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Inverse Problems and Experiments: a Fruitful Symbiosis

This paper offers a partial overview of some techniques and results emanating from inverse problems or signal processing with application to the development or enhancement of the exploitation of data obtained by experiments on physical systems pertaining to the aerospace domain. The first part deals with linear problems encountered in various problems of signal processing, whereas the second one addresses two specific image-based identification techniques. Finally, the conclusion strives to identify some future trends.

Introduction: Experiments, Simulation and Inverse Problems

From its very beginning, experiment has been indissolubly related to measurements and instrumentation, but it is only since a few decades ago that numerical processing, simulation and modeling have been deeply incorporated into the field. Symmetrically, the development of computational physics and mathematics in the mid-fifties opened the way to signal processing, and ten years later to addressing what was known as inverse problems (IP) but was studied until the eighties by a very small scientific community only due to a lack of computational power.

Since that period, a continuous development of the “accompaniment” of experiments by computation and modeling has led to new experiments, unprecedented accuracy, and measurements of “hidden” quantities. At the same time, the volume of experimental data has given rise to new capabilities for the identification of physical or geometrical parameters, and also to the emergence of validation procedures for the simulation means. Table 1 illustrates some of the mutual benefits that experiments, simulation and inverse problems have gained by collaborating.

In this paper, a partial overview will be given of some techniques and results emanating from the signal processing theory and the inverse problem community, which have enhanced, made possible or increased the performance level of experiments. Emphasis will also be placed on the notion of a priori information in inverse problems, this information being extracted from all of the knowledge accumulated through experiments. First, linear problems arising essentially from signal processing are addressed. Then, two problems coping with image capturing are investigated, since they are considered to be of utmost importance in aeronautical applications: the first deals with Particle Image Velocimetry for fluid flows, and the second one deals with full-field displacement measurement by digital image correlation in mechanics.

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Table 1 – Cross-fertilization between Experimentation, Simulation and Inverse Problems
Linear problems

Many examples can be found of linear inverse problems arising from experimental instrumentation or, more directly, from the goal aimed at in the investigation of the reality. Most of the examples fall into the field of Image and Signal Processing: signal deconvolution, image restoration, computed tomography, microwave imaging, and fluorescence imaging, as well as adaptive optics [24] and SAR [65] imaging. The general form of linear inverse problems is written as:

\[ g = Hf + \varepsilon \] (1)

where \( g \) is the measured data (output signal, blurred image, etc.), \( f \) is the unknown (input signal, original image) \( H \) can be called the forward operator (as it appears in the usual “direct problem”), and \( \varepsilon \) is the noise or the error. Whereas seeking \( g \) when the pair \((f, H)\) is given is a direct or forward problem, seeking \( f \) with given \((g, H)\) is called the estimation or inverse problem, and seeking \( H \) (or parameters describing it) with given \((f, g)\) is the identification problem.

As an example, in the deconvolution problem, \( H \) is a convolution with kernel \( h \):

\[ H(f) = \int h(t-x)f(x)dx \] (2)

\( H \) involves the impulse response of an instrument (called point spread function (PSF) in imaging, it is the spatial domain version of the imaging system transfer function). When the PSF is totally unknown we have the myopic deconvolution problem, in which both the object and linear for large ones ([14],[23]).

The operator \( H \) generally arises from the modeling of the underlying physics, and can sometimes be a matter of design, like in radar where the \( H \) operator is designated as the sensing operator. Note that the general form (1) also encompasses the Bayesian inference approach ([17], [18], [47]) where the pdf of the signal is now the unknown.

Once discretized (or sometimes directly resulting from the physics), \( H \) is a (rectangular) matrix, a Toeplitz matrix for deconvolution for example, and \((f, g)\) will be vectors of respective lengths \( N \) and \( M \) (where generally \( N > M \)).

Ill-posedness and regularization

Inverse problems are generically ill-posed. From the continuous viewpoint, the ill-posedness arises either from the non-existence of a solution, the non-(finite) uniqueness of the solution or the non-continuity of the solution with respect to the data ([33], [44]). Box 1 gives some details about the ill-posedness of the Fredholm integral equation of the first kind, frequently arising in inverse problems. In the finite dimensional context, these concepts move to a non-invertible (or rectangular with \( N > M \)) matrix \( H \), and severe bad conditioning.

In order to deal with this difficulty, the general procedure is regularization, which was introduced in the sixties ([53], [63]); that is, loosely speaking, accepting to modify the operator or the model in such a way that well-posedness is recovered. The price to pay is a loss of accuracy.

