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Discretization error for the maximum of a Gaussian field

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Abstract

A Gaussian field X defined on a square S of \mathbb{R}^2 is considered. We assume that this field is only observed at some points of a regular grid with spacing $\frac{1}{n}$. We are interested in the normalized discretization error $n^2(M - M_n)$, with M the global maximum of X over S and M_n the maximum of X over the observation grid. The density of the location of the maximum is given using Rice formulas and its regularity is studied. Joint densities with the value of the field and the value of the second derivative are also given. Then, a kind of Slepian model is used to study the field behavior around the unique point where the maximum is attained, called t^* . We show that the normalized discretization error can be bounded by a quantity that converges in distribution to a uniform variable. The set where this uniform variable lies principally depends on the second derivative of the field at t^* . The bound is a function of this quantity which is approached by finite differences in practice. The bound is applied both on simulated and real data. Real data are used in positioning by satellites systems quality assessment.

1 Introduction

The maximum of a random field is an important variable that has been extensively studied, see [Adler and Taylor, 2007], [Azaïs and Wschebor, 2009] and references therein. It plays an important role in spatial statistics see, for example, [Worsley et al., 1996] [Cressie, 2015].

In most of the cases, instead of the true maximum, we observe the maximum on a grid. There are very few results that permit to evaluate the difference between these two quantities. When we consider a stochastic process defined on an interval $[0, T]$ with T tending to infinity, the papers [Hüsler, 2004], [Piterbarg, 2004] compare the extremal behavior of the maximum on the whole segment and on a grid that can be more or less dense.

In this paper we consider a random field in dimension 2 (for simplicity) defined on a fixed set S . Our asymptotic results are obtained as the mesh of the grid tends to zero, see Th. 2 and Th. 3. Our tools are the following.

First, we study the density of the location of the maximum and in particular the regularity of the density. For that goal, we use Rice formulas and we obtain, under stronger hypotheses, results that go beyond the results of [Samorodnitsky and Shen, 2013], see Th. 1. Note that [Rychlik and Sjö, 2002] studies the joint distribution of maximum position and value in dimension 1. Note also the result of [Pimentel, 2014] that concerns the Brownian motion.

Second, we use a kind of Slepian model: a Taylor expansion at a random point. Our situation is simpler than in the classical Slepian situation [Slepian, 1963], [Leadbetter et al., 1983] since we consider the **unique** point where the maximum is attained and we don't have to consider a crossing or a maximum "chosen at random" using Palm distribution as it is the case in the classical studies.

The organization the paper is the following: in Section 2 we introduce preliminary results; Section 3 is devoted to the study of the density of the location of the maximum, stating our first main result; Section 4 stated our main results on discretization error; Section 5 is devoted to numerical applications, after a Monte-Carlo experiment, we describe a true case on positioning by satellite error where a specification is given in terms of the maximal error on a geographical zone while measurements are performed on a grid; some extra proofs are given in the appendix.

1.1 Hypotheses and notation

In all the paper $X(\cdot)$ is a stationary Gaussian random field defined on \mathbb{R}^2 . We aim at comparing the maximum M of $X(\cdot)$ on S with its maximum on a grid with mesh tending to zero. By a time and space scaling, and without loss of generality, we can assume that $S = [0, 1]^2$, and that for each $t = (t_1, t_2) \in \mathbb{R}^2$, $\mathbb{E}(X(t)) = 0$ and $\text{Var}(X(t)) = 1$. We define the partition $S = S_2 \cup S_1 \cup S_0$. S_0 is the union of the four vertices, S_1 is the union of the four edges and S_2 is the interior of S . We use the following notation:

- $\Gamma(h) = \text{Cov}(X(t), X(t+h))$, the covariance function of $X(\cdot)$
- $p_{X(t)}$, the probability density function of $X(t)$
- $G_n = \frac{1}{n}\mathbb{Z}^2 \cap [0, 1]^2$, the grid with mesh $\frac{1}{n}$
- $t^* = \underset{t \in S}{\text{argmax}} X(t)$, the point where $X(\cdot)$ achieves its maximum over S
- $t_n^* = \underset{t \in G_n}{\text{argmax}} X(t)$, the point where $X(\cdot)$ achieves its maximum over G_n
- $M = \max_{t \in S} X(t)$, the maximum of $X(\cdot)$ over S
- $M_n = \max_{t \in G_n} X(t)$, the maximum of $X(\cdot)$ over G_n

As a consequence the main goal of this paper is to give bound to $M - M_n$.

- $X'(t) = \left(\frac{\partial X(t)}{\partial t_1}, \frac{\partial X(t)}{\partial t_2} \right)^\top$
- $X''(t) = \begin{pmatrix} X''_{1,1}(t) & X''_{1,2}(t) \\ X''_{2,1}(t) & X''_{2,2}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 X(t)}{\partial t_1^2} & \frac{\partial^2 X(t)}{\partial t_1 \partial t_2} \\ \frac{\partial^2 X(t)}{\partial t_1 \partial t_2} & \frac{\partial^2 X(t)}{\partial t_2^2} \end{pmatrix}$
- X'' stands for $X''(t^*)$
- $\|u\|_{X''} = \sqrt{u^\top (-X'')u}$, the norm associated to the matrix $-X''$
- $\|u\|$ is the usual Euclidean norm
- $t_{n,X''} = \operatorname{argmin}_{t \in G_n} \|t^* - t\|_{X''}$, the point of G_n closest to t^* for the norm $\|\cdot\|_{X''}$
- $X'_{S_1}(t)$ and $X''_{S_1}(t)$ are the tangential first and second derivatives on S_1
- \bar{t}_n is the point of $G_n \cap S_1$ closest to t^* (for $\|\cdot\|_{X''}$ or $\|\cdot\|$ equivalently)
- $M \prec 0$ means that the matrix M is definite negative
- Λ is the variance-covariance matrix of $X'(t)$ or equivalently the opposite matrix of the covariance between $X''(t)$ and $X(t)$. Remark that Λ is always definite positive: considering a stationary process on \mathbb{R} with positive variance, it is direct that the variance of its derivative cannot vanish. Applied to $X(\cdot)$ considered in one direction v , this implies that $v^\top \Lambda v > 0$.

