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# Time domain boundary elements for dynamic contact problems

Heiko Gimperlein<sup>\*†</sup>   Fabian Meyer<sup>‡</sup>   Ceyhun Özdemir<sup>§</sup>   Ernst P. Stephan<sup>§</sup>

*(dedicated to Erwin Stein on the occasion of his 85th birthday)*

## Abstract

This article considers a unilateral contact problem for the wave equation. The problem is reduced to a variational inequality for the Dirichlet-to-Neumann operator for the wave equation on the boundary, which is solved in a saddle point formulation using boundary elements in the time domain. As a model problem, also a variational inequality for the single layer operator is considered. A priori estimates are obtained for Galerkin approximations both to the variational inequality and the mixed formulation in the case of a flat contact area, where the existence of solutions to the continuous problem is known. Numerical experiments demonstrate the performance of the proposed mixed method. They indicate the stability and convergence beyond flat geometries.

*Key words:* boundary element method; variational inequality; mixed method; a priori error estimates; wave equation.

## 1 Introduction

Contact problems play an important role in numerous applications in mechanics, from fracture dynamics and crash tests to rolling car tires [35]. As the contact takes place at the interface of two materials, for time-independent problems boundary elements and coupled finite / boundary elements provide an efficient and much-studied tool for numerical simulations [23, 34]. The analysis of such problems is well-understood in the context of elliptic variational inequalities.

While contact for time-dependent problems is of clear practical relevance, neither its analysis nor rigorous boundary element methods have been much explored. There is an extensive computational literature, including [15, 26, 27, 28, 29], but analytically even the existence of solutions to these free boundary problem is only known for flat contact area [13, 30]. Some rigorous results

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have recently been obtained for Nitsche stabilized finite elements [12].

In this work we propose a time domain boundary element method for a dynamic contact problem in the case of the scalar wave equation, as a model problem for elasticity. We provide a priori error estimates for our numerical scheme in the case of a flat contact area, and our numerical experiments indicate the convergence and efficiency also for curved contact geometries. Motivating references from the time-independent setting include [3, 16].

For the precise statement of the problem, consider a Lipschitz domain  $\Omega \subset \mathbb{R}^n$  with boundary  $\Gamma = \partial\Omega$ . Let  $G$  be a bounded Lipschitz subset of  $\Gamma$ . We consider a unilateral contact problem for the wave equation for the displacement  $w : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ . It corresponds to a simplified model for a crack in  $G$  between  $\Omega$  and a non-penetrable material in  $\mathbb{R}^n \setminus \bar{\Omega}$ . The contact conditions for non-penetration are described in terms of the traction  $-\mu \frac{\partial w}{\partial \nu}|_G$  and prescribed forces  $h$ :

$$\begin{cases} w|_G \geq 0, & -\mu \frac{\partial w}{\partial \nu}|_G \geq h, \\ w|_G > 0 \implies & -\mu \frac{\partial w}{\partial \nu}|_G = h. \end{cases}$$

The full system of equations for the contact problem is given by:

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = c_s^2 \Delta w, & \text{for } (t, x) \in \mathbb{R} \times \Omega, \\ w = 0, & \text{on } \Gamma \setminus G, \\ w \geq 0, & -\mu \frac{\partial w}{\partial \nu} \geq h, & \text{on } G, \\ (-\mu \frac{\partial w}{\partial \nu} - h) w = 0, & \text{on } G, \\ w = 0, & \text{for } (t, x) \in (-\infty, 0) \times \Omega. \end{cases} \quad (1)$$

Like for time-independent contact, a formulation as a nonlinear problem on the contact area  $G$  leads to efficient numerical approximations.

In this article we formulate (1) as a variational inequality on  $G$  in terms of the Dirichlet-to-Neumann operator for the wave equation. Similar to time-independent problems, the Dirichlet-to-Neumann operator may be computed in terms of boundary integral operators. In order to improve the conservation of energy in the numerical scheme, we replace the variational inequality by an equivalent mixed system, which we discretize with a time domain Galerkin boundary element method. The resulting discretized nonlinear inequality in space-time simultaneously approximates the displacement  $w$  and the contact forces  $-\mu \frac{\partial w}{\partial \nu}$  on  $G$ . It is solved with a Uzawa algorithm, either as a time-stepping scheme or in space-time.

The resulting boundary element method is analyzed in the case of a flat contact area, a situation where the existence of solutions to the contact problem (1) is known. We obtain a priori estimates for the numerical error, both for the variational inequality and a mixed formulation. For the mixed problem, a key part of the proof is an inf-sup condition for the space-time discretization [3]. We also demonstrate the convergence of the Uzawa algorithms. Numerical experiments for

the mixed formulation confirm the theoretical results and indicate the efficiency of our approach, beyond flat geometries.

In addition to the contact problem (1), as a simpler test case we also consider a punch problem, which models a rigid body indenting an elastic half-space. The contact boundary conditions for the wave equation are given by:

$$\begin{cases} -\mu \frac{\partial w}{\partial \nu}|_G \geq 0, & w|_G \geq h, \\ -\mu \frac{\partial w}{\partial \nu}|_G > 0 \implies w|_G = h. \end{cases}$$

For the half-space they lead to a variational inequality for the single layer operator  $V$ , instead of the Dirichlet-to-Neumann operator, and we obtain similar theoretical and numerical results in this case.

Our approach also relates to the recent interest in coupled and nonlinear interface problems for wave propagation, solved by time domain boundary element methods. In particular, we refer to the fundamental articles [1, 6] for the coupling of FEM and BEM, as well as [2] for an energetic Galerkin formulation of the coupling. Reference [7] considers a nonlinear boundary problem. A first analysis of the time domain Dirichlet-to-Neumann operator goes back to [5].

The current work provides a first step towards efficient boundary elements for dynamic contact. For both stationary and dynamic contact, the relevance of adaptive methods to approximate the non-smooth solutions is well-known [23, 29, 34]. Recent advances in the a posteriori error analysis and resulting adaptive mesh refinement procedures for time domain boundary elements [19, 20] will therefore be of interest for the dynamic contact considered here, with a particular view towards tire dynamics [9].

*The article is organized as follows:* Section 2 recalls the boundary integral operators associated to the wave equation as well as their mapping properties between suitable space-time anisotropic Sobolev spaces. Section 3 reduces the contact problem to a variational inequality on the contact boundary. Section 4 discusses the existence and uniqueness of solutions to the contact problem and a related, simpler Dirichlet-to-Neumann equation in a half space. Section 5 describes the discretization and proves a priori error estimates for the contact problem as well as for the Dirichlet-to-Neumann equation. The a priori error analysis for the mixed formulation is then presented in Section 6. A simpler contact problem, which involves the single layer operator  $V$  only, is analyzed in Section 7. Section 8 derives a time stepping scheme for the Dirichlet-to-Neumann equation, which forms the basis for both a time stepping and a space-time Uzawa algorithm for the nonlinear contact problem. Section 9 presents numerical experiments based on the mixed formulation.

## 2 Boundary integral operators and Sobolev spaces

We introduce the single layer potential in time domain as

$$S\varphi(t, x) = \int_{\mathbb{R}^+ \times \Gamma} G(t - \tau, x, y) \varphi(\tau, y) d\tau ds_y ,$$

where  $(t, x) \in \mathbb{R}^+ \times \Omega$  and  $G$  is a fundamental solution to the wave equation. Specifically in 3 dimensions, it is given by

$$S\varphi(t, x) = \frac{1}{4\pi} \int_{\Gamma} \frac{\varphi(t - |x - y|, y)}{|x - y|} ds_y .$$

We similarly define the double-layer potential as

$$D\varphi(t, x) = \int_{\mathbb{R}^+ \times \Gamma} \frac{\partial G}{\partial n_y}(t - \tau, x, y) \varphi(\tau, y) d\tau ds_y .$$

For the Dirichlet-to-Neumann operator, we require the single-layer operator  $V$ , its normal derivative  $K'$ , the double-layer operator  $K$  and hypersingular operator  $W$  for  $x \in \Gamma$ ,  $t > 0$ :

$$\begin{aligned} V\varphi(t, x) &= \int_{\mathbb{R}^+ \times \Gamma} G(t - \tau, x, y) \varphi(\tau, y) d\tau ds_y , \\ K\varphi(t, x) &= \int_{\mathbb{R}^+ \times \Gamma} \frac{\partial G}{\partial n_y}(t - \tau, x, y) \varphi(\tau, y) d\tau ds_y , \\ K'\varphi(t, x) &= \int_{\mathbb{R}^+ \times \Gamma} \frac{\partial G}{\partial n_x}(t - \tau, x, y) \varphi(\tau, y) d\tau ds_y , \\ W\varphi(t, x) &= \int_{\mathbb{R}^+ \times \Gamma} \frac{\partial^2 G}{\partial n_x \partial n_y}(t - \tau, x, y) \varphi(\tau, y) d\tau ds_y . \end{aligned}$$

**Remark 1.** When  $\Omega = \mathbb{R}_+^n$ , the normal derivative of  $G$  vanishes on  $\Gamma = \partial\mathbb{R}_+^n$ . Therefore,  $K\varphi = K'\varphi = 0$  in this case.

The boundary integral operators are considered between space-time anisotropic Sobolev spaces  $H_\sigma^s(\mathbb{R}^+, \tilde{H}^r(\Gamma))$ , see [21] or [24]. To define them, if  $\partial\Gamma \neq \emptyset$ , first extend  $\Gamma$  to a closed, orientable Lipschitz manifold  $\tilde{\Gamma}$ .

On  $\Gamma$  one defines the usual Sobolev spaces of supported distributions:

$$\tilde{H}^r(\Gamma) = \{u \in H^r(\tilde{\Gamma}) : \text{supp } u \subset \bar{\Gamma}\} , \quad r \in \mathbb{R} .$$

Furthermore,  $H^r(\Gamma)$  is the quotient space  $H^r(\tilde{\Gamma})/\tilde{H}^r(\tilde{\Gamma} \setminus \bar{\Gamma})$ .

To write down an explicit family of Sobolev norms, introduce a partition of unity  $\alpha_i$  subordinate to a covering of  $\tilde{\Gamma}$  by open sets  $B_i$ . For diffeomorphisms  $\varphi_i$  mapping each  $B_i$  into the unit cube  $\subset \mathbb{R}^n$ , a family of Sobolev norms is induced from  $\mathbb{R}^n$ :

$$\|u\|_{r, \omega, \tilde{\Gamma}} = \left( \sum_{i=1}^p \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^r |\mathcal{F}\{(\alpha_i u) \circ \varphi_i^{-1}\}(\xi)|^2 d\xi \right)^{\frac{1}{2}} .$$

The norms for different  $\omega \in \mathbb{C} \setminus \{0\}$  are equivalent and  $\mathcal{F}$  denotes the Fourier transform. They induce norms on  $H^r(\Gamma)$ ,  $\|u\|_{r,\omega,\Gamma} = \inf_{v \in \tilde{H}^r(\tilde{\Gamma} \setminus \bar{\Gamma})} \|u+v\|_{r,\omega,\tilde{\Gamma}}$  and on  $\tilde{H}^r(\Gamma)$ ,  $\|u\|_{r,\omega,\Gamma,*} = \|e_+ u\|_{r,\omega,\tilde{\Gamma}}$ .  $e_+$  extends the distribution  $u$  by 0 from  $\Gamma$  to  $\tilde{\Gamma}$ . It is stronger than  $\|u\|_{r,\omega,\Gamma}$  whenever  $r \in \frac{1}{2} + \mathbb{Z}$ .

