# Determining point distributions from their projections

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***Abstract*—Determining a set of points in $\mathbb{R}^d$ from its projection on lower dimensional spaces is a common task in data analysis. The aim of this note is to overview some general results from the 50s that might have been overlooked in the data analysis community and then to summerize our recent work on the subject in which the data is assumed to be supported on a quadratic manifold. Special attention is devoted to the spherical case.**

Projections; Cramér-Wold Theorem

## I. INTRODUCTION

The object of this paper is the determination of a point distribution in the $d$-dimensional space $\mathbb{R}^d$ from its projections on lines $\mathbb{R}\theta$ or hyperplanes $\theta^\perp$, $\theta \in \mathbb{S}^{d-1}$ (the unit sphere of $\mathbb{R}^d$).

Let us be more precise. To a finite set of points $A \subset \mathbb{R}^d$ we may associate a measure $\delta_A = \sum_{a\in A} \delta_a$ where $\delta_a$ is the $\delta$ probability measure at $a$, $\delta_a(E) = 1$ if $a \in E$ and 0 otherwise. For $\theta \in \mathbb{S}^{d-1}$, the projection $\pi^\ell_\theta(A)$ of $A$ on $\mathbb{R}\theta$ (resp. the projection $\pi^H_\theta(A)$ of $A$ on $\theta^\perp$) is

$$\begin{aligned}\pi^\ell_\theta(A) &= \big\{n(t) := |\{s \in \theta^\perp : s + t\theta \in A\}|, t \in \mathbb{R}\big\}\\ \pi^H_\theta(A) &= \big\{n(t) := |\{s \in \mathbb{R} : t + s\theta \in A\}|, t \in \theta^\perp\big\}.\end{aligned}$$

Note that, as $A$ is finite, $n(t) \neq 0$ only for finitely many $t$'s, precisely those $t$'s so that the corresponding point $t\theta$ (resp. $t \in \theta^\perp$) onto which $A$ projects. Then $n(t)$ is the number of points in $A$ that project to that point. In other words, $\pi^\ell_\theta(A), \pi^H_\theta(A)$ give both the number of points of $A$ that project to a given element of $\mathbb{R}\theta$ or $\theta^\perp$ and the position of these projections. Thus $\pi^\ell_\theta(A), \pi^H_\theta(A)$ can be identified with measures on $\mathbb{R}\theta$ or $\theta^\perp$:

$$\begin{aligned}\pi^\ell_\theta(\delta_A) &= \sum_{t\in\mathbb{R}} n(t)\delta_{t\theta}\\ \pi^H_\theta(\delta_A) &= \sum_{t\in\theta^\perp} n(t)\delta_t.\end{aligned}$$

The first one is a measure on $\mathbb{R}\theta$ while the second is a measure on $\theta^\perp$

The question we are dealing with here is thus the following:

**Problem 1.** *Given $A, B$ two finite sets and $\Theta \subset \mathbb{S}^{d-1}$. Assume that $\pi^\ell_\theta(A) = \pi^\ell_\theta(B)$ — resp. $\pi^H_\theta(A) = \pi^H_\theta(B)$— for every $\theta \in \Theta$, is $A = B$?*

Now, as is well known, a measure is uniquely determined by its Fourier transform (characteristic function in probabilistic language). Our problem can thus be reformulated as follows: is

$$\widehat{\delta_A}(\xi) = \sum_{a\in A} e^{i\langle a,\xi\rangle} \quad ,\xi \in \mathbb{R}^d$$

uniquely determined by the $\mathbb{R}\theta$-Fourier transform

$$\begin{aligned}\widehat{\pi^\ell_\theta(\delta_A)}(\eta\theta) &= \sum_{t\in\mathbb{R}} n(t)e^{it\eta}\\ &= \sum_{a\in A} e^{i\langle\eta\theta,a\rangle} = \widehat{\delta_A}(\eta\theta) \quad \eta \in \mathbb{R}\end{aligned}$$

