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One-dimensional q -state Potts model with multi-site interactions

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Abstract. A one-dimensional (1D) q -state Potts model with N sites, m -site interaction K in a field H is studied for arbitrary values of m . Exact results for the partition function and the two-point correlation function are obtained at $H = 0$. The system in a field is shown to be self-dual. Using a change of Potts variables, it is mapped onto a standard 2D Potts model, with first-neighbour interactions K and H , on a cylinder with helical boundary conditions (BC). The 2D system has a length N/m and a transverse size m . Thus the Potts chain with multi-site interactions is expected to develop a 2D critical singularity along the self-duality line, $(e^{qK} - 1)(e^{qH} - 1) = q$, when $N/m \rightarrow \infty$ and $m \rightarrow \infty$.

Keywords: Potts model, multi-site interactions, self-duality, helical boundary conditions

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1. Introduction

The standard Potts model is a lattice statistical model with pair interactions between q -state variables attached to neighbouring sites [1, 2]. Multi-site Potts models can be constructed by extending to an arbitrary number of states existing multispin Ising models for which $q = 2$. In this way, a self-dual three-site Potts model on the triangular lattice was introduced by Enting [3, 4], which corresponds to the Baxter-Wu model [5, 6] when $q = 2$. Similarly, a 2D self-dual Potts model with m -site interactions in one direction and n -site interactions in the other [7–9] follows from the Ising version with $n = 1$ [10].

Multi-site interactions may be generated from two-site interactions in a position-space renormalisation group transformation and thus have to be included in the initial Hamiltonian. In this way Schick and Griffiths have introduced a three-state Potts model on the triangular lattice with two- and three-site interactions [11]. For any value of q it can be reformulated as a standard q -state Potts model with two-site interactions on a

3-12 lattice [12]. When the three-site interactions are restricted to up-pointing triangles, the model is self-dual [13, 14] and related to a 20-vertex model [13, 15].

Extending the results of Fortuin and Kasteleyn [16] for pair interactions, a random-cluster representation for Potts models with multi-site interactions has been introduced [17–19] and exploited in Monte Carlo simulations [20].

Multi-site interactions enter naturally when the site percolation process is formulated as a Potts model in the limit $q \rightarrow 1$ [21–25]. Various Potts multi-site interactions have also been used to model conformational transitions in polypeptide chains [26–29].

With $s_j = 0, 1, \dots, q - 1$ denoting a q -state Potts variable attached to site j , a multi-site interaction can take one of the following forms

$$(a) \quad -K \prod_{j=1}^{m-1} \delta_{s_j, s_{j+1}}, \quad (b) \quad -K \delta_q \left(\sum_{l=0}^{m-1} s_{j+l} \right), \quad (1.1)$$

where $\delta_{n,n'}$ is the standard Kronecker delta and $\delta_q(n)$ is a Kronecker delta modulo q . When $K > 0$ the ground state is q -times degenerate in the first case (the standard one) whereas the degeneracy depends on m and is given by q^{m-1} in the second case. As an example, when $q = m = 3$ the degenerate ground states are the following ones:

$$(a) \quad \begin{Bmatrix} 000 \\ 111 \\ 222 \end{Bmatrix}, \quad (b) \quad \begin{Bmatrix} 000 & 012 & 210 \\ 111 & 120 & 021 \\ 222 & 201 & 102 \end{Bmatrix} \quad (1.2)$$

In the present work we generalize for q -state Potts variables some results recently obtained for the 1D Ising model with multispin interactions [30]. The Hamiltonian of the q -state Potts chain takes the following form:

$$-\beta \mathcal{H}_N[\{s\}] = K \sum_j \left[q \delta_q \left(\sum_{l=0}^{m-1} s_{j+l} \right) - 1 \right] + H \sum_j [q \delta_q(s_j) - 1], \quad \beta = (k_B T)^{-1}. \quad (1.3)$$

We assume ferromagnetic interactions $K \geq 0$ and $H \geq 0$, too. The Kronecker delta modulo q is given by:

$$\delta_q(s) = \frac{1}{q} \sum_{k=0}^{q-1} \exp \left(\frac{2i\pi ks}{q} \right) = \begin{cases} 1 & \text{when } s = 0 \pmod{q} \\ 0 & \text{otherwise} \end{cases}. \quad (1.4)$$

Introducing the Potts spins [31, 32]

$$\sigma_j = \exp \left(\frac{2i\pi s_j}{q} \right), \quad (1.5)$$

the Hamiltonian in (1.3) can be rewritten as †:

$$-\beta \mathcal{H}_N[\{\sigma\}] = K \sum_j \sum_{k=1}^{q-1} \prod_{l=0}^{m-1} \sigma_{j+l}^k + H \sum_j \sum_{k=1}^{q-1} \sigma_j^k. \quad (1.6)$$

† One may also express the Potts interaction using clock angular variables (see appendix A).

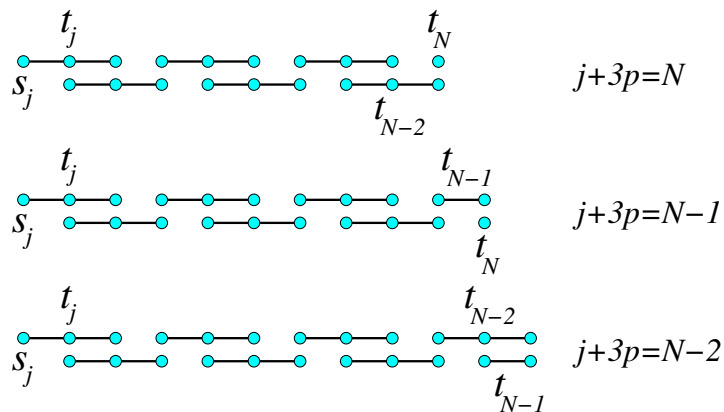


Figure 1. t -variables entering into the expression (2.3) of s_j for $m = 3$ and different values of the distance $N - j$ from the end of the chain. The t -variables, defined in (2.2), are the sums of m s -variables (circles) with the convention $s_i = 0$ when $i > N$. The second lines are subtracted from the first so that only s_j is remaining.

When $q = 2$, $\sigma_j = \pm 1$, $k = 1$ and the Ising multispin Hamiltonian studied in [30] is recovered, which *a posteriori* justifies the choice of interaction (b) in (1.3).

The zero-field partition function of the Potts chain with m -site interaction K is obtained for free BC in section 2 and for periodic BC in section 3. The periodic BC result allows a determination of the eigenvalues of \mathbf{T}^m where \mathbf{T} is the site-to-site transfer-matrix. The two-site correlation function is calculated in section 4. In section 5 the system with periodic BC is shown to be self-dual when the external field H is turned on. In section 6 the system with free BC is mapped onto a standard 2D Potts model with first-neighbour interactions K and H , length N/m and transverse size m . The mapping of 1D Potts models with m -site and n -site interactions is discussed in section 7. The conclusion in section 8 is followed by 4 appendices.

2. Zero-field partition function for free BC

With free BC the zero-field Hamiltonian of a chain with N Potts spins, with m -site interaction K , takes the following form

$$-\beta\mathcal{H}_N^{(f)}[\{s\}] = K \sum_{j=1}^{N-m+1} \left[q\delta_q \left(\sum_{l=0}^{m-1} s_{j+l} \right) - 1 \right] \quad (2.1)$$

when written in terms of the Potts variables s_j . Let us introduce the new Potts variables $t_j = 0, \dots, q - 1$ defined as

$$t_j = \sum_{l=0}^{m-1} s_{j+l} \pmod{q}, \quad j = 1, \dots, N, \quad (2.2)$$

with the convention $s_i = 0$ when $i > N$ in (2.2). Note that the relationship between old

and new variables is one-to-one with the inverse transformation given by (see figure 1):

$$s_j = \sum_{r=0}^p (t_{rm+j} - t_{r(m+j+1)}) \pmod{q}, \quad j + pm = N - l, \quad l = 0, \dots, m - 1. \quad (2.3)$$

Using (2.2) in (2.1) one obtains a system of $N - m + 1$ non-interacting Potts spins in a field K with

$$-\beta\mathcal{H}_N^{(f)}[\{t\}] = K \sum_{j=1}^{N-m+1} [q\delta_q(t_j) - 1]. \quad (2.4)$$

The canonical partition function is easily obtained and reads:

$$\begin{aligned} \mathcal{Z}_N^{(f)} &= \text{Tr}_{\{t\}} e^{-\beta\mathcal{H}_N^{(f)}[\{t\}]} = \prod_{j=1}^{N-m+1} \text{Tr}_{t_j} e^{K[q\delta_q(t_j)-1]} \prod_{j=N-m+2}^N \text{Tr}_{t_j} 1 \\ &= q^{m-1} [e^{(q-1)K} + (q-1)e^{-K}]^{N-m+1}. \end{aligned} \quad (2.5)$$

Note that although only $N - m + 1$ new variables enter into the expression of the transformed Hamiltonian (2.4), one has to trace over the N Potts variables t_j in (2.5). When $q = 2$ the Ising result (equation (2.6) in [30]) is recovered.

