



HAL
open science

Mean-field type Quadratic BSDEs

Hélène Hibon, Ying Hu, Shanjian Tang

► **To cite this version:**

Hélène Hibon, Ying Hu, Shanjian Tang. Mean-field type Quadratic BSDEs. Numerical Algebra, Control and Optimization, 2023, 13 (3-4), pp.392-412. 10.3934/naco.2022009 . hal-01512453v2

HAL Id: hal-01512453

<https://hal.science/hal-01512453v2>

Submitted on 27 Aug 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Mean-field type Quadratic BSDEs

Hélène Hibon* Ying Hu[†] Shanjian Tang[‡]

August 27, 2017

Abstract

In this paper, we give several new results on solvability of a quadratic BSDE whose generator depends also on the mean of both variables. First, we consider such a BSDE using John-Nirenberg's inequality for BMO martingales to estimate its contribution to the evolution of the first unknown variable. Then we consider the BSDE having an additive expected value of a quadratic generator in addition to the usual quadratic one. In this case, we use a deterministic shift transformation to the first unknown variable, when the usual quadratic generator depends neither on the first variable nor its mean, the general case can be treated by a fixed point argument.

1 Introduction

Let $\{W_t := (W_t^1, \dots, W_t^d)^*, 0 \leq t \leq T\}$ be a d -dimensional standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\{\mathcal{F}_t, 0 \leq t \leq T\}$ the augmented natural filtration of the standard Brownian motion W .

In this paper, we study the existence and uniqueness of an adapted solution of the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (1.1)$$

When f does not depend on (\bar{y}, \bar{z}) , BSDE (1.1) is the classical one, and it is extensively studied in the literature, see the pioneer work of Bismut [1, 2] as well as Pardoux and Peng [16]. When $f(t, y, \bar{y}, z, \bar{z})$ is scalar valued and quadratic in z while it does not depend on (\bar{y}, \bar{z}) , BSDE (1.1) is the so-called quadratic BSDE and has been studied

*IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France

[†]IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (ying.hu@univ-rennes1.fr) and School of Mathematical Sciences, Fudan University, Shanghai 200433, China. Partially supported by Lebesgue center of mathematics "Investissements d'avenir" program - ANR-11-LABX-0020-01, by ANR CAESARS - ANR-15-CE05-0024 and by ANR MFG - ANR-16-CE40-0015-01.

[‡]Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China (e-mail: sjtang@fudan.edu.cn). Partially supported by National Science Foundation of China (Grant No. 11631004) and Science and Technology Commission of Shanghai Municipality (Grant No. 14XD1400400).

by Kobylanski [13], Briand and Hu [4, 5]. BSDE (1.1) (called mean-field type BSDE) arises naturally when studying mean-field games, etc. We refer to [6] for the motivation of its study. When the generator f is uniformly Lipschitz in the last four arguments, BSDE (1.1) is shown in a straightforward manner to have a unique adapted solution, and the reader is referred to Buckdahn et al. [6] for more details. For the general generator $f(s, y, \bar{y}, z, \bar{z})$ depending quadratically on z , BSDE (1.1) is a quadratic one involving both $\mathbb{E}[Y]$ and $\mathbb{E}[Z]$. The comparison principle (see [13]) is well known to play a crucial role in the study of quadratic BSDEs (see [13]). Unfortunately, the comparison principle fails to hold for BSDE (1.1) (see, e.g. [6] for a counter-example for comparison with Lipschitz generators), the derivation of its solvability is not straightforward. Up to our best knowledge, no study on quadratic mean-field type BSDEs is available. To tackle the difficulty of lack of comparison principle, we use the John-Nirenberg inequality for BMO martingales to address the solvability.

Furthermore, we study the following alternative of mean-field type BSDE, which admits a quadratic growth in the mean of the second unknown variable $\mathbb{E}[Z]$:

$$Y_t = \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])] ds - \int_t^T Z_s \cdot dW_s, \quad (1.2)$$

where f_2 is allowed to grow quadratically in both Z and $\mathbb{E}[Z]$, the function f_1 also admits a quadratic growth in the second unknown variable for the scalar case.

To deal with the additive expected value of f_2 , Cheridito and Nam [8] introduced Krasnoselskii fixed point theorem to conclude the existence and uniqueness, by observing that the range of the expected value of f_2 is (locally) compact. Here we observe the following fact: the expected value of f_2 has no contribution to the second unknown variable Z if f_1 depends neither on the first variable Z nor on its mean. Hence we use the shift transformation to remove the expectation of f_2 . In the general case, we apply the same kind of technique and the contraction mapping principle.

Let us close this section by introducing some notations. Denote by $\mathcal{S}^\infty(\mathbb{R}^n)$ the totality of \mathbb{R}^n -valued \mathcal{F}_t -adapted essentially bounded continuous processes, and by $\|Y\|_\infty$ the essential supremum norm of $Y \in \mathcal{S}^\infty(\mathbb{R}^n)$. It can be verified that $(\mathcal{S}^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space. Let $M = (M_t, \mathcal{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$, and for $p \in [1, \infty)$ we set

$$\|M\|_{BMO_p(\mathbb{P})} := \sup_{\tau} \left| \mathbb{E}_{\tau} \left[\left(\langle M \rangle_{\tau}^{\infty} \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \right|_{\infty} \quad (1.3)$$

where the supremum is taken over all stopping times τ . The class $\{M : \|M\|_{BMO_p} < \infty\}$ is denoted by BMO_p , which is written as $BMO_p(\mathbb{P})$ whenever it is necessary to indicate the underlying probability, and observe that $\|\cdot\|_{BMO_p}$ is a norm on this space and $BMO_p(\mathbb{P})$ is a Banach space.

Denote by $\mathcal{E}(M)$ the stochastic exponential of a one-dimensional local martingale M and by $\mathcal{E}(M)_s^t$ that of $M_t - M_s$. Denote by $\beta \cdot M$ the stochastic integral of a scalar-valued adapted process β with respect to a local continuous martingale M .

For any real $p \geq 1$, $\mathcal{S}^p(\mathbb{R}^n)$ denotes the set of \mathbb{R}^n -valued adapted and càdlàg processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{\mathcal{S}^p(\mathbb{R}^n)} := \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right]^{1/p} < +\infty,$$

and $\mathcal{M}^p(\mathbb{R}^{d \times n})$ denotes the set of adapted processes $(Z_t)_{t \in [0, T]}$ with values in $\mathbb{R}^{d \times n}$ such that

$$\|Z\|_{\mathcal{M}^p(\mathbb{R}^{d \times n})} := \mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{p/2} \right]^{1/p} < +\infty.$$

The rest of our paper is organized as follows. In Section 2, we study BSDE (1.1) when $f(t, y, \bar{y}, z, \bar{z})$ is scalar valued and quadratic in z , and uniformly Lipschitz in (y, \bar{y}, \bar{z}) , and prove by the contraction mapping principle that BSDE (1.1) has a unique solution. In Section 3, we study scalar-valued BSDE (1.2) when $f_2(t, y, \bar{y}, z, \bar{z})$ is both quadratic in z and \bar{z} , and f_1 is quadratic in z . Finally, in Section 4, we study BSDE (1.2) in the multi-dimensional case, where we suppose that $f_2(t, y, \bar{y}, z, \bar{z})$ is both quadratic in z and \bar{z} , and f_1 is Lipschitz in z and \bar{z} .

2 Quadratic BSDEs with a mean term involving the second unknown variable

In this section we consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (2.1)$$

We first recall the following existence and uniqueness, a priori estimate for one-dimensional BSDEs.

Lemma 2.1. *Assume that (i) the function $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ has the following growth and locally Lipschitz continuity in the last two variables:*

$$\begin{aligned} |f(s, y, z)| &\leq g_s + \beta|y| + \frac{\gamma}{2}|z|^2, \quad y \in \mathbb{R}, z \in \mathbb{R}^d; \\ |f(s, y, z_1) - f(s, y, z_2)| &\leq C|y_1 - y_2| + C(1 + |z_1| + |z_2|)|z_1 - z_2|, \quad y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d; \end{aligned} \quad (2.2)$$

(ii) the process $f(\cdot, y, z)$ is \mathcal{F}_t -adapted for each $y \in \mathbb{R}, z \in \mathbb{R}^d$; and (iii) $g \in L^1(0, T)$. Then for bounded ξ , the following BSDE

$$Y_t = \xi + \int_t^T [f(s, Y_s, Z_s)] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (2.3)$$

has a unique solution (Y, Z) such that Y is (essentially) bounded and $Z \cdot W$ is a BMO martingale. Furthermore, we have

$$e^{\gamma|Y_t|} \leq \mathbb{E}_t \left[e^{\gamma e^{\beta(T-t)} \xi + \gamma \int_t^T |g(s)| e^{\beta(s-t)} ds} \right].$$

The following lemma plays an important role in our subsequent arguments. It indicates that following the proof of [12, Theorem 3.3, page 57] can give a more precise dependence of the two constants c_1, c_2 on $\beta \cdot M$.