Basically, regularization is performed on a variational form of the inverse problem, in which \( f \) is sought as the minimum of a functional (usually chosen as a norm of the gap with respect to the measured data):

\[ f = \text{ArgMin}_p J(p) \] (3)

The regularization is achieved by adding a stabilizing functional \( S(p) \) with a (small) scalar regularization parameter \( \alpha \), which is a compromise parameter between the fidelity to the data and the desired properties of the function \( f \) that are enforced via the stabilizing functional.

\[ J_\alpha(p) = J(p) + \alpha S(p) \] (4)

Again, the choice of the stabilizing functional relies on a priori knowledge about the underlying physics. Usual choices for \( S \) are either based on regularity requirements on \( f \), or guided by the vicinity of a given value \( f_0 \) of \( f \). Historically, the stabilizing functional was the quadratic norm of the gradient of \( f \), in order to smooth out the spatial or temporal oscillations of the signal frequently encountered in ill-posed problems, where the noise in the data is dramatically amplified in the solution. However, since this turns out to be too smoothing, and especially for edge-preserving applications, the total variation regularization or L1 stabilizing functional are used ([21], [54]), or even more sophisticated L1-L2 functionals: quadratic for small gradients and linear for large ones ([14],[23]).
Although the choice of the regularization parameter is often made on the basis of trial and error procedures, it can also rely on complementary information about the measurement errors. The discrepancy principles ([10], [50]), for example, propose roughly speaking to determine the parameter $\alpha$ just by choosing the smallest value that makes the fidelity term of the functional be of the same order of magnitude as that of the measurement error: that is, not requiring more fidelity than the error level for the data. A last call to experience for the solution of inverse problems can be the choice of the minimization algorithm starting point, usually called the initial guess.

However, the most striking example of the use of a priori information in solving inverse problems is the Bayesian inference approach, where the philosophy is to consider the experiment as a means for adapting the a priori knowledge that we have about the situation at hand. More precisely, the Bayes theorem is used to build the posterior probability distribution of the parameters, given the prior distribution and the result of the experiment. In the linear case, the formulation obtained is very similar to the deterministic regularized least-square formulation (Gaussian prior corresponding to the weighted L2 fidelity norm, and the Laplacean prior to L1) with the noticeable difference being that there is no regularization parameter per se. The balance between the corresponding fidelity term and regularization term is provided naturally, given by the covariance matrices.

An additional benefit of the probabilistic approaches is that the question of the model used for extracting information or parameters out of the measured data can be addressed. Indeed, using models leads inevitably in an approximation or a systematic error, and to quoting G. Box [15]: "The most that can be expected from any model is that it can supply a useful approximation to reality: All models are wrong; some models are useful". This is the reason why some authors plead for the incorporation of uncertainties (and thus for a probabilistic, generally Bayesian, approach [39]): A model that is wrong can only be useful if we acknowledge the fact that it is wrong […] model discrepancy is an important part of uncertainty quantification and must not be ignored, even though it may be hard to account for [17].

**Compressed sensing**

Modeling and using a priori information also play a prominent role in the recent approach of compressed sensing [19]. One of its salient features is the identification of an orthonormal basis, say $\{\psi_i\}$, where the unknown $f$ has an $S$-sparse expansion, that is, a low number $S$ of non-zero (or significant) components:

$$ f = \sum_{i=1}^{S} f_i \psi_i $$

(5)

The second fundamental premise of compressed sensing is incoherence, because it enables the possibility of performing the determination of $f$ with very few samples of the data $g$. More precisely, if the basis for "representation" of the data $g$ is denoted by $\{\varphi_i\}$, the coherence in $\mathbb{IR}^n$ between the "sensing" basis $\{\varphi_i\}$ and the representation basis $\{\psi_i\}$, is

$$ \mu(\varphi, \psi) = \sqrt{n} \max_{1 \leq i, j \leq n} |\varphi_i, \psi_j| $$

(6)

A lower value of $\mu$ (note that $1 \leq \mu \leq \sqrt{n}$) corresponds to a lower correlation between any two elements of each basis [28]. Classical examples of pairs with low coherence are the Fourier basis-Spikes basis ($\mu = 1$), which represents maximal incoherence, or the Noiselets basis-Haar wavelets basis ($\mu = \sqrt{2}$). Coherence is involved in a quite strong result stating that the number of data sufficient for a good identification of $f$ can be significantly lower than the Nyquist rate given by the Shannon theory: for an $IR^d$ signal that is $S$-sparse in the basis $\{\psi_i\}$ and a set of $m$ measurements selected in the basis uniformly at random, then

$$ m \geq C \mu S \log n \Rightarrow \{f^\dagger\} = \text{is exact} $$

(7)

where $f^\dagger$ is the solution of the following $l_1$ minimization problem, constrained by the $m$ measurements of (1):

$$ \min_{\{f\} \in IR^n} \|f - \varphi_i g_i\|_1 \text{subject to } g_i = \langle H f, \varphi_i \rangle, i = 1, \ldots, m $$

(8)

The result (6) being true within an "overwhelming probability" [19].

