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Konstantinos Constantine Kyritsis

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(1992)

ORDINAL REAL NUMBERS 2. The arithmetization of order types .

By Dr. Constantine E. Kyritsis

Department of Mathematics and Computer Science

University of Portsmouth

Software Laboratory

Electrical and Computer Engineering

National Technical University of Athens

<http://www.softlab.ntua.gr/~kyritsis>

kyritsis@softlab.ntua.gr

§0 **Introduction** . In this second paper on ordinal real numbers are proved, the main (elementary) properties of them. It is proved that the ordinal real numbers R_α of characteristic α , is the maximal field of characteristic α (maximality) and that it is , according to the theory of Artin-Schreier, a real closed field. (It turned out ,after the work was completed and by thinking aside, that they are also Archimedean complete (see [Glazal A. 1937]), formal power series fields with real coefficients , Dedekind complete (see [Massaza, Carla 1971]), and Pythagorean fields).

It is also proved a classification theorem which is analogous to the Hölder theorem for the Archimedean linearly ordered fields. In particular it is proved that any linearly ordered field of characteristic α contains the field Q_α of ordinal rational numbers of characteristic α , as a dense subfield and it is contained in the field R_α of ordinal real numbers of characteristic α , as a subfield ..As it is known, the linear segments of elementary euclidean geometry can be defined as special order-types with Archimedean property, and Archimedean (Hilbert) completeness through axioms (see e.g. for a not ancient approach the Hilbert axiomatisation in [Hilbert D.1977] ch 1). It can be

proved to be order isomorphic with subsets of the real number field R . This is well-known and it can be called , the elementary arithmetisation of the order-types of Euclidean linear segments . On this fact is based the Cartesian idea of analytic geometry. This was an important turning point in the developments of the ideas and techniques of mathematics , of the discrete nature of numbers and continuous nature of geometry. The basic principle is that *the continuum is developed from the discrete* and not vice versa! An instance of this principle is the development of images and animation in computers through pixels and bits! It is surprising that in one of the consequences of the theory of ordinal real numbers, it is proved a far more advanced and complete result for the whole category of order types that has as corollary the previous important and elementary arithmetisation. Although more advanced, the result remains in the context of elementary theory of ordinal real numbers .In this result any order type can be “discretised” or “arithmetised” through the ordinal numbers.

The process of definition of the maximal fields R_α , from the minimal (double well ordered) monoids $N_\alpha = \alpha$, of principal ordinal numbers, we call K-fundamental densification. It is proved that any order-type is order isomorphic to a subset of some field R_α . Thus any order-type is constructible by K-fundamental densification from ordinal numbers. This is called the K-arithmetisation of the order-types. Although in the way it is presented, this result is softly obtained, throws new light to the relation of ordinal numbers and order-types, this relation turns out to be similar to the elementary relation of numbers and line segments in geometry. Also it, holds a second kind of arithmetisation, the binary arithmetisation which we state in the same paragraph.

§1 On the topology of linearly ordered fields. Local deepness, α -sequences.

The ordering of any linearly ordered field F defines a well known topology: the order-topology denoted by $T_<$; In this topology, as it is known, the field F is a topological field.

This topology has very good separation properties; it is a T_1 - T_5 topology, that is a completely normal topology (see for instance [Lynn A. Steen-Seebach J.A. Jr 1970] § 39 p. 66-68, also see [Munkres J.R. 1975] Chapters I, II)

The previously described order topology is also called the locally convex topology compatible with the order (see [Nachbin L. 1976] Ch I, II). (The convexity defined by the order).

Definition 1. Let X be a topological space. Let $p \in X$. The least ordinal α such that it exists a (local) base denoted by B_p of open neighbourhoods of the point p which is an α -sequence such that if $x < y < \alpha$, $U_x \subset U_y$, is called local deepness of X at p .

We notice that the concept of local deepness is very close to the concept of local weight of a topological space, where instead of ordinal we have an initial ordinal that is a cardinal number (see [Kuratowski K. 1966] V-I p. 53-54).

