



HAL
open science

ORDINAL REAL NUMBER 3. The techniques of transfinite real , surreal, ordinal real, numbers ; unification .

Konstantinos Constantine Kyritsis

► **To cite this version:**

Konstantinos Constantine Kyritsis. ORDINAL REAL NUMBER 3. The techniques of transfinite real , surreal, ordinal real, numbers ; unification .. [Research Report] National Technical University of Athens. 1995, <http://users.softlab.ntua.gr/kyritsis/PapersInMaths/InfinityandStochastics/OR3.htm>. hal-01512985

HAL Id: hal-01512985

<https://hal.science/hal-01512985>

Submitted on 24 Apr 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

(1992)

**ORDINAL REAL NUMBER 3. The techniques of transfinite real , surreal,
ordinal real, numbers ; unification .**

By Dr. Constantine E. Kyritsis

Department of Mathematics and Computer Science

University of Portsmouth

Software Laboratory

Electrical and Computer Engineering

National Technical University of Athens

<http://www.softlab.ntua.gr/~kyritsis>

kyritsis@softlab.ntua.gr

§ 0 Introduction . In this third paper on ordinal real numbers, it is proved that the three different techniques and Hierarchies of transfinite real number-fields, of surreal numbers, and of ordinal real numbers , give by inductive limit ,or union, the same class of numbers ,already known as the class No .It can be characterized, simply, as the smallest (linearly ordered field which is a ω) class and contains every linearly ordered set- field as a subfield . This class ,and also the category of linearly ordered set-fields, we call " **the linearly ordered Newton-Leibniz realm of numbers**". It is obvious that without the set theory of G. Cantor as it is formalized, for instance, by Zermelo-Frankel and a correct thinking about the infinite, this "realm of numbers" would not be definable.

Nevertheless we could follow a still different approach. If in the completion of rational numbers to real numbers we ramify the equivalence relation of convergent sequences to others that include not only where the sequences converge (if they converge at the same point) but also how fast (if they converge in the same way e.g finally equal ,an attribute related also to computer algorithms complexity) ,then we get non-linearly ordered topological fields that contain ordinal numbers (certainly up to ω^a , $a = \omega^\omega$) and are closer to practical applications. This approach does not involve the random variables at all, but involves directly sequences of rational numbers as "*Newtonian fluxions*". This creation can be considered as a model of such linearly ordered fields (up-to-characteristic ω^a , $a = \omega^\omega$) ,when these linearly ordered fields are defined axiomatically. This gives also a construction of the real numbers with a set which is countable. This does not contradict that all models of the real numbers are isomorphic as the field- isomorphism is not in this case also an ϵ -isomorphism so the Cauchy-real numbers and such a model still have different cardinality.

If we want to define in this way *all* the Ordinal real numbers , then it is still possible but then this would give also a device for a model of all the ZFC-set theory! And such a model is indeed possible: By taking again sequences of non-decreasing (in the inclusion) finite sets of

ZFC, and requiring that any property, relation or operation if it is to hold for this set-sequence it must hold finally for each term of the sequence and finite set. In other words we take a minimality for every set of it, relative to the axiom of infinite. It is easy to prove that the (absolute) cardinality of such a model is at most 2^{\aleph_0} , that is at most the cardinality of the continuum. We could conceive such a model as the way a computer with its algorithms, data bases tables etc would represent sets of ZFC in a logically consistent way. Needles to say that a similar model of ZFC set theory could be given within Euclidean geometry! (e.g. after the Hilbert's axiom system, and in addition accepting only the natural numbers). There is no contradiction with the 2nd-incompleteness theorem of Gödel as the argument to prove that it is a model of ZFC-set theory is already outside ZFC-set theory (as are also the arguments of Gödel, or of Lowenheim-Skolem that gives a countable model of ZFC-set theory).