The free energy can be decomposed as follows

$$\mathcal{F}_N^{(f)} = -k_B T \ln \mathcal{Z}_N^{(f)} = Nf_b + \mathcal{F}_s(m), \quad (2.6)$$

where the bulk free energy per site

$$f_b = -k_B T \ln [e^{(q-1)K} + (q-1)e^{-K}], \quad (2.7)$$

does not depend on m whereas the surface contribution

$$\mathcal{F}_s(m) = (m-1)k_B T \ln \left\{ \frac{\exp[(q-1)K] + (q-1)\exp(-K)}{q} \right\}, \quad (2.8)$$

is m -dependent.

3. Zero-field partition function for periodic BC

Let us now evaluate the partition function for a periodic chain with N sites and $m > 1$. To simplify the discussion we consider only the case where N is a multiple of m . Then the Hamiltonian takes the following form

$$-\beta\mathcal{H}_{N=pm}^{(p)}[\{s\}] = K \sum_{j=1}^{N=pm} \left[q\delta_q \left(\sum_{l=0}^{m-1} s_{j+l} \right) - 1 \right], \quad (3.1)$$

with $s_{N+j} = s_j$. Making use of the change of variables (2.2), it can be rewritten as:

$$-\beta\mathcal{H}_{N=pm}^{(p)}[\{t\}] = K \sum_{j=1}^{N=pm} [q\delta_q(t_j) - 1]. \quad (3.2)$$

With periodic BC the correspondence between $\{s\}$ and $\{t\}$ Potts configurations is no longer one-to-one and the new variables have to satisfy a set of $m - 1$ constraints [30, 33–35].

There are several $\{s\}$ configurations leading to the same $\{t\}$. One of these configurations, $\{s'\}$, is obtained by changing s_j into

$$s'_j = s_j + \Delta_j \pmod{q}, \quad j = 1, \dots, N. \quad (3.3)$$

where the shifts $\Delta_j = 0, \dots, q-1$ have to satisfy some constraint. Let us first consider $t_1 = \sum_{l=0}^{m-1} s_{l+1} \pmod{q}$. One can freely choose the first $m-1$ shifts (q^{m-1} choices) and t_1 keeps its value when Δ_m is such that $\sum_{l=0}^{m-1} \Delta_{l+1} = 0 \pmod{q}$. Since t_j and t_{j+1} have the shifts Δ_{j+l} ($l = 1, \dots, m-1$) in common, the value of Δ_j leaving t_j invariant is equal to the value of Δ_{j+m} leaving t_{j+1} invariant. When $N = pm$ a periodic repetition with period m of the first m shifts acting on $\{s\}$ leaves $\{t\}$ invariant. Thus there are q^{m-1} Potts configurations $\{s'\}$ leading to the same $\{t\}$ ‡. When $\{s\}$ is a ground-state configuration, q^{m-1} gives the ground-state degeneracy.

In the following we shall make use of the Potts spin variables:

$$\tau_j = \exp\left(\frac{2i\pi t_j}{q}\right) = \prod_{l=0}^{m-1} \sigma_{j+l}, \quad \tau_j^* = \exp\left(-\frac{2i\pi t_j}{q}\right) = \tau_j^{q-1}, \quad \tau_j \tau_j^* = \tau_j^q = 1. \quad (3.4)$$

According to (1.4) one has:

$$\text{Tr}_{\tau_j} \tau_j^k = \sum_{t_j=0}^{q-1} \exp\left(\frac{2i\pi k t_j}{q}\right) = q \delta_q(k). \quad (3.5)$$

For later use, note that the Boltzmann factor

$$e^{-\beta \mathcal{H}_N^{(p)}[\{t\}]} = \prod_{j=1}^{N=pm} [e^{-K} + (e^{(q-1)K} - e^{-K}) \delta_q(t_j)], \quad (3.6)$$

can be rewritten as

$$\begin{aligned} e^{-\beta \mathcal{H}_N^{(p)}[\{\tau\}]} &= e^{-NK} \prod_{j=1}^{N=pm} \left[1 + \frac{e^{qK} - 1}{q} \sum_{k=0}^{q-1} \tau_j^k \right] \\ &= \left(\frac{e^{-K}}{q}\right)^N \prod_{j=1}^{N=pm} \left[e^{qK} + q - 1 + (e^{qK} - 1) \sum_{k=1}^{q-1} \tau_j^k \right], \end{aligned} \quad (3.7)$$

using (1.4) and (3.4),

Let us consider the product of Potts spins

$$P_i = \prod_{r=0}^{p-1} \tau_{rm+i} \tau_{rm+i+1}^*, \quad i = 1, \dots, m-1. \quad (3.8)$$

Making use of

$$\tau_{rm+i} \tau_{rm+i+1}^* = \sigma_{rm+i} \left(\prod_{l=1}^{m-1} \sigma_{rm+i+l} \sigma_{rm+i+l}^* \right) \sigma_{(r+1)m+i}^* = \sigma_{rm+i} \sigma_{(r+1)m+i}^* \quad (3.9)$$

and taking into account the periodic BC, one obtains the constraints

$$P_i = \prod_{r=0}^{p-1} \sigma_{rm+i} \sigma_{(r+1)m+i}^* = 1, \quad i = 1, \dots, m-1, \quad (3.10)$$

‡ Note that the initial configuration, $\{s\}$, corresponding to $\Delta_j = 0 \forall j$, is taken into account.

to be satisfied by the τ -configurations in (3.8). When $m > 2$ other constraints can be constructed, for instance from $\tau_{rm+i}\tau_{rm+i+2}^*$, but these are automatically satisfied since they can be written as products of the fundamental ones: $\tau_{rm+i}\underbrace{\tau_{rm+i+1}^*\tau_{rm+i+1}}_1\tau_{rm+i+2}^*$.

Thus with the new Potts spin variables, taking the constraints into account, the partition function is given by:

$$\mathcal{Z}_{N=pm}^{(p)} = q^{m-1} \text{Tr}_{\{\tau\}} e^{-\beta\mathcal{H}_N^{(p)}[\{\tau\}]} \prod_{i=1}^{m-1} \delta_{P_i,1}. \quad (3.11)$$

To go further we need an explicit expression for the Kronecker delta, $\delta_{P_i,1}$. Consider the geometric series

$$f(X) = \sum_{k=0}^{q-1} X^k = \frac{1-X^q}{1-X}, \quad (3.12)$$

it vanishes when X is a q th root of unity other than 1 and is equal to q when $X = 1$. Since P_i in (3.8) is a q th root of unity, the constraint can be written as (cf. (1.4))

$$\delta_{P_i,1} = \frac{1}{q} \sum_{k=0}^{q-1} P_i^k = \frac{1}{q} \sum_{k=0}^{q-1} \prod_{r=0}^{p-1} \tau_{rm+i}^k \tau_{rm+i+1}^{q-k}, \quad (3.13)$$

where (3.4) has been used. The partition function in (3.11) now takes the following form:

$$\mathcal{Z}_{N=pm}^{(p)} = \text{Tr}_{\{\tau\}} e^{-\beta\mathcal{H}_N^{(p)}[\{\tau\}]} \prod_{i=1}^{m-1} \left(1 + \sum_{k=1}^{q-1} P_i^k \right), \quad P_i^k = \prod_{r=0}^{p-1} \tau_{rm+i}^k \tau_{rm+i+1}^{q-k}. \quad (3.14)$$

The first product has the following expansion:

$$\prod_{i=1}^{m-1} \left(1 + \sum_{k=1}^{q-1} P_i^k \right) = 1 + \sum_{i,k} P_i^k + \sum_{i < i', k, k'} P_i^k P_{i'}^{k'} + \sum_{i < i' < i'', k, k', k''} P_i^k P_{i'}^{k'} P_{i''}^{k''} + \dots + \prod_i P_i^{q-1}. \quad (3.15)$$

The expression of P_i^k in (3.14) is periodic with period m . There are two consecutive Potts spins contributing to the product for each period and the sum of their exponents vanishes modulo q . Besides 1 the expansion (3.15) generates terms containing from $l = 2$ to m spins for each period with $\binom{m}{l}$ possible spatial configurations $\{\alpha_l\}$. These spatial configurations are labelled by the l spin exponents, each varying from 1 to $q-1$ with a sum which remains vanishing modulo q in the products, due to the Potts spins properties (3.4). As shown in appendix B, for l spins the number ν_l of allowed exponent distributions is given by:

$$\nu_l = \frac{1}{q} [(q-1)^l + (-1)^l (q-1)]. \quad (3.16)$$

Combining these results, the expansion can be written as

$$\prod_{i=1}^{m-1} \left(1 + \sum_{k=1}^{q-1} P_i^k \right) = 1 + \sum_{l=2}^m \sum_{\alpha_l=1}^m \sum_{\beta_l=1}^m \nu_l \Xi_{\alpha_l}^{\beta_l}, \quad (3.17)$$

where $\Xi_{\alpha_l}^{\beta_l}$ is a product for each period of l Potts spins in configuration α_l with an exponent distribution β_l .

