



## Further remarks on input-output linearization of SISO time-varying delay systems

Ihab Haidar, Florentina Nicolau, Jean-Pierre Barbot, Woihida Aggoune

### ► To cite this version:

Ihab Haidar, Florentina Nicolau, Jean-Pierre Barbot, Woihida Aggoune. Further remarks on input-output linearization of SISO time-varying delay systems. 2017 SIAM Conference on Control and Its Applications, Jul 2017, Pittsburgh, United States. hal-01511422

**HAL Id: hal-01511422**

**<https://hal.science/hal-01511422>**

Submitted on 21 Apr 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Further remarks on input-output linearization of SISO time-varying delay systems

Ihab Haidar<sup>\*</sup>   Florentina Nicolau<sup>†</sup>   Jean-Pierre Barbot<sup>‡</sup>   Woihida Aggoune<sup>§</sup>

## Abstract

In this paper, the problem of input-output linearization of nonlinear single-input single-output time-varying delay systems (with delays in the input and the output) is discussed and illustrated via several examples. Sufficient conditions have been recently developed by the authors. Here, we propose to discuss these conditions and give some insight allowing to improve them.

*Keywords:* Nonlinear Control Systems, Delay systems, Input-output linearization, Lie derivative

## 1 Introduction

In nonlinear control theory, several methods such the equilibrium points analysis [3] and Lyapunov–Krasovskii approach [13], aim to study the asymptotic behavior of time-delay systems. The input-output linearization approach offers interesting tools in this context. This is based on the possibility of defining, locally (around a convenient state) and starting from a sufficiently smooth output, a suitable coordinate transformation permitting an equivalent linear representation of a subsystem (and maybe the overall system) in the new coordinates. This approach is widely studied in the case of delay-free (see, e.g., [4, 12, 17], and references therein) and constant-delay control systems (see, e.g., [2, 5, 8, 9, 11, 14, 15, 21], see also the more recent paper [1], where, motivated by some observability problems of nonlinear constant-delay systems, necessary and sufficient conditions allowing an equivalent linear weakly observable time-delay system representation have been presented.

The input-output linearization problem of nonlinear time-varying delay systems with delayed input is considered in [10]. In that paper, the authors propose an extension of the Lie derivative in order to compute the relative degree of time-varying delay systems and design a coordinate transformation and a feedback that input-output linearize the behavior.

Whereas in [10], we give sufficient conditions which guarantee the existence of a suitable linearizing coordinate transformation and linearizing feedback, the goal of this paper is to illustrate those results by several examples, to understand why each condition is important and to which extent it is necessary. The interest of the paper is three fold. Firstly, it allows us to understand that the main difficulty when dealing with input-output linearization of time-delay systems is the causality and the boundedness problems of the expected linearizing feedback. Secondly, it enables us to better understand the role played by a time-varying delay. We will see (for instance, in the presented examples) that the consecutive derivatives of the considered time-varying delay appear explicitly in the expression of the linearizing feedback. Thirdly, it presents two set of examples showing the efficiency and conservativeness of our conditions: the first four examples studied in this paper highlight the sufficiency of such conditions, while the last two ones show that they are not necessary (and in the same time, give a measure to which extent they are necessary). Finally, the paper gives (through the discussion of different tutorial examples) several ideas on how to improve the sufficient

---

<sup>\*</sup>Quartz EA 7393, ENSEA, 6 Avenue du Ponceau 95014, Cergy-Pontoise cedex, France. Email: ihab.haidar@ensea.fr

<sup>†</sup>Quartz EA 7393, ENSEA. Email: florentina.nicolau@ensea.fr

<sup>‡</sup>Quartz EA 7393, ENSEA, & Non-A, INRIA-Lille, Nord Europe, 40 avenue Halley, 59650, Villeneuve d'Ascq, France. Email: barbot@ensea.fr

<sup>§</sup>Quartz EA 7393, ENSEA. Email: woihida.aggoune@ensea.fr

conditions in order to obtain less conservative conditions guaranteeing the existence of suitable linearizing coordinate and feedback transformations.

The paper is organized as follows. We first give some notations and recall the definition (introduced by the authors in [10]) extended Lie derivative for time-varying delay systems. In Section 3, we present our main result that gives an explicit coordinate transformation and a feedback that partially or completely linearize the system, while in Section 4, we illustrate our results by several examples. We highlight via examples the importance of each condition of our main theorem and show that our conditions are sufficient, but not necessary. Finally, we present some conclusions.

## 2 Extended Lie derivative for time-varying delay systems

The Lie derivative definition for time-varying delay systems (see [10]), which is a generalization of that introduced in [6] for constant-delay systems, is recalled. Let us, firstly, give some notations and definitions.

