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# Mean-field type Quadratic BSDEs

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April 23, 2017

## Abstract

In this paper, we give several new results on solvability of a quadratic BSDE whose generator depends also on the mean of both variables. First, we consider such a BSDE using John-Nirenberg's inequality for BMO martingales to estimate its contribution to the evolution of the first unknown variable. Then we consider the BSDE having an additive expected value of a quadratic generator in addition to the usual quadratic one. In this case, we use a deterministic shift transformation to the first unknown variable, when the usual quadratic generator depends neither on the first variable nor its mean, the general case can be treated by a fixed point argument.

## 1 Introduction

Let  $\{W_t := (W_t^1, \dots, W_t^d)^*, 0 \leq t \leq T\}$  be a  $d$ -dimensional standard Brownian motion defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Denote by  $\{\mathcal{F}_t, 0 \leq t \leq T\}$  the augmented natural filtration of the standard Brownian motion  $W$ .

In this paper, we study the existence and uniqueness of an adapted solution of the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (1.1)$$

When  $f$  does not depend on  $(\bar{y}, \bar{z})$ , BSDE (1.1) is the classical one, and it is extensively studied in the literature, see the pioneer work of Bismut [1, 2] as well as Pardoux and Peng [16]. When  $f(t, y, \bar{y}, z, \bar{z})$  is scalar valued and quadratic in  $z$  while it does not depend on  $(\bar{y}, \bar{z})$ , BSDE (1.1) is the so-called quadratic BSDE and has been studied

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by Kobylanski [13], Briand and Hu [4, 5]. BSDE (1.1) (called mean-field type BSDE) arises naturally when studying mean-field games, etc. We refer to [6] for the motivation of its study. When the generator  $f$  is uniformly Lipschitz in the last four arguments, BSDE (1.1) is shown in a straightforward manner to have a unique adapted solution, and the reader is referred to Buckdahn et al. [6] for more details. For the general generator  $f(s, y, \bar{y}, z, \bar{z})$  depending quadratically on  $z$ , BSDE (1.1) is a quadratic one involving both  $\mathbb{E}[Y]$  and  $\mathbb{E}[Z]$ . The comparison principle (see [13]) is well known to play a crucial role in the study of quadratic BSDEs (see [13]). Unfortunately, the comparison principle fails to hold for BSDE (1.1) (see, e.g. [6] for a counter-example for comparison with Lipschitz generators), the derivation of its solvability is not straightforward. Up to our best knowledge, no study on quadratic mean-field type BSDEs is available. To tackle the difficulty of lack of comparison principle, we use the John-Nirenberg inequality for BMO martingales to address the solvability.

Furthermore, we study the following alternative of mean-field type BSDE, which admits a quadratic growth in the mean of the second unknown variable  $\mathbb{E}[Z]$ :

$$\begin{aligned} Y_t = & \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])] ds \\ & - \int_t^T Z_s \cdot dW_s, \end{aligned} \quad (1.2)$$

where  $f_2$  is allowed to grow quadratically in both  $Z$  and  $\mathbb{E}[Z]$ , the function  $f_1$  also admits a quadratic growth in the second unknown variable for the scalar case.

To deal with the additive expected value of  $f_2$ , Cheridito and Nam [8] introduced Krasnoselskii fixed point theorem to conclude the existence and uniqueness, by observing that the range of the expected value of  $f_2$  is (locally) compact. Here we observe the following fact: the expected value of  $f_2$  has no contribution to the second unknown variable  $Z$  if  $f_1$  depends neither on the first variable  $Z$  nor on its mean. Hence we use the shift transformation to remove the expectation of  $f_2$ . In the general case, we apply the same kind of technique and the contraction mapping principle.

Let us close this section by introducing some notations. Denote by  $\mathcal{S}^\infty(\mathbb{R}^n)$  the totality of  $\mathbb{R}^n$ -valued  $\mathcal{F}_t$ -adapted essentially bounded continuous processes, and by  $\|Y\|_\infty$  the essential supremum norm of  $Y \in \mathcal{S}^\infty(\mathbb{R}^n)$ . It can be verified that  $(\mathcal{S}^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$  is a Banach space. Let  $M = (M_t, \mathcal{F}_t)$  be a uniformly integrable martingale with  $M_0 = 0$ , and for  $p \in [1, \infty)$  we set

$$\|M\|_{BMO_p(\mathbb{P})} := \sup_{\tau} \left| \mathbb{E}_{\tau} \left[ \left( \langle M \rangle_{\tau}^{\infty} \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \right|_{\infty} \quad (1.3)$$

where the supremum is taken over all stopping times  $\tau$ . The class  $\{M : \|M\|_{BMO_p} < \infty\}$  is denoted by  $BMO_p$ , which is written as  $BMO_p(\mathbb{P})$  whenever it is necessary to indicate the underlying probability, and observe that  $\|\cdot\|_{BMO_p}$  is a norm on this space and  $BMO_p(\mathbb{P})$  is a Banach space.

Denote by  $\mathcal{E}(M)$  the stochastic exponential of a one-dimensional local martingale  $M$  and by  $\mathcal{E}(M)_s^t$  that of  $M_t - M_s$ . Denote by  $\beta \cdot M$  the stochastic integral of a scalar-valued adapted process  $\beta$  with respect to a local continuous martingale  $M$ .

For any real  $p \geq 1$ ,  $\mathcal{S}^p(\mathbb{R}^n)$  denotes the set of  $\mathbb{R}^n$ -valued adapted and càdlàg processes  $(Y_t)_{t \in [0, T]}$  such that

$$\|Y\|_{\mathcal{S}^p(\mathbb{R}^n)} := \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right]^{1/p} < +\infty,$$

and  $\mathcal{M}^p(\mathbb{R}^{d \times n})$  denotes the set of adapted processes  $(Z_t)_{t \in [0, T]}$  with values in  $\mathbb{R}^{d \times n}$  such that

$$\|Z\|_{\mathcal{M}^p(\mathbb{R}^{d \times n})} := \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right]^{1/p} < +\infty.$$

The rest of our paper is organized as follows. In Section 2, we study BSDE (1.1) when  $f(t, y, \bar{y}, z, \bar{z})$  is scalar valued and quadratic in  $z$ , and uniformly Lipschitz in  $(y, \bar{y}, \bar{z})$ , and prove by the contraction mapping principle that BSDE (1.1) has a unique solution. In Section 3, we study scalar-valued BSDE (1.2) when  $f_2(t, y, \bar{y}, z, \bar{z})$  is both quadratic in  $z$  and  $\bar{z}$ , and  $f_1$  is quadratic in  $z$ . Finally, in Section 4, we study BSDE (1.2) in the multi-dimensional case, where we suppose that  $f_2(t, y, \bar{y}, z, \bar{z})$  is both quadratic in  $z$  and  $\bar{z}$ , and  $f_1$  is Lipschitz in  $z$  and  $\bar{z}$ .

## 2 Quadratic BSDEs with a mean term involving the second unknown variable

In this section we consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (2.1)$$

We first recall the following existence and uniqueness, a priori estimate for one-dimensional BSDEs.

**Lemma 2.1.** *Assume that (i) the function  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  has the following growth and locally Lipschitz continuity in the last two variables:*

$$\begin{aligned} |f(s, y, z)| &\leq g_s + \beta|y| + \frac{\gamma}{2}|z|^2, \quad y \in \mathbb{R}, z \in \mathbb{R}^d; \\ |f(s, y, z_1) - f(s, y, z_2)| &\leq C|y_1 - y_2| + C(1 + |z_1| + |z_2|)|z_1 - z_2|, \quad y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d; \end{aligned} \quad (2.2)$$

*(ii) the process  $f(\cdot, y, z)$  is  $\mathcal{F}_t$ -adapted for each  $y \in \mathbb{R}, z \in \mathbb{R}^d$ ; and (iii)  $g \in L^1(0, T)$ . Then for bounded  $\xi$ , the following BSDE*

$$Y_t = \xi + \int_t^T [f(s, Y_s, Z_s)] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (2.3)$$

*has a unique solution  $(Y, Z)$  such that  $Y$  is (essentially) bounded and  $Z \cdot W$  is a BMO martingale. Furthermore, we have*

$$e^{\gamma|Y_t|} \leq \mathbb{E}_t \left[ e^{\gamma e^{\beta(T-t)} \xi + \gamma \int_t^T |g(s)| e^{\beta(s-t)} ds} \right].$$

The following lemma plays an important role in our subsequent arguments. It indicates that following the proof of [12, Theorem 3.3, page 57] can give a more precise dependence of the two constants  $c_1, c_2$  on  $\beta \cdot M$ .