Lemma 2.2. *For $K > 0$, there are constants $c_1 > 0$ and $c_2 > 0$ depending only on K such that for any BMO martingale M , we have for any one-dimensional BMO martingale N such that $\|N\|_{BMO_2(\mathbb{P})} \leq K$,*

$$c_1 \|M\|_{BMO_2(\mathbb{P})} \leq \|\widetilde{M}\|_{BMO_2(\widetilde{\mathbb{P}})} \leq c_2 \|M\|_{BMO_2(\mathbb{P})} \quad (2.4)$$

where $\widetilde{M} := M - \langle M, N \rangle$ and $d\widetilde{\mathbb{P}} := \mathcal{E}(N)_0^\infty d\mathbb{P}$.

2.1 Main Results

We make the following three assumptions. Let C be a positive constant.

(A1) Assume that there are positive constants C and γ and $\alpha \in [0, 1)$ such that the function $f : \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ has the following linear-quadratic growth and globally-locally Lipschitz continuity: for $\forall(\omega, s, y_i, \bar{y}_i, z_i, \bar{z}_i) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$ with $i = 1, 2$,

$$\begin{aligned} |f(\omega, s, y, \bar{y}, z, \bar{z})| &\leq C(2 + |y| + |\bar{y}| + |\bar{z}|^{1+\alpha}) + \frac{1}{2}\gamma|z|^2; \\ |f(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - f(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| &\leq C \left\{ |y_1 - y_2| + |\bar{y}_1 - \bar{y}_2| + (1 + |\bar{z}_1|^\alpha + |\bar{z}_2|^\alpha)|\bar{z}_1 - \bar{z}_2| \right. \\ &\quad \left. + (1 + |z_1| + |z_2|)|z_1 - z_2| \right\}; \end{aligned} \quad (2.5)$$

The process $f(\cdot, y, \bar{y}, z, \bar{z})$ is \mathcal{F}_t -adapted for each (y, \bar{y}, z, \bar{z}) .

(A2) The terminal condition ξ is uniformly bounded by C .

We have the following two theorems.

The first one is a result concerning local solutions. For this, let us introduce some notations. For $\varepsilon > 0$, and $r_\varepsilon > 0$, we define the ball \mathbb{B}_ε by

$$\mathbb{B}_\varepsilon := \left\{ (Y, Z) : \quad Y \in \mathcal{S}^\infty, \quad Z \cdot W \in BMO_2(\mathbb{P}), \right. \\ \left. \|Y\|_{\infty, [T-\varepsilon, T]} + \|Z \cdot W\|_{BMO_2, [T-\varepsilon, T]} \leq r_\varepsilon \right\}.$$

Theorem 2.3. *Let assumptions (A1) and (A2) be satisfied with $\alpha \in [0, 1)$. Then, for any bounded ξ , there exist $\varepsilon > 0$ and $r_\varepsilon > 0$ such that the following BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (2.6)$$

has a unique local solution (Y, Z) in the time interval $[T - \varepsilon, T]$ with $(Y, Z) \in \mathbb{B}_\varepsilon$.

Example 2.4. *The condition on f means that f is of linear growth with respect to (y, \bar{y}) , and of $|\bar{z}|^{1+\alpha}$ growth. For example, For $\alpha \in (0, 1)$,*

$$f(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |\bar{y}| + \frac{1}{2}|z|^2 + |\bar{z}|^{1+\alpha}.$$

The second theorem is a result about global solutions.

Theorem 2.5. *Let assumption (A2) be satisfied. Moreover, assume that there is a positive constant C such that the function $f : \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ has the following linear-quadratic growth and globally-locally Lipschitz continuity: for $\forall(\omega, s, y_i, \bar{y}_i, z_i, \bar{z}_i) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$ with $i = 1, 2$,*

$$\begin{aligned} |f(s, 0, 0, 0, 0) + h(\omega, s, y, \bar{y}, z, \bar{z})| &\leq C(1 + |y| + |\bar{y}|), \\ |f(\omega, s, 0, 0, z_1, 0) - f(\omega, s, 0, 0, z_2, 0)| &\leq C(1 + |z_1| + |z_2|)|z_1 - z_2|, \\ |h(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - h(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| &\leq C(|y_1 - y_2| + |\bar{y}_1 - \bar{y}_2| + |z_1 - z_2| + |\bar{z}_1 - \bar{z}_2|) \end{aligned} \quad (2.7)$$

where for $(\omega, s, y, \bar{y}, z, \bar{z}) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$,

$$h(s, y, \bar{y}, z, \bar{z}) := f(s, y, \bar{y}, z, \bar{z}) - f(s, 0, 0, z, 0). \quad (2.8)$$

The process $f(\cdot, y, \bar{y}, z, \bar{z})$ is \mathcal{F}_t -adapted for each $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^2 \times (\mathbb{R}^d)^2$.

Then, the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (2.9)$$

has a unique adapted solution (Y, Z) on $[0, T]$ such that Y is bounded. Furthermore, $Z \cdot W$ is a $BMO(\mathbb{P})$ martingale.

Example 2.6. *The inequality (2.7) requires that f is bounded with respect to the last variable \bar{z} . The following function*

$$f(s, y, \bar{y}, z, \bar{z}) = 1 + s + |y| + |\bar{y}| + \frac{1}{2}|z|^2 + |\sin(\bar{z})|, \quad (s, y, \bar{y}, z, \bar{z}) \in [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2,$$

satisfies such an inequality.

2.2 Local solution: the proof of Theorem 2.3

We prove Theorem 2.3 (using the contraction mapping principle) in the following three subsections: in Subsection 2.2.1, we construct a map (which we call quadratic solution map) in a Banach space; in Subsection 2.2.2, we show that this map is stable in a small ball; and in Subsection 2.2.3, we prove that this map is a contraction.

2.2.1 Construction of the map

For a pair of bounded adapted process U and BMO martingale $V \cdot W$, we consider the following quadratic BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (2.10)$$

As

$$|f(s, y, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s])| \leq C(2 + |\mathbb{E}[U_s]| + |\mathbb{E}[V_s]|^{1+\alpha}) + C|y| + \frac{1}{2}\gamma|z|^2,$$

in view of Lemma 2.1, it has a unique adapted solution (Y, Z) such that Y is bounded and $Z \cdot W$ is a BMO martingale. Define the quadratic solution map $\Gamma : (U, V) \mapsto \Gamma(U, V)$ as follows:

$$\Gamma(U, V) := (Y, Z), \quad \forall (U, V \cdot W) \in \mathcal{S}^\infty \times BMO_2(\mathbb{P}).$$

It is a transformation in the Banach space $\mathcal{S}^\infty \times BMO_2(\mathbb{P})$.

Let us introduce here some constants and a quadratic (algebraic) equation which will be used in the next two subsections.

Define

$$C_\delta := e^{\frac{6}{1-\alpha}\gamma CT e^{CT} + \frac{1-\alpha}{2}(\frac{3}{1-\alpha}\gamma C e^{CT})^{\frac{2}{1-\alpha}}(\frac{1+\alpha}{2\delta})^{\frac{1+\alpha}{1-\alpha}}T}, \quad (2.11)$$

$$\beta := \frac{1}{2}(1-\alpha)C^{\frac{2}{1-\alpha}}(2(1+\alpha))^{\frac{1+\alpha}{1-\alpha}}; \quad (2.12)$$

$$\begin{aligned} \mu_1 &:= (1-\alpha) \left(1 + \frac{1-\alpha}{(1+\alpha)\gamma} \right) = 1-\alpha + \frac{(1-\alpha)^2}{(1+\alpha)\gamma}; \\ \mu_2 &:= \frac{1}{2}(1+\alpha) \left(1 + \frac{1-\alpha}{(1+\alpha)\gamma} \right) = \frac{1}{2}(1+\alpha) + \frac{1-\alpha}{2\gamma}; \end{aligned} \quad (2.13)$$

$$\mu := (\beta + C\mu_1)\gamma^{\frac{2}{\alpha-1}} + 2C\mu_2. \quad (2.14)$$

Consider the following standard quadratic equation of A :

$$\delta A^2 - (1 + 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta)A + 4\gamma^{-2}e^{\gamma|\xi|_\infty} + 4\mu C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}\varepsilon = 0.$$