Examples of topological spaces such that every point has local deepness, are the ξ^* -uniform topological spaces as they are defined in [Cohen L.W. Goffman C. 1949] pp 66 conditions 1.2.3.4.

As in the case of fields that are classes, we may permit topological spaces that are classes and the open sets is a class of subclasses closed to union and finite intersection. For such spaces, the local deepness may be Ω_1 that is the class of all ordinal numbers.

Proposition 2. Let X a topological space and α , a limit ordinal such that every point has local-deepness α let $A \subset X$. It holds that $x \in \overline{A} \iff$ there is a β -sequence $\{x_s | s < \beta\}$ from elements of A such that $\lim_{s < \beta} x_s = x$. In other words topological convergence in X can be treated with β -sequences where $\beta = \text{car}(\alpha)$ is the upper character of α (see [N.L. Alling 1987] ch 1 §1.30 pp 29)

The proof is almost direct and to save space we shall not give it here.

Proposition 3. Let a field denoted by F of ordinal characteristic α , where α is a limit ordinal. Then every point $x \in F$ in the order-topology has local-deepness $\text{car}(\alpha)$, where $\text{car}(\alpha)$ is the upper character of α (see [N.L. Alling 1987] ch 1 §1.30 pp 29).

The proof is again direct and outside the scope of the paper.

Corollary 4. Convergence in the order-topology of a field of ordinal characteristic α , can be treated with β -sequences $\beta \geq \text{car}(\alpha)$

Needless to say, that in the case in which the topological space is a class and the local deepness is Ω_1 , then convergence can be treated with Ω_1 -sequences.

§2 The Holder- type classification .

Lemma 5. In every field of characteristic α the field $Q\alpha$ is a dense subset.

Proof. Let a field of characteristic α , which we denote by $F\alpha$. By the theorem 17 of [Kyritsis C. OR1] the field $Q\alpha$ is a subfield of $F\alpha$. Let us suppose that it is not dense in $F\alpha$. Then there are two elements $x, y \in F\alpha$ $x < y$, such that there is no element of $Q\alpha$ in the interval $[x, y]$. Then the element $z = y - x$ is $Q\alpha$ -infinitesimal.

This holds because $Q\alpha = \bar{L}(x) \cup \bar{R}(x) = \bar{L}(y) \cup \bar{R}(y)$ where $\bar{L}(x) = \{v | v \in Q\alpha, v \leq x\}$, $\bar{R}(x) = \{v | v \in Q\alpha, x \leq v\}$ and $\bar{L}(y), \bar{R}(y)$ similarly. But by the hypothesis $\bar{L}(x) = \bar{L}(y), \bar{R}(x) = \bar{R}(y)$. Thus $Q\alpha = \bar{L}(x) \cup \bar{R}(y)$ and every element of $Q\alpha$ can be written as $r_2 - r_1$ where $r_2 \in \bar{R}(y)$ and $r_1 \in \bar{L}(x)$. Also we have that $0 < y - x < r_2 - r_1$.

Then $y - x$ is a $Q\alpha$ -infinitesimal and $\frac{1}{y - x}$ is a $Q\alpha$ -infinite element of $F\alpha$, thus $\frac{1}{y - x} > \alpha$, contradiction since $\text{Char } F\alpha = \alpha$.

Then there are not two element $y, x \in F\alpha$ $x < y$ with no element of $Q\alpha$ in $[x, y]$, and $Q\alpha$ is dense in $F\alpha$. Q E D

Remark. Thus every field $F\alpha$ of characteristic α is a Weil completion of the field $Q\alpha$ of ordinal rational numbers (see [Weil A.] ChIII Definition 2 but applied not only to local fields).

Theorem 6 (Maximality or completeness up-to-characteristic).

The field $R\alpha$ is the maximal field, of characteristic α . In the sense that every field of characteristic α is contained as subfield of $R\alpha$ (more precisely $R\alpha$ contains an order preserving monomorphic image of the field). The field $R\alpha$ is the unique fundamentally complete field of characteristic α .