We assume the hypothesis (H) on the field $X(\cdot)$: almost surely, the sample paths of $X(\cdot)$ are of class \mathcal{C}^2 , the covariance $\Gamma(h) \neq \pm 1$, for $h > 0$ and for each $t \in S$, the distribution of $X''(t)$ is nondegenerate.

2 Preliminary results

The following lemmas ensure that t^* and $\|\cdot\|_{X''}$ are well defined.

Lemma 1. *The maximum of $X(\cdot)$ over S is almost surely achieved at a single point.*

Proof. The considered process is Gaussian and continuous on the compact set S . Because of (H)

$$\text{for all } s \neq t, \mathbb{P}\{X(s) = X(t)\} = 0.$$

Apply the result due to [Tsirelson, 1975] with a nice proof in [Lifshits, 1983] (Theorem 3). \square

Lemma 2. *Almost surely (a.s.), there exist no points in S_2 such that $X'(t) = 0$ and $\det(X''(t)) = 0$.*

Almost surely, there exist no points in S_1 such that $X'_{S_1}(t) = 0$ and $X''_{S_1}(t) = 0$.

Proof. We give the proof of the first statement. The process $X'(\cdot)$ that goes from S to \mathbb{R}^2 has \mathcal{C}^1 paths. In addition, for all t , $X(t)$ has a bounded density. By [Azaïs and Wschebor, 2009], Proposition 6.5 with condition b) satisfied by (H) , there is almost surely no point $t \in S$ such that $X'(t) = 0$, $\det(X''(t)) = 0$.

The proof of the second statement is similar. \square

Lemma 2 implies that $-X'' = -X''(t^*)$ is a.s. a positive definite matrix.

Lemma 3. *For $t \in S_2$ and $x \in \mathbb{R}$ the distribution of $X''(t)$ conditional to $\{X(t) = x, X'(t) = 0\}$ admits the following representation*

$$X''(t) = R - \Lambda x, \quad (1)$$

where R is a centered Gaussian random matrix the distribution of which does not depend on x or t .

Proof. Remark that because of stationarity, $X'(t)$ and $X''(t)$ are independent. So it suffices to compute the distribution of $X''(t)$ conditional to $X(t) = x$. This last distribution is given by classical regression formulas yielding (1). \square

3 Density of the argmax

We present now the first main result of the paper.

Theorem 1 (Density of the argmax). *Let μ be the measure that is the sum of three components:*

- the counting measure on S_0 ,
- λ_1 , the one-dimensional Lebesgue measure on S_1 ,
- λ_2 , the two-dimensional Lebesgue measure on S_2 .

Then the random variable t^* admits a density with respect to μ expressed as:

$$\begin{aligned} p_{t^*}(t) = & \mathbf{1}_{t \in S_0} \mathbb{P}(\forall s \in S, X(s) \leq X(t)) \\ & + \mathbf{1}_{t \in S_1} \mathbb{E} \left(|X''_{S_1}(t)| \mathbf{1}_{A_{X(t), S}} \middle| X'_{S_1}(t) = 0 \right) \times p_{X'_{S_1}(t)}(0) \\ & + \mathbf{1}_{t \in S_2} \mathbb{E} \left(|\det(X''(t))| \mathbf{1}_{A_{X(t), S}} \middle| X'(t) = 0 \right) \times p_{X'(t)}(0), \end{aligned} \quad (2)$$

where $A_{x, S} = \{\forall s \in S : X(s) \leq x\}$.

This density is continuous on S_1 and S_2 . The densities restricted to S_1 or S_2 can be prolonged continuously on \bar{S}_1 and $\bar{S}_2 = S$ respectively.

Remarks:

- A by-product of Theorem 1 is that it permits to compute $\mathbb{P}(t^* \in S_i)$ for $i = 0, 1, 2$.
- This theorem has been stated in our restricted framework for coherence with the rest of the paper but it can be extended, at a cost of heavy notation, to higher dimension on more general parameter sets, as stratified manifolds of [Adler and Taylor, 2007].
- The WAFO Matlab toolbox ([Brodtkorb et al., 2000] and [WAFO-group, 2000]) furnishes some heuristic expressions for application to wave analysis, like the computation of the joint distribution of crest height and position. These expressions are similar to those of (2).

Proof. We first prove the existence of the density and consider three cases depending on the location of t^* .

The event $\{t^* \in S_0\}$ is clearly the union of four events that are almost surely disjoint $\{\forall s \in S; X(s) \leq X(t)\}$, for $t \in S_0$. This gives the first term in (2).

Let now consider the case $t^* \in S_2$. Let B be a compact set of S_2 . Let $M(B)$ be the number of global maxima of $X(\cdot)$ in B . More precisely,

$$M(B) = \#\{t \in B; \forall s \in S, X(s) \leq X(t)\}.$$

We have the equalities $\mathbb{P}(t^* \in B) = \mathbb{P}(M(B) = 1) = \mathbb{E}(M(B))$. This last quantity can be computed by a Rice formula exactly as in Theorem 7.2 of [Azaïs and Wschebor, 2009]. See Appendix A for more details.

$$\mathbb{E}(M(B)) = \int_{t \in B} \mathbb{E} \left(|\det(X''(t))| \mathbb{1}_{X(s) - X(t) \leq 0, \forall s \in S} \middle| X'(t) = 0 \right) \times p_{X'(t)}(0) dt,$$

Since B is arbitrary, this gives the result.