The discriminant of the quadratic equation reads

$$\begin{aligned} \Delta &:= (1 + 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta)^2 - 4\delta \left[4\gamma^{-2}e^{\gamma|\xi|_\infty} + 4\mu C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}\varepsilon \right] \\ &= (1 - 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta)^2 - 16\mu\delta C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}\varepsilon. \end{aligned} \quad (2.15)$$

Take

$$\begin{aligned} \delta &:= \frac{1}{8}\gamma^2 e^{-\gamma|\xi|_\infty}, \quad \varepsilon \leq \min \left\{ \frac{1}{3C}e^{-CT}, \frac{1}{8\mu C_\delta}\gamma^{-2}e^{\gamma(1-\frac{3e^{CT}}{1-\alpha})|\xi|_\infty} \right\}, \\ A &:= \frac{1 + 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta - \sqrt{\Delta}}{2\delta} = \frac{3 - 2\sqrt{\Delta}}{4\delta} \leq \frac{3}{4\delta} = 6\gamma^{-2}e^{\gamma|\xi|_\infty}, \end{aligned} \quad (2.16)$$

and we have

$$\begin{aligned} \Delta \geq 0, \quad 1 - \delta A = \frac{1 + 2\sqrt{\Delta}}{4} > 0, \\ \gamma^{-2} e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3\gamma e^{CT}}{1-\alpha}|\xi|_\infty}}{1 - \delta A} \varepsilon + \frac{1}{4} A = \frac{1}{2} A. \end{aligned} \quad (2.17)$$

Throughout this section, we base our discussion on the time interval $[T - \varepsilon, T]$.

We shall prove Theorem 2.3 by showing that the quadratic solution map Γ is a contraction on the closed convex set \mathcal{B}_ε defined by

$$\mathcal{B}_\varepsilon := \left\{ (U, V) : \begin{aligned} &U \in \mathcal{S}^\infty, \quad V \cdot W \in BMO_2(\mathbb{P}), \\ &\|V \cdot W\|_{BMO_2}^2 \leq A, \quad e^{\frac{2}{1-\alpha}\gamma|U|_\infty} \leq \frac{C_\delta e^{\frac{3\gamma e^{CT}}{1-\alpha}|\xi|_\infty}}{1 - \delta A} \end{aligned} \right\} \quad (2.18)$$

(where (U, V) is defined on $\Omega \times [T - \varepsilon, T]$) for a positive constant ε (to be determined later).

2.2.2 Estimation of the quadratic solution map

We shall show the following assertion: $\Gamma(\mathcal{B}_\varepsilon) \subset \mathcal{B}_\varepsilon$, that is,

$$\Gamma(U, V) \in \mathcal{B}_\varepsilon, \quad \forall (U, V) \in \mathcal{B}_\varepsilon. \quad (2.19)$$

Step 1. Exponential transformation.

Define

$$\phi(y) := \gamma^{-2} [\exp(\gamma|y|) - \gamma|y| - 1], \quad y \in \mathbb{R}. \quad (2.20)$$

Then, we have for $y \in \mathbb{R}$,

$$\phi'(y) = \gamma^{-1} [\exp(\gamma|y|) - 1] \operatorname{sgn}(y), \quad \phi''(y) = \exp(\gamma|y|), \quad \phi''(y) - \gamma|\phi'(y)| = 1. \quad (2.21)$$

Using Itô's formula, we have for $t \in [T - \varepsilon, T]$,

$$\begin{aligned} &\phi(Y_t) + \frac{1}{2} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] \\ &\leq \phi(|\xi|_\infty) + C \mathbb{E}_t \left[\int_t^T |\phi'(Y_s)| (2 + |Y_s| + |\mathbb{E}[U_s]| + |\mathbb{E}[V_s]|^{1+\alpha}) ds \right]. \end{aligned} \quad (2.22)$$

Since (in view of the definition of notation β in (2.12))

$$C|\phi'(Y_s)| |\mathbb{E}[V_s]|^{1+\alpha} \leq \frac{1}{4} |\mathbb{E}[V_s]|^2 + \beta |\phi'(Y_s)|^{\frac{2}{1-\alpha}},$$

we have

$$\begin{aligned}
& \phi(Y_t) + \frac{1}{2} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] \\
& \leq \phi(|\xi|_\infty) + C \mathbb{E}_t \left[\int_t^T |\phi'(Y_s)| (2 + |Y_s| + |\mathbb{E}[U_s]|) ds \right] \\
& \quad + \beta \mathbb{E}_t \left[\int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] + \frac{1}{4} E_t \left[\int_t^T |E[V_s]|^2 ds \right] \\
& \leq \phi(|\xi|_\infty) + C E_t \left[\int_t^T |\phi'(Y_s)| ((1 + |Y_s|) + (1 + |\mathbb{E}[U_s]|)) ds \right] \\
& \quad + \beta \mathbb{E}_t \left[\int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] + \frac{1}{4} \mathbb{E}_t \left[\int_t^T |E[V_s]|^2 ds \right].
\end{aligned} \tag{2.23}$$

In view of the inequality for $x > 0$,

$$1 + x \leq \left(1 + \frac{1 - \alpha}{\gamma(1 + \alpha)} \right) e^{\frac{\gamma(1+\alpha)}{1-\alpha}x},$$

we have

$$\begin{aligned}
& C \mathbb{E}_t \left[\int_t^T |\phi'(Y_s)| ((1 + |Y_s|) + (1 + |\mathbb{E}[U_s]|)) ds \right] \\
& \leq C \mathbb{E}_t \left[\int_t^T |\phi'(Y_s)| \left(1 + \frac{1 - \alpha}{\gamma(1 + \alpha)} \right) \left(e^{\gamma \frac{1+\alpha}{1-\alpha} |Y_s|} + e^{\gamma \frac{1+\alpha}{1-\alpha} |\mathbb{E}[U_s]|} \right) ds \right].
\end{aligned} \tag{2.24}$$

Since (by Young's inequality)

$$|\phi'(Y_s)| \left(e^{\gamma \frac{1+\alpha}{1-\alpha} |Y_s|} + e^{\gamma \frac{1+\alpha}{1-\alpha} |\mathbb{E}[U_s]|} \right) \leq (1 - \alpha) |\phi'(Y_s)|^{\frac{2}{1-\alpha}} + \frac{1 + \alpha}{2} \left(e^{\frac{2}{1-\alpha} \gamma |Y_s|} + e^{\frac{2}{1-\alpha} \gamma |\mathbb{E}[U_s]|} \right),$$

in view of the definition of the notations μ_1 and μ_2 in (2.13), we have

$$\begin{aligned}
& C E_t \left[\int_t^T |\phi'(Y_s)| ((1 + Y_s) + (1 + |E[U_s]|)) ds \right] \\
& \leq C \mu_1 E_t \left[\int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] + C \mu_2 E_t \left[\int_t^T \left(e^{\frac{2\gamma}{1-\alpha} |Y_s|} + e^{\frac{2\gamma}{1-\alpha} |E[U_s]|} \right) ds \right].
\end{aligned} \tag{2.25}$$

In view of inequality (2.23), we have

$$\begin{aligned}
& \phi(Y_t) + \frac{1}{2} E_t \left[\int_t^T |Z_s|^2 ds \right] \\
& \leq \phi(|\xi|_\infty) + (\beta + C \mu_1) E_t \left[\int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] \\
& \quad + C \mu_2 E_t \left[\int_t^T \left(e^{\frac{2\gamma}{1-\alpha} |Y_s|} + e^{\frac{2\gamma}{1-\alpha} |E[U_s]|} \right) ds \right] + \frac{1}{4} \int_t^T |E[V_s]|^2 ds \\
& \leq \phi(|\xi|_\infty) + \left[C \mu_2 + \gamma^{\frac{2}{\alpha-1}} (\beta + C \mu_1) \right] E_t \left[\int_t^T e^{\frac{2\gamma}{1-\alpha} |Y_s|} ds \right] \\
& \quad + C \mu_2 E_t \left[\int_t^T e^{\frac{2\gamma}{1-\alpha} |U_s|} ds \right] + \frac{1}{4} E \left[\int_t^T |V_s|^2 ds \right].
\end{aligned} \tag{2.26}$$

Step 2. Estimate of $e^{\gamma|Y|_\infty}$.