It is not directly apparent that so different techniques and ideas would have such an underlying unity. It is, also, surprising that, although the Hessenberg operations were very early known in the theory of ordinal numbers, (at least since 1906, see [Gleizal A. 1937]) no one went far enough to define through them, fields in a way similar to the way that the real numbers are defined from the natural numbers. Although G.Cantor, himself was conceiving the ordinals as a natural continuation of the natural numbers (see [Frankell A. A. 1953] introduction pp 3), as it is known, he rejected the attempts to define infinitesimals through them. (see [Frankell A. A. 1953] ch ii § 7.7 pp 120). We could speculate that an underlying reason for this, might be that, his set-theory was already strongly attacked and was facing the danger of final rejection, and these were good enough reasons to avoid the additional charge that his theory "opened the door" to infinitesimals. In spite of this, there are many who might consider that although the present results are coming now, nevertheless it is too late and, they might speculate, for this long delay (more than eighty years) and diversion of ideas and technique, nevertheless on the same subject, we could suspect systematic obstructions, that came outside the mathematics. Nevertheless, there are others who consider that it is too early for such a development, and especially for an analysis on such numbers. It seems that it has never been published any **"partially ordered Newton-Leibniz realm of numbers"** (in other words a category of transcendental extensions of the real numbers, that are partially ordered fields and complete in the order topology) with reasonably "good" properties for a classification. In this paper we use the surreal numbers, as they are definable in the Zermelo-Frankel set theory, through the binary trees, directly as a class, and not as union of some set fields. (The original technique of J.H.Conway). I met J.H.Conway during 1992 at Philadelphia in the USA, I talked to him about the new developments in this area of research and I gave to him the present work but as he told me he had more than a decade that for the last time he had active interest in the subject. It is somehow necessary to make use of classes instead of sets; since, for the kind of "induction" that the J.H.Conway uses, we prove that it is reduced to the usual transfinite

induction on the height of the elements of the trees; but in their union as a class and not for each one of them separately as a set; in the latter case in which the trees are sets the induction fails. The key-point is to prove that for every cut that J.H.Conway uses it does really exist a unique element of the trees of least height. "simplest number" as it is used to be called). This is a very crucial point, for the whole technique of the surreal numbers, to work, and it seems that it has been obscured, by not paying sufficient attention to it

The author has initially included also the non-standard real numbers in the classification. As they are also linearly ordered fields and the present classification is of all linearly ordered fields it was natural to include them. There were experts in non-standard analysis that were glad about it. Nevertheless there were experts that insisted that according to the initial definition of A.Robinson and not of later definitions, it was not claimed that the non-standard real numbers were sets inside Zermelo-Frankel system. Only if Zermelo-Frankel system was used to model meta-mathematics also the they would be also sets. This was nevertheless different as such sets would model of meta-mathematical entities different than the sets that are models of mathematical and not meta-mathematical entities. Because of their arguments and in spite the fact that this made some other researchers of non-standard mathematics unhappy, the author prefers in this first publication about ordinal real numbers not to include the non-standard real numbers in the unification. Any definition nevertheless that has the non-standard real numbers as ordinary sets of Zermelo-Frankel set theory, would naturally lead to a straightforward proof that such fields are always subfields of some field of ordinal real numbers! The author has already produced pages with this proof that is based on the premise that I mentioned.

§ 2. The surreal numbers .

In this paragraph we define the class No of surreal numbers inside the ZF-set theory. We use the binary trees (see [Conway J.H. 1976] appendix to part zero pp 65 and [Kuratwiski K.-Mostowski A. 1968] Ch IX §1, §2). The crucial point is to prove that for the cuts defined by J.H.Conway in these trees it does really exist a unique element strictly greater than all the elements of the left section and strictly smaller than all the elements of the right section (the "simplest number"). Through this the Conway-induction is reduced to the usual transfinite induction on the height of the elements of the tree. As we shall see this works for the union of all trees as a class but fails for each one set-tree. For the definition of the tree, binary tree, height, levels of the tree, H_ξ -set see [Kuratwiski K.-Mostowski A. 1968] Ch IX §1, §2 Theorem 2, . The binary tree of height α we denote by D_α . More precisely we are interested for the trees of the next definition.

Definition 1. Let α be an ordinal . We define $\overset{\circ}{D}_\alpha = \{x/x \in D_\alpha \text{ such that there is } \beta < \alpha \text{ such that for the element } x \text{ as a zero-one sequence } x = \{x_\xi/\xi < \alpha\} \text{ holds that } x_\beta = 1 \text{ and } x_\xi = 0 \text{ for } \xi > \beta\}$.