Remark. This theorem is analogous to the well-known Holders theorem $\zeta\eta\psi\eta\sigma\alpha\tau\epsilon\zeta$ that every linearly ordered Archimedean field is a subfield of the field of real numbers. In other words the field of real numbers is the maximal Archimedean linearly ordered field. The previous property of the ordinal real numbers R_α relative to their characteristic, we call maximality or completeness up-to characteristic.

But as an erroneous application of terms R is also the minimal Cauchy complete field of characteristic ω and this also applies for the fields $R\alpha$ in the sense that a completion of a linearly ordered field of characteristic α must be the field $R\alpha$.

Proof. Let any field of ordinal characteristic α denoted by $F\alpha$. By theorem 17 of [Kyritsis C. 1991], the field $Q\alpha$ is contained in $F\alpha$: $Q\alpha \subseteq F\alpha$. Let $x \in F\alpha$. Let $(L(x), R(x))$ be the cut that x defines on $Q\alpha$ ($L(x) = \{v | v \in Q\alpha, v < x\}$, $R(x) = \{v | v \in Q\alpha, x < v\}$). Since $Q\alpha$ is dense in $F\alpha$ (Lemma 5). There is a Cauchy α -sequence $\{x_n | n \in \omega(\alpha)\}$ of elements of $Q\alpha$ that converges in $F\alpha$ to x (all topologies are the order-topologies). Hence $Q\alpha \subseteq F\alpha \subseteq R\alpha$ and the field $R\alpha$ is a maximal field of characteristic α ; but also the field $R\alpha$ is actually a minimal Cauchy complete field of characteristic α in the sense that the (strong) Cauchy

completion \hat{F}_α of any field F_α of characteristic α contains the field $R_\alpha: Q_\alpha \subseteq F_\alpha$ has as a consequence that $R_\alpha \subseteq \hat{F}_\alpha$. Thus if F_α is complete then $R_\alpha \subseteq F_\alpha$, $F_\alpha \subseteq R_\alpha$ hence $F_\alpha = R_\alpha$ Q.E.D.

The theory of Artin-Schreier of real closed fields has an excellent application to the ordinal real numbers.

Corollary 7. The fields of ordinal real numbers R_α are real closed fields.

Proof. Direct from Theorem 6, and remark 5 of [Kyritsis C.1991] Q.E.D.

Post written Remark A. The author developed the theory of ordinal real numbers during 1990-1992. He had used the name "transfinite real numbers" without being aware that this term had been introduced by A. Glayzal during 1937 for his theory of linearly ordered fields. From the moment he fell upon the work of A. Glayzal (see [Glayzal A. 1937]) in the bibliography of the Book of N.L. Alling (see [N.L. Alling 1987]) he changed the title to "Ordinal Real Numbers". After the work had been completed, the author realised, by thinking aside, a quite unexpected and not unhappy fact: That the fields of ordinal real numbers are algebraically and order isomorphic to the fields of transfinite real numbers of Glayzal. This can be deduced by the fact that the fields of transfinite real numbers are exactly all the Archimedean complete fields (see [Glayzal A. 1937] theorems 4,8,9) and by the maximality of the ordinal real numbers (theorem 6). Thus if R_α is a field of ordinal real numbers of characteristic α , any Archimedean (linearly ordered field) extension of it, it shall have the same characteristic with R_α . It seems that it can be proved, that any cofinal (coterminal) linearly ordered field extension, is of the same characteristic. By the maximality of R_α (theorem 6) it shall have to coincide with R_α . In other words the fields of ordinal real numbers are Archimedean complete fields (although they may be non-Archimedean). But this is a characteristic property of the fields of transfinite real numbers of Glayzal.

Thus they are order and field isomorphic with fields of transfinite real numbers. Conversely, let any field $R(\lambda)$ of transfinite real numbers of Archimedean base λ . Let us denote by α its ordinal characteristic. Let us suppose that there is an order and field extension of it with the same characteristic. Then it has to be an Archimedean extension of $R(\lambda)$. By the Archimedean completeness of the transfinite real numbers, it has to coincide with the $R(\lambda)$. Thus the transfinite real numbers are also complete up-to-characteristic.