We now define a class of space-time anisotropic Sobolev spaces:

**Definition 2.** For  $\sigma > 0$  and  $s, r \in \mathbb{R}$  define

$$\begin{aligned} H_\sigma^s(\mathbb{R}^+, H^r(\Gamma)) &= \{u \in \mathcal{D}'_+(H^r(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(H^r(\Gamma)) \text{ and } \|u\|_{s,r,\sigma,\Gamma} < \infty\} , \\ H_\sigma^s(\mathbb{R}^+, \tilde{H}^r(\Gamma)) &= \{u \in \mathcal{D}'_+(\tilde{H}^r(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(\tilde{H}^r(\Gamma)) \text{ and } \|u\|_{s,r,\sigma,\Gamma,*} < \infty\} . \end{aligned}$$

$\mathcal{D}'_+(E)$  resp.  $\mathcal{S}'_+(E)$  denote the spaces of distributions, resp. tempered distributions, on  $\mathbb{R}$  with support in  $[0, \infty)$ , taking values in  $E = H^r(\Gamma), \tilde{H}^r(\Gamma)$ . The relevant norms are given by

$$\begin{aligned} \|u\|_{s,r,\sigma} &:= \|u\|_{s,r,\sigma,\Gamma} = \left( \int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2s} \|\hat{u}(\omega)\|_{r,\omega,\Gamma}^2 d\omega \right)^{\frac{1}{2}} , \\ \|u\|_{s,r,\sigma,*} &:= \|u\|_{s,r,\sigma,\Gamma,*} = \left( \int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2s} \|\hat{u}(\omega)\|_{r,\omega,\Gamma,*}^2 d\omega \right)^{\frac{1}{2}} . \end{aligned}$$

For  $|r| \leq 1$  the spaces are independent of the choice of  $\alpha_i$  and  $\varphi_i$ . We further introduce the set of nonnegative distributions  $H_\sigma^r(\mathbb{R}^+, \tilde{H}^s(G))^+$ .

The boundary integral operators obey the following mapping properties between these spaces:

**Theorem 3** ([21]). The following operators are continuous for  $r \in \mathbb{R}$ :

$$\begin{aligned} V &: H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) , \\ K' &: H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) , \\ K &: H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) , \\ W &: H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) . \end{aligned}$$

Furthermore, as noted by Bamberger and Ha Duong [4],  $V$  and  $W$  are coercive in a weak norm when composed with a time derivative:  $\|\phi\|_{0,-\frac{1}{2},\sigma,*}^2 \lesssim \langle V\phi, \partial_t \phi \rangle$ , resp.  $\|\psi\|_{0,\frac{1}{2},\sigma,*}^2 \lesssim \langle W\psi, \partial_t \psi \rangle$ . This is not sufficient for the analysis of variational inequalities, which no longer hold after differentiation in time.

When  $\Omega = \mathbb{R}_+^n$ , Fourier methods yield improved estimates for  $V$  and  $W$ , see also Section 4:

**Theorem 4** ([25], pp. 503-506). The following operators are continuous for  $r, s \in \mathbb{R}$ :

$$\begin{aligned} V &: H_\sigma^{r+\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^s(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{s+1}(\Gamma)) , \\ W &: H_\sigma^r(\mathbb{R}^+, \tilde{H}^s(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{s-1}(\Gamma)) . \end{aligned}$$

Theorems 3 and 4 imply the corresponding mapping properties for the composition with the restriction  $p_Q$  to  $Q = \mathbb{R} \times G$ . For example, from Theorem 3 we obtain  $p_Q V : H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{\frac{1}{2}}(G))$  and  $p_Q V : H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G)) \hookrightarrow H_\sigma^{r+1}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{\frac{1}{2}}(G))$ .

Also, the crucial coercivity without composition with  $\partial_t$  can be shown in this flat geometry. In particular,  $\|\phi\|_{-\frac{1}{2},-\frac{1}{2},\sigma,*}^2 \lesssim_\sigma \langle p_Q V \phi, \phi \rangle$ .

### 3 Boundary integral formulation

For some  $\sigma > 0$ , we write  $w_\sigma = e^{-\sigma t} w$ , resp.  $h_\sigma = e^{-\sigma t} h$ . Using appropriate units, we may also assume  $c_s = 1$ . Multiplying (1) by  $e^{-\sigma t}$ , we then obtain

$$\begin{cases} \left( \frac{\partial}{\partial t} + \sigma \right)^2 w_\sigma = \Delta w_\sigma, & \text{for } (t, x) \in \mathbb{R} \times \Omega, \\ w_\sigma = 0, & \text{on } \Gamma \setminus G, \\ w_\sigma \geq 0, \quad -\mu \frac{\partial w_\sigma}{\partial \nu} \Big|_G \geq h_\sigma, & \text{on } G, \\ \left( -\mu \frac{\partial w_\sigma}{\partial \nu} \Big|_G - h_\sigma \right) w_\sigma \Big|_G = 0, & \text{on } G, \\ w_\sigma = 0, & \text{for } (t, x) \in (-\infty, 0) \times \Omega. \end{cases} \quad (2)$$

The Dirichlet-to-Neumann operator, defined as

$$\mathcal{S}_\sigma w_\sigma|_\Gamma := -\mu \frac{\partial w_\sigma}{\partial \nu} \Big|_\Gamma, \quad (3)$$

allows to reduce the contact problem (1) to an equivalent inequality in terms of the trace  $u_\sigma := w_\sigma|_\Gamma$  on the boundary: Find  $u_\sigma$  with  $\text{supp } u_\sigma \subset Q_0 = \mathbb{R}^+ \times G$  such that

$$u_\sigma \geq 0, \quad \mathcal{S}_\sigma u_\sigma \geq h_\sigma, \quad (\mathcal{S}_\sigma u_\sigma - h_\sigma) u_\sigma = 0 \text{ on } Q = \mathbb{R} \times G. \quad (4)$$

More precisely, in terms of the restriction  $p_Q$  to  $Q$ , we obtain the following weak formulation as a variational inequality for suitably smooth  $h_\sigma$ : Find  $u_\sigma \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$  such that:

$$u_\sigma \geq 0 \quad \text{and} \quad \forall v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G)) \text{ with } v \geq 0: \quad \langle p_Q \mathcal{S}_\sigma u_\sigma, v - u_\sigma \rangle \geq \langle h_\sigma, v - u_\sigma \rangle. \quad (5)$$

### 4 Existence and uniqueness for contact problems

We now restrict ourselves to the half space  $\Omega = \mathbb{R}_+^n$ , where sharp continuity and coercivity results allow to show the wellposedness of the contact problem.

We illustrate the use of Fourier methods in this geometry. As shown in [10, 25] the Dirichlet-Neumann operator  $\mathcal{S}_\sigma$  is a pseudo-differential operator on  $\mathbb{R}_t \times \partial\Omega_x = \mathbb{R} \times \mathbb{R}^{n-1} \times \{0\}$ :

$$\mathcal{S}_\sigma u_\sigma(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it\xi_0 + ix\xi'} (-i\mu\Gamma(\xi_0 + i\sigma, \xi')) \hat{u}_\sigma(\xi_0, \xi') d\xi_0 d\xi',$$

with symbol  $\Gamma(\xi_0 + i\sigma, \xi') = \sqrt{(\xi_0 + i\sigma)^2 - |\xi'|^2}$ . From this explicit formula, one notices the estimates ([25], p. 499):

$$\begin{aligned} \frac{\sigma^{\frac{1}{2}}}{|\xi_0 + i\sigma|^{\frac{1}{2}}} (|\xi_0 + i\sigma|^2 + |\xi'|^2)^{\frac{1}{2}} &\leq |\Gamma(\xi_0 + i\sigma, \xi')| \leq (|\xi_0 + i\sigma|^2 + |\xi'|^2)^{\frac{1}{2}}, \\ \text{Im } \Gamma(\xi_0 + i\sigma, \xi') &\geq \frac{\sigma}{|\xi_0 + i\sigma|} (|\xi_0 + i\sigma|^2 + |\xi'|^2)^{\frac{1}{2}}. \end{aligned}$$

They translate into the following coercivity and mapping properties:

**Theorem 5.**  $p_Q \mathcal{S}_\sigma : H_\sigma^s(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G)) \rightarrow H_\sigma^s(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$  continuously and  $\|\phi\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *}^2 \lesssim_\sigma \langle p_Q \mathcal{S}_\sigma \phi, \phi \rangle \leq C \|\phi\|_{0, \frac{1}{2}, \sigma, *}^2$ .

These estimates are a basic ingredient to prove the well-posedness of the contact problem:

**Theorem 6** ([13], p. 450). *Let  $h \in H_{\sigma}^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ . Then there exists a unique solution  $u_{\sigma} \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$  of (5).*

In terms of the original problem (2) we obtain:

**Theorem 7** ([13], p. 451). *Let  $h \in H_{\sigma}^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ . Then there exists a unique  $w(\cdot, x_n) \in C(\overline{\mathbb{R}}_{x_n}^+; H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, H^{\frac{1}{2}}(\mathbb{R}^{n-1})) \cap H_{\sigma}^0(\mathbb{R}^+, H^1(\mathbb{R}^n))$  satisfying (2).*

We also note the (simpler) existence of solutions to the corresponding equality from [32], p. 48:

**Theorem 8.** *Let  $h \in H_{\sigma}^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ . Then there exists a unique  $u_{\sigma} \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$  which solves:*

$$\langle p_Q \mathcal{S}_{\sigma} u_{\sigma}, v \rangle = \langle h, v \rangle \quad \forall v \in H_{\sigma}^{-\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G)).$$

## 5 Discretization and a priori error estimates

We assume that  $n = 2$  or  $n = 3$ , and if  $\Gamma$  is not polygonal, we approximate it by a piecewise polygonal curve resp. surface. The approximation is again denoted by  $\Gamma$ . When  $n = 3$ , we may assume that the surface  $\Gamma$  is composed of  $N$  closed triangular faces  $S_i$  such that  $\Gamma = \cup_{i=1}^N S_i$ . When  $n = 2$ , we assume  $\Gamma = \cup_{i=1}^N S_i$  is composed of line segments  $S_i$ . The triangulation  $\mathcal{T}_S := \{S_1, \dots, S_N\}$  is assumed to be quasi-uniform and compatible with the area of contact  $G$ : If  $S_i \cap G \neq \emptyset$ ,  $\text{int } S_i \subset G$ .

Associated to the triangulation, we obtain the space  $V_h^q(\Gamma)$  of piecewise polynomial functions of degree  $q$ . Due to the compatibility of the meshes we have  $V_h^q(G) \subset V_h^q(\Gamma)$ . Moreover we define  $\tilde{V}_h^q(G)$  as the subspace of those functions in  $V_h^q(G)$ , which vanish on  $\partial G$  for  $q \geq 1$ .

For the time discretization we consider a uniform decomposition of the time interval  $[0, \infty)$  into subintervals  $[t_{n-1}, t_n)$  with time step  $\Delta t$ , such that  $t_n = n\Delta t$  ( $n = 0, 1, \dots$ ). Associated to this mesh, the space  $V_t^p$  consists of piecewise polynomial functions of degree of  $p$  (continuous and vanishing at  $t = 0$  if  $p \geq 1$ ).