The partition function in (3.14) splits in two parts:

$$\mathcal{Z}_{N=pm}^{(p)} = \underbrace{\text{Tr}_{\{\tau\}} e^{-\beta \mathcal{H}_N^{(p)}[\{\tau\}]}_A + \sum_{l=2}^m \sum_{\alpha_l=1}^{\binom{m}{l}} \sum_{\beta_l=1}^{\nu_l} \underbrace{\text{Tr}_{\{\tau\}} e^{-\beta \mathcal{H}_N^{(p)}[\{\tau\}]}_{\Xi_{\alpha_l}^{\beta_l}}}_B. \quad (3.18)$$

In A , according to (3.5), each of the pm factors in (3.7) contributes to the trace by:

$$\frac{e^{-K}}{q} \text{Tr}_{\tau_j} \left[e^{qK} + q - 1 + (e^{qK} - 1) \sum_{k=1}^{q-1} \tau_j^k \right] = e^{-K} (e^{qK} + q - 1). \quad (3.19)$$

In B , for each period, $\Xi_{\alpha_l}^{\beta_l}$ contains l supplementary spin terms of the form $\tau_j^{k'}$ with $k' = 1, 2, \dots, q-1$. Thus the trace involves $p(m-l)$ factors given by (3.19) and pl factors of the form

$$\frac{e^{-K}}{q} \text{Tr}_{\tau_j} \left[(e^{qK} + q - 1) \tau_j^{k'} + (e^{qK} - 1) \sum_{k=1}^{q-1} \tau_j^{k+k'} \right] = e^{-K} (e^{qK} - 1), \quad (3.20)$$

where the non-vanishing contribution comes from the term $q - k'$ in the sum over k according to (3.5). Collecting the different contributions to the partition function, we finally obtain

$$\begin{aligned} \mathcal{Z}_{N=pm}^{(p)} &= \sum_{l=0}^m \binom{m}{l} \nu_l [e^{(q-1)K} + (q-1)e^{-K}]^{p(m-l)} [e^{(q-1)K} - e^{-K}]^{pl} \\ &= [e^{(q-1)K} + (q-1)e^{-K}]^N \left[1 + \sum_{l=2}^m \binom{m}{l} \nu_l \left(\frac{e^{qK} - 1}{e^{qK} + q - 1} \right)^{pl} \right], \end{aligned} \quad (3.21)$$

where ν_l , given by (3.16), is such that $\nu_0 = 1$ and $\nu_1 = 0$. For $q = 2$

$$\nu_l = \begin{cases} 1 & \text{when } l \text{ is even} \\ 0 & \text{when } l \text{ is odd} \end{cases} \quad (3.22)$$

and the Ising result, equation (3.13) in [30], is recovered.

Let \mathbf{T} be the transfer matrix from $|\sigma_j \sigma_{j+1} \dots \sigma_{j+m-2}\rangle$ to $|\sigma_{j+1} \sigma_{j+2} \dots \sigma_{j+m-1}\rangle$. As discussed in appendix C, its m th power is real and symmetric. The real eigenvalues of \mathbf{T}^m , ω_l , and their degeneracy, g_l , can be deduced from the expression (3.21) of the partition function (see (C.6)).

4. Zero-field correlation function

In this section the zero-field correlation function is obtained for free BC and $m > 1$. The correlations between the Potts variables at sites i and i' are evaluated by taking the thermal average of the following expression:

$$\frac{q \delta_q(s_i - s_{i'}) - 1}{(q-1)}. \quad (4.1)$$

It is equal to one when the two sites are in the same state and has a vanishing average in a fully disordered system. Making use of (1.4) and (1.5) the numerator in (4.1) can be expressed in terms of Potts spins as:

$$q\delta_q(s_i - s_{i'}) - 1 = \sum_{k=1}^{q-1} \exp\left[\frac{2i\pi k(s_i - s_{i'})}{q}\right] = \sum_{k=1}^{q-1} \sigma_i^k \sigma_{i'}^{*k}. \quad (4.2)$$

Let us first suppose that $i' = i + rm$. Taking into account (3.9) one may write

$$\sigma_i \sigma_{i+rm}^* = \prod_{r'=0}^{r-1} \tau_{r'm+i} \tau_{r'm+i+1}^*, \quad (4.3)$$

and the correlation function takes the following form:

$$\mathcal{G}_N^{(f)}(i, i+rm) = \left\langle \frac{q\delta_q(s_i - s_{i+rm}) - 1}{(q-1)} \right\rangle = \text{Tr}_{\{\tau\}} \frac{e^{-\beta\mathcal{H}_N^{(f)}[\{\tau\}]} \sum_{k=1}^{q-1} \prod_{r'=0}^{r-1} \tau_{r'm+i}^k \tau_{r'm+i+1}^{*k}}{(q-1)\mathcal{Z}_N^{(f)}}. \quad (4.4)$$

Following the same steps that led to (3.7), the Boltzmann factor for free BC can be written as

$$\begin{aligned} e^{-\beta\mathcal{H}_N^{(f)}[\{\tau\}]} &= \left[\frac{e^{(q-1)K} + (q-1)e^{-K}}{q} \right]^{N-m+1} \prod_{j=1}^{N-m+1} \left[1 + \frac{e^{qK} - 1}{e^{qK} + q - 1} \sum_{k'=1}^{q-1} \tau_j^{k'} \right] \\ &= \frac{\mathcal{Z}_N^{(f)}}{q^N} \prod_{j=1}^{N-m+1} \left[1 + \frac{e^{qK} - 1}{e^{qK} + q - 1} \sum_{k'=1}^{q-1} \tau_j^{k'} \right] \end{aligned} \quad (4.5)$$

where the expression of $\mathcal{Z}_N^{(f)}$ in (2.5) has been used §. Inserting this expression in (4.4), one obtains:

$$\begin{aligned} \mathcal{G}_N^{(f)}(i, i+rm) &= \frac{1}{q^N(q-1)} \sum_{k=1}^{q-1} \text{Tr}_{\{\tau\}} \prod_{r'=0}^{r-1} \tau_{r'm+i}^k \tau_{r'm+i+1}^{*k} \times \\ &\quad \times \prod_{j=1}^{N-m+1} \left[1 + \frac{e^{qK} - 1}{e^{qK} + q - 1} \sum_{k'=1}^{q-1} \tau_j^{k'} \right], \end{aligned} \quad (4.6)$$

The trace over $\{\tau\}$ contains r factors with $j = r'm + i$ of the form

$$\text{Tr}_{\tau_{r'm+i}} \left[\tau_{r'm+i}^k + \frac{e^{qK} - 1}{e^{qK} + q - 1} \sum_{k'=1}^{q-1} \tau_{r'm+i}^{k+k'} \right] = q \frac{e^{qK} - 1}{e^{qK} + q - 1}, \quad (4.7)$$

the only non-vanishing contribution coming from the second term for $k' = q - k$ according to (3.5). The same result is obtained for the r factors with $j = r'm + i + 1$ and $k' = k$. The trace over the remaining $N - 2r$ Potts spins contributes a factor q^{N-2r} , the sum over k gives $q - 1$, so that, finally:

$$\mathcal{G}_N^{(f)}(i, i+rm) = \left[\frac{e^{qK} - 1}{e^{qK} + q - 1} \right]^{2r} = \exp\left(-\frac{rm}{\xi}\right). \quad (4.8)$$

§ Taking the trace over the N Potts spins in (4.5), all the terms in the product involving τ_j vanish and the trace over 1 gives q^N .

As expected, this expression can be rewritten in terms of transfer matrix eigenvalues (C.6) as $(\omega_2/\omega_0)^r$. The correlation length, given by

$$\xi = \frac{m}{2} \left[\ln \left(\frac{e^{qK} + q - 1}{e^{qK} - 1} \right) \right]^{-1}, \quad (4.9)$$

diverges at the zero-temperature critical point when $K \rightarrow \infty$.