The case $t^* \in S_1$ can be divided into four sub-cases depending on the considered edge. The proof follows the same line as the case $t^* \in S_2$.

We now prove the continuity of the density. Let consider the case $t^* \in S_2$. The stationarity of $X(\cdot)$ allows to apply a translation by t :

$$p_{t^*}(t) = \mathbb{E} \left(|\det(X''(0))| \mathbb{1}_{A_{X(0), S-t}} \middle| X'(0) = 0 \right) \times p_{X'(0)}(0).$$

Note that this function is defined on S .

Let $\{t_n, n \in \mathbb{N}\} \subset S_2^{\mathbb{N}}$ be such that $t_n \rightarrow t_\infty \in \bar{S}_2$. Set $A_n = p_{t^*}(t_n) - p_{t^*}(t_\infty)$. We have to prove that $A_n \rightarrow 0$. Using Cauchy-Schwarz inequality and conditioning by $X(0)$ we have:

$$\begin{aligned} A_n^2 &\leq \int_{x \in \mathbb{R}} \mathbb{E} \left(\det^2(X''(0)) \middle| X(0) = x, X'(0) = 0 \right) \times p_{X(0), X'(0)}(x, 0) dx \\ &\quad \times \int_{x \in \mathbb{R}} \mathbb{E} \left((\mathbb{1}_{A_{x, S-t_n}} - \mathbb{1}_{A_{x, S-t_\infty}})^2 \middle| X(0) = x, X'(0) = 0 \right) \times p_{X(0), X'(0)}(x, 0) dx, \end{aligned}$$

$p_{X(0),X'(0)}$ being the joint density of $X(0)$ and $X'(0)$.

The first term is obviously bounded as the expectation of a polynomial of a Gaussian variable (Lemma 3). For the second integral, the Ylvisaker Theorem (see [Ylvisaker, 1968]) and its extension Theorem 1.22 in [Azaïs and Wschebor, 2009] proves that, under our conditions, the distribution of $M_{S-t_\infty} = \max_{t \in S-t_\infty} (X(t))$ has no atom. As a consequence, a.s. $M_{S-t_\infty} \neq x$ and the continuity of the paths implies that

$$\mathbb{1}_{A_{x,S-t_n}} \rightarrow \mathbb{1}_{A_{x,S-t_\infty}}.$$

A dominated convergence argument implies that the integral tends to zero.

In the case $t^* \in S_1$, the continuity of the density can be proved exactly in the same fashion. \square

Theorem 1 can be extended by considering the joint density of t^* and $X(t^*)$. The proof is essentially the same as for Theorem 1 and is omitted.

Corollary 1. *With respect to the product measure $\mu \otimes \lambda$, with λ the Lebesgue measure on \mathbb{R} , the joint distribution of $(t^*, X(t^*))$ is:*

$$\begin{aligned} p_{t^*,X(t^*)}(t,x) &= \mathbb{1}_{t \in S_0} \mathbb{P}(\forall s \in S, X(s) \leq X(t) | X(t) = x) \times p_{X(t)}(x) \\ &+ \mathbb{1}_{t \in S_1} \mathbb{E} \left(|X''_{S_1}(t)| \mathbb{1}_{A_{x,S}} \middle| X(t) = x, X'_{S_1}(t) = 0 \right) \times p_{X(t),X'_{S_1}(t)}(x,0) \\ &+ \mathbb{1}_{t \in S_2} \mathbb{E} \left(|\det(X''(t))| \mathbb{1}_{A_{x,S}} \middle| X(t) = x, X'(t) = 0 \right) \times p_{X(t),X'(t)}(x,0), \end{aligned}$$

$p_{X(t),X'(t)}$ being the joint density of $X(t)$ and $X'(t)$.

In the next section we will need to study the joint distribution of t^* and X'' . Let us introduce further notation. The space of symmetric 2×2 matrices will be identified to \mathbb{R}^3 using for example the parametrization $(M_{1,1}, M_{1,2}, M_{2,2})$. Let $p_{X''(t)}$ be the Gaussian probability density function of $X''(t)$ (or $X''(0)$) using this parameterization. Let λ_3 be the Lebesgue measure on \mathbb{R}^3 . By similar tools, see Appendix B for a proof, we have

Corollary 2. *Under the hypotheses of Theorem 1, the joint density of (t^*, X'') at (t, x'') with respect to $\mu \otimes \lambda_3$ is given by*

$$\begin{aligned} p_{t^*,X''}(t,x'') &= \mathbb{1}_{t \in S_0} p_{X''(t)}(x'') \mathbb{P}(\forall s \in S, X(s) \leq X(t) | X''(t) = x'') \\ &+ \mathbb{1}_{t \in S_1} p_{X''(t)}(x'') |x''_{S_1}| \mathbb{E} \left(\mathbb{1}_{A_{X(t),S}} \middle| X'_{S_1}(t) = 0, X''(t) = x'' \right) p_{X'_{S_1}(t)}(0) \\ &+ \mathbb{1}_{t \in S_2} p_{X''(t)}(x'') |\det(x'')| \mathbb{E} \left(\mathbb{1}_{A_{X(t),S}} \middle| X'(t) = 0, X''(t) = x'' \right) p_{X'(t)}(0). \end{aligned} \tag{3}$$

where x''_{S_1} is the element of x'' giving the value of $X''_{S_1}(t)$.