It is readily seen that for a coherence equal to one, the number of needed data is of the order $S \log n$. If we now turn to noisy data, the question of robustness with respect to noise becomes central. The version of (8) is now a relaxed version, known as the LASSO or $l_1$-$l_2$ regularization ([29], [62], [51], [24], [27], [67]), where advantage is taken of the knowledge of the noise level $\epsilon$.

$$ \min_{\{f\} \in IR^n} \|f - \varphi_i g_i\|_1 \text{subject to } \|f - \varphi_i g_i\|_2 < \epsilon $$

(9)

Using the concept of the isometry constant $\delta$, for the matrix $H$ and any integer $S$ defined by the smallest number such that:

$$ 1 - \delta_S \leq \frac{\|P_S f\|_2}{\|f\|_2} \leq 1 + \delta_S \ \forall \ S - \text{sparse} \ x $$

(10)

[20] showed that the solution $f^\dagger$ of (8) has the following error inequality, when $\delta_{2S} < \sqrt{2} - 1$:

$$ \|f^\dagger - f\|_2 \leq C_0 \frac{\epsilon}{\sqrt{S}} \|f - f_S\|_2 + C_1 \epsilon $$

(11)

This means that the error in the reconstruction is just supplemented, in the case of noise, with a term directly proportional to the noise level. Furthermore, [20] claims that the constants $C_0$ and $C_1$ are "small" (typically, for $\delta_{2S} = \frac{1}{4}$, $C_0$ and $C_1 < 6$).

Sparsity can also be enforced within the Bayesian approach to linear inverse problems [47], and also has a wide field of application in the identification of 3D vector fields ([25], [48]).

**Completion and interpolation of velocity fields obtained by PIV**

According to [64], "The great challenges of flow dynamics require the validation of physical models with relevant length ranges spanning dozens of orders of magnitude, from a few Angstroms, at which atoms and molecules collide and bond (particularly in combustion), to a meter range (the size of a combustion chamber or a wing). Optical diagnostics are ideal for addressing this enormous challenge as they can span these different scales, offering both a global view through imaging and a microscopic view through spectroscopy. Moreover, these techniques are non-intrusive, so that they do not disturb the system or bias the measurements as other probing techniques do."
Unlike Particle Tracking Velocimetry [2], dealing with low seeding density and sparse estimation of the displacement, PIV is based either on direct correlation techniques between two successive images (including fast algorithms on GPUs, like the iterative Lucas-Kanade optical flow algorithm FOLKI [22]), or on image correlation techniques using the optic flow conservation equation. This last equation just expresses the fact that, under the assumption of conservation of the luminance of the particles, the image (gray level) \( I(x,t) \) is only transformed by the velocity flow \( \mathbf{v} \) between two time instants:

\[
\partial_t I + \mathbf{v} \cdot \mathbf{\nabla} I = 0 \tag{12}
\]

It is clear that this equation only gives access to limited information about the velocity field, namely the component of \( \mathbf{v} \) parallel to the gradient of the function \( I \), \( \mathbf{v}^\perp \) remains undetermined (note in passing that no information is gained from a part of the flow with a uniform gradient in the image):

\[
\mathbf{v} = -\partial_t I \frac{\mathbf{\nabla} I}{\| \mathbf{\nabla} I \|} + \mathbf{v}^\perp \cdot \mathbf{\nabla} I = 0 \tag{13}
\]

Thus, a naive least-square formulation like the following, striving to minimize the residual of the equation (12), is doomed to fail.

\[
\mathbf{v}^{opt} = \text{ArgMin}_v J(v) = \int_{\Omega_v} (\partial_t I + \mathbf{v} \cdot \mathbf{\nabla} I)^2 \, dV \tag{14}
\]

For two dimensional domains \( \Omega \), [59] showed that regularized functionals consisting of two terms (the first measuring the closeness to the data and the second measuring desired smoothing properties of the solution):

\[
J_\alpha(v) = \int_{\Omega_v} \left( \partial_t I + \mathbf{v} \cdot \mathbf{\nabla} I \right)^2 \, dV + \alpha \int_{\Omega_v} \| \mathbf{\nabla} v \| \, dV \tag{15}
\]

or

\[
\begin{aligned}
J_\alpha(v) &= \int_{\Omega_v} \left( \partial_t I + \mathbf{v} \cdot \mathbf{\nabla} I \right)^2 \, dV + \alpha \int_{\Omega_v} \| \mathbf{\nabla} v \|_{V_0}^2 \, dV \\
\| \mathbf{\nabla} v \|_{V_0}^2 &= \mathbf{\nabla} v_1 \cdot \mathbf{\nabla} v_1 + v_i' \mathbf{\nabla} v_i + \mathbf{\nabla} v_i \cdot \mathbf{\nabla} v_i' + \mathbf{\nabla} v_i' \cdot \mathbf{\nabla} v_i \\
&= \frac{1}{\| \mathbf{\nabla} I \|} \left[ \| \mathbf{\nabla} I \| \otimes \mathbf{\nabla} I^\perp + \gamma I d I \
\right] \cdot \mathbf{\nabla} I^\perp \\
&= \epsilon_i \, \mathbf{\nabla} I
\end{aligned} \tag{16}
\]