In view of the last inequality of Lemma 2.1, we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq E_t \left[e^{\frac{3}{1-\alpha}\gamma e^{CT}(|\xi|_\infty + C \int_t^T (2 + |E[U_s]| + |E[V_s]|^{1+\alpha}) ds)} \right]. \quad (2.27)$$

Since (by Young's inequality)

$$\frac{3e^{CT}}{1-\alpha} \gamma C |E[V_s]|^{1+\alpha} \leq \frac{1-\alpha}{2} \left(\frac{3e^{CT}}{1-\alpha} \gamma C \left(\frac{1+\alpha}{2\delta} \right)^{\frac{1+\alpha}{2}} \right)^{\frac{2}{1-\alpha}} + \delta |E[V_s]|^2, \quad (2.28)$$

in view of the definition of notation C_δ in (2.11), we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq C_\delta \left[e^{\left(\frac{3}{1-\alpha}\gamma e^{CT} |\xi|_\infty + \frac{3}{1-\alpha}\gamma C e^{CT} \varepsilon |U|_\infty + \delta \int_t^T |E[V_s]|^2 ds \right)} \right]; \quad (2.29)$$

and therefore using Jensen's inequality,

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq C_\delta e^{\left(\frac{3}{1-\alpha}\gamma e^{CT} (|\xi|_\infty + C\varepsilon|U|_\infty) \right)} E \left[e^{\delta \int_t^T |V_s|^2 ds} \right]. \quad (2.30)$$

It follows from (2.16) and the definition of \mathcal{B}_ε that $\|\sqrt{\delta}V \cdot W\|_{BMO_2(\mathbb{P})}^2 \leq \delta A < 1$, applying the John-Nirenberg inequality to the BMO martingale $\sqrt{\delta}V \cdot W$, we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq \frac{C_\delta e^{\left(\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty + \frac{3e^{CT}}{1-\alpha}C\gamma\varepsilon|U|_\infty \right)}}{1 - \delta\|V \cdot W\|_{BMO_2}^2} \leq \frac{C_\delta e^{\left(\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty + \frac{3e^{CT}}{1-\alpha}C\gamma\varepsilon|U|_\infty \right)}}{1 - \delta A}. \quad (2.31)$$

Since $3e^{CT}C\varepsilon \leq 1$ (see the choice of ε in (2.16)) and $(U, V) \in \mathcal{B}_\varepsilon$, we have

$$\begin{aligned} e^{\left(\frac{3}{1-\alpha}\gamma|Y|_\infty \right)} &\leq \frac{C_\delta e^{\left(\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty + \frac{1}{1-\alpha}\gamma|U|_\infty \right)}}{1 - \delta A} \\ &\leq C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \left(C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \right)^{\frac{1}{2}} \\ &\leq \left(\frac{C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \right)^{\frac{3}{2}}, \end{aligned} \quad (2.32)$$

which gives a half of the desired result (2.19).

Step 3. Estimate of $\|Z \cdot W\|_{BMO_2}^2$.

From inequality (2.26) and the definition of notation μ in (2.14), we have

$$\frac{1}{2} E_t \left[\int_t^T |Z_s|^2 ds \right] \leq \gamma^{-2} e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \varepsilon + \frac{1}{4} A. \quad (2.33)$$

In view of (2.17), we have

$$\frac{1}{2} \|Z \cdot W\|_{BMO_2}^2 \leq \gamma^{-2} e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \varepsilon + \frac{1}{4} A = \frac{1}{2} A. \quad (2.34)$$

The other half of the desired result (2.19) is then proved.

2.2.3 Contraction of the quadratic solution map

For $(U, V) \in \mathcal{B}_\varepsilon$ and $(\tilde{U}, \tilde{V}) \in \mathcal{B}_\varepsilon$, set

$$(Y, Z) := \Gamma(U, V), \quad (\tilde{Y}, \tilde{Z}) := \Gamma(\tilde{U}, \tilde{V}).$$

That is,

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, E[U_s], Z_s, E[V_s]) ds - \int_t^T Z_s dW_s, \\ \tilde{Y}_t &= \xi + \int_t^T f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) ds - \int_t^T \tilde{Z}_s dW_s. \end{aligned} \quad (2.35)$$

We can define the vector process β in an obvious way such that

$$\begin{aligned} |\beta_s| &\leq C(1 + |Z_s| + |\tilde{Z}_s|), \\ f(s, Y_s, E[U_s], Z_s, E[V_s]) - f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) &= (Z_s - \tilde{Z}_s)\beta_s. \end{aligned} \quad (2.36)$$

Then $\tilde{W}_t := W_t - \int_0^t \beta_s ds$ is a Brownian motion under the equivalent probability measure \tilde{P} defined by

$$d\tilde{P} := \mathcal{E}(\beta \cdot W)_0^T dP,$$

and from the above-established a priori estimate, there is $K > 0$ such that $\|\beta \cdot W\|_{BMO_2}^2 \leq K^2 := 3C^2T + 6C^2A$.

In view of the following equation

$$\begin{aligned} Y_t - \tilde{Y}_t + \int_t^T (Z_s - \tilde{Z}_s) d\tilde{W}_s \\ = \int_t^T \left[f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) - f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \right] ds, \end{aligned} \quad (2.37)$$

taking square and then the conditional expectation with respect to \tilde{P} (denoted by \tilde{E}_t) on both sides of the last equation, we have the following standard estimates:

$$\begin{aligned} &|Y_t - \tilde{Y}_t|^2 + \tilde{E}_t \left[\int_t^T |Z_s - \tilde{Z}_s|^2 ds \right] \\ &= \tilde{E}_t \left[\left(\int_t^T \left(f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) - f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \right) ds \right)^2 \right] \\ &\leq C^2 \tilde{E}_t \left[\left(\int_t^T \left(|Y_s - \tilde{Y}_s| + |E[U_s - \tilde{U}_s]| + (1 + |E[V_s]|^\alpha + |E[\tilde{V}_s]|^\alpha) |E[V_s - \tilde{V}_s]| \right) ds \right)^2 \right] \\ &\leq 3C^2(T-t)^2 (|U - \tilde{U}|_\infty^2 + |Y - \tilde{Y}|_\infty^2) \\ &\quad + 3C^2 \int_t^T (1 + |E[V_s]|^\alpha + |E[\tilde{V}_s]|^\alpha)^2 ds \int_t^T |E[V_s - \tilde{V}_s]|^2 ds \\ &\leq 3C^2(T-t)^2 (|U - \tilde{U}|_\infty^2 + |Y - \tilde{Y}|_\infty^2) \\ &\quad + 9C^2 \int_t^T (1 + |E[V_s]|^{2\alpha} + |E[\tilde{V}_s]|^{2\alpha}) ds \int_t^T |E[V_s - \tilde{V}_s]|^2 ds \end{aligned} \quad (2.38)$$

We have for $t \in [T - \varepsilon, T]$,

$$\begin{aligned}
& \int_t^T (1 + |E[V_s]|^{2\alpha} + |E[\tilde{V}_s]|^{2\alpha}) ds \\
& \leq \int_t^T (1 + E[|V_s|^2]^\alpha + E[|\tilde{V}_s|^2]^\alpha) ds \\
& \leq \varepsilon + \varepsilon^{1-\alpha} E \left[\int_t^T |V_s|^2 ds \right]^\alpha + \varepsilon^{1-\alpha} E \left[\int_t^T |\tilde{V}_s|^2 ds \right]^\alpha \\
& \leq \varepsilon^{1-\alpha} \left(T^\alpha + 2 + \alpha E \int_t^T |V_s|^2 ds + \alpha E \int_t^T |\tilde{V}_s|^2 ds \right) \\
& \leq \varepsilon^{1-\alpha} \left(T^\alpha + 2 + \alpha \|V \cdot W\|_{BMO_2(\mathbb{P})}^2 + \alpha \|\tilde{V} \cdot W\|_{BMO_2(\mathbb{P})}^2 \right) \\
& \leq \varepsilon^{1-\alpha} (T^\alpha + 2 + 2\alpha A).
\end{aligned} \tag{2.39}$$

Concluding the above estimates, we have for $t \in [T - \varepsilon, T]$,

$$\begin{aligned}
& |Y_t - \tilde{Y}_t|^2 + \tilde{E}_t \left[\int_t^T |Z_s - \tilde{Z}_s|^2 ds \right] \\
& \leq 3C^2 \varepsilon^2 (|U - \tilde{U}|_\infty^2 + |Y - \tilde{Y}|_\infty^2) \\
& \quad + 9C^2 (T^\alpha + 2 + 2\alpha A) \varepsilon^{1-\alpha} \|(V - \tilde{V}) \cdot W\|_{BMO_2(\mathbb{P})}^2.
\end{aligned} \tag{2.40}$$

In view of estimates (2.4), noting that $1 - 3C^2 \varepsilon^2 \geq \frac{2}{3}$, we have for $t \in [T - \varepsilon, T]$,

$$\begin{aligned}
& |Y - \tilde{Y}|_\infty^2 + 3c_1^2 \|(Z - \tilde{Z}) \cdot W\|_{BMO_2(\mathbb{P})}^2 \\
& \leq 9C^2 \varepsilon^2 |U - \tilde{U}|_\infty^2 + 27C^2 (T^\alpha + 2 + 2\alpha A) \varepsilon^{1-\alpha} \|(V - \tilde{V}) \cdot W\|_{BMO_2(\mathbb{P})}^2.
\end{aligned} \tag{2.41}$$

It is then standard to show that there is a very small positive number ε such that the quadratic solution map Γ is a contraction on the previously given set \mathcal{B}_ε , by noting that $A \leq 6\gamma^{-2} e^{\gamma|\xi|_\infty}$ from (2.16). The proof is completed by choosing a sufficiently small $r_\varepsilon > 0$ such that $\mathbb{B}_\varepsilon \subset \mathcal{B}_\varepsilon$.