(since there are $n(t)$ points in $A$ such that $\langle\eta\theta, a\rangle = t\eta$) or by the $\theta^\perp$-Fourier transform

$$\begin{aligned}\widehat{\pi^H_\theta(\delta_A)}(\eta) &= \sum_{t\in\theta^\perp} n(t)e^{i\langle t,\eta\rangle}\\ &= \sum_{a\in A} e^{i\langle\eta,a\rangle} = \widehat{\delta_A}(\eta) \quad \eta \in \theta^\perp.\end{aligned}$$

The reader may have recognized the Fourier-Slice Theorem for the Radon transform. This allows us to reformulate Problem 1 as:

**Problem 2.** *Given $A, B$ two finite sets and $\Theta \subset \mathbb{S}^{d-1}$. Assume that $\widehat{\delta_A} = \widehat{\delta_B}$ on $\mathbb{R}\theta$ — resp. on $\theta^\perp$– for every $\theta \in \Theta$, is $A = B$?*

As such, our problem can be seen as a spectral estimation problem. The first consequence of this change of view is that, if $\Theta = \mathbb{S}^{d-1}$ then the set of $\mathbb{R}\theta$'s or of $\theta^\perp$'s covers $\mathbb{R}^d$. We then immediately obtain the following thereom:

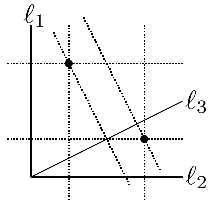


Fig. 1. The knot points in the proof of Theorem I.2

**Theorem I.1** (Cramér-Wold [CW])**.** *Given $A, B$ two finite sets.*

*If $\widehat{\delta_A} = \widehat{\delta_B}$ on $\mathbb{R}\theta$ — resp. on $\theta^\perp$– for every $\theta \in \mathbb{S}^{d-1}$ then $A = B$.*

*If $\pi^\ell_\theta(A) = \pi^\ell_\theta(B)$ — resp. $\pi^H_\theta(A) = \pi^H_\theta(B)$— for every $\theta \in \mathbb{S}^{d-1}$ then $A = B$.*

The actual theorem is more general as it deals with arbitrary probability measures, not only with point distributions. Intuitively, a finite number of projections should be sufficient to determine finite point distributions. This has been proved by G. Hajós (published in a paper of A. Rényi [Re]) in dimension $d = 2$ and in full generality by A. Heppes [He]:

**Theorem I.2** (Hajós-Rényi-Heppes [Re], [He])**.** *Let $\theta_1, \ldots, \theta_{k+1} \in \mathbb{S}^{d-1}$ be such that the hyperplanes $\theta_1^\perp, \ldots, \theta_{k+1}^\perp$ are all distinct. If $A, B$ have cardinality at most $k$ and are such that $\pi^H_{\theta_j}(A) = \pi^H_{\theta_j}(B)$ for $j = 1, \ldots, k+1$ then $A = B$.*

Let us reproduce the simple and elegant argument here.

*Proof.* Let $A \subset \mathbb{R}^d$ and set $A_j = \pi^H_{\theta_j}(A)$. For each $\alpha \in A_j$, consider the line $\alpha + \mathbb{R}\theta_j$ issued from this point in the direction $\theta_j$ orthogonal to $\theta_j^\perp$. We will say that a point $x \in \mathbb{R}^d$ is a *knot point* if $x$ is the intersection of at least $k+1$ such lines.

Note first that every point of $A$ is a knot point, since its projection on $\theta_j^\perp$ is in $A_j$ for every $j$. Let us prove that every knot point is in $A$. To do so, let $x$ be a knot point and let $\alpha_1, \ldots, \alpha_{k+1}$ be the projections of $x$ on $\theta_1^\perp, \ldots, \theta_{k+1}^\perp$. As each line $\alpha_j + \mathbb{R}\theta_j$ contains at least one of the $k$ points of $A$, the pigeon hole principle implies that at least two of these lines contain the same point of $A$. But their only intersection point is $x$ thus $x \in A$.