are \( H^1(\Omega) \)-elliptic (hence convex) for \( \alpha > 0 \) and have a unique minimizer, which depends continuously on the data. The second regularized functional appears to be more physically grounded (since it turns out that it actually regularizes only the vector field component in the direction parallel to the intensity \( I \), a component which is loosely controlled by the optical flow equation). Nevertheless, the first order Euler conditions for stationarity involve far more complicated boundary conditions for the solution field than for the first functional.

Estimates of fluid flow velocity fields are often corrupted, however, due to various deficiencies of the imaging process, making the physical interpretation of the measurements questionable. Some authors have proposed to deal with vector field estimates from any method and return a “physically plausible denoised version” thereof [66], whereas some more physical considerations can be introduced in the regularization added functional ([36] and the review paper [37]). By designing variational PIV methods, [55] showed how physically consistent flows should be estimated from PIV image sequences utilizing a distributed-parameter control approach. This has been extended in ([56], [57]) to a dynamic setting based on the vorticity transport equation formulation of the Navier-Stokes equation. [68] proposed to achieve spatial and temporal super-resolution of a measured flow. Applications are numerous in aerodynamics, for example in [26] for the analysis of turbulent jets, or for the interaction between shocks and boundary layer, [58].

Apart from the question of noise, which cannot be avoided, one can also be interested in extending the identification of the velocity field within the flow away from a “measurement box” or "measurement slice" \( \Omega \) and in identifying other parameters describing the flow, or involved in its modeling. Two approaches can be proposed, sometimes sharing the final form of the algorithms, but with a different reasoning behind them: the optimal control approach [46] and the data assimilation approach [45]. Figure 1 describes the kind of results that can be achieved.

**An optimal control approach for a model problem**

Let \( \Omega \) be the total volume (or surface) where the velocity field \( \mathbf{v} \) has to be identified, and let \( \Omega_c \) be the subdomain where the velocity measurement is made. The inflow and outflow boundaries are denoted by \( \Gamma \) and let us denote the obstacle around which the flow is studied by \( \omega \) (Figure 2).

In order to then enhance the velocity identification or estimation \( \mathbf{v} \), and to extend the estimation outside of \( \Omega_c \), the following functional is built involving two auxiliary fields as control variables: a volume force density \( \mathbf{f} \) in \( \Omega \) and a surface velocity field \( \mathbf{g} \) on \( \Gamma^* \)

\[
F(\mathbf{f}, \mathbf{g}) = H(\mathbf{v}(\mathbf{f}, \mathbf{g}), \mathbf{f}, \mathbf{g}) \tag{17}
\]

with the velocity field \( \mathbf{v} \) over the entire domain \( \Omega_{\mathbf{v}} \) being related to the pair \( (\mathbf{f}, \mathbf{g}) \) by a state equation \( E(\mathbf{v}, \mathbf{f}, \mathbf{g}) = 0 \); for example, considering the simplest one:
Let us make a few remarks related to this formulation of the identification problem. First, the state equation is chosen in such a way that, given a pair of (regular) functions \((f, g)\), a unique velocity field \(v\), and is written as:

\[
F(v, w, \mu; f, g) = H(v, f, g) - \int_\Omega v\nu \nabla w dV + \int_{\partial \Omega} \mu (v - V^R) - g) dS
\]

The equation for the adjoint field \(w\) is obtained by stationarity of the Lagrangian with respect to \(v\), and is written as:

\[
\begin{align*}
-\nu \Delta w &= H^*(v(f, g), f, g) \text{ in } \Omega \\
\nu \nabla w \cdot n &= -\nu v(f, g).n |_\partial \Omega
\end{align*}
\]

The (variational) data assimilation approach

Taking advantage of the measurements performed on a system, the general idea behind the data assimilation is to mix the information gained from the experiment, including the related uncertainties, the information gained from the modeling of the underlying physics, possibly including model uncertainties or imperfections and, lastly, a priori information on the system, substantially on its initial state (background information). The objective was initially to build a more precise estimation of the state, present or future, of the system. There are broadly two main families of methodologies for achieving this goal: filtering methods based on the (statistical) estimation theory [40] and variational methods [45] originally designed for meteorological forecasting applications. The general form of the functional to be minimized in data assimilation along a time interval of duration \(T\) is the following [32]:

\[
J(g, u) = \int_0^T [H(v(g, u) - m)^2 dt + \int_\Omega |g|^2 dV + \int_\Omega |u - w|^2 dV
\]