We call the set $\overset{\circ}{D}_\alpha$ the open full-binary tree of height α .

We also remind that if for the height α , holds that $\aleph(\alpha)$ is a cofinal to α regular aleph: $\aleph(\alpha) = \aleph_{\text{cf}(\alpha)} = \aleph_\xi$ the open full-binary tree is an H_ξ set, (see [Kuratwiski K.-Mostowski A. 1968]

ChIX §2 Theorem 2, the proof works also for trees $\overset{\circ}{D}_\alpha$ where $\aleph(\alpha) = \aleph_{\text{cf}(\alpha)}$)

Lemma 2. For every pair of subsets L, R of the open-full-binary tree $\overset{\circ}{D}_\alpha$ of height the ordinal α , such that $\aleph(\alpha)$ is a regular aleph, and holds that: for every $l \in L, r \in R, l < r$,

and $\aleph(L), \aleph(R) < \aleph(\alpha)$, there is exactly one element x_0 of least height in $\overset{\circ}{D}_\alpha$ such that $l < x_0 < r$ for every $l \in L, r \in R$.

Proof. Let $D(L) = \{x | x \in \overset{\circ}{D}_\alpha \text{ such that there exists } l \in L \text{ with } x \leq l\}$ and $I(R) = \{x | x \in \overset{\circ}{D}_\alpha \text{ such that there exists } r \in R \text{ with } r \in x\}$ that is $D(L), I(R)$ are the decreasing and increasing lower and upper half subsets of $\overset{\circ}{D}_\alpha$ determined by L, R , in the linear ordering of $\overset{\circ}{D}_\alpha$ as a tree (see [Kuratwiski K.-Mostowski A. 1968] Ch IX §1 Lemma A). Let the set $M = \{x | x \in \overset{\circ}{D}_\alpha$ and for every $l \in D(L), r \in I(R)$ it holds that $l < x < r\}$. By the H_ξ property of $\overset{\circ}{D}_\alpha$ it holds that $M \neq \emptyset$. Let $A = \{\beta | \beta \text{ is an ordinal number such that there is } x \in M \text{ with } x \in T_\beta \text{ where } T_\beta \text{ is the } \beta\text{-level of } D\alpha \text{ in other words there is } x \in M \text{ of height } \beta\}$. Let $\alpha_0 = \min A$. Let $D\alpha_0(L), I\alpha_0(R)$ the subsets of $D(L), I(R)$ of elements of height less than α_0 , and let $M\alpha_0 \subseteq M$ the subset of M that consists of elements of height α_0 . Suppose that the set $M\alpha_0$ contains two elements x, y with e.g. $x \leq y$. We will prove that $M\alpha_0$ contains only one element.

Let $x' = \{x_\beta | \beta < \alpha_0\}$ that is that part of the α_0 -sequence x with terms of indices less than α_0 . And the same also with $y' = \{y_\beta | \beta < \alpha_0\}$. Then there is l_x or r_x and l_y or r_y respectively in $D\alpha_0(L), I\alpha_0(R)$ such that they are equal with x', y' . If $x=r_x$ then, if the α_0 -term of x is 0 or 1, in both cases $x > r_x$, contradiction. Hence there is no such r_x and also such r_y . Then $l_x=x', l_y=y'$ and $l_x \leq l_y$. The α_0 -term of x and y might be 0 or 1. The only possible cases are $\{x = (l_x, 0), y = (l_y, 0)\}, \{x = (l_x, 0), y = (l_y, 1)\}, \{x = (l_x, 1), (l_y, 1)\}, \{x = (l_x, 1), y = (l_y, 0)\}$ where with the parenthesis we symbolize the α_0 -

sequence which is the elements x, y . Let us suppose that $x \neq y$ and $\overset{\circ}{D}_\beta(x) = \{x_\beta | \beta < \delta\}$, the part of the α_0 -sequence with terms with indices less than δ , with $\delta \leq \alpha$. Let the least