4 Normalized discretization error

This section is dedicated to results concerning the observation grid and then, the normalized discretization error $n^2(M - M_n)$. Before stating the second main result, Theorem 2, we begin with some preliminary lemmas. Their proofs are given in Appendices C and D.

Lemma 4. *As $n \rightarrow \infty$, we have the almost sure convergences: $t_n^* \rightarrow t^*$, and $t_{n,X''} \rightarrow t^*$.*

Lemma 5. *Conditionally to $\{t^* \in S_2\}$ and to X'' , as $n \rightarrow \infty$, we get:*

$$n(t^* - t_{n,X''}) \xrightarrow{\mathcal{D}} \mathcal{U}(\mathcal{V}(0)).$$

where the \mathcal{D} superscript denotes the convergence in distribution and $\mathcal{U}(\mathcal{V}(0))$ the uniform distribution on the Voronoï cell around 0 in \mathbb{Z}^2 , for the norm $\|\cdot\|_{X''}$.

We state now the second main result of the paper.

Theorem 2. *Under our hypotheses, as $n \rightarrow \infty$ the discretization error $n^2(M - M_n)$ is bounded by a quantity which converges in distribution to a mixture of three components:*

- with probability $\mathbb{P}(t^* \in S_0)$, it is zero
- with probability $\mathbb{P}(t^* \in S_1)$, it is $\frac{1}{2} \|\mathcal{U}([-1/2, 1/2])\|_{X''_{S_1}}^2$
- with probability $\mathbb{P}(t^* \in S_2)$, it is $\frac{1}{2} \|\mathcal{U}(\mathcal{V}(0))\|_{X''}^2$.

Remarks:

- In the case $t^* \in S_2$, there is no need to bound the discretization error since we can directly prove the convergence toward $\frac{1}{2} \|\mathcal{U}(\mathcal{V}(0))\|^2$. In the case $t^* \in S_1$, is more delicate since $t^* \in S_1$ does not imply $t_n^* \in S_1$. We use a third point, \bar{t}_n , which belongs to S_1 . This point allows to obtain a convergence result for the quantity $M - X(\bar{t}_n)$, which is greater than the discretization error.
- As in this theorem, the following theorem (Theorem 3) demands an estimation of the second derivative matrix X'' . As it is explained in applications (Section 5) it is rather easy to approach it by finite differences as defined in Equation (8).

Proof. The proof for $t^* \in S_0$ is trivial.

We begin with the proof for the case $t^* \in S_2$. In this case we directly obtain the convergence of the discretization error.

The sample paths of $X(\cdot)$ are of class \mathcal{C}^2 so the Taylor expansion of X at t^* is given by

$$X(t^* + h) = X(t^*) - \frac{1}{2}h^\top (-X''(t^*))h + o(\|h\|^2).$$

By Lemma 4, $t_n^* \rightarrow t^*$ and $t_{n,X''} \rightarrow t^*$, thus

$$X(t^*) - X(t_n^*) \simeq \frac{1}{2}\|t^* - t_n^*\|_{X''}^2, \quad (4)$$

$$X(t^*) - X(t_{n,X''}) \simeq \frac{1}{2}\|t^* - t_{n,X''}\|_{X''}^2. \quad (5)$$

By definition of t_n^* , $X(t^*) - X(t_n^*)$ is a minimum on the grid G_n , thus

$$X(t^*) - X(t_n^*) \leq X(t^*) - X(t_{n,X''}). \quad (6)$$

By definition of $t_{n,X''}$, $\|t^* - t_{n,X''}\|_{X''}^2$ is a minimum on G_n , thus

$$\|t^* - t_{n,X''}\|_{X''}^2 \leq \|t^* - t_n^*\|_{X''}^2. \quad (7)$$

Putting together these four equations we see that the four terms involved are equivalent. The proof is finished with Lemma 5.

If $t^* \in S_1$, consider \bar{t}_n , the closest point of t^* in $G_n \cap S_1$, for the norm associated to the matrix $-X''_{S_1}(t^*)$. Then $M - M_n \leq M - X(\bar{t}_n)$. The convergence of this last quantity can be proved using the same approach, in dimension one. \square

The expression in Theorem 2 is uneasy to use in practical applications. For example, we never know where t^* lies (the three cases $t^* \in S_0, S_1$ or S_2 cannot be differentiated from observations). The following theorem gives an explicit bound for the discretization error, it is based on a worst case approach. The proof can be found in appendix E.

Theorem 3. *Under our hypotheses, the we have*

$$n^2(M - M_n) \leq -\frac{1}{8} \frac{X''_{1,1}X''_{2,2} (X''_{1,1} + X''_{2,2} + 2X''_{1,2})}{X''_{1,1}X''_{2,2} - X''_{1,2}^2}$$

Remarks:

- All the approximations we made (Theorems 2 and 3 and approximation by finite differences in Eq. (8)) are particularly precise when the level M reached by the field is high. Indeed, in this case $-X''$ is close to the deterministic matrix $M \times \Lambda$.
- In the case of a Gaussian process on $[0, 1]$, we can give the more precise bound

$$n^2(M - M_n) \leq -\frac{1}{8}X''.$$

5 Applications

5.1 Numerical simulations

Isotropic Gaussian random fields are generated using the R-package RandomFields [Schlather et al., 2015b] and [Schlather et al., 2015a]. We use normalized random fields with Gaussian covariance function $\Gamma(h) = \exp(-\|h\|^2)$. Fields are simulated on the square $[0, 5]^2$.