Let us now consider the case where $i' - i$ is not a multiple of m . Using the Potts spin variables (1.5) and (3.4) the inverse transformation in (2.3) translates into:

$$\sigma_i = \prod_{r'=0}^p \tau_{r'm+i} \tau_{r'm+i+1}^*, \quad i+l = N - pm, \quad l = 0, \dots, m-1. \quad (4.10)$$

In the same way let

$$\sigma_{i'}^* = \prod_{r'=0}^{p-r} \tau_{r'm+i'}^* \tau_{r'm+i'+1}, \quad i'+l' = N - pm + rm, \quad l' = 0, \dots, m-1, \quad (4.11)$$

with, in both cases, $\tau_j = 1$ when $j > N$. Since $i' - i = rm + l - l'$, we need $l \neq l'$. In the product $\sigma_i \sigma_{i'}^*$, the last factor contributed by σ_i is either τ_{N-l+1}^* or τ_N when $l = 0$ whereas for $\sigma_{i'}^*$ it is either $\tau_{N-l'+1}$ or τ_N^* when $l' = 0$. Thus these factors cannot all disappear in the product when $l \neq l'$. At least one of them leads to a vanishing trace over $\{\tau\}$ in the correlation function since the product over j in the Boltzmann factor (4.5) ends at $N - m + 1$. It follows that:

$$\mathcal{G}_N^{(f)}(i, i') = 0, \quad i' - i \neq rm. \quad (4.12)$$

When $m = 2$ and $q = 2$ this argument no longer applies. With $m = 2$ the τ_j and the τ_j^* always appear twice in the product $\sigma_i \sigma_{i'}^*$ for values of $j \geq i'$. Accordingly, the correlation function does not vanish since $\tau_j^2 = \tau_j^{*2} = 1$ when $q = 2$. The difference between $q = 2$ and $q > 2$ when $m = 2$ can be understood by looking at the behaviour of the correlations in the ground state. For $q = 2$ there are 2 degenerate ground states which, using Potts variables, are given by 00000... and 11111... so that $\langle 2\delta_q(s_i - s_{i'}) - 1 \rangle = 1$. When $q = 3$, for example, there are 3 degenerate ground states, 00000..., 12121... and 21212..., leading to $\langle 3\delta_q(s_i - s_{i'}) - 1 \rangle = 0$ when $i' - i$ is odd.

5. Self-duality under external field

In this section standard methods [4, 7, 36] are used to show that the Potts chain with multi-site interactions and periodic BC is self-dual under external field.

According to (1.3), the partition function is given by:

$$\mathcal{Z}_N^{(p)}(K, H) = e^{-N(K+H)} \text{Tr}_{\{s\}} \prod_{j=1}^N \exp \left[qK\delta_q \left(\sum_{l=0}^{m-1} s_{j+l} \right) \right] \exp [qH\delta_q(s_j)]. \quad (5.1)$$

Introducing the auxiliary function

$$C(X, x) = e^{qX} - 1 + q\delta_q(x), \quad (5.2)$$

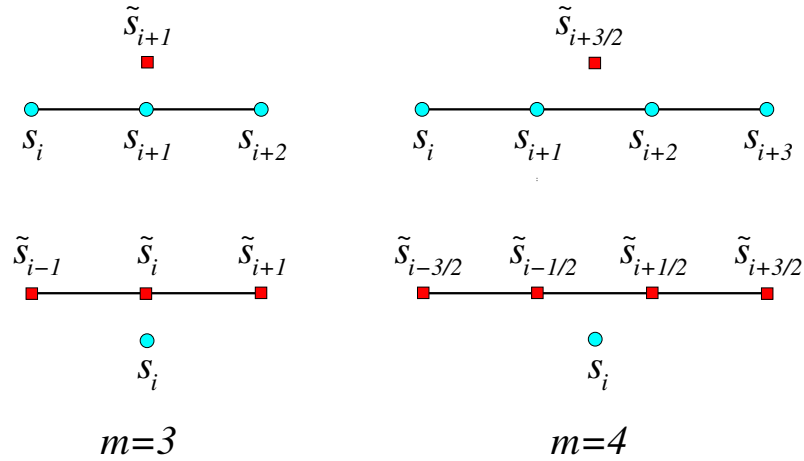


Figure 2. Position of the dual Potts variables (squares) entering in the definitions (5.6) of u_i and v_i relative to the original ones (circles) for odd and even values of m .

one obtains the identity:

$$e^{qX\delta_q(y)} = 1 + (e^{qX} - 1) \delta_q(y) = \frac{1}{q} \sum_{x=0}^{q-1} C(X, x) \exp\left(\frac{2i\pi xy}{q}\right). \quad (5.3)$$

Thus the partition function can be rewritten as:

$$\begin{aligned} \mathcal{Z}_N^{(p)}(K, H) &= \frac{e^{-N(K+H)}}{q^N} \text{Tr}_{\{s\}} \prod_{j=1}^N \sum_{u_j=0}^{q-1} \sum_{v_j=0}^{q-1} C(K, u_j) C(H, v_j) \\ &\quad \times \frac{1}{q} \exp\left[\frac{2i\pi}{q} \left(v_j s_j + u_j \sum_{l=0}^{m-1} s_{j+l}\right)\right]. \end{aligned} \quad (5.4)$$

Regrouping the factors containing s_i in the last exponential and reordering the sums, one obtains

$$\begin{aligned} \mathcal{Z}_N^{(p)}(K, H) &= \frac{e^{-N(K+H)}}{q^N} \text{Tr}_{\{u, v\}} \prod_{j=1}^N C(K, u_j) C(H, v_j) \prod_{i=1}^N \frac{1}{q} \sum_{s_i=0}^{q-1} \exp\left(\frac{2i\pi s_i w_i}{q}\right) \\ &= \frac{e^{-N(K+H)}}{q^N} \text{Tr}_{\{u, v\}} \prod_{j=1}^N C(K, u_j) C(H, v_j) \prod_{i=1}^N \delta_q(w_i), \end{aligned} \quad (5.5)$$

where w_i stands for $v_i + \sum_{l=0}^{m-1} u_{i-l}$.

Non-vanishing contributions to the partition function correspond to configurations of $\{u\}$ and $\{v\}$ such that $w_i = 0 \pmod{q} \forall i$. Introducing dual q -state Potts variables $\{\tilde{s}\}$, this condition is automatically satisfied when u_i and v_i take the following forms

$$u_i = -\tilde{s}_{i+(m-1)/2} \pmod{q}, \quad v_i = \sum_{l=0}^{m-1} \tilde{s}_{i+l-(m-1)/2} \pmod{q}, \quad (5.6)$$

such that:

$$w_i = \sum_{l=0}^{m-1} \tilde{s}_{i+l-(m-1)/2} - \sum_{l=0}^{m-1} \tilde{s}_{i-l+(m-1)/2} \pmod{q}$$

$$= \sum_{l=0}^{m-1} \tilde{s}_{i+l-(m-1)/2} - \sum_{l'=0}^{m-1} \tilde{s}_{i-(m-1-l')+(m-1)/2} = 0 \pmod{q}. \quad (5.7)$$

The dual lattice coincides with the original lattice when m is odd. It is shifted by half a lattice spacing when m is even (see figure 2).

