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Comparison and Convergence of Two Topological Models for 3D Image Segmentation

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Abstract. In this paper we compare two topological models of 3D segmented images representation. These models are based on a collaboration between a topological representation and a geometrical representation of the regions of the segmented image. In both models the description of the topology lays on topological maps with a combinatorial representation. A geometrical embedding of maps is used to describe the geometry of regions. Both models differ in the way of defining this embedding. The aim of this paper is to compare these two models from the point of view of their interest for image segmentation, and to explore the possibility and the interest of making both these models converge.

1 Introduction

A 3D image is a 3-dimensional set of elements called voxels. A voxel can be seen either as a valuated point or as a valuated unit cube. The second approach is more convenient when considering the decomposition of an image into volumetric objects. In this approach, the space is recursively decomposed into 3D, 2D, 1D and 0D unit elements called respectively *3-cells*, *2-cells*, *1-cells*, and *0-cells*. A voxel is then a valuated *3-cell*. An object of the 3D segmented image is defined as a set V of 6-connected voxels (2 voxels are 6-connected if they share a common *2-cell*), and the boundary of V is the set $B(V)$ of faces of voxels defined by $B(V) = \{b = v_1 \cap v_2, \text{ with } v_1 \in V \text{ and } v_2 \notin V\}$. Such voxel faces are called *s-cells* (for *surface cells*). This definition of boundary is consistent from a topological point of view and can be easily extended to the decomposition of an image into several objects, and to images of higher dimensions.

The problem of image segmentation is to define and implement some process for decomposing images into several objects (or regions) according to some homogeneity criteria. This problem has been widely explored in dimension 2, and it has been shown that topological maps provide a powerful framework for split and merge segmentation [2, 4]. On the other hand, the segmentation of 3D image is most of the time restricted to the extraction of a foreground component from a background. In order to extend 2D segmentation methods to 3-dimensional images it is necessary to provide the set of 3D images with a structure allowing:

1. The topological and geometrical representation of the segmented image: definition of boundary, neighborhood, surface shape, etc.
2. The extraction of features involved in the segmentation process: boundary and volume reconstruction, determination of adjacency (such as adjacent or included volume, common boundary), etc.
3. The updates of the representation required by the segmentation process, mainly when splitting a volume into sub-volume or when merging adjacent volumes.

Two models have been proposed for solve this problem which are both based on a topological representation of the decomposition, associated with a geometry embedding. In both models the topology is described by 3D topological maps and based on a combinatorial representation, also called combinatorial maps. These models differ in the way of defining the geometrical embedding of the topological map and the correspondence between topological and geometrical levels. One of them uses a global geometrical embedding [3, 7] and the other one a local and hierarchical geometrical embedding [1, 6]. In this paper we call the first one *topological map with global embedding* (in short GE topological map) and the second one *topological map with hierarchical local embedding* (in short HLE topological map). The aim of this paper is to compare both models and to investigate the possibility and the interest of making these models converge in the context of 3D image segmentation.

2 Combinatorial Representation of the Topology

In both models the topological representation is based on 3D combinatorial maps or 3-maps. Combinatorial maps [9, 5] are a mathematical model allowing the topological representation of a space subdivision in any dimension. They were first introduced in 2D as a planar graph representation model [8, 11], then extended in dimension n to represent orientable or not-orientable quasi-manifold [10]. They encode the space subdivision and the incidence relations between subdivision elements. A combinatorial map is a set of atomic elements, called *darts*, that represent the space subdivision, and functions between these elements that represent the incidence relations. Intuitively, a dart corresponds to a minimal oriented element of the boundary of the decomposition.