**Lemma 2.2.** *For  $K > 0$ , there are constants  $c_1 > 0$  and  $c_2 > 0$  depending only on  $K$  such that for any BMO martingale  $M$ , we have for any one-dimensional BMO martingale  $N$  such that  $\|N\|_{BMO_2(\mathbb{P})} \leq K$ ,*

$$c_1 \|M\|_{BMO_2(\mathbb{P})} \leq \|\widetilde{M}\|_{BMO_2(\widetilde{\mathbb{P}})} \leq c_2 \|M\|_{BMO_2(\mathbb{P})} \quad (2.4)$$

where  $\widetilde{M} := M - \langle M, N \rangle$  and  $d\widetilde{\mathbb{P}} := \mathcal{E}(N)_0^\infty d\mathbb{P}$ .

## 2.1 Main Results

We make the following three assumptions. Let  $C$  be a positive constant.

(A1) Assume that there are positive constants  $C$  and  $\gamma$  and  $\alpha \in [0, 1)$  such that the function  $f : \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$  has the following linear-quadratic growth and globally-locally Lipschitz continuity: for  $\forall(\omega, s, y_i, \bar{y}_i, z_i, \bar{z}_i) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$  with  $i = 1, 2$ ,

$$\begin{aligned} |f(\omega, s, y, \bar{y}, z, \bar{z})| &\leq C(2 + |y| + |\bar{y}| + |\bar{z}|^{1+\alpha}) + \frac{1}{2}\gamma|z|^2; \\ |f(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - f(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| &\leq C \left\{ |y_1 - y_2| + |\bar{y}_1 - \bar{y}_2| + (1 + |\bar{z}_1|^\alpha + |\bar{z}_2|^\alpha) |\bar{z}_1 - \bar{z}_2| \right. \\ &\quad \left. + (1 + |z_1| + |z_2|) |z_1 - z_2| \right\}; \end{aligned} \quad (2.5)$$

The process  $f(\cdot, y, \bar{y}, z, \bar{z})$  is  $\mathcal{F}_t$ -adapted for each  $(y, \bar{y}, z, \bar{z})$ .

(A2) The terminal condition  $\xi$  is uniformly bounded by  $C$ .

We have the following two theorems.

The first one is a result concerning local solutions. For this, let us introduce some notations. For  $\varepsilon > 0$ , and  $r_\varepsilon > 0$ , we define the ball  $\mathbb{B}_\varepsilon$  by

$$\begin{aligned} \mathbb{B}_\varepsilon := \left\{ (Y, Z) : \quad Y \in \mathcal{S}^\infty, \quad Z \cdot W \in BMO_2(\mathbb{P}), \right. \\ \left. \|Y\|_{\infty, [T-\varepsilon, T]} + \|Z \cdot W\|_{BMO_2, [T-\varepsilon, T]} \leq r_\varepsilon \right\}. \end{aligned}$$

**Theorem 2.3.** *Let assumptions (A1) and (A2) be satisfied with  $\alpha \in [0, 1)$ . Then, for any bounded  $\xi$ , there exist  $\varepsilon > 0$  and  $r_\varepsilon > 0$  such that the following BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (2.6)$$

has a unique local solution  $(Y, Z)$  in the time interval  $[T - \varepsilon, T]$  with  $(Y, Z) \in \mathbb{B}_\varepsilon$ .

**Example 2.4.** The condition on  $f$  means that  $f$  is of linear growth with respect to  $(y, \bar{y})$ , and of  $|\bar{z}|^{1+\alpha}$  growth. For example, For  $\alpha \in (0, 1)$ ,

$$f(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |\bar{y}| + \frac{1}{2}|z|^2 + |\bar{z}|^{1+\alpha}.$$

The second theorem is a result about global solutions.

**Theorem 2.5.** Let assumption (A2) be satisfied. Moreover, assume that there is a positive constant  $C$  such that the function  $f : \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$  has the following linear-quadratic growth and globally-locally Lipschitz continuity: for  $\forall(\omega, s, y_i, \bar{y}_i, z_i, \bar{z}_i) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$  with  $i = 1, 2$ ,

$$\begin{aligned} |f(s, 0, 0, 0, 0) + h(\omega, s, y, \bar{y}, z, \bar{z})| &\leq C(1 + |y| + |\bar{y}|), \\ |f(\omega, s, 0, 0, z_1, 0) - f(\omega, s, 0, 0, z_2, 0)| &\leq C(1 + |z_1| + |z_2|)|z_1 - z_2|, \\ |h(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - h(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| &\leq C(|y_1 - y_2| + |\bar{y}_1 - \bar{y}_2| + |z_1 - z_2| + |\bar{z}_1 - \bar{z}_2|) \end{aligned} \quad (2.7)$$

where for  $(\omega, s, y, \bar{y}, z, \bar{z}) \in \Omega \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$ ,

$$h(s, y, \bar{y}, z, \bar{z}) := f(s, y, \bar{y}, z, \bar{z}) - f(s, 0, 0, z, 0). \quad (2.8)$$

The process  $f(\cdot, y, \bar{y}, z, \bar{z})$  is  $\mathcal{F}_t$ -adapted for each  $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^2 \times (\mathbb{R}^d)^2$ .

Then, the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (2.9)$$

has a unique adapted solution  $(Y, Z)$  on  $[0, T]$  such that  $Y$  is bounded. Furthermore,  $Z \cdot W$  is a  $BMO(\mathbb{P})$  martingale.

**Example 2.6.** The inequality (2.7) requires that  $f$  is bounded with respect to the last variable  $\bar{z}$ . The following function

$$f(s, y, \bar{y}, z, \bar{z}) = 1 + s + |y| + |\bar{y}| + \frac{1}{2}|z|^2 + |\sin(\bar{z})|, \quad (s, y, \bar{y}, z, \bar{z}) \in [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2,$$

satisfies such an inequality.

## 2.2 Local solution: the proof of Theorem 2.3

We prove Theorem 2.3 (using the contraction mapping principle) in the following three subsections: in Subsection 2.2.1, we construct a map (which we call quadratic solution map) in a Banach space; in Subsection 2.2.2, we show that this map is stable in a small ball; and in Subsection 2.2.3, we prove that this map is a contraction.

### 2.2.1 Construction of the map

For a pair of bounded adapted process  $U$  and BMO martingale  $V \cdot W$ , we consider the following quadratic BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (2.10)$$

As

$$|f(s, y, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s])| \leq C(2 + |\mathbb{E}[U_s]| + |\mathbb{E}[V_s]|^{1+\alpha}) + C|y| + \frac{1}{2}\gamma|z|^2,$$

in view of Lemma 2.1, it has a unique adapted solution  $(Y, Z)$  such that  $Y$  is bounded and  $Z \cdot W$  is a BMO martingale. Define the quadratic solution map  $\Gamma : (U, V) \mapsto \Gamma(U, V)$  as follows:

$$\Gamma(U, V) := (Y, Z), \quad \forall (U, V \cdot W) \in \mathcal{S}^\infty \times BMO_2(\mathbb{P}).$$

It is a transformation in the Banach space  $\mathcal{S}^\infty \times BMO_2(\mathbb{P})$ .

Let us introduce here some constants and a quadratic (algebraic) equation which will be used in the next two subsections.

Define

$$C_\delta := e^{\frac{6}{1-\alpha}\gamma C T e^{CT} + \frac{1-\alpha}{2}(\frac{3}{1-\alpha}\gamma C e^{CT})^{\frac{2}{1-\alpha}}(\frac{1+\alpha}{2\delta})^{\frac{1+\alpha}{1-\alpha}}T}, \quad (2.11)$$

$$\beta := \frac{1}{2}(1-\alpha)C^{\frac{2}{1-\alpha}}(2(1+\alpha))^{\frac{1+\alpha}{1-\alpha}}; \quad (2.12)$$

$$\begin{aligned} \mu_1 &:= (1-\alpha) \left( 1 + \frac{1-\alpha}{(1+\alpha)\gamma} \right) = 1-\alpha + \frac{(1-\alpha)^2}{(1+\alpha)\gamma}; \\ \mu_2 &:= \frac{1}{2}(1+\alpha) \left( 1 + \frac{1-\alpha}{(1+\alpha)\gamma} \right) = \frac{1}{2}(1+\alpha) + \frac{1-\alpha}{2\gamma}; \end{aligned} \quad (2.13)$$

$$\mu := (\beta + C\mu_1) \gamma^{\frac{2}{\alpha-1}} + 2C\mu_2. \quad (2.14)$$

Consider the following standard quadratic equation of  $A$ :

$$\delta A^2 - (1 + 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta)A + 4\gamma^{-2}e^{\gamma|\xi|_\infty} + 4\mu C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}\varepsilon = 0.$$