Introducing the dual Potts variables in (5.5), one obtains:

$$\mathcal{Z}_N^{(p)}(K, H) = \frac{e^{-N(K+H)}}{q^N} \text{Tr}_{\{\tilde{s}\}} \prod_{j=1}^N C(K, \tilde{s}_{j+(m-1)/2}) C\left(H, \sum_{l=0}^{m-1} \tilde{s}_{j+l-(m-1)/2}\right). \quad (5.8)$$

Let us rewrite the auxiliary function C as:

$$C(X, x) = D(X) \exp\left[\tilde{Y}(q\delta_q(x) - 1)\right] = D(X) e^{-\tilde{Y}} \left[1 + (e^{q\tilde{Y}} - 1)\delta_q(x)\right]. \quad (5.9)$$

A comparison with (5.2) leads to

$$D(X) = e^{\tilde{Y}} (e^{qX} - 1), \quad (e^{qX} - 1)(e^{q\tilde{Y}} - 1) = q. \quad (5.10)$$

Making use of these relations, with $\tilde{Y} = \tilde{H}$ when $X = K$ and $\tilde{Y} = \tilde{K}$ when $X = H$, the following duality relations for the couplings are obtained:

$$(e^{qK} - 1)(e^{q\tilde{H}} - 1) = q, \quad (e^{qH} - 1)(e^{q\tilde{K}} - 1) = q. \quad (5.11)$$

The partition function (5.8) is now given by:

$$\mathcal{Z}_N^{(p)}(K, H) = \frac{e^{-N(K+H)} e^{N(\tilde{K}+\tilde{H})}}{q^N} (e^{qK} - 1)^N (e^{qH} - 1)^N \mathcal{Z}_N^{(p)}(\tilde{K}, \tilde{H}). \quad (5.12)$$

Using (5.11), this can be put in the more symmetric form:

$$\frac{e^{N(K+H)}}{[(e^{qK} - 1)(e^{qH} - 1)]^{N/2}} \mathcal{Z}_N^{(p)}(K, H) = \frac{e^{N(\tilde{K}+\tilde{H})}}{[(e^{q\tilde{K}} - 1)(e^{q\tilde{H}} - 1)]^{N/2}} \mathcal{Z}_N^{(p)}(\tilde{K}, \tilde{H}). \quad (5.13)$$

Taking the product of the duality relations in (5.11) and separating the original and dual parts gives

$$\frac{(e^{qK} - 1)(e^{qH} - 1)}{q} = \frac{q}{(e^{q\tilde{K}} - 1)(e^{q\tilde{H}} - 1)}, \quad (5.14)$$

so that the line $(e^{qK} - 1)(e^{qH} - 1) = q$ in the (K, H) -plane, which is invariant in the duality transformation, is a self-duality line.

6. Mapping on a 2D q -state Potts model when $H \neq 0$

Let us consider a Potts chain with N spins, $m > 1$ and free BC. According to (1.6), the Hamiltonian of the system in an external field H is given by:

$$-\beta\mathcal{H}_N[\{\sigma\}] = K \sum_{j=1}^{N-m+1} \sum_{k=1}^{q-1} \prod_{l=0}^{m-1} \sigma_{j+l}^k + H \sum_{j=1}^N \sum_{k=1}^{q-1} \sigma_j^k. \quad (6.1)$$

Let us define new Potts spins $\{\zeta\}$ and Potts variables $\{z\}$ such that:

$$\zeta_j = \exp\left(\frac{2i\pi z_j}{q}\right) = \prod_{i=j}^N \sigma_i, \quad z_j = 0, \dots, q-1. \quad (6.2)$$

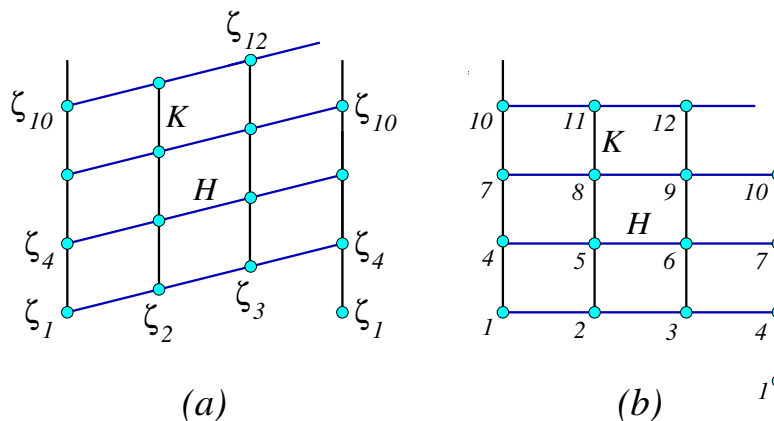


Figure 3. (a) Under the change of spin variables (6.2) the 1D Potts model with m -spin interactions in an external field is mapped onto a 2D standard Potts model on a cylinder with helical BC. H and K are the first-neighbour interactions along the helix and parallel to its axis, respectively. The helix has a length N/m and m spins per turn. The helicity factor is $1/m$. (b) The same lattice in a rectangular representation.

Using (3.4) one obtains

$$\sigma_j = \begin{cases} \zeta_j \zeta_{j+1}^*, & j < N \\ \zeta_j, & j = N \end{cases}, \quad \prod_{l=0}^{m-1} \sigma_{j+l} = \begin{cases} \zeta_j \zeta_{j+m}^*, & j < N - m + 1 \\ \zeta_j, & j = N - m + 1 \end{cases}, \quad (6.3)$$

and the correspondence with the original variables is one-to-one. The Hamiltonian (6.1) now takes the following form:

$$-\beta \mathcal{H}_N[\{\zeta\}] = K \sum_{j=1}^{N-m} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+m}^{*k} + H \sum_{j=1}^{N-1} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+1}^{*k} + K \sum_{k=1}^{q-1} \zeta_{N-m+1}^k + H \sum_{k=1}^{q-1} \zeta_N^k. \quad (6.4)$$

Alternatively, using

$$\sum_{k=1}^{q-1} \zeta_j^k \zeta_{j'}^{*k} = \sum_{k=0}^{q-1} \exp \left[\frac{2i\pi k}{q} (z_j - z_{j'}) \right] - 1 = q\delta_{z_j, z_{j'}} - 1, \quad \sum_{k=1}^{q-1} \zeta_j^k = q\delta_{z_j, 0} - 1, \quad (6.5)$$

the following standard form is recovered:

$$-\beta \mathcal{H}_N[\{z\}] = K \sum_{j=1}^{N-m} (q\delta_{z_j, z_{j+m}} - 1) + H \sum_{j=1}^{N-1} (q\delta_{z_j, z_{j+1}} - 1) + K (q\delta_{z_{N-m+1}, 0} - 1) + H (q\delta_{z_N, 0} - 1). \quad (6.6)$$

Thus the 1D Potts model with m -site interaction K in a field H is mapped onto an anisotropic 2D Potts model, with standard first-neighbour interactions, on a cylinder with helical BC (see figure 3). The interaction is K parallel to the cylinder axis and H along the helix. Local fields K and H are acting on two of the end spins. The length of the system is $\ell = N/m$, there are m spins per turn and the helicity factor is equal to $1/m$.

In the limit $\ell = N/m \rightarrow \infty$ the free energy of the 1D Potts chain with multi-site interactions K under external field H develops a 2D Potts critical singularity along

the self-duality line, $(e^{qK} - 1)(e^{qH} - 1) = q$, when $m \rightarrow \infty$ ||. Exact expressions for the bulk free energy per site have been obtained for the 2D Potts model on its critical line [13, 37, 38]. Taking into account the difference in the form of the interactions, the critical free energy per site is given by [38]

$$\beta f_b(K, H) = \lim_{m \rightarrow \infty} \lim_{N/m \rightarrow \infty} -\frac{\ln \mathcal{Z}_N}{N} = K + H + \psi, \quad \psi = -\frac{1}{2} \ln q - \phi(x_K) - \phi(x_H), \quad (6.7)$$

where

$$x_K = q^{-1/2} (e^{qK} - 1), \quad x_H = q^{-1/2} (e^{qH} - 1), \quad (6.8)$$

and $x_K x_H = 1$ for the critical system. The transition is second-order when $q \leq 4$ and first-order when $q > 4$ [37]. The expression of the function $\phi(x)$ in the different regimes can be found in [38].

Note that successive derivatives of the free energy with respect to H , leading to the magnetization and the susceptibility for the Potts chain, give the contributions of one type of bonds to the internal energy and the specific heat of the 2D Potts model. The derivatives with respect to K are of the same nature for both systems. It follows that along the critical line, in the thermodynamic limit ($\ell \rightarrow \infty, m \rightarrow \infty$), the thermal and magnetic critical behaviours of the 1D Potts model with multi-site interactions in a field, are both governed by the thermal sector of 2D Potts model. When $q > 4$ the discontinuities of the magnetization and the internal energy add up to give the latent heat of the 2D system. When $q \leq 4$ the thermal and magnetic critical exponents of the second-order phase transition are the 2D thermal Potts exponents [39–41].