Definition 1. A 3-map, is an 4-tuple $M = (\mathbf{D}, \delta_1, \delta_2, \delta_3)$ where:

1. \mathbf{D} is a finite set of darts;
2. δ_1 is a permutation on \mathbf{D} ;
3. δ_2, δ_3 , and $\delta_1\delta_3$ are involutions on \mathbf{D} ;

The incidence relations can be represented in different ways with the permutation and the two involutions. The 3D combinatorial map used in the GE model is defined by $M_{GE} = (\mathbf{D}, \gamma, \sigma, \alpha)$. The permutation γ links darts that belong to adjacent faces around a same edge. The involution σ links two darts incident to

the same vertex and that belong to a same face. Finally the involution α links two opposite darts of the same edge and that belong to a same face.

The combinatorial map used in the HLE model is defined by $M_{HLE} = (\mathbf{D}, \beta_1, \beta_2, \beta_3)$. The permutation β_1 links darts that belong to consecutive edges around a same face. The involution β_2 links two darts incident to the same edge and that belong to a same volume. Finally the involution β_3 is the same that the involution α and links two darts incident to the same face and that belong to the same edge.

These two representations are equivalent. Indeed, both set of darts \mathbf{D} are equal and all permutations of each model can be obtained by a combination of permutations of the other model. The conversion from a M_{GE} map into the corresponding M_{HLE} map, is given by: $\gamma = \beta_3\beta_2$, $\sigma = \beta_3\beta_1$, $\alpha = \beta_3$ and the inverse conversion is given by: $\beta_1 = \sigma\alpha$, $\beta_2 = \gamma^{-1}\alpha$, $\beta_3 = \alpha$.

In Fig. 1, we present an example of a 3D object and the corresponding combinatorial maps used in both models.

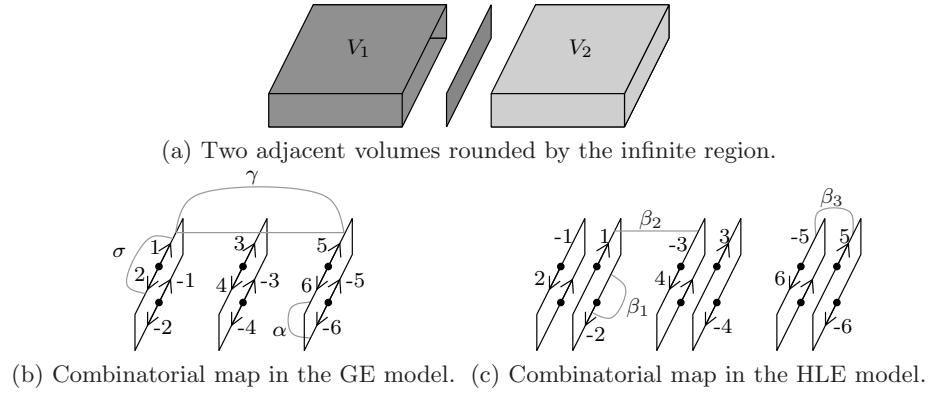


Fig. 1. The topological elements of the 3D subdivision are: 3 volumes, 6 oriented faces, 2 edges, and 2 vertices. The relations between darts are partially represented.

Topological cells (vertex, edges, faces, volumes) are implicitly represented in 3-maps. Each cell corresponds to the set of darts that are incident to the cell. These sets of darts are the connected components of particular 2-maps that represent the partitions of the space into vertices, edges, faces and volumes:

	Vertices	Edges	Faces	Volumes
GE model	$(\mathbf{D}, \gamma, \sigma)$	$(\mathbf{D}, \gamma, \alpha)$	$(\mathbf{D}, \sigma, \alpha)$	$(\mathbf{D}, \gamma^{-1}\sigma, \gamma^{-1}\alpha)$
HLE model	$(\mathbf{D}, \beta_2\beta_1, \beta_3\beta_1)$	$(\mathbf{D}, \beta_2, \beta_3)$	$(\mathbf{D}, \beta_1, \beta_3)$	$(\mathbf{D}, \beta_1, \beta_2)$

When a volume is totally included into another one V , the boundary of V is composed of more than one oriented closed surface. Each closed surface is associated with a connected component of the 3-maps. One of its closed surface

has a positive orientation (the exterior boundary of V), the others have negative orientation (the interior boundaries). We need to keep the relative positions of these surfaces that is not represented in the combinatorial map. A classical solution consists in using an *inclusion tree* that gives associations between a volume and its cavities and that allows to preserve the inclusion relation.