value of δ , be denoted by δ_0 such that $\overset{\circ}{D}_\beta(x) = \overset{\circ}{D}_\beta(y), \delta_0 \leq \alpha_0$ and $X\delta_0 \neq Y\delta_0$. If holds that $0 = X\delta_0 \neq Y\delta_0 \neq 1$ because $x < y$. In the sequent, let $z = (\overset{\circ}{D}_\beta(x) = \overset{\circ}{D}_\beta(y) \beta < \delta_0, 1)$. Then $x < z \leq y$. If $\delta_0 = \alpha_0$ then $x=y$ because $X\alpha_0 = Y\alpha_0 = 1$. Then $\delta_0 < \alpha_0$ and also $z < y$ and $x < z < y$ and the height of z is $\delta_0 < \alpha_0$ contradiction. Hence $x = y$, and $M\alpha_0$ contains only one element. It also holds that if we restrict to $D_c(L), I_c(R)$

where $D_c(L) = DC(L) \cap \overset{\circ}{D}_c, I_c(R) = I(R) \cap \overset{\circ}{D}_c$ (and L, R have height $< c$), then $\alpha_0 \leq c$ by the H_ξ -property of $\overset{\circ}{D}_c$ if c is also such that $\aleph(\alpha) = \aleph_{cf(\alpha)}$ Q.E.D.

Definition 3. The open full binary tree $\overset{\circ}{D}_\alpha$ of height α , such that $\aleph(\alpha)$ is a cofinal to α , regular aleph, I call regular open full-binary tree.

The property of the previous lemma of a regular open full-binary tree I call H_ξ -leveled Dedekind completeness.

We remark that the class of regular alephs is unbounded (see [Kuratwiski K.-Mostowski A. 1968] p. 275 relation 5) Thus the class of ordinals α such that $\aleph(\alpha) = \aleph_{cf(\alpha)}$ is unbounded.

The next definition is the definition of the class of surreal numbers in the ZF-set theory and it depends as we mentioned on the lemma 2. As it is seen, in the hypotheses of the lemma 2 the cardinality of halves of the cut is bounded by $\aleph(\alpha)$. If it is to include all possible cuts of the

tree $\overset{\circ}{D}_\alpha$ then the lemma 2 will give the element x_0 in some tree $\overset{\circ}{D}_\beta$, of sufficient greater height, thus outside the original tree D_α . This is why we mentioned that the definition of surreal numbers (with the original technique of J.H.Conway) does not apply to the trees $\overset{\circ}{D}_\alpha$ separately.

Definition 4. Let $U \overset{\circ}{D}_\alpha = No$ be the union of all regular open full-binary trees. It is a class (after axiom A2. (see [Cohn P.M. 1965] p1-36)) Operations may be defined in this linearly ordered class according to the formulae of Lemma 2 in [Kyritsis C.1991 Alt. or Free etc.] II, that hold for every linearly ordered field that is:

1. let α be an ordinal with $\aleph(\alpha) = \aleph_{cf(\alpha)}$ and L, R subsets of $\overset{\circ}{D}_\alpha$ such that for every $l \in L, r \in R$ holds that $l < r$. Then there exists a regular aleph β such that $L, R \subseteq \overset{\circ}{D}_\beta$ and $\aleph(L), \aleph(R) < \aleph(\beta)$. Then there is by lemma 2 a unique element $x_0 \in \overset{\circ}{D}_\beta$ of least height such that $l < x_0 < r$ for every $l \in L, r \in R$, we denote this element by $\{L|R\}$ and we write $x_0 = \{L|R\}$. We note that although $L, R \subseteq \overset{\circ}{D}_\alpha$, it holds that $x_0 \in \overset{\circ}{D}_\beta$ and $\alpha < \beta$.