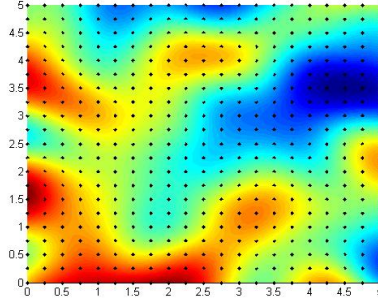


Figure 1: Example of simulated field with grid G_n for $n = 20$ (diamonds).

As said before, in practice t^* and $X''(t^*)$ are unknown and Theorem 3 can not be applied directly. Since $X(\cdot)$ paths are \mathcal{C}^2 , $X''(t^*)$ can be approached by $X''(t_n^*)$. This matrix is estimated using finite differences by:

$$\begin{aligned} X''_{1,1} &\simeq n^2 [X(t_n^* - (1/n, 0)) - 2X(t_n^*) + X(t_n^* + (1/n, 0))], \\ X''_{2,2} &\simeq n^2 [X(t_n^* - (0, 1/n)) - 2X(t_n^*) + X(t_n^* + (0, 1/n))], \\ X''_{1,2} &\simeq n^2 [X(t_n^* + (1/n, 1/n)) + X(t_n^* - (1/n, 1/n)) \\ &\quad - X(t_n^* + (-1/n, 1/n)) - X(t_n^* + (1/n, -1/n))]. \end{aligned} \quad (8)$$

In these simulations, S is taken equal to the entire simulation square $[0, 5]^2$. Then, t_n^* sometimes belongs to S_1 and the finite differences estimator can not be used. In these cases, the bound is not computed and a missing value is returned. In the following section, we present an application to real data and we set up an *ad hoc* area of study in order to minimize the number of missing values returned. Figure 2 shows an example of simulation results for two different grid meshes.

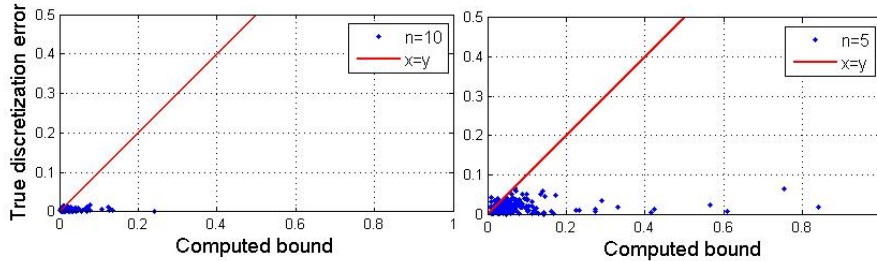


Figure 2: Computed bound versus true discretization error for simulated data. Results for 300 fields for $n = 10$ (left) and $n = 5$ (right). Red lines correspond to the set $\{y = x\}$.

5.2 Application to positioning by satellite data

The discretization question addressed in this paper is encountered in positioning by satellite augmentation system (SBAS) like EGNOS for the European Union or WAAS for USA [sit, a] [sit, b]. Such systems complement positioning systems (like Galileo or GPS) to improve some of their specifications by using additional data to compute positioning corrections or quality information. Here we are interested in two of these specifications: the positioning accuracy and the integrity. Integrity is, roughly speaking, the system ability to furnish confidence interval or threshold for the correction provided and to alert the user in a given time when these corrections are corrupted.

Our bound is applied to a data used in EGNOS to evaluate his performances, called GIVDe (Grid Ionospheric Vertical Delay error). It is the error of estimation of the vertical positioning error, i.e. the difference between the vertical error estimation furnished by EGNOS and a vertical error reference furnished by IGS (International GNSS Service). It is available on some points of a virtual grid located at 350km of altitude. An example of available data is depicted in Figure 3. To compute correction data for his position, the user has to perform an interpolation of data of this grid. Then, the estimation error behavior within a grid cell is important to asses the integrity feature of the system. The bound presented in this paper is developed for this purpose.

The monitored/not monitored status of a point may vary over time. We set a restricted area of study in order to consider points with a high observation rate over time and with a neighborhood also frequently observed (necessary for the finite differences approximation, Eq. (8)). Since we work on a sphere, we use the great circle distance between points. Figure 4 presents the example of the days 60 and 100 of 2013. Missing points correspond to

missing data or issues in the second derivative matrix estimation (missing neighbor point or incoherent result).

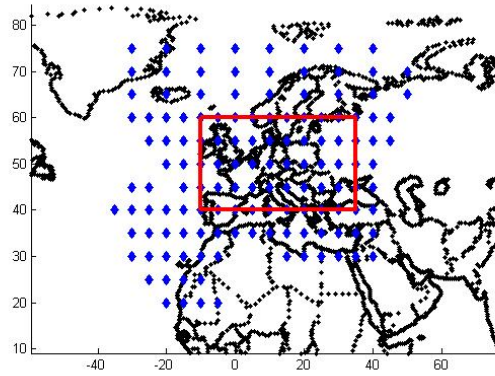


Figure 3: Projection of points where data are available for a given day (diamonds) and restricted area of study (red rectangle). Latitude and longitude in degrees (equirectangular projection).