Let  $\mathcal{T}_T = \{[0, t_1), [t_1, t_2), \dots, [t_{N-1}, T)\}$  the time mesh for a finite subinterval  $[0, T)$ . In space-time we consider the tensor product of the approximation spaces,  $V_h^q$  and  $V_t^p$ , associated to the space-time mesh  $\mathcal{T}_{S,T} = \mathcal{T}_S \times \mathcal{T}_T$ , and we write

$$V_{t,h}^{p,q} := V_t^p \otimes V_h^q. \quad (6)$$

We analogously define

$$\tilde{V}_{t,h}^{p,q} := V_t^p \otimes \tilde{V}_h^q \quad (7)$$

We further define the subspace  $K_{t,h}^+ \subset V_{t,h}^{p,q}$  as the subspace of nonnegative piecewise polynomials.

For  $u_{\Delta t, h} \in V_{t, h}^{p, q}$  we thus may write

$$u_{\Delta t, h}(t, x) = \sum_{i=0}^{N_t} \sum_{j=0}^{N_s} c_j^i \beta_{\Delta t}^i(t) \xi_h^j(x) .$$

in terms of the basis functions  $\beta_{\Delta t}^i$  in time and  $\xi_h^j$  in space.

We use the following notation for piecewise linear or constant functions:

- $\gamma_{\Delta t}^n(t)$  for the basis of piecewise constant functions in time,
- $\beta_{\Delta t}^n(t)$  for the basis of piecewise linear functions in time,
- $\psi_h^i(x)$  for the basis of piecewise constant functions in space,
- $\xi_h^i(x)$  for the basis of piecewise linear functions in space.

We recall the formulation as a continuous variational inequality:

Find  $u \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$  such that

$$\langle p_Q \mathcal{S}_{\sigma} u_{\sigma}, v - u_{\sigma} \rangle \geq \langle h, v - u_{\sigma} \rangle \quad (8)$$

holds for all  $v \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$ . The discretized variational inequality reads as follows.

Find  $u_{\Delta t, h} \in \tilde{K}_{t, h}^+$  such that

$$\langle p_Q \mathcal{S}_{\sigma} u_{\Delta t, h}, v_{\Delta t, h} - u_{\Delta t, h} \rangle \geq \langle h, v_{\Delta t, h} - u_{\Delta t, h} \rangle \quad (9)$$

holds for all  $v_{\Delta t, h} \in \tilde{K}_{t, h}^+$ .

Note that we assume that  $\mathcal{S}_{\sigma}$  is computed exactly. We refer to [5] for the challenges of analyzing its discretization. While  $\sigma > 0$  is required for the theoretical analysis, practical computations use  $\sigma = 0$  [4, 20].

Using a conforming ansatz space, we are able to derive an a priori estimate for the variational inequality. It is the hyperbolic counterpart of the elliptic estimate proved by Falk [17].

**Theorem 9.** *Let  $h \in H_{\sigma}^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$  and let  $u \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$ ,  $u_{\Delta t, h} \in \tilde{K}_{t, h}^+ \subset H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$  be the solutions of (8), resp. (9). Then the following estimate holds:*

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 \lesssim_{\sigma} \inf_{\phi_{\Delta t, h} \in \tilde{K}_{t, h}^+} (\|h - p_Q \mathcal{S}_{\sigma} u\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star} + \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2). \quad (10)$$

*Proof.* Rewriting (8) and (9), we note that

$$\langle p_Q \mathcal{S}_{\sigma} u, u \rangle \leq \langle h, u - \phi \rangle + \langle p_Q \mathcal{S}_{\sigma} u, \phi \rangle \quad (11)$$

and

$$\langle p_Q \mathcal{S}_\sigma u_{\Delta t, h}, u_{\Delta t, h} \rangle \leq \langle h, u_{\Delta t, h} - \phi_{\Delta t, h} \rangle + \langle p_Q \mathcal{S}_\sigma u_{\Delta t, h}, \phi_{\Delta t, h} \rangle. \quad (12)$$

Using the coercivity in the  $\|\cdot\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}$ -norm (Theorem 5), we obtain

$$\begin{aligned} & \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 \\ & \lesssim_\sigma \langle p_Q \mathcal{S}_\sigma(u - u_{\Delta t, h}), u - u_{\Delta t, h} \rangle \\ & \leq \langle h, u - \phi \rangle + \langle h, u_{\Delta t, h} - \phi_{\Delta t, h} \rangle + \langle p_Q \mathcal{S}_\sigma u_{\Delta t, h}, \phi_{\Delta t, h} \rangle + \langle p_Q \mathcal{S}_\sigma u, \phi \rangle \\ & \quad - \langle p_Q \mathcal{S}_\sigma u, u_{\Delta t, h} \rangle - \langle p_Q \mathcal{S}_\sigma u_{\Delta t, h}, u \rangle \\ & = \langle h, u - \phi \rangle + \langle h, u_{\Delta t, h} - \phi_{\Delta t, h} \rangle + \langle p_Q \mathcal{S}_\sigma u, \phi - u_{\Delta t, h} \rangle \\ & \quad + \langle p_Q \mathcal{S}_\sigma u_{\Delta t, h}, \phi_{\Delta t, h} - u \rangle. \end{aligned}$$

We rewrite

$$\langle p_Q \mathcal{S}_\sigma u_{\Delta t, h}, \phi_{\Delta t, h} - u \rangle = \langle p_Q \mathcal{S}_\sigma u - u_{\Delta t, h}, u - \phi_{\Delta t, h} \rangle - \langle p_Q \mathcal{S}_\sigma u, u - \phi_{\Delta t, h} \rangle,$$

so that

$$\begin{aligned} \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 & \lesssim_\sigma \langle h - p_Q \mathcal{S}_\sigma u, u - \phi_{\Delta t, h} \rangle + \langle h - p_Q \mathcal{S}_\sigma u_{\Delta t, h}, u_{\Delta t, h} - \phi \rangle \\ & \quad + \langle p_Q \mathcal{S}_\sigma u - u_{\Delta t, h}, u - \phi_{\Delta t, h} \rangle. \end{aligned}$$

Because of the conforming discretization, we may choose  $\phi = u_{\Delta t, h}$  and conclude

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 \lesssim_\sigma \langle h - p_Q \mathcal{S}_\sigma u, u - \phi_{\Delta t, h} \rangle + \langle p_Q \mathcal{S}_\sigma u - u_{\Delta t, h}, u - \phi_{\Delta t, h} \rangle. \quad (13)$$

We estimate both terms by duality:

$$\begin{aligned} \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 & \lesssim_\sigma \|h - p_Q \mathcal{S}_\sigma u\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star} \\ & \quad + \|p_Q \mathcal{S}_\sigma(u - u_{\Delta t, h})\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, \star}. \end{aligned}$$

From the mapping properties of  $p_Q \mathcal{S}_\sigma$ , Theorem 5, one then sees that

$$\begin{aligned} \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 & \lesssim_\sigma \|h - p_Q \mathcal{S}_\sigma u\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star} \\ & \quad + \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star} \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, \star}. \end{aligned}$$

We conclude with the help of Young's inequality

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 \lesssim_\sigma \|h - p_Q \mathcal{S}_\sigma u\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star} + \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2.$$

Taking the infimum over all  $\phi_{\Delta t, h}$  yields the assertion.  $\square$

The theorem implies explicit convergence rates, using approximation properties in anisotropic Sobolev spaces as stated e.g. in [22]:

**Corollary 10.** *Let  $u \in H_{\sigma}^{\frac{1}{2}+\epsilon}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}+\epsilon}(G))$  for some  $\epsilon > 0$ . There exists a constant  $C > 0$  (independent of  $\Delta t$  and  $h$ ) such that*

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 \leq C(h^{\epsilon} + (\Delta t)^{\frac{1}{2}+\epsilon})\|u\|_{\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon, \sigma} + C(h^{2\epsilon} + (\Delta t)^{2\epsilon})\|u\|_{\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon, \sigma}^2.$$

*Proof.* We estimate the first term  $\|u - \phi_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}$  on the right hand side in Theorem 9 by  $\|u - \phi_{\Delta t, h}\|_{0, \frac{1}{2}, \sigma, \star}$ . Then we apply Proposition 3.56 in [22].  $\square$

We also consider the continuous and discrete variational equalities for the Dirichlet-to-Neumann operator.

Find  $u \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$  such that

$$\langle p_Q \mathcal{S}_{\sigma} u_{\sigma}, v \rangle = \langle h, v \rangle \quad (14)$$

holds for all  $v \in H_{\sigma}^{-\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$ .

Find  $u_{\Delta t, h} \in \tilde{V}_{t, h}^{p, q}$  such that

$$\langle p_Q \mathcal{S}_{\sigma} u_{\Delta t, h}, \phi_{\Delta t, h} \rangle = \langle h, \phi_{\Delta t, h} \rangle \quad (15)$$

holds for all  $\phi_{\Delta t, h} \in \tilde{V}_{t, h}^{p, q}$ .

An a priori estimate for the variational equality is obtained from the previous arguments as a special case.

**Theorem 11.** *Let  $u \in H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$  and  $u_{\Delta t, h} \in \tilde{V}_{t, h}^{p, q} \subset H_{\sigma}^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$  be the solutions of (14), resp. (15). We have the following a priori estimate:*

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star} \lesssim_{\sigma} \inf_{\phi_{\Delta t, h} \in \tilde{V}_{t, h}^{p, q}} \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, \star}.$$

*Proof.* For the conforming test space  $\tilde{V}_{t, h}^{p, q}$  we have

$$\langle p_Q \mathcal{S}_{\sigma} u, \phi_{\Delta t, h} \rangle = \langle h, \phi_{\Delta t, h} \rangle,$$

for all  $\phi_{\Delta t, h} \in \tilde{V}_{t, h}^{p, q}$ . Therefore (13) becomes

$$\begin{aligned} \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 &\lesssim_{\sigma} \langle h - p_Q \mathcal{S}_{\sigma} u, u - \phi_{\Delta t, h} \rangle + \langle p_Q \mathcal{S}_{\sigma}(u - u_{\Delta t, h}), u - \phi_{\Delta t, h} \rangle \\ &= \langle p_Q \mathcal{S}_{\sigma}(u - u_{\Delta t, h}), u - \phi_{\Delta t, h} \rangle \\ &\leq \|p_Q \mathcal{S}_{\sigma}(u - u_{\Delta t, h})\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, \star}. \end{aligned}$$

The continuity of  $p_Q \mathcal{S}_{\sigma}$ ,  $\|p_Q \mathcal{S}_{\sigma}(u - u_{\Delta t, h})\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma} \lesssim \|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}$ , yields the assertion.  $\square$

Analogous to Corollary 10 we derive the following rate of convergence.

**Corollary 12.** *Let  $u \in H_{\sigma}^{\frac{1}{2}+\epsilon}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}+\epsilon}(G))$  for some  $\epsilon > 0$ . There exists a constant  $C > 0$  (independent of  $\Delta t$  and  $h$ ) such that*

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, \star}^2 \leq C(h^{2\epsilon} + (\Delta t)^{2\epsilon})\|u\|_{\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon, \sigma}^2.$$

## 6 Mixed formulation

To assure conservation of energy in the numerical approximation, it proves useful to impose the constraints on the displacement only indirectly. We therefore reformulate the variational inequality as an equivalent mixed system. The Lagrange multiplier  $\lambda = \mathcal{S}_\sigma u - h$  in this formulation provides a measure to which extent the variational inequality is not an equality; physically, it corresponds to the contact forces and indicates the contact area within the computational domain.