The choice of the state equation results clearly from a compromise between the simplicity of the state equation and the physics that it contributes to the problem; here, the simplest state equation has been retained (no inertial terms, no incompressibility constraint): the arbitration will be returned by the experience.
where the following ingredients appear:

- \( \nu \) is the flow, governed by a state equation: 
  \( \partial_t \nu + M ( \mathbf{g}, \mathbf{u} ; \nu ) = 0 \), \( \nu ( x, 0 ) = \nu ( x ) \) involving the initial state \( \mathbf{u} \), taken here as a control variable, and \( g \), the perturbation of the inflow, taken as a control variable (like in Equation 18) as well;

- \( H \) is the observation operator, which maps the flow \( \nu \) to the measurements \( m \), so that the first term of the functional \( J \) is simply a fidelity term to the data. The norm \( \| \cdot \|_2 \) used to measure the gap to the data in the data assimilation process, is built form the knowledge of measurement errors and is a covariance-based norm. In a finite dimensional setting, it reduces to the Manahohis norm generated by the inverse of the covariance matrix \( R \): 
  \[ \| \nu - m \|_2 = \langle R^{-1} \nu, \nu \rangle. \]

- The second term appearing in the functional term is a regularization term on the control variable \( g \). Again, the norm \( \| \cdot \|_2 \) is a covariance based norm.

- The last term involves the control variable \( \mathbf{u} \) constituted by the initial condition of the flow, and here the regularization is related to the closeness of a background (a priori) information \( \mathbf{u}_B \). The norm \( \| \cdot \|_2 \) is grounded in a finite dimensional setting to the covariance matrix \( B \) in the initial state.

As mentioned before, the probabilistic approach of variational data assimilation can be viewed as a regularization variational least-square identification method, but with a different reasoning behind it in the building of the regularization terms, which are based on a priori information (covariance matrix, backgrounds) and the peculiar feature that the regularization parameters are directly incorporated into the stabilizing functional terms via the choice of the covariance matrices. This last choice is nevertheless a delicate matter, with major concerns about cross-correlation and dependency.

In the previous example, a control variable has been added to the initial condition \( \mathbf{u} \) which is the only control variable in the so-called 4D-Var data assimilation method. For several years, there has been a growing demand for simultaneous identification of parameters and the estimation of state variables. Beyond the interest of exploiting large amounts of data for enhancing the simulation models (identification of model parameters) or the description of the experiments (identification of boundary conditions), a great potential can be expected either in having simulations or performing analysis for systems that do not yet exist, or even in revisiting the process of simulation itself. Joint state and parameter estimation can be simply addressed by augmenting the state variable space with the parameters \( \{ \rho \} \) and correspondingly the state equation on the parameters with the simple equation \( \dot{\rho} = 0 \) ([52], [60]). Apart from the issue of defining the related covariance operators, the adaptation of the previous formulation is straightforward. Nevertheless, this approach can be questionable on two points. First, the augmented state equation mixes quantities and equations of a very different nature, and the parameter evolution equation is somehow artificial given that the parameters are, by nature, fixed parameters. However, considered now as a function of time, it simply has to be expected that it will tend to a stationary value \( \rho^\star \), in order for this value to be able to be recognized as the “true value” of the parameter. Secondly, for situations where it is impossible to have significant a priori statistical information about the parameters, it will be impossible to build the covariance operator and to calibrate the level of uncertainties in relation to one of the other control variables.

This is the reason why some other approaches have been proposed ([34], [3]), based on a hierarchical identification process (with a first deterministic step and a related functional energy error functional \( E \)), and with the choice of the covariance operator as the Hessian of \( E \). As an extension of the parameter identification within a predetermined model, [1] proposed to estimate conjointly a flow and the parameters of its POD-reduced model.

Expanding, inside a body, measured surface fields

Advances in the development of digital cameras, image correlation techniques (DIC) and infrared cameras now make it possible to have measurement means for full-field surface displacements or temperatures that are cheap and relatively easy to manage and, more importantly, leading to very large amounts of information [61]; an example of estimated surface displacements on a cracked mock-up is given in Figure 3. Nevertheless, the use of these surface data is still largely restricted either to qualitative estimation or to quantitative analysis based on a plane mechanical or thermal state, or on homogeneous-through-the-thickness assumptions [12]. Aimed at a true 3D quantitative imaging process, the problem of reconstruction of the fields inside the solid from images obtained on parts of its boundary must be addressed.