2. If $x, y \in \overset{\circ}{D}_\alpha$ and we denote the height of x, y by $h(x), h(y)$ and by $L(x), L(y), R(x), R(y)$ the sets

$$L(x) = \left\{ v \mid v \in \overset{\circ}{D}_\alpha, h(v) < h(x) \text{ and } v < x \right\} \quad L(y) = \left\{ v \mid v \in \overset{\circ}{D}_\alpha, h(v) < h(y) \text{ and } v < y \right\},$$

$$R(x) = \left\{ v \mid v \in \overset{\circ}{D}_\alpha, h(v) < h(x) \text{ and } x < v \right\} \quad R(y) = \left\{ v \mid v \in \overset{\circ}{D}_\alpha, h(v) < h(y) \text{ and } y < v \right\}$$

Then the operations are defined through simultaneous two-variable transfinite induction in the form of the lemma 2,3 in [Kyritsis C. 1991 Free etc.], for the heights of the trees $\overset{\circ}{D}_\alpha$ where for the initial segments of ordinals we substitute the corresponding trees of No (For every ordinal $\beta < \alpha$ such that $N(\beta) = N_{cf(\beta)}$ corresponds a tree $\overset{\circ}{D}_\beta$). Thus the function of operation is defined not on $w(\alpha)^2$ but on $\overset{\circ}{D}_\alpha^2$. For the addition, the next rule is used $x+y = \{L(x)+y \cup x+L(y) \mid x+R(y)\} \cup R(x)+y$.

3. The opposite is defined by:

$$-x = \{-R(x) \mid -L(x)\}$$

4. Multiplication is defined by

$$x \cdot y = \{L(x) \cdot y + x \cdot L(y) - L(x) \cdot L(y) \cup R(x) \cdot y + x \cdot R(y) - R(x) \cdot R(y) \mid \\ L(x) \cdot y + x \cdot R(y) - L(x) \cdot R(y) \cup R(x) \cdot y + x \cdot L(y) - R(x) \cdot L(y)\}.$$

This definition presupposes the definition of addition.

5. Inverse is defined by

$$x^{-1} = \left\{ 0, \frac{1 + (R(x) - x)L(x)}{R(x)} \cup \frac{1 + (L(x) - x)R(x)}{L(x)} \mid \frac{1 + (L(x) - x)L(y)}{L(x)} \cup \frac{1 + (R(x) - x)R(y)}{R(x)} \right\}$$

As it is proved in [Conway J.H. 1976] Ch0, 1 the set No is a linearly ordered c -field. The characteristic of No is easily proved to be Ω_1 , we call this c -field, c -field of surreal numbers. According to Definition 3 No is an H_ε -leveled Dedekind complete field.

§ 3 The unification .

In this paragraph we prove that all the three different techniques and hierarchies of transfinite real ,of surreal ,of ordinal real numbers give by inductive limit or union the same class of numbers .We have already proved that $CR = \Omega_1 R = C^* R$. (see corollary 10) and it remains to prove $No = CR$.

Lemma 5 . *It holds that $CR = \Omega_1 R = C^* R \subseteq No$.*

Proof . Let an open full binary tree $\overset{\circ}{D} \alpha$ of height the principal ordinal α .Then $\overset{\circ}{D} \alpha \subseteq No$,and the field-inherited operations in the initial segment $W(\alpha)$ are the Hessenberg operations (see [Conway J.H. 1976] ch 2 § ""containment of the ordinals "note pp 28 and also [Kyritsis C.1991 Alt] the characterisation theorem).If α was not a principal ordinal, the $W(\alpha)$ would not be closed to the Hessenberg operations .Thus the $N_\alpha, Z_\alpha, Q_\alpha$ are contained in No ,since what it is used to define them from $W(\alpha)$ is only the field operations .The Q_α is a field and from the fact that No is closed to extensions of its set-subfields (see[Conway J.H. 1976] ch 4 theorem 28)we deduce that the field of real numbers R_α is contained in No , for every principal ordinal number α .Thus $\cup R_\alpha = \Omega_1 R \subseteq No$.Q.E.D.