According to (6.3) the two-spin correlation function of the original 1D system

$$\mathcal{G}_N^{(f)}(i, i') = \frac{1}{q-1} \sum_{k=1}^{q-1} \langle \sigma_i^k \sigma_{i'}^{*k} \rangle, \quad (6.9)$$

becomes a four-spin correlation function in 2D:

$$\mathcal{G}_N^{(f)}(i, i') = \frac{1}{q-1} \sum_{k=1}^{q-1} \langle \zeta_i^k \zeta_{i+1}^{*k} \zeta_{i'}^{*k} \zeta_{i'+1}^k \rangle, \quad i < i' < N. \quad (6.10)$$

When $H = 0$, the 2D lattice breaks into m independent spin chains and when $i' = i + rm$ a four-spin average becomes a product of two-spin averages on two neighbouring chains (see figure 3):

$$\langle \zeta_i^k \zeta_{i+1}^{*k} \zeta_{i+rm}^{*k} \zeta_{i+rm+1}^k \rangle = \langle \zeta_i^k \zeta_{i+rm}^{*k} \rangle \langle \zeta_{i+1}^k \zeta_{i+rm+1}^{*k} \rangle^*. \quad (6.11)$$

Actually these averages do not depend on k and each factor corresponds to the correlation function for two spins at a distance r on a Potts chain with standard first-neighbour interactions

$$\langle \zeta_i \zeta_{i+rm}^* \rangle = \left(\frac{e^{qK} - 1}{e^{qK} + q - 1} \right)^r, \quad (6.12)$$

from which (4.8) is recovered. When $i' \neq i + rm$, provided m and q are not both equal to two, the four-spin average in (6.10) always involve some vanishing factor.

|| The external fields acting on end spins do not affect the bulk behaviour.

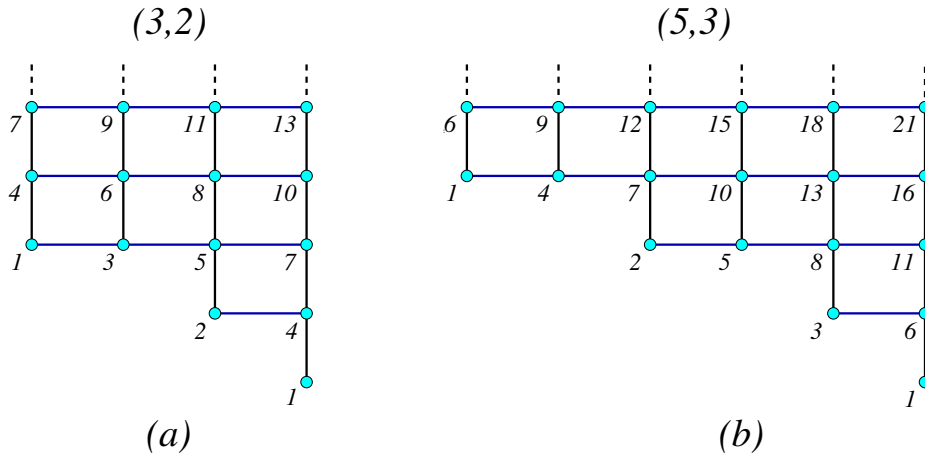


Figure 4. Examples in the rectangular representation of 2D lattices associated with the transformed Hamiltonian (7.2) when m and n are mutually primes. The helicity factor is n/m .

7. Other multi-site Potts models

We consider now a 1D Potts model with free BC and two types of multi-site interactions \P . In this (m,n) Hamiltonian, with $m > n > 1$, the external field term is replaced by a n -site interaction:

$$-\beta\mathcal{H}_N[\{\sigma\}] = K \sum_{j=1}^{N-m+1} \sum_{k=1}^{q-1} \prod_{l=0}^{m-1} \sigma_{j+l}^k + L \sum_{j=1}^{N-n+1} \sum_{k=1}^{q-1} \prod_{l=0}^{n-1} \sigma_{j+l}^k. \quad (7.1)$$

The change of variables (6.2) leads to the following transformed Hamiltonian:

$$-\beta\mathcal{H}_N[\{\zeta\}] = K \sum_{j=1}^{N-m} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+m}^{*k} + L \sum_{j=1}^{N-n} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+n}^{*k} + K \sum_{k=1}^{q-1} \zeta_{N-m+1}^k + L \sum_{k=1}^{q-1} \zeta_{N-n+1}^k. \quad (7.2)$$

For the (m,n) Hamiltonian, in the rectangular lattice representation (figure 4), the horizontal interaction L couples spins ζ_j and ζ_{j+n} thus generating n chains of connected sites with $j = 0, 1, \dots, n-1 \pmod{n}$. When m and n are mutually primes these chains are connected by vertical interactions between spins ζ_j and ζ_{j+m} . Starting from site j one reaches site $j + mn$ via either m horizontal steps or n vertical steps. Thus the 2D lattice has helical BC, m steps per turn and the helicity factor is n/m .

Let us now consider the case where m and n have a greatest common factor f so that $m = fm'$, $n = fn'$, with m' and n' mutually primes (figure 5). Then among the n horizontal chains of connected spins with $j = 0, 1, \dots, n-1 \pmod{n}$ the f chains with $j = 0, 1, \dots, f-1 \pmod{f}$ belong to f distinct 2D lattices since with $m = 0 \pmod{f}$ there are no vertical interconnections. Starting from site j one can reach site $j + fn'm'$ through either m' horizontal steps or n' vertical steps on the same lattice.

\P This type of Hamiltonian is also self-dual as shown more generally in [4] for a simple hypercubic lattice.

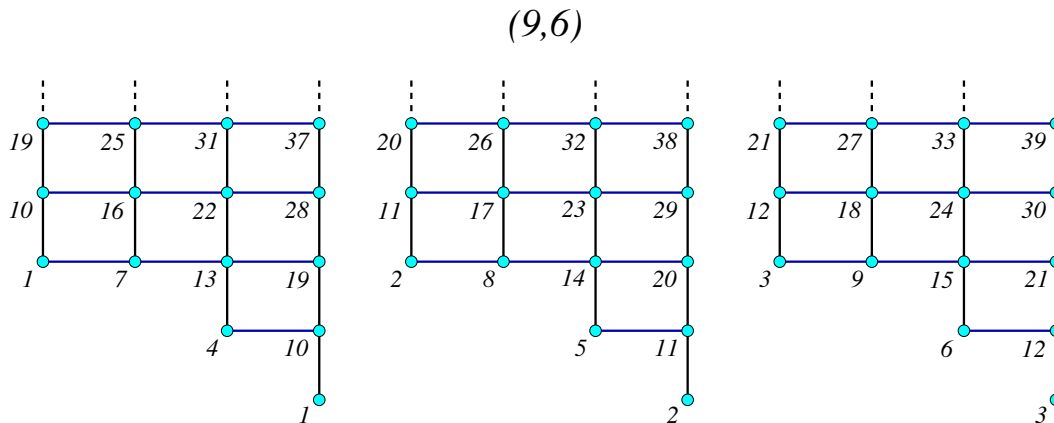


Figure 5. When $m = fm'$ and $n = fn'$, with m' and n' mutually primes, the Hamiltonian (7.2) splits into f non-interacting parts to which correspond f independent 2D lattices with helical BC and m' spins per turn. The expressions of the lattice length, $\ell = N/m$, and the helicity factor, $n/m = n'/m'$, remain unchanged.

The f distinct 2D lattices, with length N/m , have helical BC, m' steps per turn and their helicity factor remains equal to $n/m = n'/m'$.

Note that Potts chains with more complex multi-site interactions can be mapped onto triangular or honeycomb lattices as shown in appendix D.

8. Conclusion

In this work we have used some spin transformation to obtain exact results for the zero-field partition functions and the two-spin correlation function of a q -state Potts chain with multi-site interactions. We have shown that the model is self-dual under external field. With another spin transformation, the Potts chain with m -site interaction K in a field H has been mapped onto a standard 2D q -state Potts model with first-neighbour interactions K and H . The 2D system with N spins has a length $\ell = N/m$, a transverse size m and helical BC in the transverse direction.

Thus the Potts chain in a field develops a critical singularity on the self-duality line, $(e^{qK} - 1)(e^{qH} - 1) = q$, as $\ell \rightarrow \infty$ and $m \rightarrow \infty$, i.e., in the thermodynamic limit for the 2D system. Along this line the thermal and magnetic critical behaviours of the Potts chain are both governed by the thermal critical behaviour of the 2D Potts model. The transition is first-order when $q > 4$ and second-order when $q \leq 4$.

A numerical exploration of the finite-size scaling behaviour on the self-duality line would be of interest. The development of the critical singularities with increasing values of N and m should be studied for some fixed values of the aspect ratio $\ell/m = N/m^2$.