3 The Model of GE Topological Maps

The model of topological maps with global geometrical embedding is based on the global encoding of the geometry of the boundary of the segmented image and on the definition of the geometrical analogous of topological dart. This definition is used to build the correspondence between the topological representation of the segmented image and the global geometric description of regions and region boundaries.

Let I be a 3D segmented image I of size $\Delta_x \times \Delta_y \times \Delta_z$. The *s-cells* of I are encoded by a 3D array B_I of size $(\Delta_x + 1) \times (\Delta_y + 1) \times (\Delta_z + 1)$. The entry $B_I(i, j, k)$ is a tuple of three booleans, each one corresponding to one of the three faces of the voxel of center (i, j, k) which are adjacent to the voxel vertex of coordinate $(i - \frac{1}{2}, j - \frac{1}{2}, k - \frac{1}{2})$ (see Fig. 2(a)). The array $B(I)$ is called the *boundary image* of I .

In order to make the boundary image correspond with the topological maps we have to decompose it into simply connected patches of surface (sets of *s-cells* homeomorphic to a topological disc). By this way it is possible to associate each surface patch with a face of a topological map. To decompose the boundary into simply connected patches, we cut the surface by drawing closed paths of adjacent *1-cells* along the boundary. Each cutting is encoded by marking the related *1-cell*.

The decomposition of the boundary into patches requires two different cuttings. The first one is the *inter-region cutting*. It consists in cutting the boundary along the border of any maximal part of surface shared by two volumes. The *1-cells* to be marked by inter-object cutting are easy to find because they are the *1-cells* adjacent to more than two *s-cells*. The second cutting is the *intra-object cutting*. It is used to decompose the boundary components resulting from the inter-object cutting. A boundary element has to be cut once if it is homeomorphic to a sphere, twice if it is homeomorphic to a one hole torus, and so on. This cutting is based on topological region growing. The algorithm used to perform this region growing requires conditions on the segmented images that lead us to restrict this model to weakly well composed segmented images.

The result of the cuttings is a set of *1-cells* of the boundaries of the objects of segmented image. These *1-cells* are called *l-cells*. Each connected component of these *l-cells* forms a graph. We call *p-cells* the nodes of this graph.

The topological map is built from the boundary image by traversing the boundary and associating each surface patch with a face of the surface map. Since the topological encoding lays on topological dart it is necessary to define the geometrical analogous of a dart. A *geometrical dart* is a tuple $\langle p, l, s \rangle$

such that p is a p -cell, l is l -cell adjacent to p and s is a s -cell adjacent to l (see Fig. 2(b)).

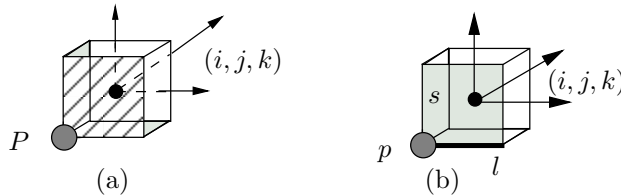


Fig. 2. (a) Each entry of the boundary image encodes the boundary of the segmented image at the neighborhood of the voxel vertex of coordinates resulting from a translation of the voxel center by $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. (b) The tuple $\langle p, l, s \rangle$ is a geometrical dart.

For this construction we consider the $\alpha\sigma\gamma$ representation. The construction of the map is done by following each chain of l -cells linking two p -cells with respect to the orientation of the surface. By this way it is possible to associate the two geometrical darts at the ends of such a chain. If one of these geometrical darts (say g) is associated with the topological dart d the other one (say g') is associated with $\alpha(d)$. Then we consider the sequence of geometrical darts encountered when turning around the l -cell of g and the sequence of geometrical darts encountered when turning in the opposite sense around the l -cell of g' . We can verify that both sequences have the same length. Let $g'_1 \dots g'_k$ and $g''_1 \dots g''_k$ be these sequences. We associate each g'_i with a new dart d_i and g''_i with $\alpha(d_i)$. Each sequence $d_1 \dots d_k$ is a cycle of γ . The permutation σ can be initialized from the neighborhood of each geometrical dart.