2.3 Global solution: the proof of Theorem 2.5

Let us first note that there exists a constant $\tilde{C} > 0$ such that $|\xi|^2 \leq \tilde{C}$ and

$$|2xh(s, y, \bar{y}, z, \bar{z}) + 2xf(s, 0, 0, 0, 0)| \leq \tilde{C} + \tilde{C}|x|^2 + \tilde{C}(|y|^2 + |\bar{y}|^2)$$

for any $(x, y, \bar{y}, z, \bar{z}) \in R \times R^2 \times (R^d)^2$. Let $\alpha(\cdot)$ be the unique solution of the following ordinary differential equation:

$$\alpha(t) = \tilde{C} + \int_t^T \tilde{C} ds + \int_t^T (2\tilde{C} + \tilde{C})\alpha(s) ds, \quad t \in [0, T].$$

It is easy to see that $\alpha(\cdot)$ is a continuous decreasing function and we have

$$\alpha(t) = \tilde{C} + \int_t^T \tilde{C}[1 + 2\alpha(s)] ds + \tilde{C} \int_t^T \alpha(s) ds, \quad t \in [0, T].$$

Define

$$\lambda := \sup_{t \in [0, T]} \alpha(t) = \alpha(0).$$

As $|\xi|^2 \leq \tilde{C} \leq \lambda$, Theorem 2.3 shows that there exists $\eta_\lambda > 0$ which only depends on λ , such that BSDE has a local solution (Y, Z) on $[T - \eta_\lambda, T]$ and it can be constructed through the Picard iteration.

Consider the Picard iteration:

$$\begin{aligned} Y_t^0 &= \xi + \int_t^T Z_s^0 dW_s; \quad \text{and for } j \geq 0, \\ Y_t^{j+1} &= \xi + \int_t^T [f(s, 0, 0, Z_s^{j+1}, 0) - f(s, 0, 0, 0, 0)] ds \\ &\quad + \int_t^T [f(s, 0, 0, 0, 0) + f(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j]) - f(s, 0, 0, Z_s^j, 0)] ds \\ &\quad - \int_t^T Z_s^{j+1} dW_s \\ &= \xi + \int_t^T [f(s, 0, 0, 0, 0) + h(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j])] ds \\ &\quad - \int_t^T Z_s^{j+1} d\tilde{W}_s^{j+1}, \quad t \in [T - \eta_\lambda, T], \end{aligned}$$

where

$$f(s, 0, 0, Z_s^{j+1}, 0) - f(s, 0, 0, 0, 0) = Z_s^{j+1} \beta_s^{j+1}$$

and the process

$$\tilde{W}_t^{j+1} = W_t - \int_0^t \beta_s^{j+1} \mathbf{1}_{[T - \eta_\lambda, T]}(s) ds$$

is a Brownian motion under an equivalent probability measure P^{j+1} which we denote by \tilde{P} with loss of generality, and under which the expectation is denoted by \tilde{E} . Using Itô's formula, it is straightforward to deduce the following estimate for $r \in [T - \eta_\lambda, t]$,

$$\begin{aligned} &\tilde{E}_r[|Y_t^{j+1}|^2] + \tilde{E}_r\left[\int_r^T |Z_s^{j+1}|^2 ds\right] \\ &= \tilde{E}_r[|\xi|^2] + \tilde{C} \tilde{E}_r \int_t^T 2Y_s^{j+1} [f(s, 0, 0, 0, 0) + h(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j])] ds \\ &\leq \tilde{C} + \tilde{C} \int_t^T \tilde{E}_r[|Y_s^{j+1}|^2] ds + \tilde{C} \int_t^T \{\tilde{E}_r[|Y_s^j|^2] + |E[Y_s^j]|^2 + 1\} ds. \end{aligned} \quad (2.42)$$

In what follows, we show by induction the following inequality:

$$|Y_t^j|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T]. \quad (2.43)$$

In fact, it is trivial to see that $|Y_t^0|^2 \leq \alpha(t)$, and let us suppose $|Y_t^j|^2 \leq \alpha(t)$ for $t \in [T - \eta_\lambda, T]$. Then, from (2.42),

$$\tilde{E}_r[|Y_t^{j+1}|^2] \leq \tilde{C} + \tilde{C} \int_t^T [1 + 2\alpha(s)] ds + \tilde{C} \int_t^T \tilde{E}_r[|Y_s^{j+1}|^2] ds, \quad t \in [T - \eta_\lambda, T].$$

From the comparison theorem, we have

$$\widetilde{E}_r[|Y_t^{j+1}|^2] \leq \alpha(t), \quad t \in [T - \eta_\lambda, T].$$

Setting $r = t$, we have

$$|Y_t^{j+1}|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T].$$

Therefore, inequality (2.43) holds.

As $Y_t = \lim_j Y_t^j$, our constructed local solution (Y, Z) in $[T - \eta_\lambda, T]$ satisfies then the following estimate:

$$|Y_t|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T], \quad \text{and} \quad |Y_t|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T].$$

In particular, $|Y_{T-\eta_\lambda}|^2 \leq \alpha(T - \eta_\lambda) \leq \lambda$.

Taking $T - \eta_\lambda$ as the terminal time and $Y_{T-\eta_\lambda}$ as terminal value, Theorem 2.3 shows that BSDE has a local solution (Y, Z) on $[T - 2\eta_\lambda, T - \eta_\lambda]$ through the Picard iteration. Once again, using the Picard iteration and the fact that $|Y_{T-\eta}|^2 \leq \alpha(T - \eta)$, we deduce that $|Y_t|^2 \leq \alpha(t)$, for $t \in [T - 2\eta_\lambda, T - \eta_\lambda]$. Repeating the preceding process, we can extend the pair (Y, Z) to the whole interval $[0, T]$ within a finite steps such that Y is uniformly bounded by λ . We now show that $Z \cdot W$ is a $BMO(P)$ martingale.

Identical to the proof of inequality (2.22), we have

$$\begin{aligned} & \phi(Y_t) + \frac{1}{2} E_t \left[\int_t^T |Z_s|^2 ds \right] \\ & \leq \phi(|\xi|_\infty) + C E_t \left[\int_t^T |\phi'(Y_s)| (1 + |Y_s| + E[|Y_s|]) ds \right] \\ & \leq \phi(|\xi|_\infty) + C \phi'(\lambda) E_t \left[\int_t^T (1 + \lambda + \lambda) ds \right]. \end{aligned} \tag{2.44}$$

Consequently, we have

$$\begin{aligned} \|Z \cdot W\|_{BMO_2(\mathbb{P})}^2 &= \sup_\tau E_\tau \left[\int_\tau^T |Z_s|^2 ds \right] \\ &\leq 2\phi(|\xi|_\infty) + 4C\phi'(\lambda)(1 + \lambda)T. \end{aligned} \tag{2.45}$$

Finally, we prove the uniqueness. Let (Y, Z) and $(\widetilde{Y}, \widetilde{Z})$ be two adapted solutions. Then, we have (recall that β is defined by (?))

$$\begin{aligned} Y_t - \widetilde{Y}_t &= \int_t^T \left[h(s, Y_s, E[Y_s], \widetilde{Z}_s, E[Z_s]) - h(s, \widetilde{Y}_s, E[\widetilde{Y}_s], \widetilde{Z}_s, E[\widetilde{Z}_s]) \right] ds \\ &\quad - \int_t^T (Z_s - \widetilde{Z}_s) d\widetilde{W}_s, \quad t \in [0, T]. \end{aligned} \tag{2.46}$$