Appendix A. Clock angular variables

Using (1.4) the Potts multi-site interaction in (1.3) can be rewritten as

$$\begin{aligned} q\delta_q\left(\sum_{l=0}^{m-1} s_{j+l}\right) - 1 &= \sum_{k=1}^{q-1} \exp\left(\frac{2i\pi k}{q} \sum_{l=0}^{m-1} s_{j+l}\right) = \frac{1}{2} \sum_{k=1}^{q-1} \left[\exp\left(\frac{2i\pi k}{q} \sum_{l=0}^{m-1} s_{j+l}\right) + \text{c.c.} \right] \\ &= \sum_{k=1}^{q-1} \cos\left(\frac{2\pi k}{q} \sum_{l=0}^{m-1} s_{j+l}\right) \end{aligned} \quad (\text{A.1})$$

or, introducing the clock angular variable $\theta_j = 2\pi s_j/q = 0, 2\pi/q, \dots, 2\pi(q-1)/q$,

$$q\delta_q\left(\sum_{l=0}^{m-1} s_{j+l}\right) - 1 = \sum_{k=1}^{q-1} \cos\left(k \sum_{l=0}^{m-1} \theta_{j+l}\right). \quad (\text{A.2})$$

Similarly for the field term $q\delta_q(s_j) - 1 = \sum_{k=1}^{q-1} \cos(k\theta_j)$.

Appendix B. Calculation of ν_l

Let us consider a term in the expansion (3.15) with l spins per period:

$$\prod_{r=0}^{p-1} \prod_{i=1}^l \tau_{rm+j_i}^{k_i}. \quad (\text{B.1})$$

For the number of distinct distributions of the exponents $k_i = 1, \dots, q-1$, such that $\sum_{i=1}^l k_i = 0 \pmod{q}$, we find:

$$\begin{aligned} \nu_l &= \sum_{k_1, k_2, \dots, k_l=1}^{q-1} \delta_q\left(\sum_{i=1}^l k_i\right) = \frac{1}{q} \sum_{k=0}^{q-1} \underbrace{\prod_{i=1}^l \sum_{k_i=1}^{q-1} \exp\left(\frac{2i\pi k k_i}{q}\right)}_{q\delta_q(k)-1} = \frac{1}{q} \sum_{k=0}^{q-1} [q\delta_q(k) - 1]^l \\ &= \frac{1}{q} [(q-1)^l + (-1)^l (q-1)]. \end{aligned} \quad (\text{B.2})$$

Thus $\nu_0 = 1$ and $\nu_1 = 0$, independent of q . For $q = 2$, due to the fact that $k_i = 1$ for Ising spins, one obtains $\nu_{2k+1} = 0$ and $\nu_{2k} = 1$.

Note that the value of ν_l in (B.2) leads to a total number of terms in (3.15) given by

$$1 + \sum_{l=2}^m \binom{m}{l} \nu_l = \frac{1}{q} \sum_{l=0}^m \binom{m}{l} [(q-1)^l + (-1)^l (q-1)] = \frac{1}{q} \sum_{l=0}^m \binom{m}{l} (q-1)^l = q^{m-1} \quad (\text{B.3})$$

as required.

As an illustration let us look for the form of the expansion when $m = 3$ and $q = 4$. With x, y, z standing for the product of p spins, $\prod_{r=0}^{p-1} \tau_{rm+j}$, with respectively, $j = 1, 2, 3$, so that $x^q = y^q = z^q = 1$, we obtain:

$$\begin{aligned} (1 + xy^3 + x^2y^2 + x^3y)(1 + yz^3 + y^2z^2 + y^3z) &= 1 \\ &+ (xy^3 + x^2y^2 + x^3y) + (yz^3 + y^2z^2 + y^3z) + (zx^3 + z^2x^2 + z^3x) \\ &+ (xyz^2 + yzx^2 + zxy^2 + x^3y^3z^2 + y^3z^3x^2 + z^3x^3y^2). \end{aligned} \quad (\text{B.4})$$

On the right-hand side the terms in brackets correspond to the different exponent distributions for the same spin configuration. The values $\nu_2 = 3$ and $\nu_3 = 6$ are in agreement with (B.2).

Appendix C. Transfer matrix at $H = 0$

Before considering general values of q and m , let us study the properties of the transfer matrix of a 3-state Potts model with 3-site interactions at $H = 0$.

In the basis $\{|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle\}$, the transfer matrix of the Hamiltonian (3.1) from $|s_j s_{j+1}\rangle$ to $|s_{j+1} s_{j+2}\rangle$, takes the following form:

$$\mathbf{T} = \begin{pmatrix} e^{2K} & e^{-K} & e^{-K} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-K} & e^{-K} & e^{2K} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-K} & e^{2K} & e^{-K} \\ e^{-K} & e^{-K} & e^{2K} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-K} & e^{2K} & e^{-K} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{2K} & e^{-K} & e^{-K} \\ e^{-K} & e^{2K} & e^{-K} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2K} & e^{-K} & e^{-K} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-K} & e^{-K} & e^{2K} \end{pmatrix}. \quad (\text{C.1})$$

It is asymmetric and has complex eigenvalues:

$$\begin{aligned} \lambda_0 &= e^{2K} + 2e^{-K}, & \lambda_{2,k} &= [(e^{2K} + 2e^{-K})(e^{2K} - e^{-K})^2]^{1/3} e^{2ik\pi/3}, \quad k = 0, 1, 2, \\ \lambda_3 &= e^{2K} - e^{-K}. \end{aligned} \quad (\text{C.2})$$

Both $\lambda_{2,k}$ and λ_3 are doubly degenerate. The oscillating behaviour is linked to the periodicity of the degenerate ground states. With $a = e^{6K} + 2e^{-3K}$, $b = e^{3K} + e^{-3K} + 1$, the cube of \mathbf{T} , corresponding to a transfer by one period from s_j to s_{j+3} , leads to the symmetric matrix

$$\mathbf{T}^3 = \begin{pmatrix} a & b & b & b & 3 & b & b & b & 3 \\ b & a & b & b & b & 3 & 3 & b & b \\ b & b & a & 3 & b & b & b & 3 & b \\ b & b & 3 & a & b & b & b & 3 & b \\ 3 & b & b & b & a & b & b & b & 3 \\ b & 3 & b & b & b & a & 3 & b & b \\ b & 3 & b & b & b & 3 & a & b & b \\ b & b & 3 & 3 & b & b & b & a & b \\ 3 & b & b & b & 3 & b & b & b & a \end{pmatrix}, \quad (\text{C.3})$$

with real eigenvalues

$$\omega_0 = (e^{2K} + 2e^{-K})^3, \quad \omega_2 = (e^{2K} + 2e^{-K})(e^{2K} - e^{-K})^2, \quad \omega_3 = (e^{2K} - e^{-K})^3. \quad (\text{C.4})$$

ω_0 is non degenerate and the two last eigenvalues are, respectively, six times and two times degenerate.

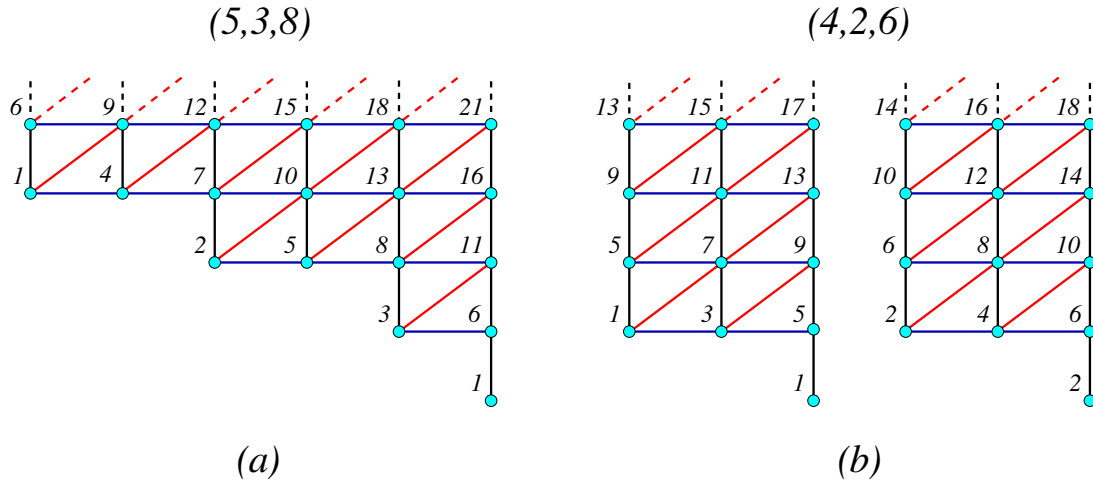


Figure D1. Rectangular representation of 2D lattices associated with the transformed Hamiltonian (D.2) when (a) m and n are mutually primes and (b) m and n have a greatest common factor f . The first-neighbour interactions are K , L and M in the vertical, horizontal and diagonal directions, respectively.