This construction associates each geometrical dart with a topological one, and conversely associates each topological dart with its geometrical embedding. Since any topological element, whatever the dimension, is represented by a topological dart, it is possible to retrieve the geometrical embedding of any topological element from the associated geometrical dart.

4 The Model of HLE Topological Maps

The main principle of the HLE topological maps definition consists in a progressive simplification of a combinatorial map that represents all the voxels of the image. For this simplification, we mainly use the *merging operation* which takes two adjacent i -cells and merge them in a unique i -cell.

The first combinatorial map, which is the starting point of the simplification process, just represents all the voxels of the image (called *level 0 map*). This map has an implicit canonical embedding (the usual geometry of the voxel grid). We are going to simplify progressively this map by using this implicit geometry. In order to simplify, we use the term of *coplanar faces* to design faces that

have coplanar embedding (and the same for *collinear edges*). First we merge all adjacent volumes of level 0 map that belong to the same region of the image (Fig. 3(b)). This removes all the interior faces and we so obtain the boundaries of all the regions of the image. Then we merge all adjacent and coplanar faces that are separated by a degree 1 or 2 edge³. This simplification allows us to remove useless edges that are in the middle of plane faces (Fig. 3(c)).

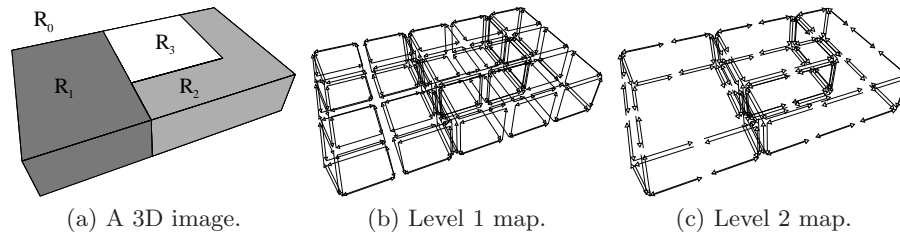


Fig. 3. A 3D region segmented image and the corresponding levels 1 and 2 maps.

Then we merge all adjacent and collinear edges that are separated by a degree 2 vertex⁴ (Fig. 4(a)). In order to obtain the minimal representation of a 3D image that does not depend on the geometry but only on the topology, we need to simplify the non-coplanar faces and the non-collinear edges by similar merging operations. So, starting from the level 3 map, we merge all adjacent and non-coplanar faces that are separated by a degree 1 or 2 edge (Fig. 4(b)), and at last we merge all adjacent and non-collinear edges that are separated by a degree 2 vertex (Fig. 4(c)).

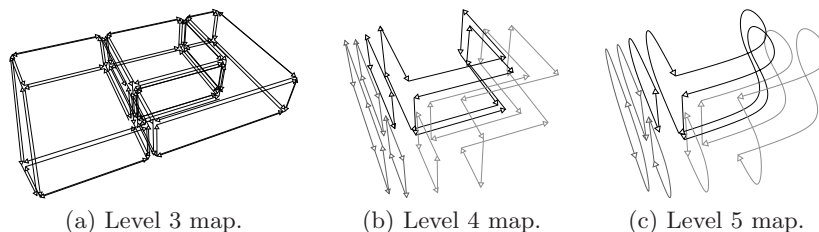


Fig. 4. Levels 3, 4 and 5 maps.

The face merging can lead to the disconnection of the map into several connected components, for faces that have more than one border. After this disconnection, it is not possible to place the different connected components relating

³ The degree of an edge is the number of incident faces.

⁴ The degree of a vertex is the number of incident edges.

to each others: we have lost topological information. In order to solve this problem, a solution consists in adding particular edges that keep the map connected. These edges are called *fictive edges* because they do not represent the border of a face unlike the other edges.