Lemma 6 . *For every regular open full binary tree $\overset{\circ}{D} \alpha$, it holds that $\overset{\circ}{D} \alpha \subseteq R_\beta$, for some sufficiently big principal ordinal number β . (With the inclusion is meant that the restriction of ordering of R_α in the tree, coincides with the ordering of the tree).*

Proof . We shall prove it by transfinite induction .It holds for the trees of finite height. The transfinite induction shall be on the transfinite sequence of all ordinal numbers such that $\aleph(\alpha) = \aleph_{cf(\alpha)}$ and $\aleph(\alpha)$ is a regular aleph. Let us suppose that it holds for all such ordinal numbers of $W(\alpha)$, and $\aleph(\alpha) = \aleph_{cf(\alpha)}$ and $\aleph(\alpha)$ is a regular aleph .Then

: $\overset{\circ}{D} \alpha \subseteq \cup_{s \in W(\alpha)} \overset{\circ}{D} s \cup \cup_{s \in W(\alpha)} R_{\beta(s)} \subseteq R_{\beta(\alpha)}$ hence $\overset{\circ}{D} \alpha \subseteq R_{\beta(\alpha)}$ where $\beta(\alpha)$ is a principal ordinal with $\beta(\alpha) > \lim_{s \in W(\alpha)} \beta(s)$. Q.E.D.

From the previous lemma we get that $\cup \overset{\circ}{D} \alpha = No \subseteq \Omega_1 R$,thus :

The unification theorem 7

It holds that the classes of transfinite real numbers CR , of surreal numbers No , of ordinal real numbers $\Omega_1 R$,coincide ,and it is the smallest class (and linearly ordered c -field) ,that contains all linearly ordered set-fields as subfields.

We can have obviously analogous statements for the other classes of numbers (complex , quaternion e.t.c.). After the previous theorem, the binary arithmetisation of the order-types, stated in [Kyritsis C. 1991] II ,theorem 11, is directly provable. We remark that because the levels of the open full binary trees have the property that any upper (lower bounded set has supremum (infimum) ,(see [Kuratowski K. -Mostowski A 1968] ch ix §1, § 2 theorem 2),and after the Hilbert and fundamental (Cauchy) completeness of the ordinal real

numbers, and remark after definition 13 and ω -normal form according to [Frankel A.A. 1953] ch 3 theorem 21 ,and after corollary 21 in [Kyritsis C. 1991] ,II , we also get:

Theorem 8. *The class of numbers $CR=\Omega_1R=No$ has leveled formal power series representation, leveled Hilbert completeness, leveled fundamental (Cauchy) completeness, leveled H_ξ Dedekind completeness ,leveled supremum completeness and representation with ω -normal forms.*

Bibliography

- [N. L. Alling 1987] Foundations of analysis over surreal number fields North-Holland Math.Studies V. 141 1987 .
- [Artin E. Schreier O.1927] Algebraische konstruktion reellerkorper, Abh. Math. Sem.Univ. Hamburg 5 (1927) pp 85-99 .
- [Artin E. - Tate J.1967] Class Field Theory Benjamin 1967.
- [Baer Reinold,1970] Dichte,Archimedizitat und Starrheit geordneter Korper, Mathematische Annalen 188 pp 165-205, 1970.
- [Baker Alan1975] Trancendental Number Theory Cambridge University Press, 1975.
- [Bourbaki N.1952] Elemente de Mathematique algebre, chapitre III Hermann Paris 1948, chapitre VI Hermann Paris 1952.
- [Brown Ron 1971] Real places and Ordered Fields Rocky Mountain Journal of Mathematics , Vol 1 ,pp 633-636, 1971.
- [Brown R.,Craven T.C., Pelling M.J.1986] Ordered fields satisfying Rolle's Theorem. Illinois Journal of Mathematics Vol 30, n 1 Spring 1986 pp 66-78.
- [Clifford A. H.1954] Note on Hahn's theorem on ordered abelian groups. Proc. of the Amer. Math. Soc. 5 (1954) pp 860-863.
- [Cohen L.W.- Goffman C.1949] Theory of transfinite Convergence.Transact. of the Amer. Math. Soc. 66 (1949) pp 65-74.
- [Cohn P.M.1965] Universal Algebva Harper - Row 1965 .
- [Conway J.H.1976] On numbers and games Academic press 1976 .
- [Cuesta Dutardi N.1954] Algebra Ordinal Rev. Acad. Cientis Madrid 4 (1954) pp 103-145 .
- [Dugundji J.1966] Topology, Allyn and Bacon inc. 1966.
- [Ehreshmann Ch.1956] Categories et structure Dunod 1956
- [Ehrlich P.1988] An alternative construction of Conway's Ordered Field No ,Algebra Universalis 25 (1988) pp 7-16 .
- [Ehrlich P.] The Dedekind completion of No ,submitted to Algebra Universalis.
- [Endler O.1972] Valuation Theory, Springer 1972.
- [Erdos P.-Gillman L.-] An isomorphism theorem for real closed fields