Note that this proof is algorithmic, but this algorithm requires generically to solve $\sim k^k$ equations. □

Note that the theorem is best possible: let $\Omega$ be a regular planar $2k$-gon and number the vertices consecutively (see Firgure 2). Let $A$ be those that are numbered evenly and $B$ those that are numbered oddly. Let $\theta_j$, $j = 1, \ldots, k$ be the directions of the edges. Then $A$ and $B$ have same projections on $\theta_j^\perp$.

Of course, the set of directions is very particular, and one would be tempted to think that some set of directions might allow to distinguish finite sets. This is not the case:

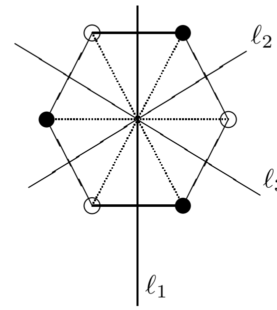


Fig. 2. A regular $2k$-gon shows optimality in Theorem I.2

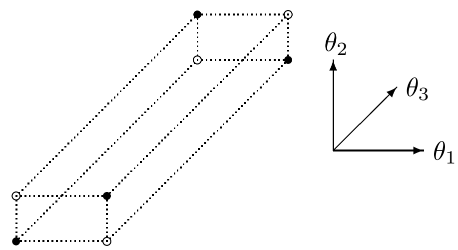


Fig. 3. The construction in Theorem I.3.

**Theorem I.3** (Hajós-Rényi-Heppes [Re], [He])**.** *Let $\theta_1, \ldots, \theta_k$ be a set of directions. There exist two distinct sets $A \neq B$ such that $\pi^H_{\theta_j}(A) = \pi^H_{\theta_j}(B)$ for $j = 1, \ldots, k$.*

*Proof.* Let us consider the sets

$$\begin{aligned} A &= \left\{ \sum_{j=1}^k 2^j \varepsilon_j \theta_j,\ \varepsilon_j \in \{0,1\},\ \sum_{j=1}^k \varepsilon_j \text{ even} \right\} \\ B &= \left\{ \sum_{j=1}^k 2^j \varepsilon_j \theta_j,\ \varepsilon_j \in \{0,1\},\ \sum_{j=1}^k \varepsilon_j \text{ odd} \right\}. \end{aligned}$$

The coefficient $2^j$ guaranteed that all points $\sum_{j=1}^k 2^j \varepsilon_j \theta_j$, $\varepsilon_j \in \{0,1\}$ are distinct so that $A$ and $B$ have both $2^{k-1}$ elements and have no element in common. Now, to each point in $A$, its projection on $\theta_j^\perp$ corresponds to the point in $B$ where $\varepsilon_j$ is replaced by $1 - \varepsilon_j$. □

This time, the sets $A$ and $B$ are more or less "lattice type". One may thus ask if finite sets, supported in some special manifolds, might be distinguished by fewer directions. The aim of this paper is to show that this is indeed the case for finite sets supported on quadratic manifolds.

## II. THE 2-DIMENSIONAL CASE

In this section, we restrict our attention to the planar case $\mathbb{R}^2$ so that hyperplanes are lines.

In this section, we will now assume that $A, B \subset \Gamma$ where $\Gamma = \{\gamma(t), t \in I\}$ is a *known* smooth planar curve. Typical examples we have in mind are the parabola and the circle.