Similar to the first two inequalities in (2.38), for any stopping time τ which takes values

in $[T - \varepsilon, T]$, we have

$$\begin{aligned}
& |Y_\tau - \tilde{Y}_\tau|^2 + \tilde{E}_\tau \left[\int_\tau^T |Z_s - \tilde{Z}_s|^2 ds \right] \\
&= \tilde{E}_\tau \left[\left(\int_\tau^T \left[h(s, Y_s, E[Y_s], \tilde{Z}_s, E[Z_s]) - h(s, \tilde{Y}_s, E[\tilde{Y}_s], \tilde{Z}_s, E[\tilde{Z}_s]) \right] ds \right)^2 \right] \\
&\leq C^2 \tilde{E}_\tau \left[\left(\int_\tau^T \left[|Y_s - \tilde{Y}_s| + |E[Y_s - \tilde{Y}_s]| + |E[Z_s - \tilde{Z}_s]| \right] ds \right)^2 \right] \tag{2.47} \\
&\leq 6C^2 \varepsilon^2 |Y - \tilde{Y}|_\infty^2 + 3C^2 \varepsilon \tilde{E}_\tau \left[\int_\tau^T |Z_s - \tilde{Z}_s|^2 ds \right] + 3C^2 \varepsilon E \left[\int_{T-\varepsilon}^T |Z_s - \tilde{Z}_s|^2 ds \right] \\
&\leq 6C^2 \varepsilon^2 |Y - \tilde{Y}|_\infty^2 + 3C^2 \varepsilon \|(Z - \tilde{Z}) \cdot \tilde{W}\|_{BMO_2(\tilde{P})}^2 + 3C^2 \varepsilon \|(Z - \tilde{Z}) \cdot W\|_{BMO_2(\mathbb{P})}^2 \\
&\leq 6C^2 \varepsilon^2 |Y - \tilde{Y}|_\infty^2 + 3C^2(1 + c_2^2) \varepsilon \|(Z - \tilde{Z}) \cdot W\|_{BMO_2(\mathbb{P})}^2.
\end{aligned}$$

Therefore, we have (on the interval $[T - \varepsilon, T]$)

$$\begin{aligned}
& \left| Y - \tilde{Y} \right|_\infty^2 + c_1^2 \left\| (Z - \tilde{Z}) \cdot W \right\|_{BMO_2(\mathbb{P})}^2 \\
&\leq 6C^2 \varepsilon^2 \left| Y - \tilde{Y} \right|_\infty^2 + 3C^2(1 + c_2^2) \varepsilon \left\| (Z - \tilde{Z}) \cdot W \right\|_{BMO_2(\mathbb{P})}^2. \tag{2.48}
\end{aligned}$$

Note that since

$$|\beta| \leq C(1 + |Z| + |\tilde{Z}|),$$

the two generic constants c_1 and c_2 only depend on the sum

$$\|Z \cdot W\|_{BMO_2(\mathbb{P})}^2 + \|\tilde{Z} \cdot W\|_{BMO_2(\mathbb{P})}^2.$$

Then when ε is sufficiently small, we conclude that $Y = \tilde{Y}$ and $Z = \tilde{Z}$ on $[T - \varepsilon, T]$. Repeating iteratively with a finite of times, we have the uniqueness on the given interval $[0, T]$.

3 The expected term is additive and has a quadratic growth in the second unknown variable

Let us first consider the following quadratic BSDE with mean term:

$$Y_t = \xi + \int_t^T [f_1(s, Z_s) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds - \int_t^T Z_s dB_s, \quad t \in [0, T] \tag{3.1}$$

where $f_1 : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies: $f_1(\cdot, z)$ is an adapted process for any z , and

$$|f_1(t, 0)| \leq C, \quad |f_1(t, z) - f_1(t, z')| \leq C(1 + |z| + |z'|)|z - z'|, \tag{3.2}$$

and $f_2 : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies: $f_2(\cdot, z, \bar{z})$ is an adapted process for any z and \bar{z} , and

$$|f_2(t, z, \bar{z})| \leq C(1 + |z|^2 + |\bar{z}'|^2). \tag{3.3}$$

Proposition 3.1. *Let us suppose that f_1 and f_2 be two generators satisfying the above conditions and ξ be a bounded random variable. Then (3.1) admits a unique solution (Y, Z) such that Y is bounded and $Z \cdot W$ is a BMO martingale.*

Proof. Let us first prove the existence. We solve this equation in two steps:

Step one. First solve the following BSDE:

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, Z_s) ds - \int_t^T Z_s \cdot dW_s. \quad (3.4)$$

It is well known that this BSDE admits a unique solution (\tilde{Y}, Z) such that \tilde{Y} is bounded and $Z \cdot W$ is a BMO martingale.

Step two. Define

$$Y_t = \tilde{Y}_t + \int_t^T E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds. \quad (3.5)$$

Then

$$Y_t = \xi + \int_t^T [f_1(s, Z_s) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (3.6)$$

The uniqueness can be proved in a similar way: Let (Y^1, Z^1) and (Y^2, Z^2) be two solutions. Then set

$$\tilde{Y}_t^1 = Y_t^1 - \int_t^T E[f_2(s, Z_s^1, \mathbb{E}[Z_s^1])] ds, \quad \tilde{Y}_t^2 = Y_t^2 - \int_t^T E[f_2(s, Z_s^2, \mathbb{E}[Z_s^2])] ds.$$

(\tilde{Y}^1, Z^1) and (\tilde{Y}^2, Z^2) being solution of the same BSDE (4.4), from the uniqueness of solution to this BSDE,

$$\tilde{Y}^1 = \tilde{Y}^2, \quad Z^1 = Z^2,$$

Hence $Y^1 = Y^2$, and $Z^1 = Z^2$. □

Now we consider a more general form of BSDE with a mean term:

$$\begin{aligned} Y_t = & \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])] ds \\ & - \int_t^T Z_s \cdot dW_s. \end{aligned} \quad (3.7)$$

Here for $i = 1, 2$, $f_i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies: for any (y, \bar{y}, z, \bar{z}) , $f_i(t, y, \bar{y}, z, \bar{z})$ is an adapted process, and

$$|f_1(t, y, \bar{y}, 0, \bar{z})| + |f_2(t, 0, 0, 0, 0)| \leq C,$$

$$|f_1(t, y, z, \bar{y}, \bar{z}) - f_1(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + |\bar{z} - \bar{z}'| + (1 + |z| + |z'|)|z - z'|),$$

$$|f_2(t, y, z, \bar{y}, \bar{z}) - f_2(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + (1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|)).$$

We have the following result.

Theorem 3.2. Assume that f_1 and f_2 satisfy the above conditions, and ξ is a bounded random variable. BSDE (3.7) has a unique solution (Y, Z) such that $Y \in \mathcal{S}^\infty$ and $Z \cdot W \in BMO_2(\mathbb{P})$.

Example 3.3. The condition on f_1 means that this function should be bounded with respect to (y, \bar{y}, \bar{z}) . For example,

$$f_1(s, y, \bar{y}, z, \bar{z}) = 1 + |\sin(y)| + |\sin(\bar{y})| + \frac{1}{2}|z|^2 + |\sin(\bar{z})|,$$

$$f_2(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |y'| + \frac{1}{2}(|z| + |\bar{z}|)^2.$$

Proof. We prove the theorem by a fixed point argument. Let $U \in \mathcal{S}^\infty$, and $V \cdot W \in BMO_2(\mathbb{P})$, we define $(Y, Z \cdot W) \in \mathcal{S}^\infty \times BMO_2(\mathbb{P})$ as the unique solution to BSDE with mean:

$$\begin{aligned} Y_t &= \xi + \int_t^T [f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) + E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds \\ &\quad - \int_t^T Z_s \cdot dW_s. \end{aligned} \quad (3.8)$$

And we define the map $\Gamma : (U, V) \mapsto \Gamma(U, V) = (Y, Z)$ on $\mathcal{S}^\infty \times BMO_2(\mathbb{P})$. Set

$$\tilde{Y}_t = Y_t - \int_t^T E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds,$$

then (\tilde{Y}, Z) is the solution to

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s dB_s.$$

As $|f_1(t, y, \bar{y}, 0, \bar{z})| \leq C$, we have

$$|f_1(t, y, \bar{y}, z, \bar{z})| \leq C + \frac{C}{2}|z|^2.$$

Applying Ito's formula to $\phi_C(|\tilde{Y}|)$,

$$\phi_C(|\tilde{Y}_t|) + \int_t^T \frac{1}{2} \phi_C''(|\tilde{Y}_s|) |Z_s|^2 ds \leq \phi_C(\xi) + \int_t^T |\phi_C'(|\tilde{Y}_s|)| (C + \frac{C}{2}|Z_s|^2) ds - \int_0^T \phi'(|\tilde{Y}_s|) Z_s dW_s, \quad (3.9)$$

we can prove that there exists a constant $K > 0$ such that

$$\|Z\|_{BMO} \leq K.$$

Let (U^1, V^1) and (U^2, V^2) , and (Y^1, Z^1) and (Y^2, Z^2) be the corresponding solution, and

$$\tilde{Y}_t^1 = \xi + \int_t^T f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[V_s^1]) ds - \int_t^T Z_s^1 dB_s,$$

$$\tilde{Y}_t^2 = \xi + \int_t^T f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2]) ds - \int_t^T Z_s^2 dB_s.$$