The discriminant of the quadratic equation reads

$$\begin{aligned} \Delta &:= (1 + 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta)^2 - 4\delta \left[ 4\gamma^{-2}e^{\gamma|\xi|_\infty} + 4\mu C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}\varepsilon \right] \\ &= (1 - 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta)^2 - 16\mu\delta C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}\varepsilon. \end{aligned} \quad (2.15)$$

Take

$$\begin{aligned} \delta &:= \frac{1}{8}\gamma^2 e^{-\gamma|\xi|_\infty}, \quad \varepsilon \leq \min \left\{ \frac{1}{3C}e^{-CT}, \frac{1}{8\mu C_\delta}\gamma^{-2}e^{\gamma(1-\frac{3e^{CT}}{1-\alpha})|\xi|_\infty} \right\}, \\ A &:= \frac{1 + 4\gamma^{-2}e^{\gamma|\xi|_\infty}\delta - \sqrt{\Delta}}{2\delta} = \frac{3 - 2\sqrt{\Delta}}{4\delta} \leq \frac{3}{4\delta} = 6\gamma^{-2}e^{\gamma|\xi|_\infty}, \end{aligned} \quad (2.16)$$

and we have

$$\begin{aligned} \Delta \geq 0, \quad 1 - \delta A = \frac{1 + 2\sqrt{\Delta}}{4} > 0, \\ \gamma^{-2} e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3\gamma e^{CT}}{1-\alpha}|\xi|_\infty}}{1 - \delta A} \varepsilon + \frac{1}{4} A = \frac{1}{2} A. \end{aligned} \quad (2.17)$$

Throughout this section, we base our discussion on the time interval  $[T - \varepsilon, T]$ .

We shall prove Theorem 2.3 by showing that the quadratic solution map  $\Gamma$  is a contraction on the closed convex set  $\mathcal{B}_\varepsilon$  defined by

$$\begin{aligned} \mathcal{B}_\varepsilon := \left\{ (U, V) : \quad U \in \mathcal{S}^\infty, \quad V \cdot W \in BMO_2(\mathbb{P}), \right. \\ \left. \|V \cdot W\|_{BMO_2}^2 \leq A, \quad e^{\frac{2}{1-\alpha}\gamma|U|_\infty} \leq \frac{C_\delta e^{\frac{3\gamma e^{CT}}{1-\alpha}|\xi|_\infty}}{1 - \delta A} \right\} \end{aligned} \quad (2.18)$$

(where  $(U, V)$  is defined on  $\Omega \times [T - \varepsilon, T]$ ) for a positive constant  $\varepsilon$  (to be determined later).

### 2.2.2 Estimation of the quadratic solution map

We shall show the following assertion:  $\Gamma(\mathcal{B}_\varepsilon) \subset \mathcal{B}_\varepsilon$ , that is,

$$\Gamma(U, V) \in \mathcal{B}_\varepsilon, \quad \forall (U, V) \in \mathcal{B}_\varepsilon. \quad (2.19)$$

*Step 1. Exponential transformation.*

Define

$$\phi(y) := \gamma^{-2} [\exp(\gamma|y|) - \gamma|y| - 1], \quad y \in R. \quad (2.20)$$

Then, we have for  $y \in R$ ,

$$\phi'(y) = \gamma^{-1} [\exp(\gamma|y|) - 1] \operatorname{sgn}(y), \quad \phi''(y) = \exp(\gamma|y|), \quad \phi''(y) - \gamma|\phi'(y)| = 1. \quad (2.21)$$

Using Itô's formula, we have for  $t \in [T - \varepsilon, T]$ ,

$$\begin{aligned} \phi(Y_t) + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T |Z_s|^2 ds \right] \\ \leq \phi(|\xi|_\infty) + C \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)| (2 + |Y_s| + |\mathbb{E}[U_s]| + |\mathbb{E}[V_s]|^{1+\alpha}) ds \right]. \end{aligned} \quad (2.22)$$

Since (in view of the definition of notation  $\beta$  in (2.12))

$$C|\phi'(Y_s)| |\mathbb{E}[V_s]|^{1+\alpha} \leq \frac{1}{4} |\mathbb{E}[V_s]|^2 + \beta |\phi'(Y_s)|^{\frac{2}{1-\alpha}},$$



we have

$$\begin{aligned}
& \phi(Y_t) + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T |Z_s|^2 ds \right] \\
& \leq \phi(|\xi|_\infty) + C \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)| (2 + |Y_s| + |\mathbb{E}[U_s]|) ds \right] \\
& \quad + \beta \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] + \frac{1}{4} E_t \left[ \int_t^T |E[V_s]|^2 ds \right] \\
& \leq \phi(|\xi|_\infty) + C E_t \left[ \int_t^T |\phi'(Y_s)| ((1 + |Y_s|) + (1 + |\mathbb{E}[U_s]|)) ds \right] \\
& \quad + \beta \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] + \frac{1}{4} \mathbb{E}_t \left[ \int_t^T |\mathbb{E}[V_s]|^2 ds \right].
\end{aligned} \tag{2.23}$$

In view of the inequality for  $x > 0$ ,

$$1 + x \leq \left( 1 + \frac{1 - \alpha}{\gamma(1 + \alpha)} \right) e^{\frac{\gamma(1+\alpha)}{1-\alpha}x},$$

we have

$$\begin{aligned}
& C \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)| ((1 + |Y_s|) + (1 + |\mathbb{E}[U_s]|)) ds \right] \\
& \leq C \mathbb{E}_t \left[ \int_t^T |\phi'(Y_s)| \left( 1 + \frac{1 - \alpha}{\gamma(1 + \alpha)} \right) \left( e^{\gamma \frac{1+\alpha}{1-\alpha} |Y_s|} + e^{\gamma \frac{1+\alpha}{1-\alpha} |\mathbb{E}[U_s]|} \right) ds \right].
\end{aligned} \tag{2.24}$$

Since (by Young's inequality)

$$|\phi'(Y_s)| \left( e^{\gamma \frac{1+\alpha}{1-\alpha} |Y_s|} + e^{\gamma \frac{1+\alpha}{1-\alpha} |\mathbb{E}[U_s]|} \right) \leq (1 - \alpha) |\phi'(Y_s)|^{\frac{2}{1-\alpha}} + \frac{1 + \alpha}{2} \left( e^{\frac{2}{1-\alpha} \gamma |Y_s|} + e^{\frac{2}{1-\alpha} \gamma |\mathbb{E}[U_s]|} \right),$$

in view of the definition of the notations  $\mu_1$  and  $\mu_2$  in (2.13), we have

$$\begin{aligned}
& C E_t \left[ \int_t^T |\phi'(Y_s)| ((1 + Y_s) + (1 + |E[U_s]|)) ds \right] \\
& \leq C \mu_1 E_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] + C \mu_2 E_t \left[ \int_t^T \left( e^{\frac{2\gamma}{1-\alpha} |Y_s|} + e^{\frac{2\gamma}{1-\alpha} |E[U_s]|} \right) ds \right].
\end{aligned} \tag{2.25}$$

In view of inequality (2.23), we have

$$\begin{aligned}
& \phi(Y_t) + \frac{1}{2} E_t \left[ \int_t^T |Z_s|^2 ds \right] \\
& \leq \phi(|\xi|_\infty) + (\beta + C \mu_1) E_t \left[ \int_t^T |\phi'(Y_s)|^{\frac{2}{1-\alpha}} ds \right] \\
& \quad + C \mu_2 E_t \left[ \int_t^T \left( e^{\frac{2\gamma}{1-\alpha} |Y_s|} + e^{\frac{2\gamma}{1-\alpha} |E[U_s]|} \right) ds \right] + \frac{1}{4} \int_t^T |E[V_s]|^2 ds \\
& \leq \phi(|\xi|_\infty) + \left[ C \mu_2 + \gamma^{\frac{2}{\alpha-1}} (\beta + C \mu_1) \right] E_t \left[ \int_t^T e^{\frac{2\gamma}{1-\alpha} |Y_s|} ds \right] \\
& \quad + C \mu_2 E_t \left[ \int_t^T e^{\frac{2\gamma}{1-\alpha} |U_s|} ds \right] + \frac{1}{4} E \left[ \int_t^T |V_s|^2 ds \right].
\end{aligned} \tag{2.26}$$

*Step 2. Estimate of  $e^{\gamma|Y|_\infty}$ .*

In view of the last inequality of Lemma 2.1, we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq E_t \left[ e^{\frac{3}{1-\alpha}\gamma e^{CT}(|\xi|_\infty + C \int_t^T (2 + |E[U_s]| + |E[V_s]|^{1+\alpha}) ds)} \right]. \quad (2.27)$$