The last map obtained is the minimal map that represents the topology of the 3D image. Indeed, we can prove that all the different simplifications do not lead to a modification of any topological characteristic. Moreover, it is not possible to remove anything in this map without modifying its topological characteristics. This is why this map is called *topological map*.

There are several ways to add geometrical information to the topological map and the choice of one of this way depends on the application and the needed operations. We use here a local and hierarchical embedding which is interesting when the 3D images are composed with many large plane surfaces. To achieve this embedding, we associate each face of the topological map with an embedded 2-map (which has a straightforward linear embedding) that represents the geometry of the face (we can see Fig. 5(b) an example of such an embedding).

To obtain this embedding, we begin by linking to each face of the level 0 map a 2-map that represents the geometry of the corresponding voxel face. Then, we just need to update the embedding during each merging operations: when we merge two faces, we merge the corresponding embeddings when they are coplanar (merge operation to obtain the level 2 map) and we just sew the embedding when they are not (to obtain the level 4 map). This is the same principle for the edge merging, where we need to perform similar simplifications in the embedding maps. For this reason, the updating of the embedding can be achieved easily with a very efficient algorithm in complexity time. Moreover, this principle can be extended in upper dimension where each i -dimensional cell could be embedded with a i -map.

5 Comparison of Both Models

Both models can be compared from several points of view: power of expression, cost, reconstruction, extraction of features, point localization, and modifications of the subdivision. First we are going to present how some useful operations can be performed on both models, then we summarize the advantages of each one.

5.1 Presentation of Basic Operations in Both Models

Both models structure images decomposed into 6-connected objects. The HLE model makes possible to analyze any segmented image, whereas with the GE model there is a restriction to weekly well composed image. This restriction is not imposed by the model but by the cutting algorithm which fails on some pathological cases. These cases correspond to some configurations of edge adjacent volumes for which the topology cannot be reconstructed by a local analysis.

With the GE model, the size of the data structure is approximatively twice the one of a 256 gray level image. In the HLE model the size of data structure

depends of several image properties (mainly the planarity of faces). Conversely the cost in time of the map construction is lower with the HLE model where each voxel is processed once. In the GE model some preprocessings involved in the construction (for instance to get the weekly well formed condition) or in updates may require several processings for each voxel. Both complexities are linear but on large images the difference of run times may be noticeable.

There are two important reconstruction operations: the reconstruction of the boundary and the reconstruction of the set of voxels of a volume or of a union of volumes. Each boundary patch shared by two adjacent volumes is described by a set of topological darts. In the GE model each of these topological darts is associated with a geometrical one, each geometrical dart corresponding to a *s-cell* in the boundary image. Thus the boundary element can be reconstructed by traversing the boundary image from any of these *s-cell*. In the HLE model each topological dart is associated with an embedded 2-map. The traversal of this 2-map gives planar patches of surfaces that are decomposed in rectangles and then reconstructed.

With the GE model the reconstruction of a volume can be achieved directly in the image boundary by traversing it from a starting voxel given by a geometrical dart. With the HLE model this reconstruction needs first to reconstruct the boundary and then to perform a similar processing.

The extraction of topological features is based on the analysis of the 3-maps and thus is equivalent in both models. Topological characteristics, like the genus or the number of cavities, come directly from the maps. The topological maps also provides extraction of features like volumes adjacent to a given one, included volumes, faces shared by two adjacent volumes. These features can then be used to initiate boundary or volume reconstruction.

The point localization consists in retrieving a volume from a given voxel of the image. In the GE model it is done by traversing the boundary image until reaching a *s-cell* *s*, then by founding an *l-cell* of the surface element containing *s* and following adjacent *l-cells* until reaching a geometrical dart *g*. The volume is given by the topological dart associated with *g*. This localization needs a more heavy algorithm that consists in computing intersections between a half-line and every 2-face of the HLE.