**Theorem 13** (Mixed formulation). *Let  $h \in H_\sigma^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ . The variational inequality formulation (8) is equivalent to the following formulation:*

*Find  $(u, \lambda) \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G)) \times H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))^+$  such that*

$$\begin{cases} (a) & \langle \mathcal{S}_\sigma u, v \rangle - \langle \lambda, v \rangle = \langle h, v \rangle \\ (b) & \langle u, \mu - \lambda \rangle \geq 0, \end{cases} \quad (16)$$

*for all  $(v, \mu) \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G)) \times H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))^+$ .*

*Proof.* We first note, that (8) is equivalent to the following problem. Find  $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$  solving

$$\begin{cases} (a) & \langle \mathcal{S}_\sigma u, u \rangle = \langle h, u \rangle \\ (b) & \langle \mathcal{S}_\sigma u, v \rangle \geq \langle h, v \rangle, \end{cases} \quad (17)$$

for all  $v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$ . Setting  $v = 2u, v = 0$  in the variational inequality (8) we obtain

$$\langle \mathcal{S}_\sigma u, u \rangle \begin{matrix} \geq \\ \leq \end{matrix} \langle h, u \rangle \Rightarrow \langle \mathcal{S}_\sigma u, u \rangle = \langle h, u \rangle \quad (*)$$

If we add (\*) to (8) we obtain the second line in (17). To get (8) from (17) we subtract (17) (a) from (17b).

We now show the equivalence of (17) and (16):

(17)  $\Rightarrow$  (16): If we set  $\lambda = \mathcal{S}_\sigma u - h$  we have by (17b) :  $\langle \mathcal{S}_\sigma u - h, v \rangle \geq 0$  for all  $v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$  and therefore  $\lambda \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))^+$ . The first line in (16) holds trivially.

By (17a) we have that  $\langle \lambda, u \rangle = 0$ . Therefore,  $\langle u, \mu - \lambda \rangle = \langle u, \mu \rangle \geq 0$ , as  $u$  and  $\mu$  are positive.

(16)  $\Rightarrow$  (17): Now let  $(u, \lambda) \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G)) \times H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))^+$  be the solution to (16). Setting  $\mu = 2\lambda$  and  $\mu = 0$  yields  $\langle u, \lambda \rangle \geq 0, \langle u, \lambda \rangle \leq 0$ . Therefore  $\langle u, \lambda \rangle = 0$ .

Assume that  $u$  is not  $\geq 0$ . Then there exists  $\mu \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))^+$  such that  $\langle u, \mu \rangle < 0$ , and we obtain the contradiction

$$0 \leq \langle u, \mu - \lambda \rangle = \langle u, \mu \rangle - \langle u, \lambda \rangle = \langle u, \mu \rangle < 0 .$$

Therefore  $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$ .

We now insert  $\tilde{v} = v - u$  for  $u, v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))^+$  into (16a). Note  $\langle v - u, \lambda \rangle = \langle v, \lambda \rangle - \langle u, \lambda \rangle = \langle v, \lambda \rangle \geq 0$ . Inserting  $v - u$  in (16a), we have

$$\langle \mathcal{S}_\sigma u, v - u \rangle - \langle \lambda, v - u \rangle = \langle h, v - u \rangle ,$$

or equivalently

$$\langle \mathcal{S}_\sigma u - h, v - u \rangle = \langle \lambda, v - u \rangle \geq 0.$$

□

The discrete formulation reads as follows:

Find  $(u_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2}) \in \tilde{V}_{t_1, h_1}^{1,1} \times (V_{t_2, h_2}^{0,0})^+$  such that

$$\begin{cases} (a) \langle \mathcal{S}_\sigma u_{\Delta t_1, h_1}, v_{\Delta t_1, h_1} \rangle - \langle \lambda_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle = \langle h, v_{\Delta t_1, h_1} \rangle \\ (b) \langle u_{\Delta t_1, h_1}, \mu_{\Delta t_2, h_2} - \lambda_{\Delta t_2, h_2} \rangle \geq 0 \end{cases} \quad (18)$$

holds for all  $(v_{\Delta t_1, h_1}, \mu_{\Delta t_2, h_2}) \in \tilde{V}_{t_1, h_1}^{1,1} \times (V_{t_2, h_2}^{0,0})^+$ .

Like for the variational inequality (9), we assume that  $\mathcal{S}_\sigma$  is computed exactly. Note that, as in the elliptic case, we allow possibly different meshes for the displacement and the Lagrange multiplier. If the meshes for the Lagrange multiplier and the solution are sufficiently different, as in the elliptic case [3], we obtain a discrete inf-sup condition in the space-time Sobolev spaces:

**Theorem 14.** *Let  $C > 0$  sufficiently small, and  $\frac{\max\{h_1, \Delta t_1\}}{\min\{h_2, \Delta t_2\}} < C$ . Then there exists  $\alpha > 0$  such that for all  $\lambda_{\Delta t_2, h_2}$ :*

$$\sup_{\mu_{\Delta t_1, h_1}} \frac{\langle \mu_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2} \rangle}{\|\mu_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *}} \geq \alpha \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}.$$

*Proof.* Let  $z$  be the solution to the equation  $z - \partial_t^2 z - \Delta z = 0$  in  $\mathbb{R}^+ \times \Omega$  with boundary conditions

$$\frac{\partial z}{\partial t} = 0 \quad \text{at } t = 0, \quad \frac{\partial z}{\partial \nu} = \lambda_{\Delta t_2, h_2} \quad \text{on } \mathbb{R}^+ \times G, \quad z = 0 \quad \text{on } \mathbb{R}^+ \times \Gamma \setminus \overline{G}, \quad z \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

We know from the coercivity of the Neumann-to-Dirichlet operator that

$$\langle z, \lambda_{\Delta t_2, h_2} \rangle = \langle z, \partial_\nu z \rangle \geq C \|\partial_\nu z\|_{0, -\frac{1}{2}, \sigma}^2 = C \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}^2. \quad (19)$$

Let  $\delta$  such that  $\|z\|_{0, \frac{1}{2} + \delta, \sigma} < \infty$ . From the approximation properties we note that there exists a function  $\hat{\mu}_{\Delta t_1, h_1}$  with

$$\|z - \hat{\mu}_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *} \lesssim (\max\{h_1, \Delta t_1\})^\delta \|z\|_{0, \frac{1}{2} + \delta, \sigma}.$$

By the continuity of the Neumann-to-Dirichlet operator, the right hand side is estimated by

$$(h_1 + \Delta t_1)^\delta \|z\|_{0, \frac{1}{2} + \delta, \sigma} = (\max\{h_1, \Delta t_1\})^\delta \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2} + \delta, \sigma}.$$

Using an inverse inequality for  $\lambda_{\Delta t_2, h_2}$ , we conclude

$$\|z - \hat{\mu}_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, G, *} \lesssim \frac{(\max\{h_1, \Delta t_1\})^\delta}{(\min\{h_2, \Delta t_2\})^\delta} \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}. \quad (20)$$

From the continuity of the Neumann-to-Dirichlet operator and (20), we obtain

$$\|\hat{\mu}_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *} \leq \|z - \hat{\mu}_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *} + \|z\|_{0, \frac{1}{2}, \sigma, *} \quad (21)$$

$$\begin{aligned} &\lesssim \frac{(\max\{h_1, \Delta t_1\})^\delta}{(\min\{h_2, \Delta t_2\})^\delta} \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} + \|z\|_{0, \frac{1}{2}, \sigma, *} \\ &\lesssim \frac{(\max\{h_1, \Delta t_1\})^\delta}{(\min\{h_2, \Delta t_2\})^\delta} \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} + \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} . \end{aligned} \quad (22)$$

Using  $\hat{\mu}_{\Delta t_1, h_1}$  from above and (21), we now estimate:

$$\begin{aligned} \sup_{\mu_{\Delta t_1, h_1}} \frac{\langle \mu_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2} \rangle}{\|\mu_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *}} &\geq \frac{\langle \hat{\mu}_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2} \rangle}{\|\hat{\mu}_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *}} \gtrsim \frac{\langle \hat{\mu}_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2} \rangle}{\|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}} \\ &= \frac{1}{\|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}} (\langle z, \lambda_{\Delta t_2, h_2} \rangle - \langle z - \hat{\mu}_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2} \rangle) . \end{aligned}$$

The first term  $\langle z, \lambda_{\Delta t_2, h_2} \rangle$  is estimated from below by  $\|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}$ , with the help of (19), while for the second we have

$$\langle z - \hat{\mu}_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2} \rangle \leq \|z - \hat{\mu}_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *} \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} \lesssim \frac{(\max\{h_1, \Delta t_1\})^\delta}{(\min\{h_2, \Delta t_2\})^\delta} \|\lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma}^2, \quad (23)$$

where we used (20). The assertion follows.  $\square$

**Theorem 15.** *The discrete mixed formulation (18) admits a unique solution. The following a priori estimates hold:*

$$\|\lambda - \lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} \lesssim \inf_{\tilde{\lambda}_{\Delta t_2, h_2}} \|\lambda - \tilde{\lambda}_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} + (\Delta t_1)^{-\frac{1}{2}} \|u - u_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} , \quad (24)$$

$$\begin{aligned} \|u - u_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} &\lesssim_\sigma \inf_{v_{\Delta t_1, h_1}} \|u - v_{\Delta t_1, h_1}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, *} \\ &+ \inf_{\tilde{\lambda}_{\Delta t_2, h_2}} \left\{ \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda_{\Delta t_2, h_2}\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \right\} . \end{aligned} \quad (25)$$

*Proof.* For a fixed mesh, the weak coercivity implies that the discretization of  $\mathcal{S}_\sigma$  is positive definite. The existence therefore follows from standard results for elliptic problems.

For the a priori estimate we first note that for arbitrary  $\tilde{\lambda}_{\Delta t_2, h_2}$  the following identity holds:

$$\begin{aligned} \langle \lambda_{\Delta t_2, h_2} - \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle &= \langle \mathcal{S}_\sigma u_{\Delta t_1, h_1}, v_{\Delta t_1, h_1} \rangle - \langle h, v_{\Delta t_1, h_1} \rangle - \langle \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle \\ &= \langle \mathcal{S}_\sigma u_{\Delta t_1, h_1}, v_{\Delta t_1, h_1} \rangle - \langle \mathcal{S}_\sigma u, v_{\Delta t_1, h_1} \rangle + \langle \lambda, v_{\Delta t_1, h_1} \rangle - \langle \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle \\ &= \langle \mathcal{S}_\sigma (u_{\Delta t_1, h_1} - u), v_{\Delta t_1, h_1} \rangle + \langle \lambda - \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle, \end{aligned} \quad (26)$$

where we made use of (16) and (18). By the inf-sup condition (14) and (26), we have:

$$\begin{aligned} \alpha \|\lambda_{\Delta t_2, h_2} - \tilde{\lambda}_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} &\leq \sup_{v_{\Delta t_1, h_1}} \frac{\langle \lambda_{\Delta t_2, h_2} - \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle}{\|v_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *}} \\ &= \sup_{v_{\Delta t_1, h_1}} \frac{\langle \mathcal{S}_\sigma (u_{\Delta t_1, h_1} - u), v_{\Delta t_1, h_1} \rangle + \langle \lambda - \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle}{\|v_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *}} . \end{aligned}$$

We estimate both terms separately. From duality and an inverse inequality in time we obtain for the first term

$$\begin{aligned} |\langle \mathcal{S}_\sigma(u_{\Delta t_1, h_1} - u), v_{\Delta t_1, h_1} \rangle| &\leq \|\mathcal{S}_\sigma(u_{\Delta t_1, h_1} - u)\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma} \|v_{\Delta t_1, h_1}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, *} \\ &\lesssim \|u_{\Delta t_1, h_1} - u\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} (\Delta t_1)^{-\frac{1}{2}} \|v_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *} . \end{aligned}$$