But this is a characteristic property of the fields of the ordinal real numbers. Hence they are order and field isomorphic with fields of ordinal real numbers. Thus the ordinal real numbers should be considered as a different technique, nevertheless indispensable and more far reaching. It is the technique that everyone would like to work.

Post written Remark B. Let a field R_α of ordinal real numbers of ordinal characteristic α . It is also a field of transfinite real numbers of archimedean base λ . The set of all elements of R_α that as formal power series have support of ordinality less than $\beta \leq o(\lambda) = \text{maximum ordinality of well ordered set of } \lambda$, and which we denote

by $R_{\alpha,\beta}$ is a field, subfield of R_α . Indeed $R_{\alpha,o(\lambda)} = R_\alpha$. For the applications and especially with measurement processes, the fields $R_{\alpha,\omega}$ are of prime interest and indispensable.

Post written remark C. The facts of the previous remark, have as a consequence that the fields of ordinal real numbers are formal power series fields with coefficients in the real numbers and exponents in some order types. Thus the n -roots of their positive elements are contained in them (see [Neumann B.H. 1949] pp 211, 4.91 Corollary). In other words they are Pythagorean complete fields.

Theorem 8. (The Holder-type classification theorem).

Every field of ordinal characteristic α , denoted by F_α (where α is a principal ordinal) is contained between the fields Q_α and $R_\alpha : Q_\alpha \subseteq F_\alpha \subseteq R_\alpha$.

Proof. Contained in the proofs of the theorem 7 and lemma 5 Q.E.D.

Remark.9 The previous theorem gives that the hierarchy of ordinal real numbers has *universal embedding property* for the category of linearly ordered fields, that is every linearly ordered field has a monomorphic image in some field of the hierarchy. The hierarchy of transfinite real numbers is known to have, also, this property. Such hierarchies we call universal embedding hierarchies. Especially the hierarchy of ordinal real numbers after the classification theorem 8, we call also, universal classification hierarchy.

Remark.10 We notice that since every order type λ is order-embeddable in some transfinite real number field $R(\lambda)$ (see [Glazal A. 1937]) as Archimedean base which in its turn is embeddable in some ordinal real number field R_α , the above two hierarchies as hierarchies of order-types are universal embedding hierarchies for the category of order-types. Let an order type λ ; the least principal ordinal number α such that λ is order-embeddable (by a monomorphism) in the order-type and field R_α , is called the fundamental density of the order type λ and is denoted by $df(\lambda)$.

Remark. In the [Massaza Carla, 1971] Definition I, is defined which cuts are the Dedekind cuts in linearly ordered fields. It is proved also that the Dedekind completion $D(F)$ of a linearly ordered field F is also its Cauchy completion (in the order topology). If we take the Dedekind completion $D(R_\alpha)$ of a field of ordinal real numbers R_α , it has to be its Cauchy completion which is again the R_α . Thus the fields of the ordinal real numbers are also Dedekind complete. Conversely, let any Dedekind complete linearly field F . Let us denote with α its ordinal characteristic. Then by the Holder type classification (theorem 8) it is a subfield of the field R_α of ordinal real numbers of characteristic α . Since the Dedekind completion $D(F) = F$ coincides with the F and also with its Cauchy completion, we get that $F = R_\alpha$, because the Cauchy completion of F is the R_α . In other words the class of Dedekind complete fields coincides with the class of the fields of ordinal real numbers.

Summarising we mention that the fields of ordinal real numbers have at least four kinds of completeness that characterise them: Cauchy completeness, Dedekind completeness, completeness up-to-characteristic, Archimedean completeness. It seems that the previous four completeness can be summarised by saying that there is no cofinal (coterminal) order field extensions of them; in short they are cofinally complete, or cofinally maximal. They are also realcomplete (closed, Artin-Schreier) and Pythagorean complete.

Remark. By corollary 7 we get that the field C_α is the algebraic closure of $R_\alpha : C_\alpha = \overline{R_\alpha}$.

We close this paragraph by mentioning that an axiomatic definition of the field R_α (α is a principal ordinal) would be the following:

First axiomatic definition of R_α .

The field of ordinal real numbers R_α is the unique Fundamental (Cauchy)-complete, in the order-topology, field of characteristic α .