The main modifications involved in the segmentation process are the splitting of a volume into two or more sub-volumes and the merging of two or more adjacent volumes. Both these operations require topological and geometrical updates. The topological updates are equivalent in both models. In the GE model the geometrical update of the merge operation requires to traverse the *s-cells* of the removed boundary elements in order to erase them in the boundary image. In the HLE model, it is enough to traverse the *l-cells* of the border of these boundary elements in order to merge related embedded 2-maps. The geometrical update of the splitting of a volume into two adjacent sub-volumes consists in adding a boundary element separating these sub-volumes. This operation is the inverse of the previous one. The splitting of a volume into more than two sub-volumes

is more complex and consists in applying the initial construction described in Sec. 3 and 4, but on the split volume instead of on the whole image.

5.2 Advantages of Each Model

Since the GE and HLE models mainly differ in the way of coding the geometrical embedding, pure topological operations are totally equivalent in both of them. But for most of operations that have to access to the geometry, we can establish which is the most efficient model. So advantages and inconveniences of each one are strongly linked to the use and the kind of operations from the point of view of segmentation.

The GE model is more interesting for volume traverse because it explicitly describes each border cell and locates it in the global embedding. So while moving inside the image, it is easy to detect when a volume boundary is crossed. Therefore, point localization is quite simple both on algorithmic and time complexity aspects, unlike with the HLE model in which a localization requires heavy processings. The GE model is also more efficient for global geometrical characteristics determination like the size of a volume. Boundary and volume reconstructions are also very simple. All of these features can be computed using a simple 2D or 3D floodfill algorithm. In the other model, volumetric computation and volume reconstruction are quite difficult operations because the geometrical description is more concise and does not encode explicitly all the *s-cells*. Last, the split of an image by a discrete plane (useful operation in medical imaging) is also more efficient with a global embedding because intersections can be found locally by exploring the plane surface, whereas with the HLE there is no other way but computing the intersection of each 2-face with the plane.

On the other hand the HLE model has no restriction on segmented images, the simplicity of the construction makes its implementation easier, and the multi-level representation allows us to define multi-level operations. Actually, each operation can be decomposed in several levels, and each level is made of only a few number of basic processes. By the fact that most constituent elements of boundaries are given implicitly, local operations are lighter than with the GE model. This is particularly obvious for the volume merging operation, in which we need to remove every surfel on the deleted face. This process is not needed in the HLE model. For the same reason, the inverse operation which consists in splitting a volume by inserting a face along consecutive edges located on its boundary is more efficient with the HLE model than with the GE one.

The interest of the integration of both models in a unique framework is of course to get the advantages of both models. But the problem is to avoid getting back inconveniences of each one. It seems difficult to achieve, but could be done by allowing to switch on or off the update of several parts of the geometry to prevent some operations from getting a bad efficiency. The goal is to obtain a more generic model, more adaptable to different applications and also to gather different connected works in order to test and combine several operations and algorithms and define new ones.

6 Integration of Both Models

It follows from the previous sections that both models presents their own advantages and drawbacks. It is thus natural to try to make both models converge in order to get the advantages of each one. A first solution consists in making both models collaborate, and maintaining in parallel both data structures. A second solution, more ambitious, is to integrate both approaches in a unique model.

Conversion between both models. The collaboration of both models has been investigated by defining the operation of conversions between each representation. These conversions have been partially implemented.

The conversion operations are shared out among topological operations and geometrical operations. For both models, the topology is described by a combinatorial map and an inclusion tree. We have seen in Sec. 2 how to switch between both kind of map. So the topological conversions are obvious. But we have to notice that the topological representation of an image is not unique, because of the edges that result from the intra-object cutting (for GE) and fictive edges (for HLE) can generally be variously placed. But except for the position of these particular edges, the conversion of the representation obtained using a model gives the representation of the other one.

The conversion for the geometrical part requires particular algorithms. We study both possible directions of conversion, basing on the example in Fig. 5.