There exists a bounded adapted process β such that

$$f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[V_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2]) = \beta_s(Z_s^1 - Z_s^2).$$

Then $\tilde{W}_t := W_t - \int_0^t \beta_s ds$ is a Brown motion under the equivalent probability measure $\tilde{\mathbb{P}}$ defined by

$$d\tilde{\mathbb{P}} = \mathcal{E}_0^T(\beta \cdot W)_0^T d\mathbb{P}.$$

We have

$$\begin{aligned} & \Delta \tilde{Y}_t + \int_t^T \Delta Z_s d\tilde{W}_s \\ &= \int_t^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^2, \mathbb{E}[V_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2])] ds, \quad t \in [0, T]. \end{aligned} \quad (3.10)$$

For any stopping time τ which takes values in $[T - \varepsilon, T]$, taking square and then the conditional expectation with respect to $\tilde{\mathbb{P}}$ (denoted by $\tilde{\mathbb{E}}_\tau$), we have

$$\begin{aligned} & |\Delta \tilde{Y}_\tau|^2 + \tilde{E}_\tau \int_\tau^T |\Delta Z|^2 ds \\ &= \tilde{E}_\tau \left(\int_\tau^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^2, \mathbb{E}[V_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2])] ds \right)^2 \\ &\leq C^2 \tilde{E}_\tau \left(\left[\int_\tau^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]| + |\mathbb{E}[\Delta V_s]|) ds \right]^2 \right) \\ &\leq C^2 \varepsilon (|\Delta U|_\infty^2 + |\Delta V \cdot W|_{BMO_2(\mathbb{P})}^2). \end{aligned} \quad (3.11)$$

Therefore, we have (on the interval $[T - \varepsilon, T]$)

$$\left| \Delta \tilde{Y} \right|_\infty^2 + c_1^2 \|(\Delta Z) \cdot W\|_{BMO_2(\mathbb{P})}^2 \leq C^2 \varepsilon (|\Delta U|_\infty^2 + |\Delta V \cdot W|_{BMO_2(\mathbb{P})}^2). \quad (3.12)$$

Note that since

$$|\beta| \leq C(1 + |Z_1| + |Z_2|),$$

the generic constant c_1 and c_2 only depend on C and K .

As

$$\Delta Y_t = \Delta \tilde{Y}_t + \int_t^T E [f_2(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^2]) - f_2(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds,$$

then

$$\begin{aligned} |\Delta Y_t| &\leq |\Delta \tilde{Y}_t| + CE \left[\int_t^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]|) ds \right] \\ &\quad + CE \left[\int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E}Z_s^1| + |\mathbb{E}Z_s^2|) (|\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds \right] \end{aligned}$$

we deduce that

$$|\Delta Y_t|^2 \leq C^2 \left(|\Delta \tilde{Y}_t|^2 + E \left[\int_t^T (|\Delta U_s| + |\mathbb{E} \Delta U_s|) ds \right]^2 \right. \\ \left. E \left[\int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E}[Z_s^1]| + |\mathbb{E}[Z_s^2]|) (|\Delta Z_s| + |\mathbb{E} \Delta Z_s|) ds \right]^2 \right),$$

which implies that

$$|\Delta Y|_\infty^2 \leq C^2 (|\Delta \tilde{Y}|_\infty^2 + \varepsilon |\Delta U|_\infty^2 + |(\Delta Z) \cdot W|_{BMO_2(\mathbb{P})}^2) \\ \leq C^2 \varepsilon (|\Delta U|_\infty^2 + \|V \cdot W\|_{BMP_2(P)}^2).$$

Then when ε is sufficiently small, we conclude that the application is contracting on $[T - \varepsilon, T]$. Repeating iteratively with a finite of times, we have the existence and uniqueness on the given interval $[0, T]$. \square

4 Multi-dimensional Case

In this section, we will study the multi-dimensional case of (3.7), where f_1 is Lipschitz. Our result generalizes the corresponding one of Cheridito and Nam [8].

We first consider the following BSDE with mean term:

$$Y_t = \xi + \int_t^T [f_1(s, Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (4.1)$$

where $f_1 : \Omega \times [0, T] \times R^{d \times n} \times \mathbb{R}^{d \times n} \rightarrow R^n$ satisfies: for any z and \bar{z} , $f_1(t, z, \bar{z})$ is an adapted process, and

$$|f_i(t, 0, 0)| \leq C, \quad |f_1(t, z, \bar{z}) - f_1(t, z', \bar{z}')| \leq C(|z - z'| + |\bar{z} - \bar{z}'|), \quad (4.2)$$

and $f_2 : \Omega \times [0, T] \times R^{d \times n} \times R^{d \times n} \rightarrow R^n$ satisfies: for any z, \bar{z} , $f_2(t, z, \bar{z})$ is an adapted process, and

$$|f_2(t, 0, 0)| \leq C, \quad |f_2(t, z, \bar{z}) - f_2(t, z', \bar{z}')| \leq C(1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|). \quad (4.3)$$

Proposition 4.1. *Let us suppose that f_1 and f_2 be two generators satisfying the above conditions and $\xi \in L^2(\mathcal{F}_T)$. Then (4.1) admits a unique solution (Y, Z) such that $Y \in \mathcal{S}^2$ and $Z \in \mathcal{M}^2$.*

Proof. Let us first prove the existence. Our proof is divided into the following two steps.

Step one. First consider the following BSDE

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (4.4)$$

It admits a unique solution (\tilde{Y}, Z) such that $\tilde{Y} \in \mathcal{S}^2$ and $Z \in \mathcal{M}^2$, see Buckdahn et al. [6].

Step two. Define

$$Y_t = \tilde{Y}_t + \int_t^T E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds, \quad t \in [0, T]. \quad (4.5)$$

Then, we have for $t \in [0, T]$,

$$Y_t = \xi + \int_t^T [f_1(s, Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Z_s, \mathbb{E}[Z_s])]] ds - \int_t^T Z_s \cdot dW_s. \quad (4.6)$$

The uniqueness can be proved in a similar way: Let (Y^1, Z^1) and (Y^2, Z^2) be two solutions. Then set

$$\tilde{Y}_t^1 = Y_t^1 - \int_t^T E[f_2(s, Z_s^1, \mathbb{E}[Z_s^1])] ds, \quad \tilde{Y}_t^2 = Y_t^2 - \int_t^T E[f_2(s, Z_s^2, \mathbb{E}[Z_s^2])] ds.$$

(\tilde{Y}^1, Z^1) and (\tilde{Y}^2, Z^2) being solution of the same BSDE (4.4), from the uniqueness of solution to this BSDE,

$$\tilde{Y}^1 = \tilde{Y}^2, \quad Z^1 = Z^2,$$

Hence $Y^1 = Y^2$, and $Z^1 = Z^2$. □

Now we consider the following more general form of BSDE with a mean term:

$$\begin{aligned} Y_t = & \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])]] ds \\ & - \int_t^T Z_s \cdot dW_s, t \in [0, T]. \end{aligned} \quad (4.7)$$

Here for $i = 1, 2$, $f_i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^n$ satisfies: for any (y, \bar{y}, z, \bar{z}) , $f_i(t, y, \bar{y}, z, \bar{z})$ is an adapted process, and

$$|f_1(t, y, \bar{y}, 0, 0)| + |f_2(t, 0, 0, 0, 0)| \leq C, \quad (4.8)$$

$$|f_1(t, y, z, \bar{y}, \bar{z}) - f_1(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + |z - z'| + |\bar{z} - \bar{z}'|), \quad (4.9)$$

$$|f_2(t, y, z, \bar{y}, \bar{z}) - f_2(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + (1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|)). \quad (4.10)$$

We have the following result.