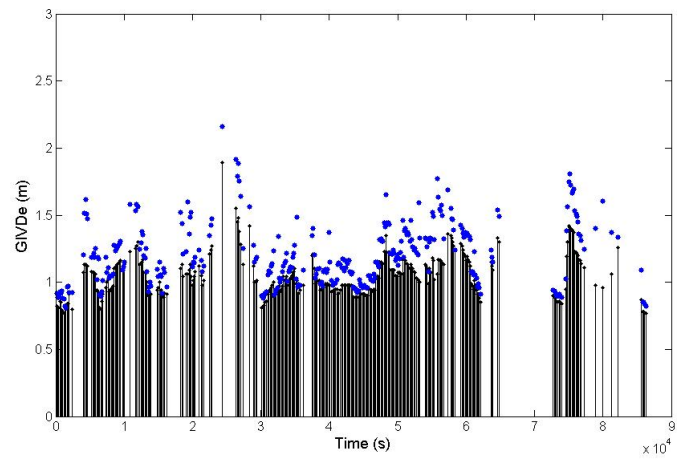
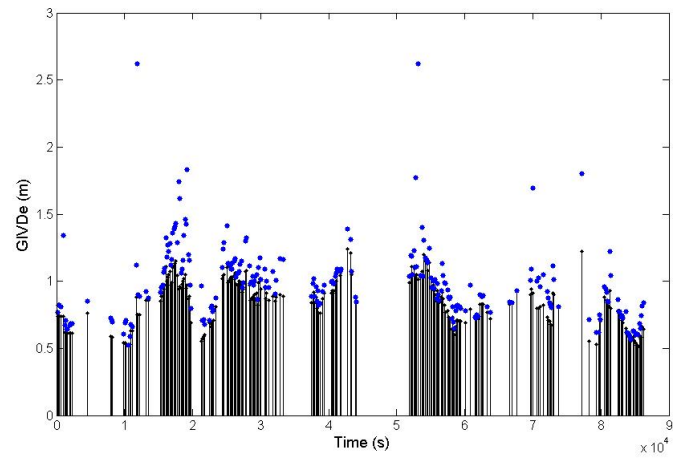


Figure 4: Observed maximum (black) and observed maximum plus computed bound (blue stars) over time for real data. Days 60 and 100 of 2013 is presented.

6 Appendix

A Details on the proof of Theorem 1

Using notation of Theorem 6.4 of [Azaïs and Wschebor, 2009], the weighted Rice Formula is applied as follow:

- $Z = X'$ on S_2 . Lemma 2 ensures that Z satisfies the hypotheses for $U = S_2$, $d = 2$ and $u = 0$.
- for each $t \in S_2$, set $W = S$ and $Y^t : W \rightarrow \mathbb{R}$ defined by:

$$Y^t(w) := X(w) - X(t).$$

Y^t verifies the a) and b) conditions for $n = 1$.

- For $k = 1, 2, \dots$, and for f a continuous function from W to \mathbb{R} , set:

$$g_k(t, f) = \left(1 - \mathcal{F}_k\left(\sup_{w \in W} f(w)\right) \right),$$

where, for $x \geq 0$, $\mathcal{F}_k(x) := \mathcal{F}(kx)$, $\mathcal{F}(x) = 0$ if $0 \leq x \leq 1/2$, $\mathcal{F}(x) = 1$ if $x \geq 1$ and \mathcal{F} is continue, monotonous and non decreasing (Figure 5).

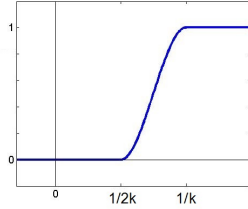


Figure 5: $\mathcal{F}_k(x)$

Finally, we obtain:

$$\mathbb{E} \left(\sum_{t \in B, X'(t)=0} g_k(t, Y^t) \right) = \int_B \mathbb{E} \left(|\det(X''(t))| g_k(t, Y^t) \Big| X'(t) = 0 \right) p_{X'(t)}(0) dt.$$

Concerning the limit of the left hand term, as $k \rightarrow +\infty$,

$$g_k(t, Y^t) \downarrow \mathbf{1}_{X(s) - X(t) \leq 0, \forall s \in S}.$$

By Lemma 2, we know that almost surely a critical point of X is non degenerated. Using the inverse function theorem, we know that the critical points of X are isolated. Since B is compact, there is an almost surely finite number of points $t \in B$ such as $X'(t) = 0$. By monotone convergence, the left hand term tends to $\mathbb{E}(M(B))$.

B Details on the proof of Corollary 2

We first treat the case $t^* \in S_2$. Let N_2 be a compact of the set of 2×2 symmetric matrices, with the parametrization described previously. Let N_1 be a compact set of S_2 . By a Rice formula as in the proof of Theorem 1

$$\begin{aligned} & \mathbb{P}\{t^* \in N_1, X'' \in N_2\} \\ &= \int_{N_1} \mathbb{E} \left(|\det(X''(s))| \mathbb{1}_{A_{X(s),S}} \mathbb{1}_{X''(s) \in N_2} \Big| X'(s) = 0 \right) p_{X'(t)}(0) ds \\ &= \int_{N_1} dt \int_{N_2} dx'' |\det(x'')| \mathbb{E} \left(\mathbb{1}_{A_{X(t),S}} \Big| X'(t) = 0, X''(t) = x'' \right) p_{X'(t), X''(t)}(0, x'') \\ & \int_{N_1} dt \int_{N_2} dx'' p_{X''(t)}(x'') |\det(x'')| \mathbb{1}_{x'' \prec 0} \mathbb{E} \left(\mathbb{1}_{A_{X(t),S}} \Big| X'(t) = 0, X''(t) = x'' \right) p_{X'(t)}(0), \end{aligned}$$

since $X'(t)$ and $X''(t)$ are independent, giving the density. Note that the term $\mathbb{1}_{x'' \prec 0}$ can be omitted since, if x'' is not definite negative, $\mathbb{E} \left(\mathbb{1}_{A_{X(t),S}} \Big| X'(t) = 0, X''(t) = x'' \right)$ vanishes. The case $t^* \in S_1$ can be treated in the same way, paying attention to the fact the coefficient of interest in the expectation is x''_{S_1} .