A similar argument applied to the second term yields:

$$|\langle \lambda - \tilde{\lambda}_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle| \leq \|\lambda - \tilde{\lambda}_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} \|v_{\Delta t_1, h_1}\|_{0, \frac{1}{2}, \sigma, *} .$$

We obtain the a priori estimate

$$\|\lambda - \lambda_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} \lesssim_\sigma \inf_{\tilde{\lambda}_{\Delta t_2, h_2}} \|\lambda - \tilde{\lambda}_{\Delta t_2, h_2}\|_{0, -\frac{1}{2}, \sigma} + (\Delta t_1)^{-\frac{1}{2}} \|u_{\Delta t_1, h_1} - u\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} . \quad (27)$$

Next we combine the Galerkin orthogonality

$$\langle \mathcal{S}_\sigma(u - u_{\Delta t_1, h_1}), V_{\Delta t_1, h_1} \rangle = \langle \lambda - \lambda_{\Delta t_2, h_2}, V_{\Delta t_1, h_1} \rangle$$

with the coercivity of the Dirichlet-to-Neumann operator to obtain

$$\begin{aligned} \|u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *}^2 &\lesssim_\sigma \langle \mathcal{S}_\sigma(u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}), u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1} \rangle \\ &= \langle \mathcal{S}_\sigma(u - v_{\Delta t_1, h_1}), u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1} \rangle + \langle \mathcal{S}_\sigma(u_{\Delta t_1, h_1} - u), u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1} \rangle \\ &= \langle \mathcal{S}_\sigma(u - v_{\Delta t_1, h_1}), u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1} \rangle \\ &\quad + \langle \tilde{\lambda}_{\Delta t_2, h_2} - \lambda + \lambda_{\Delta t_2, h_2} - \tilde{\lambda}_{\Delta t_2, h_2}, u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1} \rangle \end{aligned}$$

for all  $v_{\Delta t_1, h_1}$  and  $\tilde{\lambda}_{\Delta t_2, h_2}$ . From the mapping properties and the continuity of the dual pairing, we conclude

$$\begin{aligned} \|u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *}^2 &\lesssim \|u - v_{\Delta t_1, h_1}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, *} \|u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} \\ &\quad + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \|u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} \\ &\quad + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda_{\Delta t_2, h_2}\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} \|u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} . \end{aligned}$$

Therefore,

$$\|u_{\Delta t_1, h_1} - v_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} \lesssim_\sigma \|u - v_{\Delta t_1, h_1}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, *} + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda_{\Delta t_2, h_2}\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} .$$

It follows that

$$\|u - u_{\Delta t_1, h_1}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} \lesssim_\sigma \|u - v_{\Delta t_1, h_1}\|_{\frac{1}{2}, \frac{1}{2}, \sigma, *} + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} + \|\tilde{\lambda}_{\Delta t_2, h_2} - \lambda_{\Delta t_2, h_2}\|_{\frac{1}{2}, -\frac{1}{2}, \sigma} .$$

□

## 7 A variational inequality for the single layer operator

As a test case, we also consider the simpler punch problem, which models a rigid body indenting an elastic half-space. We consider the following formulation as a variational inequality for the single-layer operator  $V$ :

Find  $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G))^+$  such that for all  $v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G))^+$ :

$$\langle p_Q V u, v - u \rangle \geq \langle h, v - u \rangle. \quad (28)$$

As for the variational inequality for the Dirichlet-to-Neumann operator, a solution exists in the case where  $\Omega = \mathbb{R}_+^n$  is the half space:

**Theorem 16.** *Let  $\sigma > 0$  and  $h \in H_\sigma^{\frac{3}{2}}(\mathbb{R}^+, H^{\frac{1}{2}}(G))$ . Then there exists a unique classical solution  $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G))^+$  of (28).*

*Proof.* See [13], p. 456. □

The corresponding discretized variational inequality reads as follows.

Find  $u_{\Delta t, h} \in \tilde{K}_{t, h}^+$  such that:

$$\langle p_Q V u_{\Delta t, h}, v_{\Delta t, h} - u_{\Delta t, h} \rangle \geq \langle h, v_{\Delta t, h} - u_{\Delta t, h} \rangle \quad (29)$$

holds for all  $v_{\Delta t, h} \in \tilde{K}_{t, h}^+$ .

The relevant a priori estimate reads:

**Theorem 17.** *Let  $h \in H_\sigma^{\frac{3}{2}}(\mathbb{R}^+, H^{\frac{1}{2}}(G))$ , and let  $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G))^+$ ,  $u_{\Delta t, h} \in \tilde{K}_{t, h}^+$  be the solutions of (28), resp. (29). Then the following estimate holds:*

$$\|u - u_{\Delta t, h}\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma, \star}^2 \lesssim_\sigma \inf_{\phi_{\Delta t, h} \in \tilde{K}_{t, h}^+} (\|h - p_Q V u\|_{\frac{1}{2}, \frac{1}{2}, \sigma} \|u - \phi_{\Delta t, h}\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma, \star} + \|u - \phi_{\Delta t, h}\|_{\frac{1}{2}, -\frac{1}{2}, \sigma, \star}^2). \quad (30)$$

The proof proceeds analogous to the proof of Theorem 9. It replaces Theorem 5 for the Dirichlet-to-Neumann operator by the mapping properties of  $V$  in the half space, Theorem 4, and the coercivity  $\|\phi\|_{-\frac{1}{2}, -\frac{1}{2}, \sigma, \star}^2 \lesssim_\sigma \langle p_Q V \phi, \phi \rangle$  noted there.

Similarly to the contact problem, for the numerical implementation a mixed formulation of the variational inequality (28) proves useful. Its analysis is analogous to the contact problem.

**Theorem 18 (Mixed formulation).** *Let  $h \in H_\sigma^{\frac{3}{2}}(\mathbb{R}^+, H^{\frac{1}{2}}(G))$ . The variational inequality formulation (28) is equivalent to the following formulation:*

Find  $(u, \lambda) \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G)) \times H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{\frac{1}{2}}(G))^+$  such that

$$\begin{cases} (a) \quad \langle V u, v \rangle - \langle \lambda, v \rangle = \langle h, v \rangle \\ (b) \quad \langle u, \mu - \lambda \rangle \geq 0, \end{cases} \quad (31)$$

for all  $(v, \mu) \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G)) \times H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{\frac{1}{2}}(G))^+$ .

The discrete formulation reads as follows:

Find  $(u_{\Delta t_1, h_1}, \lambda_{\Delta t_2, h_2}) \in V_{t_1, h_1}^{1,1} \times (V_{t_2, h_2}^{1,1})^+$  such that

$$\begin{cases} (a) \langle Vu_{\Delta t_1, h_1}, v_{\Delta t_1, h_1} \rangle - \langle \lambda_{\Delta t_2, h_2}, v_{\Delta t_1, h_1} \rangle = \langle h, v_{\Delta t_1, h_1} \rangle \\ (b) \langle u_{\Delta t_1, h_1}, \mu_{\Delta t_2, h_2} - \lambda_{\Delta t_2, h_2} \rangle \geq 0 \end{cases} \quad (32)$$

holds for all  $(v_{\Delta t_1, h_1}, \mu_{\Delta t_2, h_2}) \in V_{t_1, h_1}^{1,1} \times (V_{t_2, h_2}^{1,1})^+$ .

## 8 Algorithmic considerations

### 8.1 Marching-on-in-time scheme for the variational equality

A first step towards the solution of the variational inequality is to find the solution of the Dirichlet-to-Neumann equation. Using the representation  $W - (K' - \frac{1}{2}I)V^{-1}(K - \frac{1}{2}I)$  for the Dirichlet-to-Neumann operator in terms of layer potentials, the Dirichlet-to-Neumann equation is equivalently reformulated as follows:

For given  $h \in H_\sigma^{\frac{3}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ , find  $u \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$ ,  $v \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G))$  such that

$$\int_0^\infty \langle Wu - (K' - \frac{1}{2})v, \phi \rangle_G dt = \int_0^\infty \langle h, \phi \rangle_G dt, \quad (33)$$

$$\int_0^\infty [\langle Vv, \Psi \rangle_G - \langle (K - \frac{1}{2})u, \Psi \rangle_G] dt = 0, \quad (34)$$

holds for all  $\phi \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{\frac{1}{2}}(G))$ ,  $\Psi \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, \tilde{H}^{-\frac{1}{2}}(G))$ .

For the discretization, let  $u_{\Delta t, h} = \sum_{m,i} c_i^m \beta_{\Delta t}^m(t) \xi_h^i(x, y) \in \tilde{V}_{t, h}^{1,1}$ ,  $v_{\Delta t, h} = \sum_{m,i} d_i^m \beta_{\Delta t}^m(t) \xi_h^i(x, y) \in V_{t, h}^{1,1}$

be ansatz functions that are linear in space and time. To obtain a stable marching-on-in-time scheme we test the first equation against constant test functions in time and the second equation against the time derivative of constant test functions. We let  $\phi_{\Delta t, h}^{ij} := \gamma_{[t_{i-1}, t_i]}(t) \xi_h^j(x, y) =: \gamma^i(t) \xi^j(x, y)$ ,  $\dot{\Psi}_{\Delta t, h}^{ij} = \dot{\gamma}^i(t) \xi^j(x, y)$  be test functions that are constant in time and linear in space. Thus, the discrete system reads as follows:

$$\begin{aligned} \int_0^\infty \langle Wu_{\Delta t, h} - (K' - \frac{1}{2})v_{\Delta t, h}, \gamma^n \xi^j \rangle_G dt &= \int_0^\infty \langle h, \gamma^n \xi^j \rangle_G dt, \\ \int_0^\infty [\langle Vv_{\Delta t, h}, \dot{\gamma}^n \xi^j \rangle_G - \langle (K - \frac{1}{2})u_{\Delta t, h}, \dot{\gamma}^n \xi^j \rangle_G] dt &= 0, \end{aligned}$$

for all  $n = 1 \dots, N_t, j = 1, \dots, N_s$ .

Setting  $I^j = I$ , if  $j = 0, 1$ ,  $I^j = 0$  otherwise and  $\hat{I}^0 = (-I)$ ,  $\hat{I}^1 = I$ ,  $\hat{I}^j = 0$  otherwise, where  $I$  is the mass matrix, we may rewrite the system as

$$\mathcal{M}^j := \begin{pmatrix} W^j & -((K^j)' - \frac{1}{2} \frac{\Delta t}{2} I^j) \\ -(K^j - \frac{1}{2} \hat{I}^j) & V^j \end{pmatrix},$$

for all  $j = 2, \dots, N_t$  (see [18] for further details). From the convolution structure in time, we obtain: For arbitrary  $n \in \{1, \dots, N_t\}$ :

$$\sum_{m=1}^{\infty} \mathcal{M}^{n-m} \begin{pmatrix} c^m \\ d^m \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{2} I(h^{n-1} + h^n) \\ 0 \end{pmatrix}. \quad (35)$$

By causality,  $\mathcal{M}^j = 0$  when  $j < 0$ , so that the sum on the left hand side ends at  $m = n$ . This results in the time stepping scheme:

$$\mathcal{M}^0 \begin{pmatrix} c^n \\ d^n \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{2} I(h^{n-1} + h^n) \\ 0 \end{pmatrix} - \sum_{m=1}^{n-1} \mathcal{M}^{n-m} \begin{pmatrix} c^m \\ d^m \end{pmatrix}. \quad (36)$$

Hence, if we save the matrices from previous time steps, we only need to calculate one new matrix  $\mathcal{M}^{n-1}$  in time step  $n$  to obtain the vector  $[c^n, d^n]^T$ .