Let us start with the parabola, or more generally, with curves that look like parabolas:

**Proposition II.1.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\Gamma = \big\{\big(t, \psi(t)\big), t \in \mathbb{R}\big\}$. Let $A, B \subset \Gamma$ be two finite sets.*

*— Let $\theta = (1,0)$. Then $\pi^\ell_\theta(A) = \pi^\ell_\theta(B)$ if and only if $A = B$.*

*— Assume that $\psi$ is strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, +\infty)$ and $t^{-1}\psi(t) \to +\infty$ when $t \to \pm\infty$ and let $\theta_1 \neq \pm\theta_2 \in \mathbb{S}^1$. Then $\pi^\ell_{\theta_j}(A) = \pi^\ell_{\theta_j}(B)$, $j = 1, 2$ if and only if $A = B$.*

*Remark* II.2. The result is actually valid for any finite measure supported on $\Gamma$. It has been proved for measures that are absolutely continuous with respect to arc length in [JK], the case of the parabola was proved by Sjölin [Sj2].

The result holds of course for any rotation of $\Gamma$ provided $(1, 0)$ is rotated by the same angle.

*Sketch of proof.* The first case is trivial as the projection $\pi^\ell_{(1,0)}$ is one-to-one from $\Gamma$ onto $\mathbb{R}(1, 0)$.

The second case is slightly more subtle assume that $A \neq B$ and let $x_0 \in A \setminus B$. As $\pi^\ell_{\theta_1}(A) = \pi^\ell_{\theta_1}(B)$, there exists $y_1 \in B \setminus A$ such that $\pi^\ell_{\theta_1}(y_1) = \pi^\ell_{\theta_1}(x_0)$. But then, there exist $x_1 \in A \setminus B$ such that $\pi^\ell_{\theta_2}(x_1) = \pi^\ell_{\theta_2}(y_1)$ from which we get $y_2 \in B \setminus A$ such that $\pi^\ell_{\theta_1}(y_2) = \pi^\ell_{\theta_1}(x_1)$.... and it is not hard to see (and was proved in [JK]) that this sequence is infinite, which contradicts the assumption that $A$ and $B$ were finite.

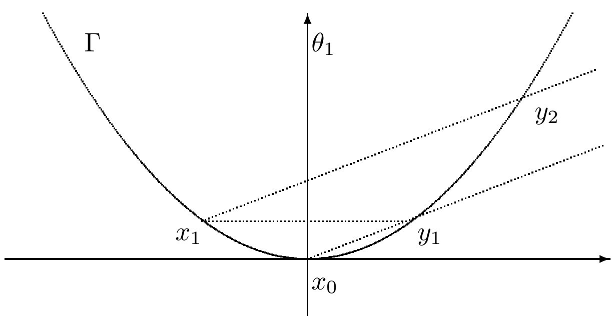


Fig. 4. The construction of $x_k, y_k$

□

For the circle, the polygonal case of Figure 2 shows that some condition needs to be imposed on the angles. This is obtained as follows:

**Theorem II.3.** *Let $\Gamma$ be a circle and $\theta_1, \theta_2 \in \mathbb{S}^1$ be two vectors with an angle that is an irrational multiple of $\pi$: $\theta := \dfrac{1}{\pi}\arccos\langle\theta_1, \theta_2\rangle \notin \mathbb{Q}$. Let $A, B \subset \Gamma$ be two finite sets. Then $\pi^\ell_{\theta_j}(A) = \pi^\ell_{\theta_j}(B)$, $j = 1, 2$, if and only if $A = B$.*

*Remark* II.4. The result is actually valid for any finite measure supported on $\Gamma$. It has been proved for measures that are absolutely continuous with respect to arc length by P. Sjölin [Sj1] and N. Lev [Le] independently. A slightly more geometric proof appears in [JK] and another proof appears in [GJ], see the next section.

*Sketch of proof.* The proof is almost the same as previously, the main point is that the operation $x_n \to x_{n+1}$ is simply the rotation by an angle $2\theta$ centered at the center of the circle since it is obtained from the composition of the reflections with respect to $\mathbb{R}\theta_1$ and $\mathbb{R}\theta_2$. Now if this angle is an irrational multiple of $\pi$, the sequence $x_n$ is infinite (and even uniformly distributed), otherwise, it is periodic. □

The argument extends to smooth closed convex curves but the precise condition on the angles $\theta_1, \theta_2$ is in general impossible to compute explicitely.