**Definition 1** Let  $\bar{\theta} > 0$  be the maximum of the time-delay function  $\theta(\cdot)$  which is supposed to be known over  $\mathbb{R}$  and takes its values in  $(0, \bar{\theta}]$ , and let the recursive relation

$$\tau_{i+1} = \tau_i - \theta \circ \tau_i, \quad \text{for } i \geq 0, \quad (1)$$

where  $\tau_0(t) \equiv t$ . We denote by  $\delta^i$  the delay operator that shifts the time from  $t$  to  $\tau_i(t)$  and is given by

$$\delta^0 \sigma(t) = \sigma(t), \quad \text{and} \quad \delta^i \sigma(t) = \sigma \circ \tau_i(t), \quad \text{for } i \geq 1, \quad (2)$$

where  $\sigma$  is a function defined on an interval containing  $[t - i\bar{\theta}, t]$ . The application of  $\delta^i$  on a composed function  $\gamma \circ \sigma(\cdot)$  is given by

$$\delta^i \gamma \circ \sigma(t) = \gamma(\delta^i \sigma(t)), \quad \text{for } i \geq 1. \quad (3)$$

Applied on the product of two functions, this delay operator acts as the following

$$\delta^i \gamma(t) \cdot \sigma(t) = (\delta^i \gamma(t)) \cdot (\delta^i \sigma(t)), \quad \text{for } i \geq 1, \quad (4)$$

i.e., the delay spreads to the right. If brackets are present, i.e., we have  $(\delta^i \gamma(t))\sigma(t)$ , then the delay affects only the first function (here  $\gamma$ ). We can now give the definition of the Lie derivative for time-varying delay systems.

**Definition 2** Let  $q$  be a positive integer. Let  $f(x, \delta) \equiv f(x, \delta x, \dots, \delta^q x)$  and  $g(x, \delta) \equiv g(x, \delta x, \dots, \delta^q x)$  be two vector fields whose components are functions of  $\delta^i x(t)$ , and  $h(x, \delta, t) \equiv h(x, \delta x, \dots, \delta^q x)$  a real valued function of  $t$  and  $\delta^i x(t)$ , for  $i = 0, \dots, q$ . Then, the derivative of  $h$  along  $f$  at  $(x, \delta, t)$  is defined as

$$L_f h(x, \delta, t) := \sum_{i=0}^q \frac{\partial h}{\partial \delta^i x} \dot{\tau}_i \delta^i f(x, \delta) + \frac{\partial h}{\partial t}(x, \delta, t). \quad (5)$$

Since  $L_f h$  is a real-valued function with delays, this operation can be recursively repeated for higher order as

$$L_f^\rho h(x, \delta, t) := \sum_{i=0}^{q\rho} \frac{\partial L_f^{\rho-1} h}{\partial \delta^i x} \dot{\tau}_i \delta^i f(x, \delta) + \frac{\partial L_f^{\rho-1} h}{\partial t}(x, \delta, t). \quad (6)$$

Notice that (5) is an extension of the Lie Backlund derivative (see, e.g., [5]). The difference resides in our consideration of time-varying delays which leads to multiplicative coefficients (the derivative of the recursive delay function  $\tau_i$ ) affecting the operator  $\delta^i$  as well as the partial derivative of  $h$  with respect to  $t$ .

We also need the reduced Lie derivative

$$\bar{L}_g h(x, \delta, t) := \sum_{i=0}^q \frac{\partial h}{\partial \delta^i x} \dot{\tau}_i \delta^i g(x, \delta). \quad (7)$$

Notice that, contrary to (5), the partial derivative of  $h$  with respect to  $t$  is not present in (7).

### 3 Input-output Linearization

Consider the single-input single-output nonlinear time-varying delay system

$$\begin{aligned}\dot{x}(t) &= f(x(t), \delta) + g_0(x(t), \delta)u(t) + g_1(x(t), \delta)\delta u(t) \\ y(t) &= h(x(t), \delta), \\ x(s) &= \varphi(s), \quad \forall s \in [-q\bar{\theta}, 0], \\ u(s) &= \nu(s), \quad \forall s \in [-\bar{\theta}, 0].\end{aligned}\tag{8}$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system at time  $t$ ,  $\delta$  is the pure time-delay operator associated to a sufficiently smooth time-varying delay  $\theta : \mathbb{R} \rightarrow (0, \bar{\theta}]$  satisfying  $\dot{\theta} < 1$ , and  $\bar{\theta}$  is a positive real number. The condition on the derivative of the delay function  $\theta$  is important for causality reason. The vector fields  $f, g_0, g_1 : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^n$  and the function  $h : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$  are sufficiently smooth, such that  $f(0, 0) = 0$  and  $h(0, 0) = 0$  (here the second argument of  $f$  and  $h$  is the delayed state  $(\delta^1 x(t), \dots, \delta^q x(t)) = (0, \dots, 0)$ , see Definition 2). The initial condition  $\varphi \in \mathcal{C}([-q\bar{\theta}, 0], \mathbb{R}^n)$  and the input  $u : [-\bar{\theta}, +\infty) \rightarrow \mathbb{R}$  is a Lebesgue measurable function. Knowing that the Lie derivatives of  $h$  may involve the initial state at a certain order of derivation, we suppose that  $\varphi$  satisfies the differential equations of (8) on a sufficiently large interval. We suppose also that system (8) is forward complete.