## 8.2 Space-time Uzawa algorithm

The solution of the discrete mixed formulation (18) may be computed using a Uzawa algorithm in space-time.

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### Algorithm 1 Space-time Uzawa algorithm

---

choose  $\rho > 0$ :

$k = 0 : y^{(0)} = \vec{0}$

**while** stopping criterion not satisfied **do**

**solve:**  $\mathcal{S}x^{(k)} = h + y^{(k)}$

**compute:**  $y^{(k+1)} = \text{Pr}_K(y^{(k)} - \rho x^{(k)})$ , where  $(\text{Pr}_K y)_i = \max\{y_i, 0\}$

$k \leftarrow k + 1$

**end while**

---

**Lemma 19.** *The space-time Uzawa algorithm converges, provided that  $0 < \rho < 2C_\sigma$ . Here  $C_\sigma$  is the coercivity constant in Theorem 5.*

*Proof.* From the algorithm and the contraction property of the projection  $\text{Pr}_K$ ,

$$\begin{aligned} \|\lambda_{\Delta t, h}^{(k+1)} - \lambda_{\Delta t, h}\|_{0,0,\sigma}^2 &= \|\text{Pr}_K(\lambda_{\Delta t, h}^{(k)} - \rho u_{\Delta t, h}^{(k)}) - \text{Pr}_K(\lambda_{\Delta t, h} - \rho u_{\Delta t, h})\|_{0,0,\sigma}^2 \\ &\leq \|\lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h} - \rho(u_{\Delta t, h}^{(k)} - u_{\Delta t, h})\|_{0,0,\sigma}^2 \\ &= \|\lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}\|_{0,0,\sigma}^2 - 2\rho \langle \lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}, u_{\Delta t, h}^{(k)} - u_{\Delta t, h} \rangle + \rho^2 \|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{0,0,\sigma}^2. \end{aligned}$$

We conclude that

$$\|\lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}\|_{0,0,\sigma}^2 - \|\lambda_{\Delta t, h}^{(k+1)} - \lambda_{\Delta t, h}\|_{0,0,\sigma}^2 \geq 2\rho \langle \lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}, u_{\Delta t, h}^{(k)} - u_{\Delta t, h} \rangle - \rho^2 \|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{0,0,\sigma}^2.$$

Further note that

$$\begin{aligned} \langle \lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}, u_{\Delta t, h}^{(k)} - u_{\Delta t, h} \rangle &= \langle \mathcal{S}_\sigma(u_{\Delta t, h}^{(k)} - u_{\Delta t, h}), u_{\Delta t, h}^{(k)} - u_{\Delta t, h} \rangle \\ &\geq C_\sigma \|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *}^2. \end{aligned}$$

As  $\|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{-\frac{1}{2}, \frac{1}{2}, \sigma, *} \geq \|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{0, 0, \sigma}$ , we conclude that

$$\|\lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}\|_{0, 0, \sigma}^2 - \|\lambda_{\Delta t, h}^{(k+1)} - \lambda_{\Delta t, h}\|_{0, 0, \sigma}^2 \geq (2\rho C_\sigma - \rho^2) \|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{0, 0, \sigma}^2.$$

The right hand side is non-negative provided  $0 < \rho < 2C_\sigma$ . We conclude that  $\|\lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}\|_{0, 0, \sigma}$  is a decreasing sequence. As  $\|\lambda_{\Delta t, h}^{(k)} - \lambda_{\Delta t, h}\|_{0, 0, \sigma} \geq 0$ , it therefore converges, and it follows that  $\|u_{\Delta t, h}^{(k)} - u_{\Delta t, h}\|_{0, 0, \sigma} \rightarrow 0$ .  $\square$

Practically, we choose the test functions of the mixed discretized formulation (18) to be piecewise constant in time to obtain a marching-on-in-time scheme. In this case, computing the coefficients  $x^{(k)} = (\{x_i^m\}_{i,m=1}^{N_s, N_t})^{(k)}$  corresponds to solving the equality for the Dirichlet-to-Neumann operator in Section 8.1, with an augmented right hand side:

$$\langle \mathcal{S}(\sum_{m,i} (x_i^m)^k), \gamma^n \phi^j \rangle = \langle \sum_{m,i} (y_i^m)^{k-1} \beta^m \xi^i, \gamma^n \phi^j \rangle + \langle \sum_{m,i} h_i^m \beta^m \xi^i, \gamma^n \phi^j \rangle, \quad (37)$$

for all  $j = 1, \dots, N_s, n = 1, \dots, N_t$ . Comparing this system with the system (36) we observe that (37) may be written as:

$$\sum_{m=1}^n \mathcal{M}^{n-m} \begin{pmatrix} (x^m)^k \\ (d^m)^k \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{2} I(h^{n-1} + h^n) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\Delta t}{2} I((y^{n-1})^{k-1} + (y^n)^{k-1}) \\ 0 \end{pmatrix}, \quad (38)$$

for  $n = 1, \dots, N_t$ .

**Remark 20.** For a Lagrange multiplier that is constant in time, i.e.  $\lambda_{\Delta t, h} = \sum_{m,l} y_l^m \gamma^m(t) \xi^l(x)$ ,

the second term on the right hand side of (38) becomes  $\Delta t \begin{pmatrix} I(y^n)^{k-1} \\ 0 \end{pmatrix}$ . This will be relevant in the following section.

### 8.3 Time-step Uzawa algorithm

The space-time Uzawa algorithm solves the whole space-time system in every Uzawa iteration. While it is provably convergent, a time-stepping Uzawa algorithm turns out to be more efficient and will be useful for future adaptive computations.

A crucial observation to derive a time-stepping algorithm is that  $L^2(Q)$  continuously embeds in  $H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ . One may therefore use piecewise constant ansatz functions in time,  $\lambda_{\Delta t, h} = \sum_{m,i} y_i^m \gamma^m(t) \phi^i(x) \in L^2(Q) \subset H_\sigma^{\frac{1}{2}}(\mathbb{R}^+, H^{-\frac{1}{2}}(G))$ , for the Lagrange multiplier.

Set

$$\tilde{I}_{i,j} = \int_{\Gamma} \phi^i \phi^j ds_x, \quad I_{i,j} = \int_{\Gamma} \xi^i \phi^j ds_x,$$

and note that  $\sum_{m,i} y_i^m \int_0^\infty \int_\Gamma \gamma^m \phi^i \gamma^n \phi^j ds_x dt = \Delta t \sum_i y_i^n \int_\Gamma \phi^i \phi^j ds_x$ . The resulting space-time variational inequality reads:

$$\begin{pmatrix} \mathcal{M}^0 \\ \mathcal{M}^1 & \mathcal{M}^0 \\ \vdots \\ \mathcal{M}^{N_t} & \dots & \mathcal{M}^0 \end{pmatrix} \begin{pmatrix} [c^1, d^1]^t \\ [c^2, d^2]^t \\ \vdots \\ [c^{N_t}, d^{N_t}]^t \end{pmatrix} - \Delta t \begin{pmatrix} [\tilde{I}y^1, 0]^t \\ [\tilde{I}y^2, 0]^t \\ \vdots \\ [\tilde{I}y^{N_t}, 0]^t \end{pmatrix} = \frac{\Delta t}{2} \begin{pmatrix} [I(h^0 + h^1), 0]^t \\ [I(h^1 + h^2), 0]^t \\ \vdots \\ [I(h^{N_t-1} + h^{N_t}), 0]^t \end{pmatrix}, \quad (39)$$

$$\langle \sum_{m,i} c_i^m \beta^m \xi^i, \mu_{\Delta t, h} - \sum_{m,i} y_i^m \gamma^m \phi^i \rangle \geq 0, \quad \forall \mu_{\Delta t, h} \in (V_{t, h}^{0,0})^+. \quad (40)$$

With  $\mu_{\Delta t, h} = \sum_{m,i} \mu_i^m \gamma^m(t) \phi^i(x)$ , we first consider the space-time variational inequality (40) for a fixed  $n = 1 \dots, N_t$ :

$$\begin{aligned} \langle \sum_{m,i} c_i^m \beta^m \xi^i, \sum_j (\mu_j^n - y_j^n) \gamma^n \phi^j \rangle &= \frac{\Delta t}{2} \langle \sum_i c_i^n \xi^i, \sum_j (\mu_j^n - y_j^n) \phi^j \rangle + \frac{\Delta t}{2} \langle \sum_i c_i^{n-1} \xi^i, \sum_j (\mu_j^n - y_j^n) \phi^j \rangle, \\ &= (c^n)^\top \hat{I}(\mu^n - y^n) + (c^{n-1})^\top \hat{I}(\mu^n - y^n). \end{aligned}$$

Here we used

$$\langle \beta^m \xi^i, \gamma^n \phi^j \rangle = \int_\Gamma \xi^i(x) \phi^j(x) \int_0^\infty \beta^m(t) \gamma^n(t) dt ds_x = \int_\Gamma \xi^i(x) \phi^j(x) \frac{\Delta t}{2} [\delta_{n,m} + \delta_{n-1,m}] ds_x,$$

with  $\delta_{n,m} = 1$  if  $n = m$  and  $= 0$  otherwise. Also  $\hat{I}_{ij} = \frac{\Delta t}{2} I_{ij}$ . Therefore (40) may be written as

$$\begin{aligned} (c^1)^\top \hat{I}(\mu^1 - y^1) + (c^0)^\top \hat{I}(\mu^1 - y^1) + (c^2)^\top \hat{I}(\mu^2 - y^2) + (c^1)^\top \hat{I}(\mu^2 - y^2) + \dots \\ + (c^{N_t})^\top \hat{I}(\mu^{N_t} - y^{N_t}) + (c^{N_t-1})^\top \hat{I}(\mu^{N_t} - y^{N_t}) \geq 0 \quad \forall \mu^j \quad \forall j, \end{aligned}$$

with  $c^0 = 0$ . Setting  $\mu = (y^1 \dots, y^{k-1}, \mu', y^{k+1}, \dots, y^{N_t})$  for  $k = 1, \dots, N_t$  yields

$$(c^k)^\top \hat{I}(\mu' - y^k) + (c^{k-1})^\top \hat{I}(\mu' - y^k) \geq 0, \quad \forall \mu',$$

and we see that the solution to the space-time variational inequality satisfies the following time-stepping scheme:

For  $k = 1, \dots, N_t$ , find  $(c^k, d^k, y^k)$  such that

$$\mathcal{M}^0 \begin{pmatrix} c^k \\ d^k \end{pmatrix} - \Delta t \begin{pmatrix} \tilde{I}y^k \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{2} I(h^{k-1} + h^k) \\ 0 \end{pmatrix} - \sum_{m=1}^{k-1} \mathcal{M}^{k-m} \begin{pmatrix} c^m \\ d^m \end{pmatrix}, \quad (41)$$

$$(c^k)^\top \hat{I}(\mu^k - y^k) \geq -(c^{k-1})^\top \hat{I}(\mu^k - y^k) \quad \forall \mu^k. \quad (42)$$

Conversely, if we have solutions to (41) and (42) for every  $k = 1, \dots, N_t$ , summing (42) yields (40). We conclude:

**Lemma 21.** *The solution to the space-time variational inequality is also a solution to the time-step variational inequality and vice versa.*

The resulting time-step Uzawa algorithm is as follows:

---

**Algorithm 2** Time-step Uzawa algorithm

---

choose  $\rho > 0$ :

**for**  $n=1, \dots, N_t$  **do**

$k=0$ :  $(y^n)^0 = \vec{0}$

**while** stopping criterion not satisfied **do**

**solve:**  $\mathcal{M}^0 \begin{pmatrix} c^n \\ d^n \end{pmatrix} - \Delta t \begin{pmatrix} \tilde{I}(y^n)^k \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{2} I(h^{n-1} + h^n) \\ 0 \end{pmatrix} - \sum_{m=1}^{n-1} \mathcal{M}^{n-m} \begin{pmatrix} c^m \\ d^m \end{pmatrix}$

**compute:**  $(y^n)^{k+1} = \max\{0, (y^n)^k + \rho((c^n) + (c^{n-1})^\top \hat{I})\}$

$k \leftarrow k + 1$

**end while**

**end for**

---

The Uzawa algorithm converges in each time step, as long as  $\mathcal{M}^0$  is positive definite, and  $\rho$  is sufficiently small.