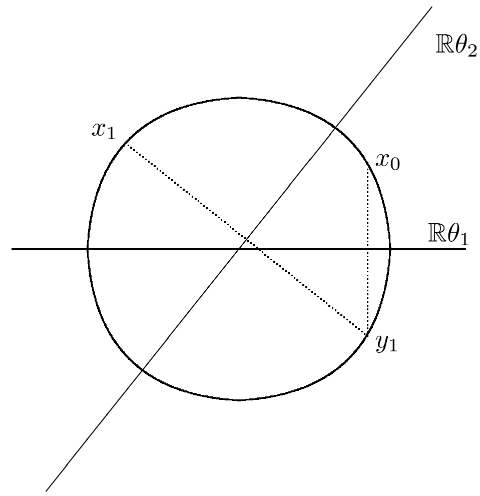


Fig. 5. The construction of $x_k, y_k$ for a closed convex curve. In the case of a circle, $x_k \to y_k$ and $y_k \to x_{x+1}$ are given by orthogonal symmetries.

Once this is understood, it becomes easy to *geometrically* guess what curves and what angles lead to uniqueness. Figure 6 provides two more examples.

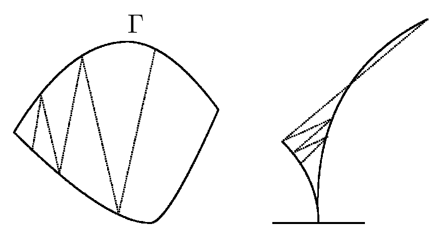


Fig. 6. Closed curve with a corner point and a cusp. The cusp is slightly more subtle as some lines may intersect the cusp three times. One needs first to eliminate this part of the curve by exploiting the fact that some lines intersect the cusp only once. For the precise argument, see [JK].

## III. The higher dimensional case for quadratic surfaces - projection on hyperplanes

In higher dimension, the situation becomes more complicated. To start, projections on hyperplanes are no longer projections on lines. As lines contain much less information than hyperplanes, it would of course be more difficult to extract information from projections on lines than from projections on hyperplanes and may even be impossible.

Further, in the previous section, we saw two very different behaviors: on the parabola, the points $x_k$ we constructed were always wandering (to infinity) while on the circle, they may be either periodic or rotating for ever. In higher dimension, both situations may occur for a given manifold e.g. on the paraboloid. However, we have been able to prove the following:

**Theorem III.1** (Gröchenig-Jaming [GJ]). *Let $Q$ be a quadratic form on $\mathbb{R}^d$, $v \in \mathbb{R}^d$, $\rho \in \mathbb{R}$ and $S = \{x \in \mathbb{R}^d \,:\, Q(x) + 2\langle v, x\rangle = \rho\}$. There exists an exceptional set $\mathcal{E} = \mathcal{E}(Q, v, \rho)$ of pairs of distinct directions such that*

*i) the set $\mathcal{E}$ has measure zero with respect to the surface measure on $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$;*

*ii) when $\theta_1, \theta_2 \in \mathbb{R}^d$ satisfy $Q(\theta_1), Q(\theta_2) \neq 0$ and $(\theta_1, \theta_2) \notin \mathcal{E}$, then, for every finite sets $A, B \subset S$ $\pi^H_{\theta_j}(A) = \pi^H_{\theta_j}(B)$, $j = 1, 2$, if and only if $A = B$.*

The full proof is too long to be reproduced here, except for the sphere:

*Sketch of proof when $S = \mathbb{S}^{d-1}$.* Consider $\delta_A, \delta_B$ the measures associated to $A$ and $B$. Then $\pi^H_{\theta_j}(A) = \pi^H_{\theta_j}(B)$ is equivalent to $\widehat{\delta_A} = \widehat{\delta_B}$ on $\theta_j^\perp$. Let us introduce $\widehat{\mu} := \widehat{\delta_A} - \widehat{\delta_B}$ and note that this is a continuous function. It is not difficult to see that this is equivalent to the fact that $\widehat{\mu}$ satisfies the invariance property

$$\widehat{\mu}(s + t\theta_j) = -\widehat{\mu}(s - t\theta_j) \qquad s \in \theta_j^\perp,\ t \in \mathbb{R}.$$