Recall that according to Definition 2,  $f(x(t), \delta)$  denotes a vector field whose components may depend on the delayed operators  $\delta^i$  up to a finite order  $q$ . In the system (8), the integer  $q$  corresponds, in fact, to the maximal delay order explicitly involved in  $f$ ,  $g_0$ ,  $g_1$  and  $h$  and it does not mean that  $f$ ,  $g_0$ ,  $g_1$  and  $h$  necessarily imply the same delay operators.

Let  $1 \leq \rho \leq n$ . We will next discuss conditions under which the variables  $z_i = L_f^{i-1}h$ , for  $i = 1, \dots, \rho$ , define a coordinates transformation and the possibility of designing a causal and bounded feedback  $u$  solution of

$$a[\delta]u(t) = -L_f^\rho h(x, \delta, t) + \delta^j v(t), \quad t \geq \tau_j^{-1}(0),\tag{9}$$

where  $a[\delta]$  is a  $m$ -degree  $\delta$ -polynomial of the form

$$a[\delta] = a^0(x, \delta, t)\delta^0 + \dots + a^m(x, \delta, t)\delta^m,\tag{10}$$

such that the closed loop subsystem corresponding to the new variables is described by

$$\begin{aligned}\dot{z}(t) &= Az(t) + B\delta^j v(t) \\ y(t) &= Cz(t),\end{aligned}\tag{11}$$

where  $A \in \mathbb{R}^{\rho \times \rho}$  and  $B, C^T \in \mathbb{R}^\rho$  are constant,  $v$  is the new control and the integer  $j$  is defined through the following assumption

**Assumption 1** *Throughout, for a  $\delta$ -polynomial  $a[\delta]$ , we denote by  $j$  the first coefficient of  $a[\delta]$  which is non identically zero, i.e., the integer  $0 \leq j \leq m$  is such that*

$$a^j(x, \delta, t) \neq 0,\tag{12}$$

$$a^i(x, \delta, t) \equiv 0, \quad \forall i < j.\tag{13}$$

Furthermore, in this paper, we do not deal with singularities, i.e., condition (12) will always be interpreted as

$$a^j(x, \delta, t) \neq 0, \quad \forall t \geq \tau_j^{-1}(0),\tag{14}$$

where  $\tau_j^{-1}(0)$  is, in fact, the time from which the control starts acting on the system. Notice that  $\tau_j^{-1}(0)$  is well defined since the time-delay function  $\theta(\cdot)$  is completely known over  $\mathbb{R}$  (see Definition 1).

**Remark 1** The  $\delta$ -polynomial  $a[\delta]$  is an element of the ring of differential time-varying delay operators. See [16] for more details about the algebraic properties of this ring.

Two problems (causality and boundedness) arise when constructing a feedback  $u$  from (9). Some of the causality problems come from the fact that the drift involves more delays than the control vector fields or vice versa. The boundedness problem derives from the fact that the expected feedback is described by a recursive<sup>1</sup> equation.

Conditions allowing the boundedness of the feedback  $u$  are thus required. This is done by the following lemma which plays a very important role, as we will see when stating our main theorem and analyzing our examples.

**Lemma 1** Let a  $\delta$ -polynomial  $a[\delta]$  of form (10), of degree  $m \geq 0$ , and  $j$  the integer defined in Assumption 1. Suppose that all  $a^i(\cdot)$ , for  $j \leq i \leq m$ , are bounded,  $v : [-m\bar{\theta}, +\infty) \rightarrow \mathbb{R}$  is such that  $-L_f^\rho h(x, \delta, t) + \delta^j v(t)$  is bounded over  $[\tau_j^{-1}(0), +\infty)$  and  $u : [-m\bar{\theta}, +\infty) \rightarrow \mathbb{R}$ , defined by (9), is bounded over the interval  $[-m\bar{\theta}, \tau_j^{-1}(0)]$ . If there exists a constant  $c > 1$  such that

$$\sup_{t \geq \tau_j^{-1}(0)} \left\| \frac{a^i(x, \delta, t)}{a^j(x, \delta, t)} \right\| \leq \frac{1}{c(m-j)}, \quad \forall i > j, \quad (15)$$

then, for every  $t \geq \tau_j^{-1}(0)$ , we have

$$\|u(t)\| \leq \frac{c}{c-1} \sup_{s \geq \tau_j^{-1}(0)} \left\| \frac{-L_f^\rho h(x, \delta, s) + \delta^j v(s)}{a^j(x, \delta, s)} \right\| + \varepsilon(t),$$

where  $\varepsilon(t)$  tends to 0 when  $t$  tends to  $+\infty$ .