Since (by Young's inequality)

$$\frac{3e^{CT}}{1-\alpha}\gamma C|E[V_s]|^{1+\alpha} \leq \frac{1-\alpha}{2} \left( \frac{3e^{CT}}{1-\alpha}\gamma C \left( \frac{1+\alpha}{2\delta} \right)^{\frac{1+\alpha}{2}} \right)^{\frac{2}{1-\alpha}} + \delta|E[V_s]|^2, \quad (2.28)$$

in view of the definition of notation  $C_\delta$  in (2.11), we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq C_\delta \left[ e^{\left( \frac{3}{1-\alpha}\gamma e^{CT}|\xi|_\infty + \frac{3}{1-\alpha}\gamma C e^{CT}\varepsilon|U|_\infty + \delta \int_t^T |E[V_s]|^2 ds \right)} \right]; \quad (2.29)$$

and therefore using Jensen's inequality,

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq C_\delta e^{\left( \frac{3}{1-\alpha}\gamma e^{CT}(|\xi|_\infty + C\varepsilon|U|_\infty) \right)} E \left[ e^{\delta \int_t^T |V_s|^2 ds} \right]. \quad (2.30)$$

It follows from (2.16) and the definition of  $\mathcal{B}_\varepsilon$  that  $\|\sqrt{\delta}V \cdot W\|_{BMO_2(\mathbb{P})}^2 \leq \delta A < 1$ , applying the John-Nirenberg inequality to the BMO martingale  $\sqrt{\delta}V \cdot W$ , we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq \frac{C_\delta e^{\left( \frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty + \frac{3e^{CT}}{1-\alpha}C\gamma\varepsilon|U|_\infty \right)}}{1 - \delta\|V \cdot W\|_{BMO_2}^2} \leq \frac{C_\delta e^{\left( \frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty + \frac{3e^{CT}}{1-\alpha}C\gamma\varepsilon|U|_\infty \right)}}{1 - \delta A}. \quad (2.31)$$

Since  $3e^{CT}C\varepsilon \leq 1$  (see the choice of  $\varepsilon$  in (2.16)) and  $(U, V) \in \mathcal{B}_\varepsilon$ , we have

$$\begin{aligned} e^{\left( \frac{3}{1-\alpha}\gamma|Y|_\infty \right)} &\leq \frac{C_\delta e^{\left( \frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty + \frac{1}{1-\alpha}\gamma|U|_\infty \right)}}{1 - \delta A} \\ &\leq C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \left( C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \right)^{\frac{1}{2}} \\ &\leq \left( \frac{C_\delta e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \right)^{\frac{3}{2}}, \end{aligned} \quad (2.32)$$

which gives a half of the desired result (2.19).

*Step 3. Estimate of  $\|Z \cdot W\|_{BMO_2}^2$ .*

From inequality (2.26) and the definition of notation  $\mu$  in (2.14), we have

$$\frac{1}{2}E_t \left[ \int_t^T |Z_s|^2 ds \right] \leq \gamma^{-2}e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \varepsilon + \frac{1}{4}A. \quad (2.33)$$

In view of (2.17), we have

$$\frac{1}{2}\|Z \cdot W\|_{BMO_2}^2 \leq \gamma^{-2}e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3e^{CT}}{1-\alpha}\gamma|\xi|_\infty}}{1 - \delta A} \varepsilon + \frac{1}{4}A = \frac{1}{2}A. \quad (2.34)$$

The other half of the desired result (2.19) is then proved.

### 2.2.3 Contraction of the quadratic solution map

For  $(U, V) \in \mathcal{B}_\varepsilon$  and  $(\tilde{U}, \tilde{V}) \in \mathcal{B}_\varepsilon$ , set

$$(Y, Z) := \Gamma(U, V), \quad (\tilde{Y}, \tilde{Z}) := \Gamma(\tilde{U}, \tilde{V}).$$

That is,

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, E[U_s], Z_s, E[V_s]) ds - \int_t^T Z_s dW_s, \\ \tilde{Y}_t &= \xi + \int_t^T f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) ds - \int_t^T \tilde{Z}_s dW_s. \end{aligned} \quad (2.35)$$

We can define the vector process  $\beta$  in an obvious way such that

$$\begin{aligned} |\beta_s| &\leq C(1 + |Z_s| + |\tilde{Z}_s|), \\ f(s, Y_s, E[U_s], Z_s, E[V_s]) - f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) &= (Z_s - \tilde{Z}_s)\beta_s. \end{aligned} \quad (2.36)$$

Then  $\tilde{W}_t := W_t - \int_0^t \beta_s ds$  is a Brownian motion under the equivalent probability measure  $\tilde{P}$  defined by

$$d\tilde{P} := \mathcal{E}(\beta \cdot W)_0^T dP,$$

and from the above-established a priori estimate, there is  $K > 0$  such that  $\|\beta \cdot W\|_{BMO_2}^2 \leq K^2 := 3C^2T + 6C^2A$ .

In view of the following equation

$$\begin{aligned} Y_t - \tilde{Y}_t + \int_t^T (Z_s - \tilde{Z}_s) d\tilde{W}_s \\ = \int_t^T \left[ f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) - f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \right] ds, \end{aligned} \quad (2.37)$$

taking square and then the conditional expectation with respect to  $\tilde{P}$  (denoted by  $\tilde{E}_t$ ) on both sides of the last equation, we have the following standard estimates:

$$\begin{aligned} &|Y_t - \tilde{Y}_t|^2 + \tilde{E}_t \left[ \int_t^T |Z_s - \tilde{Z}_s|^2 ds \right] \\ &= \tilde{E}_t \left[ \left( \int_t^T \left( f(s, Y_s, E[U_s], \tilde{Z}_s, E[V_s]) - f(s, \tilde{Y}_s, E[\tilde{U}_s], \tilde{Z}_s, E[\tilde{V}_s]) \right) ds \right)^2 \right] \\ &\leq C^2 \tilde{E}_t \left[ \left( \int_t^T \left( |Y_s - \tilde{Y}_s| + |E[U_s - \tilde{U}_s]| + (1 + |E[V_s]|^\alpha + |E[\tilde{V}_s]|^\alpha) |E[V_s - \tilde{V}_s]| \right) ds \right)^2 \right] \\ &\leq 3C^2(T-t)^2(|U - \tilde{U}|_\infty^2 + |Y - \tilde{Y}|_\infty^2) \\ &\quad + 3C^2 \int_t^T (1 + |E[V_s]|^\alpha + |E[\tilde{V}_s]|^\alpha)^2 ds \int_t^T |E[V_s - \tilde{V}_s]|^2 ds \\ &\leq 3C^2(T-t)^2(|U - \tilde{U}|_\infty^2 + |Y - \tilde{Y}|_\infty^2) \\ &\quad + 9C^2 \int_t^T (1 + |E[V_s]|^{2\alpha} + |E[\tilde{V}_s]|^{2\alpha}) ds \int_t^T |E[V_s - \tilde{V}_s]|^2 ds \end{aligned} \quad (2.38)$$

We have for  $t \in [T - \varepsilon, T]$ ,

$$\begin{aligned}
& \int_t^T (1 + |E[V_s]|^{2\alpha} + |E[\tilde{V}_s]|^{2\alpha}) ds \\
& \leq \int_t^T (1 + E[|V_s|^2]^\alpha + E[|\tilde{V}_s|^2]^\alpha) ds \\
& \leq \varepsilon + \varepsilon^{1-\alpha} E \left[ \int_t^T |V_s|^2 ds \right]^\alpha + \varepsilon^{1-\alpha} E \left[ \int_t^T |\tilde{V}_s|^2 ds \right]^\alpha \\
& \leq \varepsilon^{1-\alpha} \left( T^\alpha + 2 + \alpha E \int_t^T |V_s|^2 ds + \alpha E \int_t^T |\tilde{V}_s|^2 ds \right) \\
& \leq \varepsilon^{1-\alpha} \left( T^\alpha + 2 + \alpha \|V \cdot W\|_{BMO_2(\mathbb{P})}^2 + \alpha \|\tilde{V} \cdot W\|_{BMO_2(\mathbb{P})}^2 \right) \\
& \leq \varepsilon^{1-\alpha} (T^\alpha + 2 + 2\alpha A).
\end{aligned} \tag{2.39}$$

Concluding the above estimates, we have for  $t \in [T - \varepsilon, T]$ ,

$$\begin{aligned}
& |Y_t - \tilde{Y}_t|^2 + \tilde{E}_t \left[ \int_t^T |Z_s - \tilde{Z}_s|^2 ds \right] \\
& \leq 3C^2 \varepsilon^2 (|U - \tilde{U}|_\infty^2 + |Y - \tilde{Y}|_\infty^2) \\
& \quad + 9C^2 (T^\alpha + 2 + 2\alpha A) \varepsilon^{1-\alpha} \|(V - \tilde{V}) \cdot W\|_{BMO_2(\mathbb{P})}^2.
\end{aligned} \tag{2.40}$$