- Henrkiksen M..1955] Ann. of Math.(2) 61 (1955)pp 542-554.
- [Frankel A.A.1953] Abstract set Theory. North - Holland 1953.
- [Fuchs L.1963] Partially ordered algebraic systems Pergamon Oxford 1963.
- [Gillman L.-Jerison M.1960] Rings of continuous functions.Van Nostrand Princeton 1960.
- [Gleyzal A.1937] Transfinite real numbers. Proc. of the Nat. Acad.of scien. 23 (1937) pp 581-587.
- [Gravett K.A.H.1956] Ordered abelian groups. Quart. J. Math. Oxford 7 (1956) pp 57-63.
- [Hahn H. 1907] Uber die nichtarhimedischen Grossensysteme.S. Ber. Akad. Wiss. Wein. Math. Natur.Wkl Abt. Ila 116 (1907) pp 601-655.
- [Hausner M.-Wendel J.G.1952] Ordered Vector Spases Proc. of the Amer. Math. Soc.3 (1952) pp 977-982.
- [Hessenberg G.1906] Grundbegriffe der Mengenlehre (Abh. der Friesschen Schule, N.S. [1] Heft 4) Gottingen 220 1906).
- [Hilbert D.1977] Grundlagen der Geometry Teubner Studienbucher 1977 .
- [Hilbert D. -Ackermann W.1950] Principles of Mathematical Logic. Chelsea Pub. Comp. N.Y. 1950.
- [Kaplansky I.1942] Maximal fields with valuations Duke Math. J. 9 (1942) pp 303-321.
- [Krull W.1931] Allgemeine Bewertungs theorie. J.reine angew. Math. 176 (1931) pp 160-196.
- [Kuratowski K.1966] Topology v.I v.II Academic Press 1966.
- [Kuratowski K. -Mostowski A.1968] Set Theory North - Holland 1968.
- .
- .
- [Kyritsis C.E.1991] Algebraic characterisation of the Hessenberg operations in the ordinal numbers. (unpublished yet).
- [Kyritsis C.E.1991] Ordinal real numbers 1. The ordinal characteristic.(unpublished yet).
- [Lang S.1984] Algebra . Addison-Wesley P.C. 1984 .
- [Lam T.Y.1980] The Theory of Ordered Fields Ring theory and Algebra III. Edited by B.R. McDonald Dekker 1980 pp 1-268.
- [Laugwitz Detler 1983] Ω Calculus as a Generalization of Field Extension. An alternative approach to non-Standard analysis "Recent developments in non-standard analysis" Lecture Notes in Math 983 Springer 1983.