**Proof 1** The proof is given in [10, Section 5]. □

We stress that the boundedness problem of  $u$ , in the case of constant delays and non-delayed inputs, is extensively investigated in the literature (see, e.g., [15], see also [8, Lemma A1]).

**Definition 3** System (8) is said fully (resp. partially) linearizable with delay (resp., without delay) if  $\rho = n$  (resp.  $\rho < n$ ) and  $j \geq 1$  (resp  $j = 0$ ).

The integer  $\rho$  is in fact the relative degree of the system which can be defined by repeated differentiation of the output with respect to time  $t$  in the same way as the relative degree of finite dimensional nonlinear systems (see, e.g., [12]).

The following theorem gives an explicit coordinate transformation and a causal-bounded feedback that partially or completely linearize system (8).

**Theorem 1** Suppose that system (8) satisfies the following conditions:

1. There exists a positive integer  $\rho \leq n$  such that

$$\begin{cases} \bar{L}_{g_0} L_f^i h(x, \delta, t) = \bar{L}_{g_1} L_f^i h(x, \delta, t) = 0, \\ \quad \forall i = 0, \dots, \rho - 2, \\ \bar{L}_{g_0} L_f^{\rho-1} h(x, \delta, t) \neq 0, \text{ if } g_0 \neq 0, \text{ and} \\ \bar{L}_{g_1} L_f^{\rho-1} h(x, \delta, t) \neq 0, \text{ if } g_0 \equiv 0. \end{cases} \quad (16)$$

---

<sup>1</sup>By “recursive” we point out that the construction of  $u$  requires a recursive prediction of the values of  $v$  (which is chosen as a suitable function of the state and the desired trajectory in order to achieve the desired behavior) over the intervals  $[0, \tau_j^{-1}(0)]$  and  $[\tau_i^{-1}(0), \tau_{i+1}^{-1}(0)]$ , for  $i \geq j$ , (the integer  $j$  being defined by Assumption 1).

2. The development of  $\bar{L}_{g_0}L_f^{\rho-1}h$  and  $\bar{L}_{g_1}L_f^{\rho-1}h$  with respect to the operator  $\delta$  gives respectively

$$\begin{cases} \bar{L}_{g_0}L_f^{\rho-1}h = \sum_{i=0}^{\rho q} a_0^i(x, \delta, t)\delta^i, \\ \bar{L}_{g_1}L_f^{\rho-1}h = \sum_{i=1}^{\rho q+1} a_1^i(x, \delta, t)\delta^i \end{cases} \quad (17)$$

for which the  $\delta$ -polynomial

$$\begin{aligned} a(\delta) &= a_0^0(x, \delta, t) + \sum_{i=1}^{\rho q} (a_0^i(x, \delta, t) + a_1^i(x, \delta, t))\delta^i + a_1^{\rho q+1}(x, \delta, t)\delta^{\rho q+1} \\ &= \sum_{i=0}^{\rho q+1} a^i(x, \delta, t)\delta^i, \end{aligned} \quad (18)$$

with  $a^0 = a_0^0$ ,  $a^i = a_0^i + a_1^i$ ,  $a^{\rho q+1} = a_1^{\rho q+1}$ , satisfies Lemma 1, with  $m = \rho q + 1$ .

3. If  $\rho < n$ , then

$$\frac{\partial L_f^\rho h}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial a^k}{\partial \delta^i x} \equiv 0, \quad (19)$$

for  $0 \leq i \leq j-1$ ,  $j \leq k \leq m$ , where  $j$  is defined in Assumption 1. Moreover, if  $g_0 \neq 0$ , then  $a^j = a_0^j$ .

4. The state  $x(t)$  is known over  $[-q(\rho+1)\bar{\theta}, 0]$  and  $u(t)$  is known over  $[-(\rho q+1)\bar{\theta}, 0]$ .

Then a part of system (8) can be transformed to the  $\rho$ -th order linear input-output system (11) by applying the following coordinate transformation

$$z(t) = \Phi(x, \delta, t) = \begin{pmatrix} h(x, \delta) \\ L_f h(x, \delta, t) \\ \vdots \\ L_f^{\rho-1} h(x, \delta, t) \end{pmatrix} \quad (20)$$

and the causal and bounded feedback

$$a(\delta)u(t) = -L_f^\rho h(x, \delta, t) + \delta^j v(t), \quad t \geq \tau_j^{-1}(0). \quad (21)$$

Moreover, if  $\rho = n$ , then the system is fully linearizable with delay (in the sense of Definition 3) and if, in addition  $j = 0$  then the system is fully linearizable without delay.