One approach in dealing with this problem in mechanics is to first reformulate it within the continuous framework, taking advantage of the fact that the amount and spatial density of the information obtained using the digital image correlation techniques make it possible to consider that the complete displacement field is available on a part of the boundary and is not reduced on it to pinpoint data only. Then, it is possible secondly to formulate it as a Cauchy or Data Completion Problem, taking into account the fact that an overspecified data pair is given on a part \( \Gamma^\star \) of the boundary: the capture of displacement fields via the DIC on a stress-free boundary with unit normal \( \mathbf{n} \) gives access to the Dirichlet-Neumann pair \( ( \mathbf{u}^\star, \mathbf{n}^\star ) \). The Cauchy Problem is an archetypal ill-posed problem ([13], [33]).

![Figure 3 – Surface tangential displacement field obtained by DIC for a cracked mock-up (ONERA)](image-url)
In the generic situation, the usual boundary conditions are given on \( \Gamma_0 \). \( \Gamma_m \) is the part where, using DIC acquisition measurements, both displacement and stress vector components \((U_m, F_m) = 0)\) are available. Lastly, \( \Gamma_u \) is the remaining part of the boundary, where no boundary data is known. For the sake of simplicity, we will assume that \( \Gamma_b \) is empty, but the extension of the method to situations with non-empty \( \Gamma_b \) is straightforward. The boundary \( \Gamma_u \) is generally non-connected and can possibly contain internal surfaces such as cracks or boundaries of cavities and inclusions (see Figure 4).

Within the framework of linearized strain and for isothermal transformation, the constitutive equation for the material constituting the solid is written in an abstract format as a relation between the Cauchy stress tensor \( \sigma \) and the linearized strain tensor \( \varepsilon \), and can also involve internal variables \( \alpha \):

\[
(f, g \alpha) = \sigma \varepsilon
\]

where \( \varepsilon(u) = [\nabla u]^\text{lin} \) is the linearized strain operator. The general variational method, derived firstly in [5] for the Laplace operator (stationary isotropic conduction equation), for solving this problem is based on two steps. First, two auxiliary usual well-posed problems

\[
\begin{align*}
\text{div} \sigma &= 0 & \text{in} \Omega \\
f(\sigma, \varepsilon, \alpha) &= 0, \quad g(\sigma, \varepsilon, \alpha) &= 0 & \text{in} \Omega \\
u|_{\Gamma_m} &= u_0, \quad \sigma|_{\Gamma_m} = \sigma_0, \quad \alpha|_{\Gamma_m} = \alpha_0 & \text{in} \Omega \\
u|_{\Gamma_u} &= U_m, \quad \sigma|_{\Gamma_u} = F_m & \text{on} \Gamma_m \\
\sigma|_{\Gamma_u} = \eta & \text{on} \Gamma_u
\end{align*}
\]

Box 2 – The Bregman divergence

Introduced within the context of convex optimization ([16], [41]), the Bregman distance, or Bregman divergence, is akin to a metric, although not satisfying the triangle inequality nor symmetry. Let \( J \) be a convex function, the generalized Bregman distance between \( e_1 \) and \( e_2 \) with respect to \( J \) is the non-negative scalar:

\[
D_J(e_1, e_2) = J(e_2) - J(e_1) - \langle p, e_2 - e_1 \rangle \quad \text{for} \quad p \in \nabla J(e_1)
\]

where \( \nabla J(e) \) stands for the subdifferential of \( J \) at \( e \).
$P_i$, $i = 1,2$ are defined, each using only one of the overspecified boundary data on $\Gamma_u$ and a given normal stress vector field $\eta$ on $\Gamma_e$:

$$\begin{align*}
\text{div}\sigma_i &= 0 & \text{in } \Omega \\
f(\sigma_i, \epsilon, \alpha_i) &= 0, g(\sigma_i, \epsilon, \alpha_i) &= 0 & \text{in } \Omega \\
u_{|\Gamma_u} &= u_0, \quad \sigma_{|\Gamma_u} = 0, \quad \alpha_{|\Gamma_u} = \alpha & \text{on } \Gamma_u \\
\sigma_i, \eta_{|\Gamma_e} &= \eta & \text{on } \Gamma_e
\end{align*}$$

(29)

And respectively for $\mathcal{P}_i$ and $\mathcal{P}_2$:

$$\begin{align*}
\mathcal{P}_i \{ u_i = U_m \text{ on } \Gamma_m \} & \quad \mathcal{P}_2 \{ \sigma, n = F_m \text{ on } \Gamma_m \}
\end{align*}$$

(30)

With the additional condition (global equilibrium) on $\eta$:

$$\int_{\Omega} F_e \, dS + \int_{\partial\Omega} \eta \, dB = 0.$$  

It is clear that if a surface traction field $\eta\alpha$ on $\Gamma_e$ is such that $u\alpha = u_1 + \text{RBM}$, where $\text{RBM}$ is a Rigid Body Motion, the two problems $\mathcal{P}_1$ and $\mathcal{P}_2$ will have the same solution $(\sigma, \alpha)$. Therefore, the Cauchy problem is solved with $u\alpha$, and the solution of the Data Completion Problem is $\eta\alpha$. Furthermore, the Dirichlet-Neumann data pair on $\Gamma_e$ is $(u, \eta\alpha)$.

