j_n}(-1)^{j_{\sigma(1)}+\dots+j_{\sigma(n)}}\, v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}$$

$$= \frac{1}{2^n n!}\sum_{\mathbf{j}\in J}\sum_{\sigma\in S_n}(-1)^{j_1+\dots+j_n}(-1)^{j_1+\dots+j_n}\, v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}$$

$$= \frac{|\{0,1\}^n|}{2^n n!}\sum_{\sigma\in S_n} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)} = \mathrm{Sym}(v_1\otimes\cdots\otimes v_n). \tag{B.5}$$

It remains to show that $\sum_{\mathbf{j}} d_{\mathbf{j}}(u_{\mathbf{j}})_{PC} = 0$:

$$2^n n!\sum_{\mathbf{j}\in J} d_{\mathbf{j}}\,(u_{\mathbf{j}})_{PC}$$

$$= \sum_{\mathbf{j}\in J}\ \sum_{\mathbf{k}\in\{1,\dots,n\}^n\setminus P}(-1)^{j_1+\dots+j_n}(-1)^{j_{k_1}+\dots+j_{k_n}}\, v_{k_1}\otimes\cdots\otimes v_{k_n}$$

$$= \sum_{\mathbf{k}\in\{1,\dots,n\}^n\setminus P}\left(\sum_{\mathbf{j}\in J}(-1)^{j_1+\dots+j_n}(-1)^{j_{k_1}+\dots+j_{k_n}}\right) v_{k_1}\otimes\cdots\otimes v_{k_n}. \tag{B.6}$$

We will show that the expression in brackets vanishes. For every $\mathbf{k}\in\{1,\dots,n\}^n\setminus P$ there is at least one $s\in\{1,\dots,n\}$ such that none of the k_i is equal to s. Therefore, we can factor out

$$\sum_{\mathbf{j}\in J}(-1)^{j_1+\dots+j_n}(-1)^{j_{k_1}+\dots+j_{k_n}}$$

$$= \sum_{j_s=0}^{1}(-1)^{j_s}\sum_{\mathbf{j}\in\{0,1\}^{n-1}}(-1)^{j_1+\dots+\widehat{j_s}+\dots+j_n}(-1)^{j_{k_1}+\dots+j_{k_n}}, \tag{B.7}$$

because the remaining term on the right does not depend on j_s any more. Now $\sum_{j_s}(-1)^{j_s}=0$. $\qquad\square$

Proof of Lemma 7.2. For $\varphi,\psi\in D(\Delta_1^*)$,

$$\langle X_i(\varphi), Y_i(\psi)\rangle_{\mathbb{C}} - \langle Y_i(\varphi), X_i(\psi)\rangle_{\mathbb{C}}$$

$$= \left\langle \mathrm{e}^{\mathrm{i}\theta_i}(\alpha_i B_i + \beta_i A_i)(\varphi), \mathrm{e}^{\mathrm{i}\theta_i}(\gamma_i B_i + \delta_i A_i)(\psi)\right\rangle_{\mathbb{C}}$$

$$\quad - \left\langle \mathrm{e}^{\mathrm{i}\theta_i}(\gamma_i B_i + \delta_i A_i)(\varphi), \mathrm{e}^{\mathrm{i}\theta_i}(\alpha_i B_i + \beta_i A_i)(\psi)\right\rangle_{\mathbb{C}}$$

$$\overset{(4.7)}{=}(\alpha_i\delta_i - \beta_i\gamma_i)\langle B_i\varphi, A_i\psi\rangle_{\mathbb{C}} - (\alpha_i\delta_i - \beta_i\gamma_i)\langle A_i\varphi, B_i\psi\rangle_{\mathbb{C}}$$

$$= \langle B_i\varphi, A_i\psi\rangle_{\mathbb{C}} - \langle A_i\varphi, B_i\psi\rangle_{\mathbb{C}} \tag{B.8}$$

because the terms involving twice B_i or twice A_i cancel, and only the mixed terms survive. Summing the terms from all sources $i = 1, \ldots, N$ yields the claim. $\qquad \square$

Proof of Lemma 7.3. By assumption, $X_i(\psi) = 0$ and $X_i(\phi) = 1$ for $i = 1, \ldots, N$. Thus, from Lemma 7.2 with $\varphi = \phi$,

$$
\begin{aligned}
\sum_{i=1}^{N} Y_i(\psi) = \sum_{i=1}^{N} \langle X_i(\phi), Y_i(\psi) \rangle_{\mathbb{C}} &= \sum_{i=1}^{N} \langle X_i(\phi), Y_i(\psi) \rangle_{\mathbb{C}} - \langle Y_i(\phi), X_i(\psi) \rangle_{\mathbb{C}} \\
&= \langle \phi, -\Delta_1^* \psi \rangle_{\mathfrak{H}} - \langle -\Delta_1^* \phi, \psi \rangle_{\mathfrak{H}} \\
&= \langle \phi, (-\Delta_1^* + E_0)\psi \rangle_{\mathfrak{H}} - \langle (-\Delta_1^* + E_0)\phi, \psi \rangle_{\mathfrak{H}}.
\end{aligned}
\tag{B.9}
$$

This proves statement (a). To see why (b) is also true, observe that, since by assumption $\Delta_1^* \phi = \lambda \phi$,

$$
\begin{aligned}
2\mathrm{i}\,\mathrm{Im}\left(\sum_{i=1}^{N} Y_i(\phi)\right) = \sum_{i=1}^{N} Y_i(\phi) - \overline{Y_i(\phi)} &= \sum_{i=1}^{N} \langle X_i(\phi), Y_i(\phi) \rangle_{\mathbb{C}} - \langle Y_i(\phi), X_i(\phi) \rangle_{\mathbb{C}} \\
&= \langle \phi, -\Delta_1^* \phi \rangle_{\mathfrak{H}} - \langle -\Delta_1^* \phi, \phi \rangle_{\mathfrak{H}} \\
&= \langle \phi, (-\lambda + E_0)\phi \rangle_{\mathfrak{H}} - \langle (-\lambda + E_0)\phi, \phi \rangle_{\mathfrak{H}} = 0,
\end{aligned}
\tag{B.10}
$$

which completes the proof. $\qquad \square$

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