**Proof 2** The proof is given in [10, Section 5]. □

We will next make some observations concerning the conditions of Theorem 1. Notice that if  $g_0$  is non-identically zero, the first non zero reduced Lie derivative has to be obtained with  $g_0$ , i.e., the first coefficient  $a^j$  of  $a(\delta)$  has to stem from  $\bar{L}_{g_0}L_f^{\rho-1}h \neq 0$  only (i.e.,  $a_1^j \equiv 0$  and  $a^j = a_0^j$ ). If  $g_0$  is identically zero (i.e., if in system (8), only the delayed control is present), then the  $\delta$ -polynomial  $a(\delta)$  is clearly given by  $\bar{L}_{g_1}L_f^{\rho-1}h \neq 0$ . In both cases, the coefficients  $a_i$  of  $a(\delta)$  have to satisfy Lemma 1. In addition, in the case of partially linearizable system with delay, the coefficients  $a_i$  should contain sufficiently delayed states avoiding causality problems. The conditions of Theorem 1 guarantee the construction of a causal and bounded feedback which linearizes (partially or completely) the original system.

Notice that if the relative degree is not equal to the system dimension, then a zero-dynamics appears and must be input-to-state stable (ISS) (see, e.g., [7], [18]) in order to avoid the peaking phenomenon (see [19] for more details). The input-to-state stability of systems satisfying Theorem 1 has been discussed in [10].

Our result gives only sufficient conditions for input-output linearization. In fact, we will see in the next section that we can find examples that do not satisfy the conditions of the above theorem, but for which we can construct a suitable feedback that input-output linearizes the system.

## 4 Examples

In this section, we discuss a series of examples which show the sufficiency of the conditions of Theorem 1 and the importance of Lemma 1. The presented examples are splitted in two categories: the first one showing the sufficiency of the conditions and second one showing that these conditions are not necessary.

### 4.1 Sufficiency conditions of Theorem 1

Here, we give examples showing the efficiency of the conditions given by Theorem 1, respectively by Lemma 1. In the overall cases, we suppose that the delay function  $\theta(\cdot)$  is such that  $\dot{\theta} < 1$  (see the beginning of Section 3). In addition, we suppose that  $\dot{\theta}(t) \geq -k$ , where  $k$  is a positive constant (this is needed in order to obtain bounded coefficients  $a^i$ , for  $j \leq i \leq m$ , for the  $\delta$ -polynomial  $a[\delta]$ , as required by Lemma 1). For the sake of notation simplicity, we also introduce the constant  $k_1 = (1 + k)^2$ .

**Example 1** (*satisfying Theorem 1*)

Consider the following time-varying delay system

$$\begin{aligned}\dot{x}_1(t) &= \delta^1 u(t) \\ \dot{x}_2(t) &= k_1(4 + x_1^2(t))u(t) \\ \dot{x}_3(t) &= \delta^2 x_1(t) + x_2(t) \\ h(t) &= x_3(t),\end{aligned}\tag{22}$$

with  $x(t) = \varphi(t)$ ,  $\forall t \in [-6\bar{\theta}, 0]$ , and  $u(t) = \nu(t)$ ,  $\forall t \in [-5\bar{\theta}, 0]$ , given. Contrary to our intuition, the state  $x$  has to be known over an interval greater than that associated to the control. That is due to the fact that in order to construct  $u(t)$ , we need to know the state but also the delayed state. Here, we have  $g_0^T(t) = (0, k_1(4 + x_1^2), 0)$ ,  $g_1^T = (1, 0, 0)$  and  $f^T = (0, 0, \delta^2 x_1 + x_2)$ . A straightforward computation shows that  $\rho = 2$ ,  $\bar{L}_{g_0} L_f h = k_1(4 + x_1^2)\delta^0$ ,  $\bar{L}_{g_1} L_f h = \dot{\tau}_2 \delta^3$  and  $L_f^2 h = 0$ . Hence

$$a[\delta] = a_0^0 \delta^0 + a_1^3 \delta^3 = k_1(4 + x_1^2)\delta^0 + \dot{\tau}_2 \delta^3.\tag{23}$$

Thus, we are in the case of a partially linearizable (with a relative degree  $\rho = 2$ ) without delay (see Definition 3) and the feedback  $u$  has to verify the following relation

$$k_1(4 + x_1^2)u(t) + \dot{\tau}_2(t)\delta^3 u(t) = v(t), \quad t \geq 0.\tag{24}$$