Second axiomatic definition of R_α .

The field of ordinal real numbers R_α is the unique complete (up-to-characteristic) field of characteristic α . These definitions apply even in the case of the field of real numbers ($\alpha = \omega$).

§ 3 The arithmetisation of order-types .

Remark.As it is known the linear segments of elementary Euclidean geometry can be defined as special order-types with Archimedean property and Hilbert completeness through axioms (see e.g. for a not ancient approach the Hilbert axiomatisation in [Hilbert D 1977] ch 1).Then ,they can be proved to be order isomorphic with subsets of the real number field R . This is known as the elementary arithmetisation of the order-types of Euclidean linear segments.

Proposition 10.(the K- fundamental arithmetisation theorem of order-types.)

Every order-type λ is K-arithmetisable with ordinal numbers and has a fundamental density $df(\lambda)$ which is a principal ordinal number .

In the next paper ,after the unification theorem of the transfinite real ,surreal ,ordinal real numbers ,a second arithmetisation theorem shall be proved. Two more universal hierarchies of formal power series fields shall be, also, proved that they are universal embedding hierarchies .We state these results here. For the definition of tree ,height of a tree ,level of a tree, binary tree e.t.c.see [Kuratowski K.-Mostowski A. 1968] ch ii § 1, § 2 . The binary tree of height the ordinal α we denote with D_α . After the previously mentioned unification theorem 17 of the next paper we get that the hierarchy of binary trees is a universal embedding hierarchy for the order- types . Since the binary trees are subsets of linearly ordered fields and their elements consisting exclusively from 1's in the binary sequence, correspond to the ordinal numbers with the Hessenberg operations (see also [Conway J.H. 1976] ch 3 note pp 28 and also [Kyritsis C. 1991Alt] the characterisation theorem) this universal embedding property we call also binary arithmetisation .The least ordinal α such that an order-type λ is order embeddable in the binary tree D_α ,we call the binary density of λ ,and we denote it by $db(\lambda)$.

Theorem 11 (The binary arithmetisation theorem of order-types)

Every order-type λ is binary arithmetisable and has a binary density $db(\lambda)$ which is an ordinal number .

From the previous theorem ,by denoting a level of height α of a binary tree ,by T_α ,and giving

to the Cartesian product $\prod_{\beta < \alpha} T_\beta$ the lexicographical ordering ,we also get the next :

Corollary 12. The formal power series hierarchies $R((D_\alpha)), \mathbb{R}\left(\left(\prod_{\beta < \alpha} T_\beta\right)\right)$, are universal embedding hierarchies for the linearly ordered fields .

§ 4 Some general results on linearly ordered fields .

In this paragraph we give some results generally for the category of linearly ordered fields. To save space we shall not give the proofs, since they do not have serious difficulties,nevertheless we shall indicate how they can be obtained .

infinitesimals of the extension denoted respectively by m_F and $m_{F'}$ are isomorphic as ordered integral domains, then this isomorphism is extendable to an algebraic and order isomorphism of the fields F, F' .

Remark. The proof is direct from the definitions.

Remark. An extension F/k of the linearly ordered field k to F , is transcendental if $\text{Char } F > \text{char } k$ and then the field F is an infinite dimensional vector space over k .

Proposition 17. Let F be a linearly ordered field of characteristic $\text{char}(F) = \omega^{\omega^\alpha}$ where α is a limit ordinal. It holds that the field F in the order topology is totally disconnected.

Remark. The proof uses the existence, for every principal ordinal ω^{ω^α} , of the fields of ordinal real numbers $R^{\omega^{\omega^\alpha}}$.

Theorem 18 The classes of transfinite real numbers CR , and of ordinal real numbers $\Omega_1 R$, coincide.

Proof. Since both Hierarchies of transfinite real and ordinal real numbers have the universal embedding property (see remark 9), every transfinite real number-field is contained in some ordinal real number-field and every ordinal real-number field in some transfinite real number-field. Thus $CR \subseteq \Omega_1 R$ and $\Omega_1 R \subseteq CR$, and $CR = \Omega_1 R$. Q.E.D.