**Remark 22.** *As for the contact problem, the mixed formulation for the punch problem (31) may be discretized and solved with the above space-time or time-step Uzawa algorithms. The Dirichlet-to-Neumann operator is here replaced by the single-layer operator.*

## 9 Numerical experiments

As stated above, we set  $\sigma = 0$  in our computations and discretize the Dirichlet-to-Neumann operator as in Section 8.1.

### 9.1 Dirichlet-to-Neumann operator on unit sphere

**Example 23.** *We solve the discretized variational equality (15) for the Dirichlet-to-Neumann operator on  $\Gamma = S^2$ , with a right hand side obtained from the Neumann data of a known, radially symmetric solution to the wave equation. Specifically,*

$$h = \partial_n u(t, x) \mid_{\{|x|=1\}} = \left(-\frac{3}{4} + \cos\left(\frac{\pi}{2}(4-t)\right) + \frac{\pi}{2} \sin\left(\frac{\pi}{2}(4-t)\right) - \frac{1}{4}(\cos(\pi(4-t)) + \pi \sin(\pi(4-t)))\right)[H(4-t) - H(-t)] ,$$

where  $H$  is the Heaviside function. The solution  $u$  of the Dirichlet-to-Neumann equation  $Su = h$  corresponds to the Dirichlet data of the solution to the wave equation. Hence,

$$u(t, r) \mid_{\Gamma} = \left(\frac{3}{4} - \cos\left(\frac{\pi(4-t)}{2}\right) + \frac{1}{4} \cos(\pi(4-t))\right)[H(4-t) - H(-t)] .$$

We fix the CFL ratio  $\frac{\Delta t}{h} \approx 0.6$  and set  $T = 5$ .

Figure 1 shows the  $L^2(\Gamma)$ -norm of the exact, resp. numerical solution, while Figure 2 depicts the error in this norm. As the number of degrees of freedom increases, the  $L^2([0, T] \times \Gamma)$ -norm of

the error between the numerical approximations and the exact solution converges to 0 at a rate 0.53, as shown in Figure 3. Here, we compute the experimental convergence rate  $\alpha$  as

$$\alpha = \frac{\log E(u_1) - \log E(u_2)}{\log DOF_1 - \log DOF_2}.$$

In terms of  $h$ , the convergence rate is 1.6. The results exemplify the convergence of our proposed method to approximate the Dirichlet-to-Neumann operator.

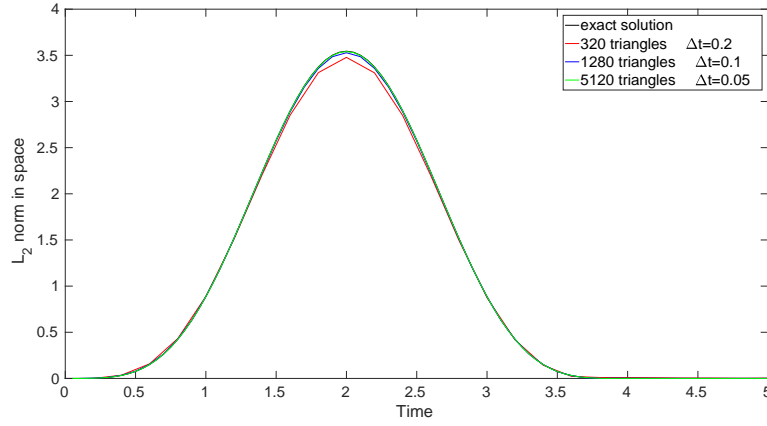


Figure 1:  $L^2(\Gamma)$ -norm of the solution to  $\mathcal{S}u = h$  for fixed  $CFL \approx 0.6$ .

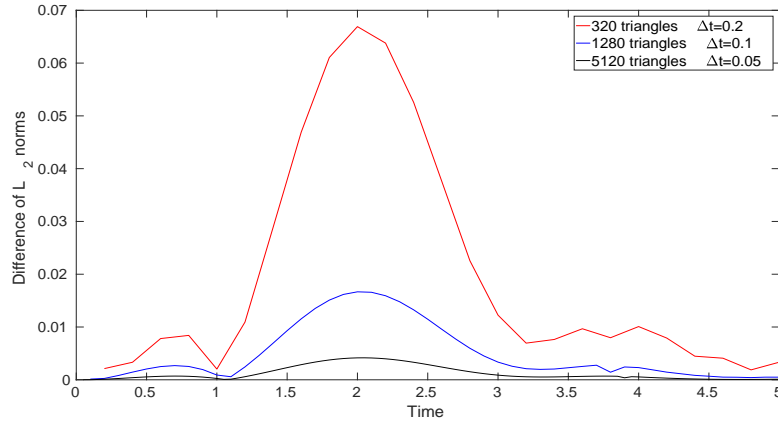


Figure 2: Absolute error  $|||u||_{L^2(\Gamma)} - ||u_{\Delta t, h}||_{L^2(\Gamma)}|$  as a function of time for fixed  $\frac{\Delta t}{h}$ .

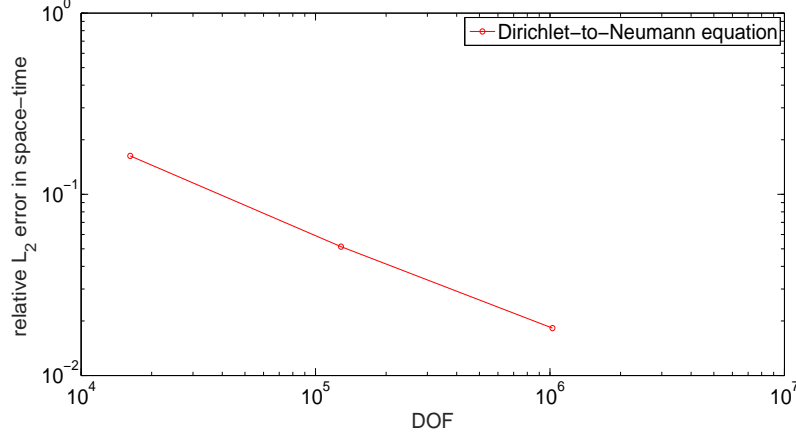


Figure 3:  $L^2([0, T] \times \Gamma)$ -error vs. degrees of freedom of the solution to  $\mathcal{S}u = h$  for fixed  $\frac{\Delta t}{h}$ .

## 9.2 Contact problem: Dirichlet-to-Neumann operator

We now consider the discretization of the nonlinear contact problem (5) for both flat and more general contact areas. In this case, no exact solutions are known, and we compare the numerical approximations to a reference solutions on an appropriately finer space-time mesh.

**Example 24.** We choose  $\Gamma = [-2, 2]^2 \times \{0\}$  with contact area  $G = [-1, 1]^2 \times \{0\}$  for times up to  $T = 5$ , with the CFL-ratio  $\frac{\Delta t}{h} = 0.7$ . The right hand side of the contact problem (5) is given by

$$h(t, x) = e^{-2t} t^4 \cos(2\pi x) \cos(2\pi y) \chi_{[-0.25, 0.25]}(x) \chi_{[-0.25, 0.25]}(y) .$$

We use the discretisation from Section 8.1, in particular with a Lagrange multiplier that is piecewise linear in space and time. The numerical solutions are compared to a reference solution on a mesh with 12800 triangles, and we use  $\Delta t = 0.05$ .

The inequality is solved using the space-time Uzawa algorithm. We stop the solver when subsequent iterates have a relative difference of less than  $10^{-11}$ .

Figure 4 shows the solution  $u_{\Delta t, h}$  to the contact problem (left column) and the corresponding Lagrange multiplier  $\lambda_{\Delta t, h}$  (right column) for several time steps. The solution gradually develops a smooth bump in the center, which gives rise to a radially outgoing wave. A nonvanishing Lagrange multiplier  $\lambda = \mathcal{S}_\sigma u - h$  indicates the emergent contact forces at the depicted times  $t = 4.25$  and  $t = 5$ .

Figure 5 considers the relative error to the reference solution in  $L^2([0, T] \times G)$ . The numerical approximations converge at a rate of  $\alpha = 0.7$  with increasing degrees of freedom. This compares to the rate of convergence for the Dirichlet-to-Neumann equation in Example 23, where also the geometry needed to be approximated. Algorithmically, the computational cost of the nonlinear solver is dominated by the cost of computing the matrix entries.

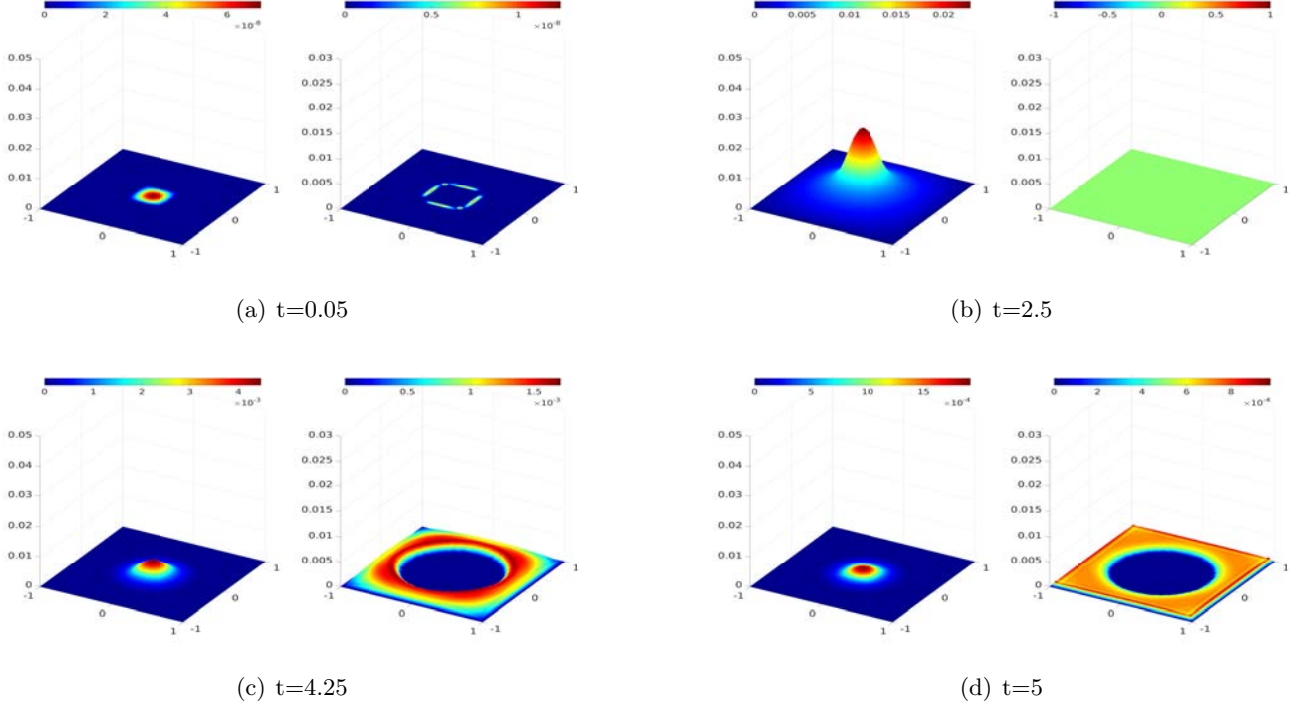


Figure 4: Evolution of  $u$  and  $\lambda$  in  $G = [-1, 1]^2 \times \{0\}$  for the contact problem, Example 24.