It is clear that Example 1 is in conformity with the statements of Theorem 1. Indeed, the integer  $\rho$  exists and equals 2 and condition (16) is satisfied. Now, since  $-k \leq \dot{\theta} < 1$ , we can easily verify that  $|\dot{\tau}_2| \leq k_1$  and it follows that condition (15) holds. Moreover, statement (3) of Theorem 1 is fulfilled. Finally, item (4) of Theorem 1 is satisfied since the state  $x(t)$  and the control  $u(t)$  are given on the intervals  $[-q(\rho + 1)\bar{\theta}, 0] = [-6\bar{\theta}, 0]$  and  $[-(\rho q + 1)\bar{\theta}, 0] = [-5\bar{\theta}, 0]$ , respectively (here  $\rho = 2$  and  $q = 2$ ). Note that,  $[-6\bar{\theta}, 0]$  and  $[-5\bar{\theta}, 0]$  are the maximal intervals on which we may need to know the initial conditions  $\varphi$  and  $\nu$ . However, in this example knowing  $\varphi$  and  $\nu$  only on  $[-2\bar{\theta}, 0]$  and  $[-3\bar{\theta}, 0]$  is needed.

**Example 2** (*not satisfying statement (3) of Theorem 1*)

Consider the following time-varying delay system

$$\begin{aligned}\dot{x}_1(t) &= u(t) \\ \dot{x}_2(t) &= k_1(4 + x_1^2(t))\delta^1 u(t) \\ \dot{x}_3(t) &= \delta^2 x_1(t) + x_2(t) \\ h(t) &= x_3(t),\end{aligned}\tag{25}$$

with  $x(t) = \varphi(t)$ ,  $\forall t \in [-6\bar{\theta}, 0]$ , and  $u(t) = \nu(t)$ ,  $\forall t \in [-5\bar{\theta}, 0]$ , given. Notice that the above system is very similar to the previous one, but the role of  $g_0$  and  $g_1$  has been interchanged. We will show that (25) is not input-output linearizable (in the sens that we cannot construct a causal feedback that linearizes the system)

and that (25) does not satisfy one of the conditions of Theorem 1.

We have  $g_0^T = (1, 0, 0)$ ,  $g_1^T = (0, k_1(4 + x_1^2), 0)$  and  $f^T = (0, 0, \delta^2 x_1 + x_2)$ . Similarly to the previous case, we obtain:

$$a(\delta) = a_1^1 \delta^1 + a_0^2 \delta^2 = k_1(4 + x_1^2) \delta^1 + \dot{\tau}_2 \delta^2 \quad (26)$$

and the feedback  $u$  has to verify the following relation

$$k_1(4 + x_1^2(t)) \delta^1 u(t) + \dot{\tau}_2(t) \delta^2 u(t) = \delta^1 v(t), \quad t \geq \tau_1^{-1}(0). \quad (27)$$

Notice that Lemma 1 holds for this example, but statement (3) of Theorem 1 is violated for two reasons:  $\frac{\partial a^1}{\partial x_1} \neq 0$  and the first non identically zero coefficient of the polynomial  $a(\delta)$  is such that  $a^1 = a_1^1$  and it stems from  $\bar{L}_{g_1} L_f h$ . The causality problem becomes now clear. Indeed, from (27), it follows that the computation of  $u(t)$  depends on the future of  $x_1$ . Or, from the first equation of (25), we see that  $u$  acts instantaneously on  $x_1$  which prevents us to assign  $u(t)$  computed by (27) to (25).

**Example 3** (not satisfying statement (3) of Theorem 1: case of non delayed inputs and delayed outputs)  
Consider the following time-varying delay system

$$\begin{aligned} \dot{x}_1(t) &= x_2^2(t) u(t) \\ \dot{x}_2(t) &= u(t) \\ \dot{x}_3(t) &= x_1(t) \\ h(t) &= x_3(t) + \delta^4 x_2(t), \end{aligned} \quad (28)$$

for which  $x(t) = \varphi(t)$ ,  $\forall t \in [-8\bar{\theta}, 0]$ , and  $u(t) = \nu(t)$ ,  $\forall t \in [-5\bar{\theta}, 0]$ , are given. Notice that in the above system the control vector field associated to the delayed input is identically equal to zero. We will show that (28) is not input-output linearizable and that it is not in conformity with Theorem 1.

We have  $g_0^T = (x_2^2, 1, 0)$ ,  $g_1^T \equiv 0$ ,  $f^T = (0, 0, x_1)$ ,

$$a(\delta) = \dot{\tau}_4 \delta^4 \quad (29)$$

and the feedback  $u$  has to verify the following relation

$$x_1(t) + \dot{\tau}_4(t) \delta^4 u(t) = \delta^4 v(t), \quad t \geq \tau_1^{-4}(0). \quad (30)$$

In this example, statement (3) of Theorem 1 is violated since  $\frac{L_f h}{\partial x_1} = 1 \neq 0$ . The causality problem derives from exactly the same arguments used in the previous example.