- [MacLane S.1939] The Universality of Formal Power Series fields.Bull. of the Amer. Math. Soc. 45 (1939) pp 880-890.
- [MacLane S. 1971] Categories for the working mathematician Springer 1971
- [Massaza ,Carla1970] On the completion of ordered fields . Practica (=Proceedings) of the Academy of Athens ,Vol 45 ,1970 (published 1971)
- [Monna A.F.1975] Analyse non-Archimedienne Springer 1970.
- [Munkress J.R.1975] Topology. Prentice Hall 1975.
- [Nachbin L.1976] Topology and Order. Robert E.Krieger P.C. N.Y. 1976.
- [Neubrunnova Anna1980] On transfinite convergence and generalised continuity. Mathematica Slovaca vol 30,1, 1980.
- [Neumann B.H.1949] On ordered division rings. Transact. of the Amer. Math. Soc. 66 (1949) pp 202-252. .
- [Prestel Alexander1980] Lectures on formally real fields .Lecture Notes 1093, Springer ,1980 .
- [Robinson A.1966] Non-Standard analysis.North-Holland 1974 (1966).
- [Robinson A.1972] On the real Closure of a Hardy Field. pp 427-433 in the Theory of Sets and topology. A Collection of papers in honour of Felix Hausdorff , VEB ,1972 .
- [Rolland,Raymond .1981] Etudes des courpures dans les groupes et corps ordonnes. Dans Geometrie Algebrique Reeles et Formes Quadratiques. Rennes 1981, Proceedings, Lecture Notes in Mathematics 959 Springer ,pp 386-405 .
- [Rudin W.1960] Fourier analysis on groups. Interscience Pub 1960 .
- [Shilling O.F.G.1950] The theory of valuations.Amer. Math. Soc. 1950.
- [Schubert H.1972] Categories Springer 1972.
- [Scott, Dana 1969,] On Completing Ordered Fields. In Applications of Model theory to Algebra, Analysis and Probability edited by W.A.J. Luxemburg pp 274-278,Holt Rinehart and Winston 1969.
- [Sirkoski R. 1948] On an ordered algebraic field. Warsaw, Towarzystwo Nankowe Warszawskie 41 (1948) pp 69-96.
- [Stone A.L.1969] Non-Standard analysis in topological algebra in Applications of Model Theory to Algebra, Analysis and Probability N.Y. (1969) PP 285-300.
- [Stroyan, K.D. and Luxemburg W.A.J.1976] Introduction to the theory of Infinitesimals N.Y.1976.

- [Lynn A.Steen- Seebach J.A. Jr.1970] Counterexamples in Topology Springer 1970 .
- [Viswanathan T.M.1977] Ordered fields and sign-changing polynomials. Journal fur reine und angewante Mathematik, 296 pp 1-9,1977.
- [Van der Waerden B.L.1970] Algebra V1 V2 Frederick Unger Pub. Co. N.Y. 1970.
- [Weil A.1967] Basic Number Theory, Springer Verlag Berlin, Heidelberg N.Y. 1967.
- [Zakon E.1955] Fractions of ordinal numbers Israel Institute of Tecnology Scient. Public. 6, 94-103 1955.
- [Zariski O. -Samuel P.1958] Commutative Algebra V.I.II Springer 1958.
- [Zervos,S.P.1961] Sur les rapportes entre la completions la cloture algebrique des corps commutatifs de caracteristique zero C.R. Acad. Sc. Paris, 5 Avril 1961 t. 282 pp 2053-2055.
- [Zervos,S.P.1991] Simple abstract and concrete considerations suggested by Thue's theorem. Mathematics revisited, mathematicians remembered Vol.1 1991.pp 1-68.

List of special symbols

- α, β, ω : Small Greek letters
- Ω_1 : Capital Greek letter omega with the subscript 1
- F^a : Capital letter F with superscript a.
- N : Capital Aleph ,the first letter of the hebrew alphabet . In the text is used a capital script. letter n .
- \oplus, \circ : cross in a circle, point in a circle .
- $N\alpha, Z\alpha, Q\alpha, R\alpha, :$ Roman capital letters with subscript small Greek letters
- $C\alpha, H\alpha$
- $^*X, ^*R$ et.c : Capital standard or roman letters with left superscript a star.

CN,CZ,CQ, :Capital standard letter c followed by capital letters, with possibly

C^*R a left superscript a star

\hat{X} : Capital tstandard letter with a cap.

Σ : Capital Greek letter sigma

$\overset{\circ}{D}_{\omega}$: Capital standard D with subscript a small Greek letter and in upper place a small zero.

LINEAR NUMBERS 3.

The technique of "free operations-fundamental completion" for the definition of a well-ordered classifying Hierarchy of Linearly ordered associative commutative number-fields. The transfinite real , the non-standard real, the surreal numbers; unification .

by

Dr.Konstantine E.Kyritsis

Computer Technology Institute

Patra Greece

Abstract

In this paper is introduced a forth new technique of "the enlargement real numbers",that in some sense includes the non-standard real numbers of A.Robinson.It is proved ,that the three different technics and hierarchies of transfinite real numbers , of the surreal numbers ,of the ordinal real numbers, give by inductive limit or union the same class of numbers .

Key words

linearly ordered commutative fields

transfinite real numbers

non-standard enlargement fields of the real numbers

surreal numbers

formal power series fields