In other words, $\widehat{\mu}$ is odd with respect to the reflection with respect to the hyperplane $\theta_j^\perp$, thus it is invariant with respect to the composition of these two reflections. Again, if $\frac{1}{\pi}\arccos\langle\theta_1, \theta_2\rangle \notin \mathbb{Q}$ this is a rotation with angle an irrational multiple of $\pi$. A continuous function that is invariant under such a transformation is constant (the orbit of a point is dense), and it vanishes on $\theta_j^\perp$, it has to be 0 everywhere. It follows that $\widehat{\delta_A} = \widehat{\delta_B}$, thus $\delta_A = \delta_B$ and then $A = B$. $\square$

*Remark* III.2. The result is not specific to point distributions but is valid for any finite measure. In the case of the sphere, it is even valid for any Schwartz-distribution.

## IV. Projections on lines of measures on three dimensional spheres

Recently, we have started a new approach which consist in using the following fact: if $\mu$ is a measure supported on $\mathbb{S}^{d-1}$, then $u = \widehat{\mu}$ is a solution of the Helmholtz equation $\Delta u + u = 0$ in $\mathbb{R}^d$. Our original problem of reconstructing point distributions on the sphere from their projections on hyperplanes is then a particular case of the following:

**Problem 3.** *Let $u$ be a solution of the Helmholtz equation $\Delta u + u = 0$ on $\mathbb{R}^d$ and let $\theta_1, \ldots, \theta_k \in \mathbb{S}^{d-1}$.*

*— Does $u = 0$ on $\theta_1^\perp, \ldots, \theta_k^\perp$ imply that $u = 0$ on $\Omega$.*

*— Does $u = 0$ on $\mathbb{R}\theta_1, \ldots, \mathbb{R}\theta_k$ imply that $u = 0$ on $\Omega$.*

For the problem considered here there is a further restriction on $u$, namely that $u$ is of the form $\widehat{\delta_A}$, that is, $u$ is a trigonometric polynomial while here we don't have such a requirement and $u$ can even be a Schwarz distribution. This new vision of the problem leads to new questions: what happens if $\mathbb{R}^d$ is replaced by a domain $\Omega \subset \mathbb{R}^d$? Can some of the Dirichlet conditions $u = 0$ on $\theta_j^\perp$ be replaced by Neumann or even Robin conditions? Can the lines and hyperplanes be replaced by more general manifolds?

We will now describe some progress we could make through this approach. To do so we will also restrict attention to dimensions $d = 2$ and $d = 3$. We refer to [FBGJ] for more general results.

The key observation is that the Helmohltz equation can be solved in polar coordinates. The solutions are then expressed as an expansion in spherical harmonics which involves Bessel functions:

$$u(r\theta) = \sum_{m=0}^{\infty}\sum_{j=1}^{N(m)} a_{m,j}\frac{J_{\nu(m)}(r)}{r^{(d-2)/2}} Y_m^j(\theta) \tag{IV.1}$$

where $J_\nu$ are the Bessel functions, $\nu(m) = m + (d-2)/2$ and $\{Y_m^j\}_{j=1,\ldots,N(m)}$ is a basis for the spherical harmonics of degree $m$ in $\mathbb{R}^d$.