**Example 4** (not satisfying Lemma 1)

This example shows that if the condition (15) of Lemma 1 is not satisfied, then even for a very simple choice of a bounded  $v$ , there is no bounded feedback  $u$  satisfying (9). For simplicity, we suppose that the delay is constant equal to 1. Consider

$$\begin{aligned} \dot{x}_1(t) &= u(t) \\ \dot{x}_2(t) &= \delta^1 u(t) \\ \dot{x}_3(t) &= x_1(t) - 2x_2(t) \\ h(t) &= x_3(t), \end{aligned} \quad (31)$$

with  $x(0) = 0$  and  $u(t) = 1$ , for  $t \in [-1, 0]$ . Notice that in this example only the control is delayed. We have

$$a(\delta) u(t) = u(t) - 2\delta^1 u(t). \quad (32)$$

Remark, from the latter equation that condition (15) of Lemma 1 is not satisfied. Now, suppose that we want to stay at  $x_3(t) = 0$  for all  $t \geq 0$ . Hence  $u$  has to verify

$$u(t) - 2\delta^1 u(t) = 0, \quad t \geq 1, \quad (33)$$

i.e., in this case  $v = 0$ . It follows from (33) that, over  $[0, 1]$ , the control  $u$  should be equal to 2. Similarly, we should have  $u(t) = 4$  over  $[1, 2]$ . By repeating this reasoning, we find recursively that over the interval  $[n, n+1]$ , the control  $u$  should be equal to  $2^{n+1}$ . Then, there is no bounded  $u$  satisfying (33).



## 4.2 Non-necessity conditions of Theorem 1

Here, we give some examples showing that the conditions of Theorem 1 are not necessary.

**Example 5** (not satisfying statement (3) of Theorem 1)

Consider the following time-varying delay system

$$\begin{aligned}\dot{x}_1(t) &= u(t) \\ \dot{x}_2(t) &= \delta^1 u(t) \\ \dot{x}_3(t) &= \delta^2 x_1(t) + 4k_1 x_2(t) \\ h(t) &= x_3(t),\end{aligned}\tag{34}$$

with  $x(t) = \varphi(t)$ ,  $\forall t \in [-6\bar{\theta}, 0]$ , and  $u(t) = \nu(t)$ ,  $\forall t \in [-5\bar{\theta}, 0]$ , given. In this example, we have  $g_0^T = (1, 0, 0)$ ,  $g_1^T = (0, 1, 0)$ ,  $f^T = (0, 0, \delta^2 x_1 + 4k_1 x_2)$  and

$$a[\delta] = a_1^1 \delta^1 + a_0^2 \delta^2 = 4k_1 \delta^1 + \dot{\tau}_2 \delta^2.\tag{35}$$

The feedback  $u$  has to verify the following relation

$$4k_1 \delta^1 u(t) + \dot{\tau}_2(t) \delta^2 u(t) = \delta^1 v(t), \quad t \geq \tau_1^{-1}(0).\tag{36}$$

This example is in conformity with Lemma 1, but does not respect statement (3) of Theorem 1. Indeed, we have  $g_0 \neq 0$  and  $a^1 = a_1^1$  (i.e., the first non identically zero coefficient of the polynomial  $a[\delta]$  stems from  $\bar{L}_{g_1} L_f h$ ). Nevertheless, remark that there is no problem of causality for computing the partially linearizing feedback  $u$  because here there is no state dependence. Notice that the values of  $u$  over  $[0, \tau^{-1}(0)]$  are not given. In fact, the definition of  $u(t)$  requires a prediction of the state over intervals  $[t, t + \tau^{-1}(0)]$ , for  $t \geq 0$ .

**Example 6** (not satisfying statement (1) of Theorem 1)

Consider the following time-varying delay system

$$\begin{aligned}\dot{x}_1(t) &= u(t) \\ \dot{x}_2(t) &= \delta^1 x_1^2(t) + \delta^1 u(t) \\ \dot{x}_3(t) &= x_2(t) \\ h(t) &= x_3(t),\end{aligned}\tag{37}$$

with  $x(t) = \varphi(t)$ ,  $\forall t \in [-3\bar{\theta}, 0]$ , and  $u(t) = \nu(t)$ ,  $\forall t \in [-3\bar{\theta}, 0]$ , given. In this example, we have  $g_0^T = (1, 0, 0)$ ,  $g_1^T = (0, 1, 0)$ ,  $f^T = (0, \delta^1 x_1^2, x_2)$ ,  $\rho = 2$ ,  $\bar{L}_{g_0} L_f h \equiv 0$ ,  $L_f^2 h = \delta^1 x_1^2$  and

$$a[\delta] = a_1^1 \delta^1 = \delta^1\tag{38}$$

The system is clearly partially linearizable with delay and the feedback  $u$  has to verify the following relation

$$\delta^1 u(t) = -\delta^1 x_1^2(t) + \delta^1 v(t), \quad t \geq \tau_1^{-1}(0).\tag{39}$$

It is obvious that Lemma 1 holds for the  $\delta$ -polynomial  $a[\delta]$ . On the other hand, the system (37) does not respect statement (1) of Theorem 1. Indeed, we have  $g_0 \neq 0$  and  $\bar{L}_{g_0} L_f h \equiv 0$ . Nevertheless, remark that there is no problem of causality for computing the partially linearizing feedback  $u$  because in (39)  $x_1$  appears with at least one delay.