The second step will therefore be to minimize a suitable gap functional $E$ between the two states $(\sigma, \alpha)$ and $(\eta\alpha, \eta\alpha)$, as a function of the sought stress vector field $\eta$ on $\Gamma_e$, leading then to the variational method:

$$\eta_{opt} = \arg\min_{\eta} G(\eta) = E[\{u, \sigma, \alpha\}|\eta], \{u, \sigma, \alpha\}|\eta]$$

(31)

Clearly, the choice of the gap $E$ is of primary importance, in view of the overall performance of the method. The gap functional takes into account first that the dimensions of the state variables are various, and second that the states involved are not any collections of the state variables, but rather obey a conservation law and a constitutive relation. What is needed is therefore a kind of metric defined on a geometric variety, rather than a distance in the vector space into which it is plunged.

It is therefore proposed to turn to Bregman divergences, which can be built from any convex function (see Box 2). More precisely, after having selected a convex function $J$ of the state variables suited to each kind of constitutive relation, the gap functional is written as:

$$E, (e_1, e_2) = \int_{\Omega} D_j(e_1, e_2) \, dV$$

(32)

Where $D_j(e_1, e_2)$ is the symmetrized Bregman divergence:

$$D_j(e_1, e_2) = D_j(e_2, e_1) + D_j(e_2, e_1) = \{\pi - p_2, e_1 - e_2\}$$

(33)

The properties of the Bregman divergence are closely linked to the Legendre-Fenchel inequality in convex analysis ([30], [38]), especially in establishing the following property, which is of utmost importance in the perspective of building a gap functional.

**Proposition**: If $J$ is strictly convex, then the following equivalences hold true:

$$D_j(e_1, e_2) = 0 \iff e_1 = e_2 \iff \partial J(e_1) = \partial J(e_2)$$

Indeed, coming back to the solution process (13) for solving the data completion problem (10), using a gap functional based on the symmetrized Bregman divergence (33) leads to a functional correctly suited for the minimization:

$$E, (e_1, e_2) = \int_{\Omega} \{\pi - p_2, e_1 - e_2\} \, dV \geq 0$$

(35)

It is worth noting that the strictly convex function $J$ does not appear anymore in the expression of the symmetrized Bregman divergence, so there is no need to compute it: it is sufficient for such a function to be able to be identified in order for the property (35) to hold true. Returning to the Data Completion Problem for extending mechanical fields inside the solid, the primal state variables are $(e, \alpha)$, and seeking convex functions $J$ generating an appropriate and effective gap functional $e_1, e_2$ one is naturally led to the convex framework of modeling for the constitutive equation. Actually, the Standard Generalized Materials [35] theory rests on the definition of a convex potential $W$ (the free Helmholtz-Gibbs energy) and a convex pseudo-potential of dissipation $\eta\psi$ (positively homogeneous of Degree 1), such that the constitutive equations appearing in (28) are written:

$$W = W(e - \eta\psi, \eta\psi) = \psi(e, \psi)$$

(36)

$$\sigma = \epsilon\psi\sigma \quad A = \frac{\partial W}{\partial \sigma}$$

Here the plastic strain $\sigma$ appears separately in the internal variable list, and the free energy is a function of the elastic strain $e = e - \sigma$ in order to comply with the usual presentations.

For linear elasticity and non-linear elasticity (in a small strain context) where neither internal variables nor dissipation pseudo-potential are involved, the variational method (31) for solving the Data Completion Problem has been applied to the extension of surfacic displacement fields for various applications, such as identification of contact areas [6], of material parameters in inclusions, of geometry of crack fronts [7], of internal pressure in cavities [9], or determination of linear fracture mechanics parameters [8]. The following gap functional has been used:

$$G(\eta) = E_{B} \{\epsilon_1([u(\eta)]), \epsilon_2([u(\eta)])\} = \int_{\Omega} \{\sigma_1 - \sigma_2\} \cdot (\epsilon_1 - \epsilon_2) \, dV$$

(37)

In the case of linear elasticity (quadratic potential $u\psi$), the symmetrized Bregman divergence turns out to be simply twice the elastic energy of the displacement field $u_i - u_{i\psi}; \quad (\sigma_1 - \sigma_2; (\epsilon_1 - \epsilon_2); = 2W(e, u_i - u_{i\psi})$.