In view of estimates (2.4), noting that  $1 - 3C^2 \varepsilon^2 \geq \frac{2}{3}$ , we have for  $t \in [T - \varepsilon, T]$ ,

$$\begin{aligned}
& |Y - \tilde{Y}|_\infty^2 + 3c_1^2 \|(Z - \tilde{Z}) \cdot W\|_{BMO_2(\mathbb{P})}^2 \\
& \leq 9C^2 \varepsilon^2 |U - \tilde{U}|_\infty^2 + 27C^2 (T^\alpha + 2 + 2\alpha A) \varepsilon^{1-\alpha} \|(V - \tilde{V}) \cdot W\|_{BMO_2(\mathbb{P})}^2.
\end{aligned} \tag{2.41}$$

It is then standard to show that there is a very small positive number  $\varepsilon$  such that the quadratic solution map  $\Gamma$  is a contraction on the previously given set  $\mathcal{B}_\varepsilon$ , by noting that  $A \leq 6\gamma^{-2}e^{\gamma|\xi|_\infty}$  from (2.16). The proof is completed by choosing a sufficiently small  $r_\varepsilon > 0$  such that  $\mathbb{B}_\varepsilon \subset \mathcal{B}_\varepsilon$ .

### 2.3 Global solution: the proof of Theorem 2.5

Let us first note that there exists a constant  $\tilde{C} > 0$  such that  $|\xi|^2 \leq \tilde{C}$  and

$$|2xh(s, y, \bar{y}, z, \bar{z}) + 2xf(s, 0, 0, 0, 0)| \leq \tilde{C} + \tilde{C}|x|^2 + \tilde{C}(|y|^2 + |\bar{y}|^2)$$

for any  $(x, y, \bar{y}, z, \bar{z}) \in R \times R^2 \times (R^d)^2$ . Let  $\alpha(\cdot)$  be the unique solution of the following ordinary differential equation:

$$\alpha(t) = \tilde{C} + \int_t^T \tilde{C} ds + \int_t^T (2\tilde{C} + \tilde{C})\alpha(s) ds, \quad t \in [0, T].$$

It is easy to see that  $\alpha(\cdot)$  is a continuous decreasing function and we have

$$\alpha(t) = \tilde{C} + \int_t^T \tilde{C}[1 + 2\alpha(s)] ds + \tilde{C} \int_t^T \alpha(s) ds, \quad t \in [0, T].$$

Define

$$\lambda := \sup_{t \in [0, T]} \alpha(t) = \alpha(0).$$

As  $|\xi|^2 \leq \tilde{C} \leq \lambda$ , Theorem 2.3 shows that there exists  $\eta_\lambda > 0$  which only depends on  $\lambda$ , such that BSDE has a local solution  $(Y, Z)$  on  $[T - \eta_\lambda, T]$  and it can be constructed through the Picard iteration.

Consider the Picard iteration:

$$\begin{aligned} Y_t^0 &= \xi + \int_t^T Z_s^0 dW_s; \quad \text{and for } j \geq 0, \\ Y_t^{j+1} &= \xi + \int_t^T [f(s, 0, 0, Z_s^{j+1}, 0) - f(s, 0, 0, 0, 0)] ds \\ &\quad + \int_t^T [f(s, 0, 0, 0, 0) + f(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j]) - f(s, 0, 0, Z_s^j, 0)] ds \\ &\quad - \int_t^T Z_s^{j+1} dW_s \\ &= \xi + \int_t^T [f(s, 0, 0, 0, 0) + h(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j])] ds \\ &\quad - \int_t^T Z_s^{j+1} d\tilde{W}_s^{j+1}, \quad t \in [T - \eta_\lambda, T], \end{aligned}$$

where

$$f(s, 0, 0, Z_s^{j+1}, 0) - f(s, 0, 0, 0, 0) = Z_s^{j+1} \beta_s^{j+1}$$

and the process

$$\tilde{W}_t^{j+1} = W_t - \int_0^t \beta_s^{j+1} \mathbf{1}_{[T - \eta_\lambda, T]}(s) ds$$

is a Brownian motion under an equivalent probability measure  $P^{j+1}$  which we denote by  $\tilde{P}$  with loss of generality, and under which the expectation is denoted by  $\tilde{E}$ . Using Itô's formula, it is straightforward to deduce the following estimate for  $r \in [T - \eta_\lambda, t]$ ,

$$\begin{aligned} &\tilde{E}_r[|Y_t^{j+1}|^2] + \tilde{E}_r\left[\int_r^T |Z_s^{j+1}|^2 ds\right] \\ &= \tilde{E}_r[|\xi|^2] + \tilde{C} \tilde{E}_r \int_t^T 2Y_s^{j+1} [f(s, 0, 0, 0, 0) + h(s, Y_s^j, E[Y_s^j], Z_s^j, E[Z_s^j])] ds \\ &\leq \tilde{C} + \tilde{C} \int_t^T \tilde{E}_r[|Y_s^{j+1}|^2] ds + \tilde{C} \int_t^T \{\tilde{E}_r[|Y_s^j|^2] + |E[Y_s^j]|^2 + 1\} ds. \end{aligned} \quad (2.42)$$

In what follows, we show by induction the following inequality:

$$|Y_t^j|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T]. \quad (2.43)$$

In fact, it is trivial to see that  $|Y_t^0|^2 \leq \alpha(t)$ , and let us suppose  $|Y_t^j|^2 \leq \alpha(t)$  for  $t \in [T - \eta_\lambda, T]$ . Then, from (2.42),

$$\tilde{E}_r[|Y_t^{j+1}|^2] \leq \tilde{C} + \tilde{C} \int_t^T [1 + 2\alpha(s)] ds + \tilde{C} \int_t^T \tilde{E}_r[|Y_s^{j+1}|^2] ds, \quad t \in [T - \eta_\lambda, T].$$

From the comparison theorem, we have

$$\widetilde{E}_r[|Y_t^{j+1}|^2] \leq \alpha(t), \quad t \in [T - \eta_\lambda, T].$$

Setting  $r = t$ , we have

$$|Y_t^{j+1}|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T].$$

Therefore, inequality (2.43) holds.

As  $Y_t = \lim_j Y_t^j$ , our constructed local solution  $(Y, Z)$  in  $[T - \eta_\lambda, T]$  satisfies then the following estimate:

$$|Y_t|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T], \quad \text{and} \quad |Y_t|^2 \leq \alpha(t), \quad t \in [T - \eta_\lambda, T].$$

In particular,  $|Y_{T-\eta_\lambda}|^2 \leq \alpha(T - \eta_\lambda) \leq \lambda$ .

Taking  $T - \eta_\lambda$  as the terminal time and  $Y_{T-\eta_\lambda}$  as terminal value, Theorem 2.3 shows that BSDE has a local solution  $(Y, Z)$  on  $[T - 2\eta_\lambda, T - \eta_\lambda]$  through the Picard iteration. Once again, using the Picard iteration and the fact that  $|Y_{T-\eta}|^2 \leq \alpha(T - \eta)$ , we deduce that  $|Y_t|^2 \leq \alpha(t)$ , for  $t \in [T - 2\eta_\lambda, T - \eta_\lambda]$ . Repeating the preceding process, we can extend the pair  $(Y, Z)$  to the whole interval  $[0, T]$  within a finite steps such that  $Y$  is uniformly bounded by  $\lambda$ . We now show that  $Z \cdot W$  is a  $BMO(P)$  martingale.