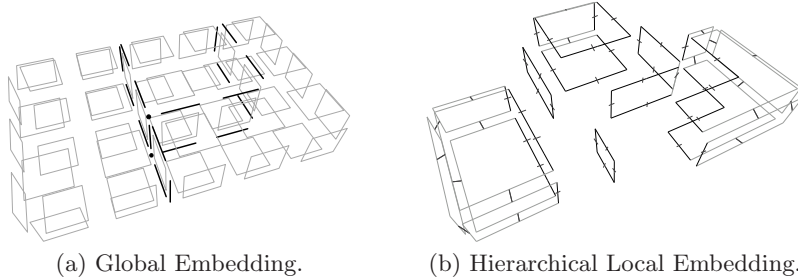


Fig. 5. GE and HLE correspondence

- Global Embedding \rightarrow Hierarchical Local Embedding
This conversion consists in traversing the s-cells of each topological face, and building a geometry equivalent to a level 3 map by merging aligned cells, first coplanar faces and then collinear edges (see Fig. 6). Mergings that would cause disconnections should not be applied. To obtain a minimal description of the geometry, it can be necessary to shift fictive edges.
- Hierarchical Local Embedding \rightarrow Global Embedding
This conversion consists in processing one by one the faces of the embedding 2-map (which we know that they are plane) and to cut them in convex

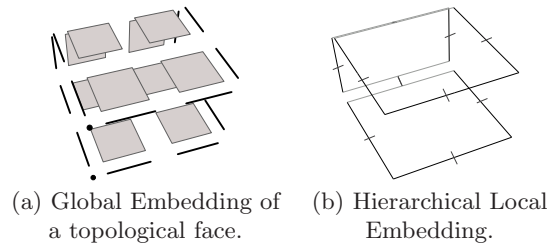


Fig. 6. Extraction of HLE.

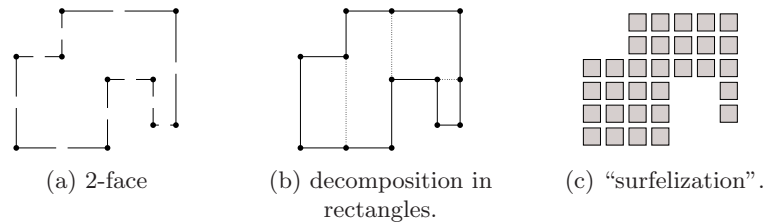


Fig. 7. Extraction of GE.

polygons, or better in rectangles, from which we can enumerate all s-cells composing the 2-face (see Fig. 7); l-cells are given by traversing the 2-map borders (see Fig. 5), and each vertex of the topological map correspond to a p-cell .

Toward a unification of both models. It follows from the previous sections that the global approach is more efficient from the point of view of geometrical implementation and reconstructions and that the local approach is more simple and more general from the point of view of the topological construction. The interest of making both models converge and to unify data structures is thus to make all the features available in a unique environment and to optimize constructions and updates.

We think that it is interesting to maintain the global embedding which provides more efficient reconstructions and which size complexity is bounded. But the construction is not bounded in time and imposes restrictions on the set of segmented images. We conjecture that these drawbacks could be drawn by using the construction by levels developed in the HLE model. By this way we hope that it will be possible to maintain the global embedding on any segmented image with a construction bounded in time. This is the future direction of this unifying work. On the other hand it will be interesting to investigate other kinds of embeddings and to compare the efficiency of each solution from the point of view of specific applications.

7 Conclusion

In this paper we have compared two representations of 3D segmented images. Both models use equivalent topological representations which are 3D topological maps described by combinatorial maps. We have given the relation between the models and the correspondence allowing to convert each model into the other one. These models differ in the way of representing the geometry of 3D segmented images. We have compared geometrical embeddings in use in each model and we have defined correspondence between themselves. Finally we have proposed to unify these models in order to get the advantages of each of them. A promising direction is to adapt the map construction used in the HLE model to improve and generalize the construction of the boundary image. Another aspect that has not been considered in this paper but that must be developed is the contribution to the unification of both models to their extension to 4D images.

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