C Proof of Lemma 4

The result is obvious for $t_{n, X''}$. We consider now the case of t_n^* . Let V be a neighborhood of t^* .

Almost surely t^* is unique so there exists $\eta > 0$ such as for all $t \notin V$, $X(t) \leq X(t^*) - \eta$. By definition of t_n^* , we get that $\forall n \in \mathbb{N}^*$, $X(t_n^*) \geq X(t_{n, X''})$ and $0 \leq X(t^*) - X(t_n^*) \leq X(t^*) - X(t_{n, X''})$. Since simple paths of X are continuous, $X(t^*) - X(t_{n, X''}) \xrightarrow{n \rightarrow \infty} 0$ and $X(t^*) - X(t_n^*) \xrightarrow{n \rightarrow \infty} 0$. This implies that $t_n^* \in V$. Since V is arbitrary, we have finished the proof.

D Proof of Lemma 5

In all the the proof we assume to be conditional to $\{X'' = x'' \prec 0\}$ and to $\{t^* \in S_2\}$. By Corollary 2 we know that the conditional density $p_{t^*|x''}$ of t^* is proportional to

$$\mathbb{E} \left(\mathbb{1}_{A_{X(0), S-t}} \Big| X'(0) = 0, X''(0) = x'' \right),$$

which is continuous by the same proof that the proof of the continuity of the density in Theorem 1. Let B a Borel set included in $\mathcal{V}(0)$. A key point is that, because of the shift invariance of \mathbb{Z} , the sets $\{B + k, k \in \mathbb{Z}\}$ are disjoint and

$$\{n(t^* - t_{n, X''}) \in B\} = \bigcup_{t \in G_n} \{n(t^* - t) \in B\}.$$

As a consequence

$$\mathbb{P}(n(t^* - t_{n,X''}) \in B) = \sum_{t \in G_n} \int_{B/n+t} p_{t^*|x''}(s) ds = \int_{B/n} \sum_{t \in G_n} p_{t^*|x''}(s+t) ds.$$

The conditional density of t^* is continuous on S compact and thus uniformly continuous with continuity modulus $\omega(\varepsilon)$. Since the cardinality of G_n is $(n+1)^2$,

$$Q_n(s) = \frac{1}{n^2} \sum_{t \in G_n} p_{t^*|x''}(s+t),$$

is also uniformly continuous with continuity modulus bounded by $4\omega(\varepsilon)$. As a consequence

$$\mathbb{P}(n(t^* - t_{n,X''}) \in B) = \int_B Q_n(s/n) ds \rightarrow Q_\infty(0) \lambda_2(B).$$

Giving the result. An analogous result can be obtained in dimension one for the vector $n(t^* - \bar{t}_n)$.

E Proof of Theorem 3

Let consider the case $t^* \in S_2$. $-X''(t^*)$ is a.s. positive-definite and symmetric so there exists a square root matrix Z which is also symmetric positive-definite. For a fixed $n \in \mathbb{N}$, $n(t^* - t_{n,X''}) \in \mathcal{V}(0)$ by definition of $t_{n,X''}$. Moreover, we have the relation

$$\min_{t_i \in G_n} \|t^* - t_i\|_{X''} = \min_{z_i \in Z(G_n)} \|Zt^* - z_i\| \quad (9)$$

with $z_i = Zt_i$. Taking the limit in n , and denoting $\mathcal{V}_{Z(\mathbb{Z}^2), \|\cdot\|}(0)$ the Voronoï cell around 0, in the Voronoï diagram of the oblique net $Z(\mathbb{Z}^2)$, for the usual Euclidean norm, Equation (9) gives that

$$\max_{v \in \mathcal{V}(0)} \|v\|_{X''} = \max_{v \in \mathcal{V}_{Z(\mathbb{Z}^2), \|\cdot\|}(0)} \|v\|.$$

This second quantity is easy to handle. Our goal is to find a bound for the Euclidean norm of a vector in $\mathcal{V}_{Z(\mathbb{Z}^2), \|\cdot\|}(0)$.

For a given value of X'' , the linear application induced by Z can be geometrically characterized by three quantities: lengths L and ℓ and angle θ defined below and depicted in Figure 6. Note that in the figure we have made a rotation that makes Oz_3 parallel to the y -axis. This does not affect distances. L and ℓ and θ are related to X'' by:

- $L^2 = \|Oz_0\|^2 = \|(1, 0)^\top\|_{X''}^2 = -X''_{1,1}$
- $\ell^2 = \|Oz_3\|^2 = \|(0, 1)^\top\|_{X''}^2 = -X''_{2,2}$

- $\|z_3 z_0\|^2 = \|(-1, 1)^\top\|_{X''}^2 = -(X''_{1,1} + X''_{2,2}) + 2X''_{1,2}$
and using law of cosine $\sin(\theta) = \frac{L^2 + \ell^2 - \|z_3 z_0\|^2}{2\ell L} = \frac{-X''_{1,2}}{\sqrt{X''_{1,1} X''_{2,2}}}$

The set of points z_i used in the computation of the Voronoi cell depends on the values of L , ℓ and θ . The description of all types of configuration is complex so we chose to give a bound valid for every configuration and sharp enough. First the figure is invariant by a central symmetry that changes θ into $-\theta$. Second we can, without changing the problem reverse the x axis, for example, before applying Z . This change Oz_0 into $-Oz_0$ and θ into $\pi - \theta$. In conclusion it is sufficient to consider the cases $\theta \in [0, \frac{\pi}{2})$ that corresponds to Figure 6.