§ 5 The A-Archimedeanity

The, at least two different, definitions of archimedeanity, that can be found for instance in [Glazal A. 1937] and in other authors as in [Conway J.H. 1976] or [Arin E. Schreier O. 1927] give us the opportunity to treat them in unified way through the concept of archimedeanity relative to a monoid.

The fact that the linearly ordered field F has characteristic ω (the least infinite ordinal) is equivalent with the statement that the field F is Archimedean according to any (classical) known definition.

Let us denote by G a linearly ordered group and by A a monoid of endomorphisms of G as a group.

It is said that x is A-Archimedean to y where $x, y \in G$ iff there are $a, b \in A$ with $a(x) \geq y$ and $b(y) \geq x$. If A is the domain Z of integers (the endomorphisms are multiplication with an integer) we simply say that x is Archimedean to y . If for every pair x, y of elements of G holds that x is A-Archimedean to y , it is said that G is A-Archimedean.

Let F be a linearly ordered field. If we consider it as an additive group, and we denote by A_1 a monoid of endomorphisms of the additive group, we get the concept of x being A-additively Archimedean to y . If we consider the multiplicative group F^* and we take a monoid, denoted by A_2 , of endomorphisms of the multiplicative group, we get the concept of x being A-multiplicatively Archimedean to y .

Let $A = A_1 \vee A_2$ be the monoid of mappings from F to F generated by the previous monoids. It is said that x is A-field-Archimedean to y iff there are $a, b \in A$ such that $a(x) \geq y$, $b(y) \geq x$.

In any extension F/k of a field K by a field F , where F, k are fields of ordinal characteristic with $\text{char } F > \text{Char } K$, if we take as A_1 , to be the multiplication with elements from the field K (considering the field F as a linear space over K), we get the concept of x

being K-additively Archimedean to y . (For $K=\mathbb{R}$ this is also known as "x is commensurate to y " see [Conway J.H. 1976] ch 3 pp 31).

If A_1 is the multiplication with integers and A_2 is power with integral exponents ,then it is simply said that x is field Archimedean to y (Known also from the A. Gleyzal's definition of Archimedeanity)

A non-Archimedean linearly ordered field denoted by F is simply a linearly ordered field for which not all pairs (x,y) of its elements are mutually additively Archimedean. (Thus $\text{char}F > \omega$) But it can be very well A-additively Archimedean for other monoids A . In particular if $\text{char}F = \alpha$ and A is the monoid of endomorphisms of the additive group of F defined by (field) multiplication with ordinals less than α , then it is A-additively Archimedean and we denote it by writing that it is α -additively Archimedean

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List of special symbols

α, β, ω : Small Greek letters

Ω_1 : Capital Greek letter omega with the subscript 1

F^a : Capital letter F with superscript a.

N : Capital Aleph ,the first letter of the hebrew alphabet . In the text is used a capital script. letter n .

\oplus, \circ : cross in a circle, point in a circle .

$N\alpha, Z\alpha, Q\alpha, R\alpha$: Roman capital letters with subscript small Greek letters

$C\alpha, H\alpha$

$^*X, ^*R$ et.c : Capital standard or roman letters with left superscript a star.

CN, CZ, CQ :Capital standard letter c followed by capital letters

C^*R , with possibly a left superscript a star

\hat{X} : Capital tstandard letter with a cap.

Σ : Capital Greek letter sigma

$\overset{\circ}{D}_\alpha$: Capital standard D with subscript a small Greek letter and in upper place a small zero.

Ordinal real numbers 2.The arithmetisation of order types .

by

Dr.Konstantin E.Kyritsis

Mathematics Dept.

University of the Aegean

Karlovassi 83200

Samos Greece

Abstract

In this paper the main results are :Proofs that the ordinal real numbers are real closed fields and complete up-to-characteristic .They are also Dedekind ,and Archemidean complete fields .They are real formal power series fields and Pythagorean complete fields It is proved and discussed the K-fundamental arithmetisationand the binary arithmetisation of the order types .

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Grothendick group

Archemidean complete fields

linearly ordered commutative fields

full binary trees