In dimension $d = 2$, the spherical harmonics just correspond to the usual Fourier basis so that

$$u(r\theta) = a_0 J_0(r) + \sum_{m=1}^{\infty}\bigl(a_m e^{im\theta} + a_{-m}e^{-im\theta}\bigr)J_m(r).$$

The key property is that, when $r \to 0$, $J_{k+1}(r) = o\bigl(J_k(r)\bigr)$. In particular, $u(0) = 0$ implies that $a_0 = 0$. Further, if $a_0 = 0$ and if $a_{\pm 1} = \cdots = a_{\pm(m-1)} = 0$ then

$$u(r, \theta) = \bigl(a_m e^{im\theta} + a_{-m}e^{-im\theta}\bigr)J_m(r) + o\bigl(J_m(r)\bigr).$$

Now, if $u(r\theta_1) = u(r\theta_2) = 0$, then $u(0) = 0$ thus $a_0 = 0$, and if $a_{\pm 1} = \cdots = a_{\pm(m-1)} = 0$ then the previous estimate implies that

$$\begin{cases} a_m e^{im\theta_1} & + & a_{-m}e^{-im\theta_1} & = & 0 \\ a_m e^{im\theta_2} & + & a_{-m}e^{-im\theta_2} & = & 0 \end{cases}.$$

The determinant of this system is $2i\sin m(\theta_1 - \theta_2) \neq 0$ if $\theta_1 - \theta_2 \notin \mathbb{Q}\pi$ thus $a_m = a_{-m} = 0$. An induction shows that $a_m = 0$ for all $m \in \mathbb{Z}$ and $u = 0$. This is another proof of Theorem II.3 and also extends to higher dimensions.

Let us now show that hyperplanes can not be replaced by lines in dimension $d = 3$. Let $\theta_1, \ldots, \theta_k$ be a finite set of directions. In this case $N(m) = 2m + 1$ and a simple dimension argument shows that, if $m \geq k/2$ there is a linear combination $Z$ of the $\{Y_m^j\}_{j=1,\ldots,2m+1}$ that vanishes at $\pm\theta_1, \ldots, \pm\theta_k$. Then $u(r\theta) = \frac{J_{\nu(m)}(r)}{\sqrt{r}} Z(\theta)$ is a non-zero solution of the Helmholtz equation that vanishes on $\mathbb{R}\theta_1, \ldots, \mathbb{R}\theta_k$. Note however that the function $u$ we constructed is *not* a trigonometric polynomial and, at this stage, it is unclear to us whether points on the sphere are uniquely determined by their projections on certain lines or not.

## V. Conclusion

In this paper, we have given an overview of the problem of unique determination of a set of points in $\mathbb{R}^d$ from their projection on lines and on hyperplanes. We have shown that, if the set of points is known to be on a (fixed) sphere, then there exists a set of a few lines that allow to determine uniquely the set of points.

## References


[CW] H. Cramér & H. Wold *Some theorems on distribution functions.* J. London Math. Soc. **11** (1936), 290–294.

[FBGJ] A. Fernández-Bertolin, K. Gröchenig & Ph. Jaming *Heisenberg Uniqueness Pairs and unique continuation for harmonic functions and solutions of the Helmholtz equation.* In preparation.

[GJ] K. Gröchenig & Ph. Jaming *The Cramer-Wold Theorem on quadratic surfaces and Heisenberg uniqueness pairs.* ArXiv:1608.06738.

[He] A. Heppes *On determination of probability distributions of more dimensions by their projections.* Acta Math. Acad. Sci. Hung. **7** (1956), 403–410.

[JK] Ph. Jaming & K. Kellay *A dynamical system approach to Heisenberg Uniqueness Pairs.* J. Anal. Math., to appear.

[Le] N. Lev *Uniqueness theorems for Fourier transforms.* Bull. Sci. Math., **135** (2011), 134–140.

[Re] A. Rényi *On projections of probability distributions* Acta Math. Acad. Sci. Hung. **2** (1952), 132–142.

[Sj1] P. Sjölin *Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin.* Bull. Sci. Math., **135** (2011), 125–133.

[Sj2] P. Sjölin *Heisenberg uniqueness pairs for the parabola.* J. Four. Anal. Appl., **19** (2013), 410–416.

[1] E. M. Stein & G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces.* Princeton University Press, 1971.