Identical to the proof of inequality (2.22), we have

$$\begin{aligned} & \phi(Y_t) + \frac{1}{2} E_t \left[ \int_t^T |Z_s|^2 ds \right] \\ & \leq \phi(|\xi|_\infty) + C E_t \left[ \int_t^T |\phi'(Y_s)| (1 + |Y_s| + E[|Y_s|]) ds \right] \\ & \leq \phi(|\xi|_\infty) + C \phi'(\lambda) E_t \left[ \int_t^T (1 + \lambda + \lambda) ds \right]. \end{aligned} \tag{2.44}$$

Consequently, we have

$$\begin{aligned} \|Z \cdot W\|_{BMO_2(\mathbb{P})}^2 &= \sup_\tau E_\tau \left[ \int_\tau^T |Z_s|^2 ds \right] \\ &\leq 2\phi(|\xi|_\infty) + 4C\phi'(\lambda)(1 + \lambda)T. \end{aligned} \tag{2.45}$$

Finally, we prove the uniqueness. Let  $(Y, Z)$  and  $(\widetilde{Y}, \widetilde{Z})$  be two adapted solutions. Then, we have (recall that  $\beta$  is defined by (?))

$$\begin{aligned} Y_t - \widetilde{Y}_t &= \int_t^T \left[ h(s, Y_s, E[Y_s], \widetilde{Z}_s, E[Z_s]) - h(s, \widetilde{Y}_s, E[\widetilde{Y}_s], \widetilde{Z}_s, E[\widetilde{Z}_s]) \right] ds \\ &\quad - \int_t^T (Z_s - \widetilde{Z}_s) d\widetilde{W}_s, \quad t \in [0, T]. \end{aligned} \tag{2.46}$$

Similar to the first two inequalities in (2.38), for any stopping time  $\tau$  which takes values

in  $[T - \varepsilon, T]$ , we have

$$\begin{aligned}
& |Y_\tau - \tilde{Y}_\tau|^2 + \tilde{E}_\tau \left[ \int_\tau^T |Z_s - \tilde{Z}_s|^2 ds \right] \\
&= \tilde{E}_\tau \left[ \left( \int_\tau^T \left[ h(s, Y_s, E[Y_s], \tilde{Z}_s, E[Z_s]) - h(s, \tilde{Y}_s, E[\tilde{Y}_s], \tilde{Z}_s, E[\tilde{Z}_s]) \right] ds \right)^2 \right] \\
&\leq C^2 \tilde{E}_\tau \left[ \left( \int_\tau^T \left[ |Y_s - \tilde{Y}_s| + |E[Y_s - \tilde{Y}_s]| + |E[Z_s - \tilde{Z}_s]| \right] ds \right)^2 \right] \tag{2.47} \\
&\leq 6C^2 \varepsilon^2 |Y - \tilde{Y}|_\infty^2 + 3C^2 \varepsilon \tilde{E}_\tau \left[ \int_\tau^T |Z_s - \tilde{Z}_s|^2 ds \right] + 3C^2 \varepsilon E \left[ \int_{T-\varepsilon}^T |Z_s - \tilde{Z}_s|^2 ds \right] \\
&\leq 6C^2 \varepsilon^2 |Y - \tilde{Y}|_\infty^2 + 3C^2 \varepsilon \|(Z - \tilde{Z}) \cdot \tilde{W}\|_{BMO_2(\tilde{P})}^2 + 3C^2 \varepsilon \|(Z - \tilde{Z}) \cdot W\|_{BMO_2(\mathbb{P})}^2 \\
&\leq 6C^2 \varepsilon^2 |Y - \tilde{Y}|_\infty^2 + 3C^2 (1 + c_2^2) \varepsilon \|(Z - \tilde{Z}) \cdot W\|_{BMO_2(\mathbb{P})}^2.
\end{aligned}$$

Therefore, we have (on the interval  $[T - \varepsilon, T]$ )

$$\begin{aligned}
& \left| Y - \tilde{Y} \right|_\infty^2 + c_1^2 \left\| (Z - \tilde{Z}) \cdot W \right\|_{BMO_2(\mathbb{P})}^2 \\
&\leq 6C^2 \varepsilon^2 \left| Y - \tilde{Y} \right|_\infty^2 + 3C^2 (1 + c_2^2) \varepsilon \left\| (Z - \tilde{Z}) \cdot W \right\|_{BMO_2(\mathbb{P})}^2. \tag{2.48}
\end{aligned}$$

Note that since

$$|\beta| \leq C(1 + |Z| + |\tilde{Z}|),$$

the two generic constants  $c_1$  and  $c_2$  only depend on the sum

$$\|Z \cdot W\|_{BMO_2(\mathbb{P})}^2 + \|\tilde{Z} \cdot W\|_{BMO_2(\mathbb{P})}^2.$$

Then when  $\varepsilon$  is sufficiently small, we conclude that  $Y = \tilde{Y}$  and  $Z = \tilde{Z}$  on  $[T - \varepsilon, T]$ . Repeating iteratively with a finite of times, we have the uniqueness on the given interval  $[0, T]$ .

### 3 The expected term is additive and has a quadratic growth in the second unknown variable

Let us first consider the following quadratic BSDE with mean term:

$$Y_t = \xi + \int_t^T [f_1(s, Z_s) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds - \int_t^T Z_s dB_s, \quad t \in [0, T] \tag{3.1}$$

where  $f_1 : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies:  $f_1(\cdot, z)$  is an adapted process for any  $z$ , and

$$|f_1(t, 0)| \leq C, \quad |f_1(t, z) - f_1(t, z')| \leq C(1 + |z| + |z'|)|z - z'|, \tag{3.2}$$

and  $f_2 : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies:  $f_2(\cdot, z, \bar{z})$  is an adapted process for any  $z$  and  $\bar{z}$ , and

$$|f_2(t, z, \bar{z})| \leq C(1 + |z|^2 + |\bar{z}|^2). \tag{3.3}$$

**Proposition 3.1.** *Let us suppose that  $f_1$  and  $f_2$  be two generators satisfying the above conditions and  $\xi$  be a bounded random variable. Then (3.1) admits a unique solution  $(Y, Z)$  such that  $Y$  is bounded and  $Z \cdot W$  is a BMO martingale.*

*Proof.* Let us first prove the existence. We solve this equation in two steps:

Step one. First solve the following BSDE:

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, Z_s) ds - \int_t^T Z_s \cdot dW_s. \quad (3.4)$$

It is well known that this BSDE admits a unique solution  $(\tilde{Y}, Z)$  such that  $\tilde{Y}$  is bounded and  $Z \cdot W$  is a BMO martingale.

Step two. Define

$$Y_t = \tilde{Y}_t + \int_t^T E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds. \quad (3.5)$$

Then

$$Y_t = \xi + \int_t^T [f_1(s, Z_s) + E[f_2(s, Z_s, \mathbb{E}[Z_s])]] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (3.6)$$

The uniqueness can be proved in a similar way: Let  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  be two solutions. Then set

$$\tilde{Y}_t^1 = Y_t^1 - \int_t^T E[f_2(s, Z_s^1, \mathbb{E}[Z_s^1])] ds, \quad \tilde{Y}_t^2 = Y_t^2 - \int_t^T E[f_2(s, Z_s^2, \mathbb{E}[Z_s^2])] ds.$$

$(\tilde{Y}^1, Z^1)$  and  $(\tilde{Y}^2, Z^2)$  being solution of the same BSDE (4.4), from the uniqueness of solution to this BSDE,

$$\tilde{Y}^1 = \tilde{Y}^2, \quad Z^1 = Z^2,$$

Hence  $Y^1 = Y^2$ , and  $Z^1 = Z^2$ . □

Now we consider a more general form of BSDE with a mean term:

$$\begin{aligned} Y_t = & \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])]] ds \\ & - \int_t^T Z_s \cdot dW_s. \end{aligned} \quad (3.7)$$

Here for  $i = 1, 2$ ,  $f_i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies: for any  $(y, \bar{y}, z, \bar{z})$ ,  $f_i(t, y, \bar{y}, z, \bar{z})$  is an adapted process, and

$$|f_1(t, y, \bar{y}, 0, \bar{z})| + |f_2(t, 0, 0, 0, 0)| \leq C,$$

$$\begin{aligned} |f_1(t, y, z, \bar{y}, \bar{z}) - f_1(t, y', z', \bar{y}', \bar{z}')| & \leq C(|y - y'| + |\bar{y} - \bar{y}'| + |\bar{z} - \bar{z}'| + (1 + |z| + |z'|)|z - z'|), \\ |f_2(t, y, z, \bar{y}, \bar{z}) - f_2(t, y', z', \bar{y}', \bar{z}')| & \leq C(|y - y'| + |\bar{y} - \bar{y}'| + (1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|)). \end{aligned}$$

We have the following result.



**Theorem 3.2.** Assume that  $f_1$  and  $f_2$  satisfy the above conditions, and  $\xi$  is a bounded random variable. BSDE (3.7) has a unique solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^\infty$  and  $Z \cdot W \in BMO_2(\mathbb{P})$ .