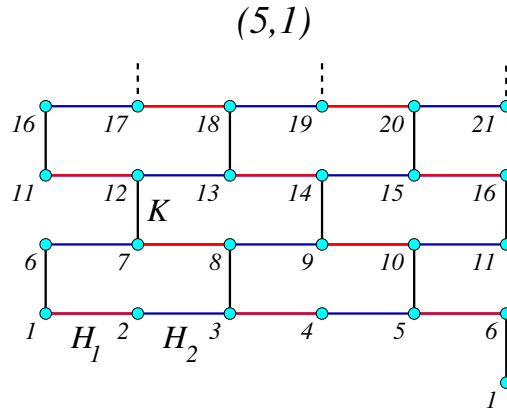


Figure D2. Rectangular representation of the honeycomb lattice associated with the transformed Hamiltonian (D.4).

For any value of m and q , the q^{m-1} eigenvalues of \mathbf{T}^m , ω_l , and their degeneracy, g_l , can be extracted from the expression of the partition function with periodic BC. Since

$$\mathcal{Z}_{N=mp}^{(p)} = \text{Tr}(\mathbf{T}^m)^p = \sum_l g_l \omega_l^p \quad (\text{C.5})$$

it follows from (3.21) that

$$\omega_l = [e^{(q-1)K} + (q-1)e^{-K}]^m \left[\frac{e^{qK} - 1}{e^{qK} + q - 1} \right]^l, \quad g_l = \binom{m}{l} \nu_l, \quad l = 0, 2, \dots, m, \quad (\text{C.6})$$

with ν_l given by (B.2).

Appendix D. Triangular and honeycomb lattices

With the $(m, n, m + n)$ Hamiltonian ($m > n$) such that

$$-\beta\mathcal{H}_N[\{\sigma\}] = K \sum_{j=1}^{N-m+1} \sum_{k=1}^{q-1} \prod_{l=0}^{m-1} \sigma_{j+l}^k + L \sum_{j=1}^{N-n+1} \sum_{k=1}^{q-1} \prod_{l=0}^{n-1} \sigma_{j+l}^k + M \sum_{j=1}^{N-m-n+1} \sum_{k=1}^{q-1} \prod_{l=0}^{m+n-1} \sigma_{j+l}^k, \quad (\text{D.1})$$

the change of variables (6.2) leads to the following transformed Hamiltonian:

$$\begin{aligned} -\beta\mathcal{H}_N[\{\zeta\}] = & K \sum_{j=1}^{N-m} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+m}^{*k} + L \sum_{j=1}^{N-n} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+n}^{*k} + M \sum_{j=1}^{N-m-n} \sum_{k=1}^{q-1} \zeta_j^k \zeta_{j+m+n}^{*k} \\ & + K \sum_{k=1}^{q-1} \zeta_{N-m+1}^k + L \sum_{k=1}^{q-1} \zeta_{N-n+1}^k + M \sum_{k=1}^{q-1} \zeta_{N-m-n+1}^k. \end{aligned} \quad (\text{D.2})$$

As shown in figure D1-a when m and n are mutually primes it corresponds to a triangular lattice Potts model with first-neighbour interactions on a cylinder with helical BC, an external fields acting on three end spins. When m and n have a greatest common factor f , as in figure D1-b, f independent triangular lattices are obtained.

Finally let us consider a 1D Potts model with m -spin interaction K ($m > 1$) starting on odd sites only, and two external fields, H_1 and H_2 , acting on odd and even sites. When $N - m$ is even the Hamiltonian can be written as:

$$-\beta\mathcal{H}_N[\{\sigma\}] = K \sum_{p=1}^{(N-m)/2+1} \sum_{k=1}^{q-1} \prod_{l=0}^{m-1} \sigma_{2p+l-1}^k + H_1 \sum_{p=1}^{\lfloor (N+1)/2 \rfloor} \sum_{k=1}^{q-1} \sigma_{2p-1}^k + H_2 \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{k=1}^{q-1} \sigma_{2p}^k. \quad (\text{D.3})$$

The transformed Hamiltonian then takes the following form

$$\begin{aligned} -\beta\mathcal{H}_N[\{\zeta\}] = & K \sum_{p=1}^{(N-m)/2} \sum_{k=1}^{q-1} \zeta_{2p-1}^k \zeta_{2p+m-1}^{*k} + K \sum_{k=1}^{q-1} \zeta_{N-m+1}^k + H_1 \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{k=1}^{q-1} \zeta_{2p-1}^k \zeta_{2p}^{*k} \\ & + \begin{cases} H_2 \sum_{p=1}^{N/2-1} \sum_{k=1}^{q-1} \zeta_{2p}^k \zeta_{2p+1}^{*k} + H_2 \sum_{k=1}^{q-1} \zeta_N^k, & N \text{ even} \\ H_1 \sum_{k=1}^{q-1} \zeta_N^k + H_2 \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{k=1}^{q-1} \zeta_{2p}^k \zeta_{2p+1}^{*k}, & N \text{ odd} \end{cases}, \end{aligned} \quad (\text{D.4})$$

which corresponds to a Potts model with first-neighbour interactions on the honeycomb lattice as shown in figure D2.

References

- [1] Potts R B 1952 *Math. Proc. Camb. Phil. Soc.* **48** 106
- [2] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235
- [3] Enting I G 1975 *J. Phys. A: Math. Gen.* **8** 1690
- [4] Deng Y, Guo W, Heringa J R, Blöte H W J and Nienhuis B 2010 *Nucl. Phys. B* **827** 406
- [5] Baxter R J and Wu F Y 1973 *Phys. Rev. Lett.* **31** 1294
- [6] Baxter R J and Wu F Y 1974 *Aust. J. Phys.* **27** 357
- [7] Turban L 1982 *J. Phys. C: Solid State Phys.* **15** L227
- [8] Alcaraz F C 1986 *Phys. Rev. B* **34** 4885
- [9] Zhang G M and Yang C Z 1993 *J. Phys. A: Math. Gen.* **26** 4907
- [10] Turban L 1982 *J. Phys. Lettres* **43** L259

- [11] Schick M and Griffiths R B 1977 *J. Phys. A: Math. Gen.* **10** 2123
- [12] Wu F Y and Stanley H E 1982 *Phys. Rev. B* **26** 6326(R)
- [13] Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc. (London)* A **358** 535
- [14] Enting I G 1978 *Aust. J. Phys.* **31** 383
- [15] Wu F Y and Lin K Y 1980 *J. Phys. A: Math. Gen.* **13** 629
- [16] Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 536
- [17] Grimmett G 1994 *J. Stat. Phys.* **75** 67
- [18] Chayes L and Machta J 1997 *Physica A* **239** 542
- [19] Chayes L and Machta J 1998 *Physica A* **254** 477
- [20] Matty M, Mansfield P, Hallinen K, Albert J and Swendsen R H 2015 *J. Stat. Mech.* P01026
- [21] Giri M R, Stephen M J and Grest G S 1977 *Phys. Rev. B* **16** 4971
- [22] Kunz H and Wu F Y 1978 *J. Phys. A: Math. Gen.* **11** L1
- [23] Essam J W 1979 *J. Math. Phys.* **20** 1769
- [24] Kasai Y, Takano T and Syosi I 1980 *Prog. Theor. Phys.* **63** 1917
- [25] Temperley H N V and Ashley S E 1982 *J. Phys. A: Math. Gen.* **15** 215
- [26] Goldstein R E 1984 *Phys. Lett.* **104A** 285
- [27] Ananikyan N S, Hajryan Sh A, Mamasakhlisov E Sh and Morozov V F 1990 *Biopolymers* **30** 357
- [28] Schreck J S and Yuan J M 2010 *Phys. Rev. B* **81** 061919
- [29] Badasyan A, Giacometti A, Podgornik R, Mamasakhlisov Y and Morozov V 2013 *Eur. Phys. J. E* **36** 46
- [30] Turban L 2016 *J. Phys. A: Math. Theor.* **49** 355002
- [31] Mittag L and Stephen M J 1971 *J. Math. Phys.* **12** 441
- [32] Sólyom J and Pfeuty P 1981 *Phys. Rev. B* **24** 218
- [33] Turban L 1993 unpublished lecture notes *Phénomènes Critiques V: Modèles Exactement Solubles* available at <http://gps.ijl.univ-lorraine.fr/webpro/turban.1>
- [34] Garrod C 1995 *Statistical Mechanics and Thermodynamics* (New York: Oxford University Press) p 517
- [35] Mueller M, Janke W and Johnston D A 2017 *Nucl. Phys. B* **914** 388
- [36] Savit R 1980 *Rev. Mod. Phys.* **52** 453
- [37] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
- [38] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic Press) p 340
- [39] den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
- [40] Black J L and Emery V J 1981 *Phys. Rev. B* **23** 429
- [41] Nienhuis B 1982 *J. Phys. A: Math. Gen.* **15** 199