## 5 Conclusion

In this paper, we have recalled our main result developed recently in the framework of input-output linearization of nonlinear systems with time-varying delays appearing in the state, the input and the output [10]. It gives sufficient conditions guaranteeing the existence of a suitable linearizing coordinate transformation and of causal and bounded linearizing feedback. Through some examples, we have discussed and clarified each condition appearing in the statement of our main result. These examples give some insights permitting us to improve these conditions in a future work.

## References

- [1] Califano, C., Marquez-Martinez, L.A. and Moog, C.H. (2013), *Linearization of time-delay systems by input-output injection and output transformation*, Automatica, 49(6), 1932–1940.
- [2] Conte, G., Moog, C.H. and Perdon, A.M. (1999), *Nonlinear control systems: an algebraic setting*, London: Springer-Verlag.
- [3] Curtain, R.F. and Zwart, H. (1995), *An introduction to infinite dimensional linear systems theory, volume 21. Springer-Verlag*.
- [4] Fliess, M. (1986), *A note on the invertibility of nonlinear input-output systems*, Sys. Contr. Lett., 8, 147–151.
- [5] Fliess, M., Lévine, J., Martin, P. and Rouchon, P. (1999), *A Lie-Backlund approach to equivalence and flatness of nonlinear systems*, IEEE Transactions on Automatic Control, 44(5), 922–937.
- [6] Fliess, M. and Mounier, H. (1998), *Controllability and observability of linear delay systems : an algebraic approach*, ESAIM: Control, Optimisation and Calculus of Variations, 3, 301–314.
- [7] Fridman, E., Dambrine, M. and Yeganefar, N. (2008), *On input-to-state stability of systems with time-delay: A matrix inequalities approach*, Automatica, 44(9), 2364–2369.
- [8] Germani, A., Manes, C. and Pepe, P. (2003), *Input-output linearization with delay cancellation for nonlinear delay systems: the problem of the internal stability*, International Journal of Robust and Nonlinear Control, 13(9), 909–937.
- [9] Germani, A., Manes, C. and Pepe, P. (1998), *Linearization and decoupling of nonlinear delay systems*, Proceedings of American Control Conference, 1948–1952.
- [10] Haidar, I., Nicolau, F., Barbot, J.P. and Aggoune, W. (2017), *Input-output linearization of SISO nonlinear time-varying delay systems*, IFAC 2017 World Congress, Toulouse, France.
- [11] Mounier, H. and Rudolph, J. (1998), *Flatness-based control of nonlinear delay systems: A chemical reactor example*, International Journal of Control, 71(5), 871–890.
- [12] Isidori, A. (1995), *Nonlinear control systems (3rd edition)*, London: Springer-Verlag.
- [13] Kolmanovskii, V. and Myshkis, A. (1992), *Applied theory of functional differential equations*, Kluwer Academic Publishers.
- [14] Oguchi, T. and Watanabe, A. (1998), *Input-output linearization of nonlinear systems with time-delays in state variables*, International Journal of Systems Science, 29, 573–578.
- [15] Oguchi, T., Watanabe, A. and Nakamizo, T. (2002), *Input-output linearization of retarded nonlinear systems by using an extension of lie derivative*, International Journal of Control, 75(8), 582–590.
- [16] Quadrat, A. and Ushirobira, R. (2016) , *Algebraic analysis for the Ore extension ring of differential time-varying delay operators*, In Proc. 22nd MTNS, 8. Minneapolis, United States. URL <https://hal.inria.fr/hal-01415256>.
- [17] Respondek, W. (1993), *Dynamic input-output linearization and decoupling of nonlinear systems*, Proc. 2nd. European Contr. conf. Groningen, 1523–1527.
- [18] Sontag, E. and Wang, Y. (1995), *On characterizations of the input-to-state stability property*, Systems Control Lett., 24(5), 351–359.
- [19] Sussmann, H. and Kokotovic, P. (1991), *The peacking phenomenon and the global stabilization of nonlinear systems*, IEEE Transactions on Automatic Control, 36(4), 424–440.

- [20] Zheng, G., Barbot, J.P., Boutat, D., Floquet, T. and Richard, J.P. (2010a), *Causal observability of nonlinear time-delay systems with unknown inputs*, Decision and Control (CDC), 2010 49th IEEE Conference.
- [21] Zheng, G., Barbot, J.P., Boutat, D., Floquet, T. and Richard, J.P. (2010b), *Finite time observation of non-linear time-delay systems with unknown inputs*, IFAC Proceedings Volumes, 43(14), 427–432.