**Example 3.3.** The condition on  $f_1$  means that this function should be bounded with respect to  $(y, \bar{y}, \bar{z})$ . For example,

$$f_1(s, y, \bar{y}, z, \bar{z}) = 1 + |\sin(y)| + |\sin(\bar{y})| + \frac{1}{2}|z|^2 + |\sin(\bar{z})|,$$

$$f_2(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |y'| + \frac{1}{2}(|z| + |\bar{z}|)^2.$$

*Proof.* We prove the theorem by a fixed point argument. Let  $U \in \mathcal{S}^\infty$ , and  $V \cdot W \in BMO_2(\mathbb{P})$ , we define  $(Y, Z \cdot W) \in \mathcal{S}^\infty \times BMO_2(\mathbb{P})$  as the unique solution to BSDE with mean:

$$\begin{aligned} Y_t &= \xi + \int_t^T [f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) + E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds \\ &\quad - \int_t^T Z_s \cdot dW_s. \end{aligned} \quad (3.8)$$

And we define the map  $\Gamma : (U, V) \mapsto \Gamma(U, V) = (Y, Z)$  on  $\mathcal{S}^\infty \times BMO_2(\mathbb{P})$ . Set

$$\tilde{Y}_t = Y_t - \int_t^T E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds,$$

then  $(\tilde{Y}, Z)$  is the solution to

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s dB_s.$$

As  $|f_1(t, y, \bar{y}, 0, \bar{z})| \leq C$ , we have

$$|f_1(t, y, \bar{y}, z, \bar{z})| \leq C + \frac{C}{2}|z|^2.$$

Applying Ito's formula to  $\phi_C(|\tilde{Y}|)$ ,

$$\phi_C(|\tilde{Y}_t|) + \int_t^T \frac{1}{2} \phi_C''(|\tilde{Y}_s|) |Z_s|^2 ds \leq \phi_C(\xi) + \int_t^T |\phi_C'(|\tilde{Y}_s|)| (C + \frac{C}{2}|Z_s|^2) ds - \int_0^T \phi'(|\tilde{Y}_s|) Z_s dW_s, \quad (3.9)$$

we can prove that there exists a constant  $K > 0$  such that

$$\|Z\|_{BMO} \leq K.$$

Let  $(U^1, V^1)$  and  $(U^2, V^2)$ , and  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  be the corresponding solution, and

$$\tilde{Y}_t^1 = \xi + \int_t^T f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[V_s^1]) ds - \int_t^T Z_s^1 dB_s,$$

$$\tilde{Y}_t^2 = \xi + \int_t^T f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2]) ds - \int_t^T Z_s^2 dB_s.$$

There exists a bounded adapted process  $\beta$  such that

$$f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[V_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2]) = \beta_s(Z_s^1 - Z_s^2).$$

Then  $\tilde{W}_t := W_t - \int_0^t \beta_s ds$  is a Brown motion under the equivalent probability measure  $\tilde{\mathbb{P}}$  defined by

$$d\tilde{\mathbb{P}} = \mathcal{E}_0^T(\beta \cdot W)_0^T d\mathbb{P}.$$

We have

$$\begin{aligned} & \Delta \tilde{Y}_t + \int_t^T \Delta Z_s d\tilde{W}_s \\ &= \int_t^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^2, \mathbb{E}[V_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2])] ds, \quad t \in [0, T]. \end{aligned} \quad (3.10)$$

For any stopping time  $\tau$  which takes values in  $[T - \varepsilon, T]$ , taking square and then the conditional expectation with respect to  $\tilde{\mathbb{P}}$  (denoted by  $\tilde{\mathbb{E}}_\tau$ ), we have

$$\begin{aligned} & |\Delta \tilde{Y}_\tau|^2 + \tilde{E}_\tau \int_\tau^T |\Delta Z|^2 ds \\ &= \tilde{E}_\tau \left( \int_\tau^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^2, \mathbb{E}[V_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2])] ds \right)^2 \\ &\leq C^2 \tilde{E}_\tau \left( \left[ \int_\tau^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]| + |\mathbb{E}[\Delta V_s]|) ds \right]^2 \right) \\ &\leq C^2 \varepsilon (|\Delta U|_\infty^2 + |\Delta V \cdot W|_{BMO_2(\mathbb{P})}^2). \end{aligned} \quad (3.11)$$

Therefore, we have (on the interval  $[T - \varepsilon, T]$ )

$$\left| \Delta \tilde{Y} \right|_\infty^2 + c_1^2 \|(\Delta Z) \cdot W\|_{BMO_2(\mathbb{P})}^2 \leq C^2 \varepsilon (|\Delta U|_\infty^2 + |\Delta V \cdot W|_{BMO_2(\mathbb{P})}^2). \quad (3.12)$$

Note that since

$$|\beta| \leq C(1 + |Z_1| + |Z_2|),$$

the generic constant  $c_1$  and  $c_2$  only depend on  $C$  and  $K$ .

As

$$\Delta Y_t = \Delta \tilde{Y}_t + \int_t^T E [f_2(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^2]) - f_2(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds,$$

then

$$\begin{aligned} |\Delta Y_t| &\leq |\Delta \tilde{Y}_t| + CE \left[ \int_t^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]|) ds \right] \\ &\quad + CE \left[ \int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E}[Z_s^1]| + |\mathbb{E}[Z_s^2]|) (|\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds \right] \end{aligned}$$

we deduce that

$$\begin{aligned} |\Delta Y_t|^2 &\leq C^2 \left( |\Delta \tilde{Y}_t|^2 + E \left[ \int_t^T (|\Delta U_s| + |\mathbb{E} \Delta U_s|) ds \right]^2 \right. \\ &\quad \left. E \left[ \int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E}[Z_s^1]| + |\mathbb{E}[Z_s^2]|) (|\Delta Z_s| + |\mathbb{E} \Delta Z_s|) ds \right]^2 \right), \end{aligned}$$

which implies that

$$\begin{aligned} |\Delta Y|_\infty^2 &\leq C^2 (|\Delta \tilde{Y}|_\infty^2 + \varepsilon |\Delta U|_\infty^2 + |(\Delta Z) \cdot W|_{BMO_2(\mathbb{P})}^2) \\ &\leq C^2 \varepsilon (|\Delta U|_\infty^2 + \|V \cdot W\|_{BMP_2(P)}^2). \end{aligned}$$

Then when  $\varepsilon$  is sufficiently small, we conclude that the application is contracting on  $[T - \varepsilon, T]$ . Repeating iteratively with a finite of times, we have the existence and uniqueness on the given interval  $[0, T]$ .  $\square$

## 4 Multi-dimensional Case

In this section, we will study the multi-dimensional case of (3.7), where  $f_1$  is Lipschitz. Our result generalizes the corresponding one of Cheridito and Nam [8].

We first consider the following BSDE with mean term:

$$Y_t = \xi + \int_t^T [f_1(s, Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T] \quad (4.1)$$

where  $f_1 : \Omega \times [0, T] \times R^{d \times n} \times \mathbb{R}^{d \times n} \rightarrow R^n$  satisfies: for any  $z$  and  $\bar{z}$ ,  $f_1(t, z, \bar{z})$  is an adapted process, and

$$|f_i(t, 0, 0)| \leq C, \quad |f_1(t, z, \bar{z}) - f_1(t, z', \bar{z}')| \leq C(|z - z'| + |\bar{z} - \bar{z}'|), \quad (4.2)$$

and  $f_2 : \Omega \times [0, T] \times R^{d \times n} \times R^{d \times n} \rightarrow R^n$  satisfies: for any  $z, \bar{z}$ ,  $f_2(t, z, \bar{z})$  is an adapted process, and

$$|f_2(t, 0, 0)| \leq C, \quad |f_2(t, z, \bar{z}) - f_2(t, z', \bar{z}')| \leq C(1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|). \quad (4.3)$$

**Proposition 4.1.** *Let us suppose that  $f_1$  and  $f_2$  be two generators satisfying the above conditions and  $\xi \in L^2(\mathcal{F}_T)$ . Then (4.1) admits a unique solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^2$  and  $Z \in \mathcal{M}^2$ .*

*Proof.* Let us first prove the existence. Our proof is divided into the following two steps.

Step one. First consider the following BSDE

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, Z_s, \mathbb{E}[Z_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T]. \quad (4.4)$$

It admits a unique solution  $(\tilde{Y}, Z)$  such that  $\tilde{Y} \in \mathcal{S}^2$  and  $Z \in \mathcal{M}^2$ , see Buckdahn et al. [6].