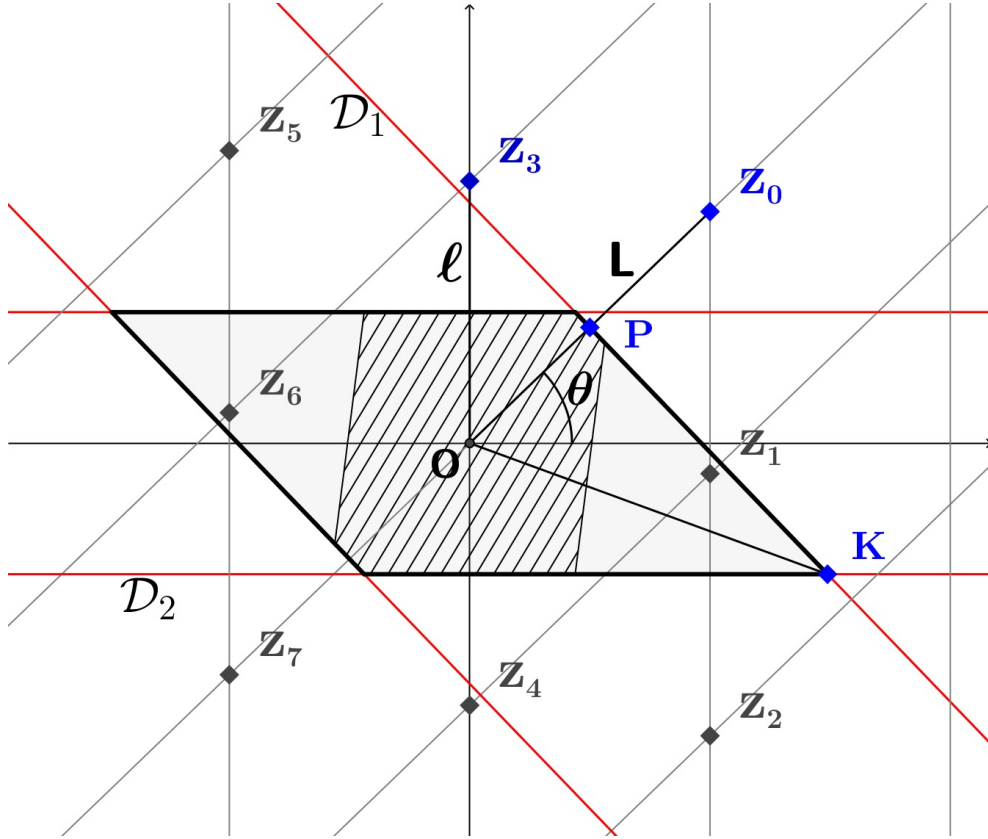


Figure 6: Example of the Voronoi cell of 0 in $Z(\mathbb{Z}^2)$ for $\|\cdot\|$ (hatched area), and corresponding bounding area (bold parallelogram). The numbering of points z_i is arbitrary except for z_0 and z_3 .

The Voronoi cell is always included in the parallelogram formed by \mathcal{D}_1 the perpendicular bisector of Oz_0 , \mathcal{D}_2 the perpendicular bisector of Oz_3 and their respective central inversions. The maximum length of a vector starting at O is the distance OK . Polar equations of lines \mathcal{D}_1 and \mathcal{D}_2 are used to

determine coordinates of K :

$$\left(\frac{L + \ell \sin(\theta)}{2 \cos(\theta)}, -\frac{\ell}{2}\right).$$

Indeed, the point P indicated in the figure has for coordinates: $(L/2 \cos(\theta), L/2 \sin(\theta))$. Let x be the length of the segment PK , then the ordinate of K is $L/2 \sin(\theta) - x \cos(\theta)$ but it is also $-\ell/2$ from which it can be deduced that

$$x \cos(\theta) = L/2 \sin(\theta) + \ell/2.$$

This implies in turn that the abscissa of K is

$$L/2 \cos(\theta) + x \sin(\theta) = \frac{L + \ell \sin(\theta)}{2 \cos(\theta)}.$$

The maximum usual Euclidean norm of a vector in the Voronoï cell around 0 in the Voronoï diagram $Z(\mathbb{Z}^2)$ is then

$$\sqrt{\left(\frac{L + \ell \sin(\theta)}{2 \cos(\theta)}\right)^2 + \ell^2/4} = \frac{1}{2} \sqrt{\frac{-X''_{1,1} X''_{2,2} (X''_{1,1} + X''_{2,2} + 2X''_{1,2})}{X''_{1,1} X''_{2,2} - X''_{1,2}^2}}.$$

Equation (9) implies that this maximum distance is equal to the maximum distance in $\mathcal{V}(0)$ for $\|\cdot\|_{X''}$.

Let now examine the case $t^* \in S_1$. This does not imply $t_n^* \in S_1$ but we have the inequality $X(t^*) - X(t_n^*) \leq X(t^*) - X(\bar{t}_n)$. The vector $n(t^* - \bar{t}_n)$ asymptotically follows a uniform distribution on $[-1/2, 1/2]$. In this case the maximum norm is $\frac{1}{2} \sqrt{-X''_{1,1}(t^*)}$ or $\frac{1}{2} \sqrt{-X''_{2,2}(t^*)}$ depending on which edge t^* lies. It is easy to see that

$$\frac{1}{2} \max\left(\sqrt{-X''_{1,1}}, \sqrt{-X''_{2,2}}\right) \leq \frac{1}{2} \sqrt{\frac{-X''_{1,1} X''_{2,2} (X''_{1,1} + X''_{2,2} + 2X''_{1,2})}{X''_{1,1} X''_{2,2} - X''_{1,2}^2}}.$$

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