For the elastoplastic case with implicit incremental formulation [4] and taking advantage of the linearity of the Bregman divergence, that is:

$$D_{\lambda, \mu} J(e_1, e_2) = \lambda D_j(e_1, e_2) + \mu D_j(e_1, e_2) \quad \forall \lambda, \mu \geq 0$$

(38)

the following single-parameter family of gap functions can be built ($0 \leq \chi \leq 1$):

$$E_{\chi\psi}(\eta) = \frac{1}{\chi} \int_{\Omega} \left(\lambda \Delta \sigma_1 - \Delta \sigma_2\right) \cdot \left(\chi \Delta \epsilon_1 - \Delta \epsilon_2\right) + (1 - 2\chi) \Delta \epsilon_1^\sigma - \Delta \epsilon_2^\sigma \right) \, dV$$

$$+ \int_{\Omega} \left(\lambda \Delta \sigma_1 - \Delta \sigma_2\right) \cdot \left(1 - 2\chi\right) \Delta \epsilon_1 - \Delta \epsilon_2 \right) \, dV$$

(39)
The condition of strict convexity of $\chi W + (1-\chi) \Psi$ can be ensured only for $\chi > 0$, because the potential $\Psi$ is not strictly convex (as a positively homogeneous function of Degree one: $\Psi(\lambda x) = \lambda \Psi(x)$ for $\lambda \geq 0$).

Within this family, the gap function obtained for $\chi = 0.5$, which balances exactly between the free energy gap and the dissipation gap, has the peculiar feature of involving only the stress and strain tensors and can then be called the Drücker Gap:

$$E_D = \int_\Omega \left( \Delta \sigma_1 - \Delta \sigma_2 \right) \cdot \left( \Delta \varepsilon_1 - \Delta \varepsilon_2 \right) dV \quad (40)$$

The Drücker Gap leads to the only computable function $G(\eta)$ by a boundary integration over the entire external boundary of the solid, as is the case in linear elasticity and hyperelasticity, using the virtual power principle. This feature has been widely used previously to improve the global performance of the solution algorithm for a linear Cauchy problem and to reduce the computational burden [6].

The following figures show the kind of results that can be achieved with respect to the extension of the mechanical field inside a solid from surface displacement measurements. The method and algorithm are summarized in Box 3. Results for linear elastic behavior were obtained with very few iterations (~ 10), whereas for strongly non-linear materials such as elastoplastic materials (elastoplasticity with linear hardening), a hundred minimizing iterations are needed.

It can be seen that either the contact surface for an elastic complex structure (Figure 5), or the residual stress after unloading for an elastoplastic solid (Figure 6) can be recovered with good accuracy.

![Displacement Field](image)

**Figure 5 – 3D Identification of the displacement on the interior surface of an elastic mock-up loaded by a flexured rod (from [6])**
The proposed approach, using the minimization formulation (31), does not explicitly incorporate any kind of regularization, although the Cauchy problem is very ill-posed. Nevertheless, the use of a conjugate gradient algorithm entails a form of regularization in the numerical applications: the value of the stopping criterion can also play the role of a regularization parameter [42]. For moderate noise in the data (under 2%), this regularization appears to be sufficient, as shown for the thermal conductivity equation [5].

Conclusions

This paper overviewed some results and techniques of inverse problems that can be used in signal processing on data emanating from various experimental situations arising in the aerospace domain. Two specific topics related to image-based identification problems have been focused on.

With regard to the near future, some leads or trends can be pointed out. Two promising, and already fruitful, ways can be identified in the perspective of combining measurements of distinct data acquisition nature. The first is the coupling between the 3D-PIV techniques and the 3D Background Oriented Schlieren (BOS), which is aimed at the reconstruction of the density field of instantaneous flows via the analysis of the light deflection through a medium with inhomogeneous optical index. The second is the simultaneous exploitation of the DIC displacement field measurements and the thermal camera temperature field measurements through nonlinear thermomechanics modeling. Originated by an interesting change of perspective, the passive approach to wave probing, which uses ambient noise or diffuse fields (nondestructive testing, seismic identification, echography, etc.) or non-cooperative sources of illumination (radar), stimulates a wide field of research. The deep learning applied to the analysis of a large amount of experimental data would also generate high hopes, although the number of really convincing results remains still low in the aerospace domain. From the numerical simulation side, progress can be expected thanks to the development of data assimilation, taking into account uncertainties both in the results and in the methods themselves, and enhancing performances through the use of model reduction techniques.

References


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Stéphane ANDRIEUX received his Engineering diploma from the École Nationale des Ponts et Chaussées, France, in 1980, and later his PhD in 1983. He joined the R&D division of Electricité de France in 1985 and led various research groups and mechanics departments there. After being the Scientific Director of the EDF R&D division, he is now the General Scientific Director of ONERA and professor at École des Ponts ParisTech. His work is focused on damage mechanics and inverse problems. In 2006, he received the Dechelle Prize from the French Academy of Sciences.