Step two. Define

$$Y_t = \tilde{Y}_t + \int_t^T E[f_2(s, Z_s, \mathbb{E}[Z_s])] ds, \quad t \in [0, T]. \quad (4.5)$$

Then, we have for  $t \in [0, T]$ ,

$$Y_t = \xi + \int_t^T [f_1(s, Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Z_s, \mathbb{E}[Z_s])]] ds - \int_t^T Z_s \cdot dW_s. \quad (4.6)$$

The uniqueness can be proved in a similar way: Let  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  be two solutions. Then set

$$\tilde{Y}_t^1 = Y_t^1 - \int_t^T E[f_2(s, Z_s^1, \mathbb{E}[Z_s^1])] ds, \quad \tilde{Y}_t^2 = Y_t^2 - \int_t^T E[f_2(s, Z_s^2, \mathbb{E}[Z_s^2])] ds.$$

$(\tilde{Y}^1, Z^1)$  and  $(\tilde{Y}^2, Z^2)$  being solution of the same BSDE (4.4), from the uniqueness of solution to this BSDE,

$$\tilde{Y}^1 = \tilde{Y}^2, \quad Z^1 = Z^2,$$

Hence  $Y^1 = Y^2$ , and  $Z^1 = Z^2$ .  $\square$

Now we consider the following more general form of BSDE with a mean term:

$$\begin{aligned} Y_t = & \xi + \int_t^T [f_1(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])]] ds \\ & - \int_t^T Z_s \cdot dW_s, t \in [0, T]. \end{aligned} \quad (4.7)$$

Here for  $i = 1, 2$ ,  $f_i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^n$  satisfies: for any  $(y, \bar{y}, z, \bar{z})$ ,  $f_i(t, y, \bar{y}, z, \bar{z})$  is an adapted process, and

$$|f_1(t, y, \bar{y}, 0, 0)| + |f_2(t, 0, 0, 0, 0)| \leq C, \quad (4.8)$$

$$|f_1(t, y, z, \bar{y}, \bar{z}) - f_1(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + |z - z'| + |\bar{z} - \bar{z}'|), \quad (4.9)$$

$$|f_2(t, y, z, \bar{y}, \bar{z}) - f_2(t, y', z', \bar{y}', \bar{z}')| \leq C(|y - y'| + |\bar{y} - \bar{y}'| + (1 + |z| + |z'| + |\bar{z}| + |\bar{z}'|)(|z - z'| + |\bar{z} - \bar{z}'|)). \quad (4.10)$$

We have the following result.

**Theorem 4.2.** Assume that  $f_1$  and  $f_2$  satisfy the inequalities (4.8)-(4.10), and  $\xi \in L^2(\mathcal{F}_T)$ . Then BSDE (4.7) has a unique solution  $(Y, Z)$  such that  $Y \in \mathcal{S}^2$  and  $Z \in \mathcal{M}^2$ .

**Example 4.3.** The condition (4.8) requires that both functions  $f_1$  and  $f_2$  should be bounded with respect to  $(y, \bar{y})$ . For example,

$$f_1(s, y, \bar{y}, z, \bar{z}) = 1 + |\sin(y)| + |\sin(\bar{y})| + |z| + |\bar{z}|,$$

$$f_2(s, y, \bar{y}, z, \bar{z}) = 1 + |y| + |\bar{y}| + \frac{1}{2}(|z| + |\bar{z}|)^2.$$

*Proof.* We prove the theorem by a fixed point argument. Let  $U \in \mathcal{S}^2$ , and  $V \in \mathcal{M}^2$ , we define  $(Y, Z \cdot W) \in \mathcal{S}^2 \times \mathcal{M}^2$  as the unique solution to BSDE with mean:

$$\begin{aligned} Y_t &= \xi + \int_t^T [f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s]) + E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds \\ &\quad - \int_t^T Z_s \cdot dW_s. \end{aligned} \quad (4.11)$$

We define the map  $\Gamma : (U, V) \mapsto \Gamma(U, V) = (Y, Z)$  on  $\mathcal{S}^2 \times \mathcal{M}^2$ . Set

$$\tilde{Y}_t = Y_t - \int_t^T E[f_2(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[Z_s])] ds, \quad t \in [0, T].$$

Then  $(\tilde{Y}, Z)$  is the solution to

$$\tilde{Y}_t = \xi + \int_t^T f_1(s, U_s, \mathbb{E}[U_s], Z_s, \mathbb{E}[V_s]) ds - \int_t^T Z_s \cdot dW_s, \quad t \in [0, T].$$

As  $|f_1(t, y, \bar{y}, 0, 0)| \leq C$ , we have

$$|f_1(t, y, \bar{y}, z, \bar{z})| \leq C(1 + |z| + |\bar{z}|).$$

Applying Ito's formula to  $|\tilde{Y}|^2$ , we have

$$|\tilde{Y}_t|^2 + \int_t^T |Z_s|^2 ds \leq |\xi|^2 + 2C \int_t^T |\tilde{Y}_s|(C + |Z_s| + |\mathbb{E}[Z]|) ds - 2 \int_t^T (Y_s, Z_s \cdot dW_s). \quad (4.12)$$

Further, using standard techniques, we can prove that there exists a constant  $K > 0$  such that

$$\|Z\|_{\mathcal{M}^2} \leq K.$$

For  $(U^i, V^i) \in \mathcal{S}^2 \times \mathcal{M}^2$ , define  $(Y^i, Z^i) = \Gamma(U^i, V^i)$  with  $i = 1, 2$ . Further, set for  $t \in [0, T]$ ,

$$\begin{aligned} \tilde{Y}_t^1 &:= \xi + \int_t^T f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[V_s^1]) ds - \int_t^T Z_s^1 \cdot dW_s, \\ \tilde{Y}_t^2 &:= \xi + \int_t^T f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[V_s^2]) ds - \int_t^T Z_s^2 \cdot dW_s. \end{aligned}$$

We have

$$\begin{aligned} &\Delta \tilde{Y}_t + \int_t^T \Delta Z_s dW_s \\ &= \int_t^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds, \quad t \in [0, T]. \end{aligned} \quad (4.13)$$

For any  $t \in [T - \varepsilon, T]$ , taking square and then expectations on both sides of the last equality, we have

$$\begin{aligned}
& \mathbb{E}[|\Delta \tilde{Y}_t|^2] + \mathbb{E} \left[ \int_t^T |\Delta Z_s|^2 ds \right] \\
&= \mathbb{E}_\tau \left( \int_t^T [f_1(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_1(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds \right)^2 \\
&\leq C^2 \mathbb{E} \left( \left[ \int_\tau^T (|\Delta U_s| + |\mathbb{E}[\Delta U_s]| + |\Delta Z_s| + |\mathbb{E}[\Delta Z_s]|) ds \right]^2 \right) \\
&\leq C^2 \varepsilon (|\Delta U|_{\mathcal{S}^2}^2 + \|\Delta Z\|_{\mathcal{M}^2}^2).
\end{aligned} \tag{4.14}$$

Therefore, we have (on the interval  $[T - \varepsilon, T]$ ) for a sufficiently small  $\varepsilon > 0$ ,

$$\left| \Delta \tilde{Y} \right|_{\mathcal{S}^2}^2 + \|\Delta Z\|_{\mathcal{M}^2}^2 \leq C^2 \varepsilon (|\Delta U|_{\mathcal{S}^2}^2). \tag{4.15}$$

As

$$\Delta Y_t = \Delta \tilde{Y}_t + \int_t^T E [f_2(s, U_s^1, \mathbb{E}[U_s^1], Z_s^1, \mathbb{E}[Z_s^1]) - f_2(s, U_s^2, \mathbb{E}[U_s^2], Z_s^2, \mathbb{E}[Z_s^2])] ds,$$

we have

$$\begin{aligned}
|\Delta Y_t| &\leq |\Delta \tilde{Y}_t| + CE \left[ \int_t^T (|\Delta U_s| + |\mathbb{E} \Delta U_s|) ds \right] \\
&\quad + CE \left[ \int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E} Z_s^1| + |\mathbb{E} Z_s^2|) (|\Delta Z_s| + |\mathbb{E} \Delta Z_s|) ds \right].
\end{aligned}$$

Moreover, we deduce that

$$\begin{aligned}
|\Delta Y_t|^2 &\leq C^2 \left( |\Delta \tilde{Y}_t|^2 + E \left[ \int_t^T (|\Delta U_s| + |\mathbb{E} \Delta U_s|) ds \right]^2 \right. \\
&\quad \left. + E \left[ \int_t^T (1 + |Z_s^1| + |Z_s^2| + |\mathbb{E} Z_s^1| + |\mathbb{E} Z_s^2|) (|\Delta Z_s| + |\mathbb{E} \Delta Z_s|) ds \right]^2 \right).
\end{aligned}$$

In view of (4.15)

$$\left| \Delta \tilde{Y} \right|_{\mathcal{S}^2}^2 + \|\Delta Z\|_{\mathcal{M}^2}^2 \leq C^2 \varepsilon (|\Delta U|_{\mathcal{S}^2}^2).$$

Then when  $\varepsilon$  is sufficiently small, we conclude that the application is contracting on  $[T - \varepsilon, T]$ . Repeating iteratively with a finite of times, we have the existence and uniqueness on the given interval  $[0, T]$ . □

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