Theorem 4.2. *Assume that f_1 and f_2 satisfy the inequalities (4.8)-(4.10), and $\xi \in L^2(\mathcal{F}_T)$. Then BSDE (4.7) has a unique solution (Y, Z) such that $Y \in \mathcal{S}^2$ and $Z \in \mathcal{M}^2$.*

Example 4.3. *The condition (4.8) requires that both functions f_1 and f_2 should be bounded with respect to (y, \bar{y}) . For example,*

$$f_1(s, y, \bar{y}, z, \bar{z}) = 1 + |\sin(y)| + |\sin(\bar{y})| + |z| + |\bar{z}|,$$

$$f_2(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |\bar{y}| + \frac{1}{2}(|z| + |\bar{z}|)^2.$$

Proof. We prove the theorem by a fixed point argument. Let $U \in \mathcal{S}^2$, and $V \in \mathcal{M}^2$, we define $(Y, Z \cdot W) \in \mathcal{S}^2 \times \mathcal{M}^2$ as the unique solution to BSDE with mean:

$$\begin{aligned} Y_t &= \xi + \int_t^T [f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds \\ &\quad - \int_t^T Z_s \cdot dW_s. \end{aligned} \quad (4.11)$$

We define the map $\Gamma : (U, V) \mapsto \Gamma(U, V) = (Y, Z)$ on $\mathcal{S}^2 \times \mathcal{M}^2$. Set

$$\tilde{Y}_t = Y_t - \int_t^T E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds, \quad t \in [0, T].$$

Then (\tilde{Y}, Z) is the solution to

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s \cdot dW_s, t \in [0, T].$$

As $|f_1(t, y, \bar{y}, 0, 0)| \leq C$, we have

$$|f_1(t, y, \bar{y}, z, \bar{z})| \leq C(1 + |z| + |\bar{z}|).$$

Applying Ito's formula to $|\tilde{Y}|^2$, we have

$$|\tilde{Y}_t|^2 + \int_t^T |Z_s|^2 ds \leq |\xi|^2 + 2C \int_t^T |\tilde{Y}_s|(C + |Z_s| + |\mathbb{E}[Z]|) ds - 2 \int_t^T (Y_s, Z_s \cdot dW_s). \quad (4.12)$$

Further, using standard techniques, we can prove that there exists a constant $K > 0$ such that

$$\|Z\|_{\mathcal{M}^2} \leq K.$$

For $(U^i, V^i) \in \mathcal{S}^2 \times \mathcal{M}^2$, define $(Y^i, Z^i) = \Gamma(U^i, V^i)$ with $i = 1, 2$. Further, set for $t \in [0, T]$,

$$\begin{aligned} \tilde{Y}_t^1 &:= \xi + \int_t^T f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[V_s^1]) ds - \int_t^T Z_s^1 \cdot dW_s, \\ \tilde{Y}_t^2 &:= \xi + \int_t^T f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2]) ds - \int_t^T Z_s^2 \cdot dW_s. \end{aligned}$$

We have

$$\begin{aligned} &\Delta \tilde{Y}_t + \int_t^T \Delta Z_s dW_s \\ &= \int_t^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds, \quad t \in [0, T]. \end{aligned} \quad (4.13)$$

For any $t \in [T - \varepsilon, T]$, taking square and then expectations on both sides of the last equality, we have

$$\begin{aligned}
& \mathbb{E}[|\Delta\tilde{Y}_t|^2] + \mathbb{E}\left[\int_t^T |\Delta Z_s|^2 ds\right] \\
&= \mathbb{E}_\tau \left(\int_t^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds \right)^2 \\
&\leq C^2 \mathbb{E} \left(\left[\int_\tau^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]| + |\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds \right]^2 \right) \\
&\leq C^2 \varepsilon (|\Delta U|_{\mathcal{S}^2} + \|\Delta Z\|_{\mathcal{M}^2}^2).
\end{aligned} \tag{4.14}$$

Therefore, we have (on the interval $[T - \varepsilon, T]$) for a sufficiently small $\varepsilon > 0$,

$$\left| \Delta\tilde{Y} \right|_{\mathcal{S}^2}^2 + \|\Delta Z\|_{\mathcal{M}^2}^2 \leq C^2 \varepsilon (|\Delta U|_{\mathcal{S}^2}^2). \tag{4.15}$$

As

$$\Delta Y_t = \Delta\tilde{Y}_t + \int_t^T E [f_2(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_2(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds,$$

we have

$$\begin{aligned}
|\Delta Y_t| &\leq |\Delta\tilde{Y}_t| + CE \left[\int_t^T (|\Delta U_s| + |\mathbb{E}\Delta U_s|) ds \right] \\
&\quad + CE \left[\int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E}Z_s^1| + |\mathbb{E}Z_s^2|) (|\Delta Z_s| + |\mathbb{E}\Delta Z_s|) ds \right].
\end{aligned}$$

Moreover, we deduce that

$$\begin{aligned}
|\Delta Y_t|^2 &\leq C^2 \left(|\Delta\tilde{Y}_t|^2 + E \left[\int_t^T (|\Delta U_s| + |\mathbb{E}\Delta U_s|) ds \right]^2 \right. \\
&\quad \left. + E \left[\int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E}Z_s^1| + |\mathbb{E}Z_s^2|) (|\Delta Z_s| + |\mathbb{E}\Delta Z_s|) ds \right]^2 \right).
\end{aligned}$$

In view of (4.15)

$$\left| \Delta\tilde{Y} \right|_{\mathcal{S}^2}^2 + \|\Delta Z\|_{\mathcal{M}^2}^2 \leq C^2 \varepsilon (|\Delta U|_{\mathcal{S}^2}^2).$$

Then when ε is sufficiently small, we conclude that the application is contracting on $[T - \varepsilon, T]$. Repeating iteratively with a finite of times, we have the existence and uniqueness on the given interval $[0, T]$. \square

References

- [1] J.-M. BISMUT, *Conjugate convex functions in optimal stochastic control*, J. Math. Anal. Appl., **44** (1973), pp. 384–404.
- [2] J.-M. BISMUT, *Linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optim., **14** (1976), pp. 419–444.
- [3] PH. BRIAND AND R. ELIE, *A simple constructive approach to quadratic BSDEs with or without delay*, Stochastic Process. Appl., **123** (2013), pp. 2921–2939.
- [4] PH. BRIAND AND Y. HU, *BSDE with quadratic growth and unbounded terminal value*, Probab. Theory Related Fields, **136** (2006), pp. 604–618.
- [5] PH. BRIAND AND Y. HU, *Quadratic BSDEs with convex generators and unbounded terminal conditions*, Probab. Theory Related Fields, **141** (2008), pp. 543–567.
- [6] R. BUCKDAHN, J. LI AND S. PENG, *Mean-field backward stochastic differential equations and related partial differential equations*, Stochastic Process. Appl., **119** (2009), pp. 3133–3154.
- [7] P. CHERIDITO AND K. NAM, *Multidimensional quadratic and subquadratic BSDEs with special structure*, Stochastics, to appear. arXiv:1309.6716v4 [math.PR], 2015.
- [8] P. CHERIDITO AND K. NAM, *BSEs, BSDEs and fixed point problems*, Ann. Probab., to appear. arXiv:1410.1247v2 [math.PR], 2015.
- [9] N. EL KAROUÏ AND S. HAMADÈNE, *BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations*, Stochast. Process. Appl., **107** (2003), 145–169.
- [10] C. FREI AND G. DOS REIS, *A financial market with interacting investors: does an equilibrium exist?*, Math. Finan. Econ., **4** (2011), pp. 161–182.
- [11] C. KARDARAS, H. XING, AND G. ZITKOVIĆ, *Incomplete stochastic equilibria with exponential utilities close to pareto optimality*, arXiv:1505.07224v1 [math.PR], 2015.
- [12] N. KAZAMAKI, *Continuous Exponential Martingales and BMO*, Lecture Notes in Mathematics, **1579**, Springer-Verlag, Berlin, 1994.
- [13] M. KOBYLANSKI, *Backward stochastic differential equations and partial differential equations with quadratic growth*, Ann. Probab., **28** (2000), pp. 558–602.
- [14] D. KRAMKOV AND S. PULIDO, *Stability and analytic expansions of local solutions of systems of quadratic bsdes with applications to a price impact model*, arXiv:1410.6144v2 [q-fin.MF], 2014.
- [15] D. KRAMKOV AND S. PULIDO, *A system of quadratic BSDEs arising in a price impact model*, Ann. Appl. Probab., to appear. arXiv:1408.0916v2 [q-fin.MF], 2014.

- [16] E. PARDOUX AND S. PENG, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett., **14** (1990), pp. 55–61.
- [17] S. PENG, *Open problems on backward stochastic differential equations*, in Control of Distributed Parameter and Stochastic Systems (Hangzhou, 1998), S. Chen, et al., eds., Kluwer Academic Publishers, Boston, 1999, pp. 265–273.
- [18] S. PENG, *Backward stochastic differential equation, nonlinear expectation and their applications*, in: Proceedings of the International Congress of Mathematicians, Volume I, pp. 393–432, Hindustan Book Agency, New Delhi, 2010.
- [19] R. TEVZADZE, *Solvability of backward stochastic differential equations with quadratic growth*, Stochastic. Process. Appl., **118** (2008), 503–515.
- [20] S. TANG, *General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations*, SIAM J. Control Optim., **42** (2003), pp. 53–75.
- [21] S. TANG, *Dynamic programming for general linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optim., **53** (2015), 1082-1106.