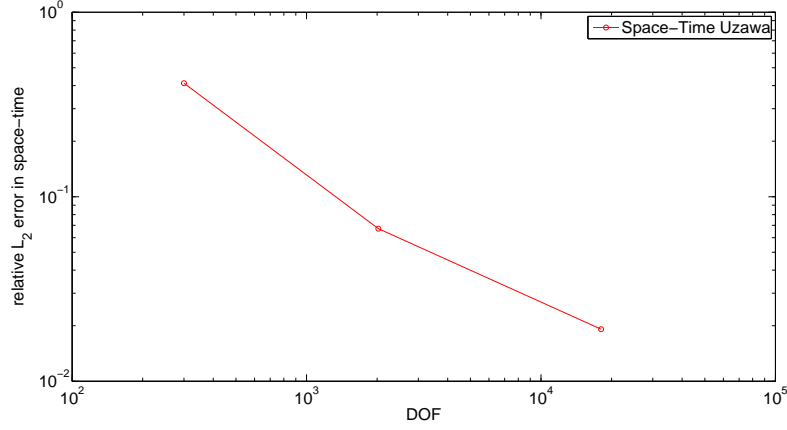


Figure 5: Relative  $L^2([0, T] \times \Gamma)$ -error vs. degrees of freedom of the solutions to the contact problem for fixed  $\frac{\Delta t}{h}$ , Example 24.

Next we compare the above space-time Uzawa algorithm with the time-step Uzawa variant. In a given time step, we stop the Uzawa iteration when subsequent iterates either have a relative difference of less than  $10^{-12}$  or if the  $\ell_\infty$ -norm is less than  $10^{-10}$ . For the space-time Uzawa algorithm we use the stopping criterion from before.

**Example 25.** We consider the contact problem (5) with the geometry and right hand side from Example 24. On a fixed mesh of 3200 triangles, and with  $\Delta t = 0.1$  and  $T = 6$ , we investigate the difference of the approximate solutions obtained from the space-time and time-step Uzawa algorithms.

Figure 6 shows the temporal evolution of the relative difference in  $L^2(\Gamma)$  between the two methods in a semi-logarithmic plot. The difference is smaller than 0.01% for all times, but increases sharply around the onset of contact shortly after time  $t = 4$ .

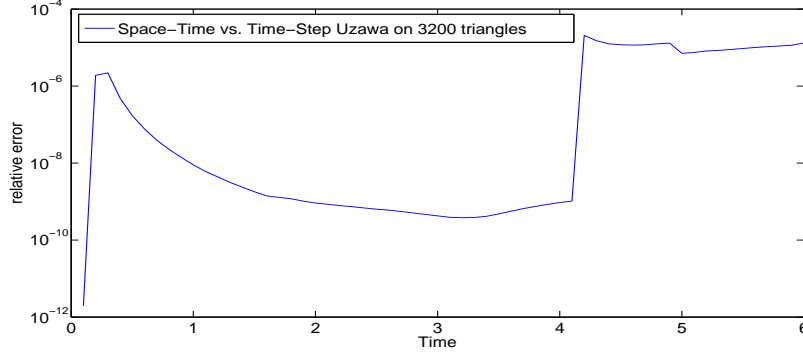


Figure 6: Relative  $L^2(\Gamma)$ -error between the solutions of the space-time and time-step Uzawa algorithms, Example 25.

A comparison of the clock times shows the computational efficiency of the time-step Uzawa algorithm: Its runtime of 455.3 cpu seconds compares to 1301.6 cpu seconds required by the space-time algorithm.

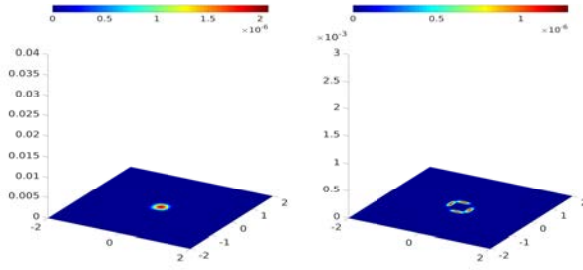
To illustrate our method for non-flat contact geometries, we consider a cube with three contact faces. Physically, one may think of a rigid cube which is tightly fixed to an elastic surrounding material on its three other faces.

**Example 26.** Let  $\Gamma$  be the surface of the cube  $[-2, 2]^3$ , with contact area  $G$  consisting of the top, front and right faces. We set  $T = 6$  and keep the CFL-ratio fixed at  $\frac{\Delta t}{h} \approx 0.7$ . On each of the contact faces we prescribe a right hand side

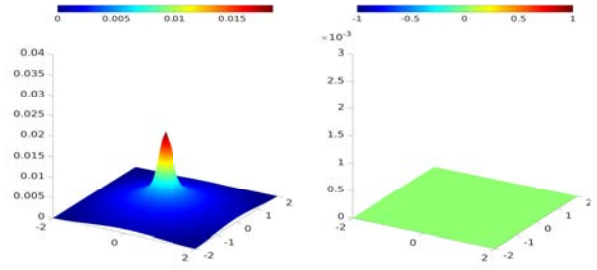
$$h(t, x) = e^{-2t} t^4 \cos(2\pi x) \cos(2\pi y) \chi_{[-0.25, 0.25]}(x) \chi_{[-0.25, 0.25]}(y) ,$$

centered in the midpoint of each face. The reference solution is computed on a mesh with 19200 triangles, for  $\Delta t = 0.1$ .

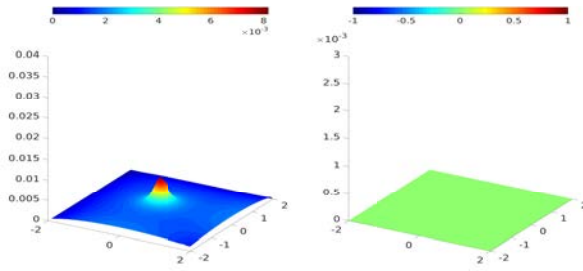
The evolution of  $u_{\Delta t, h}$  and the Lagrange multiplier  $\lambda_{\Delta t, h}$  on the top face of the cube are depicted in Figure 7. The nonzero displacement  $u_{\Delta t, h}$  spreads from the neighboring contact faces into the shown area by time  $t = 5$  and eventually leads to strong contact near the upper left corner.



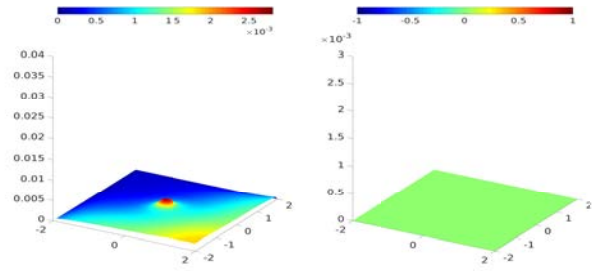
(a)  $t=0.1$



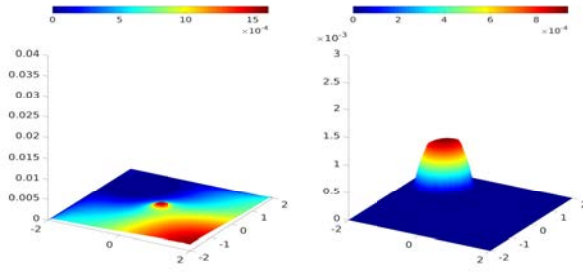
(b)  $t=3$



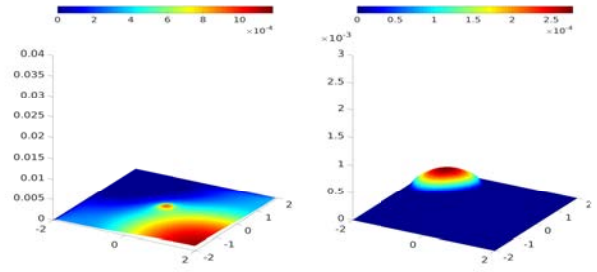
(c)  $t=4$



(d)  $t=5$



(e)  $t=5.5$



(f)  $t=6$

Figure 7: Evolution of  $u$  and  $\lambda$  on  $[-2, 2]^2 \times \{2\}$  for the contact problem on  $[-2, 2]^3$ , Example 26.

The error between the benchmark and approximate solutions on coarser meshes is shown in Figure 8. It shows a convergence with an approximate convergence rate of  $\alpha = 0.6$ .

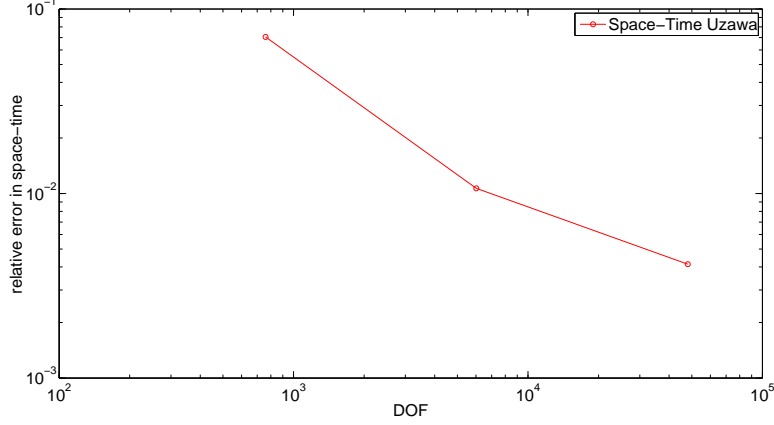


Figure 8: Relative  $L^2([0, T] \times \Gamma)$ -error vs. degrees of freedom for the contact problem for fixed  $\frac{\Delta t}{h}$ , Example 26.

### 9.3 Single-layer potential

We finally address the punch problem described by the variational inequality (28). Unlike for the problems involving the Dirichlet-to-Neumann operator, in this case we do not need to approximate the integral operator. As in the previous numerical examples, we consider both flat and more general contact geometries.

**Example 27.** For the punch problem (28) we choose  $\Gamma = [-2, 2]^2 \times \{0\}$  and the area of contact  $G = [-1.2, 1.2]^2 \times \{0\}$ . We set  $T = 5$  and keep the CFL-ratio fixed at  $\frac{\Delta t}{h} \approx 0.7$ . As right hand side we again consider  $h(t, x) = e^{-2t} t^4 \cos(2\pi x) \cos(2\pi y) \chi_{[-0.25, 0.25]}(x) \chi_{[-0.25, 0.25]}(y)$ , and we look for a numerical solution and Lagrange multiplier which are piecewise constant in time, linear in space. The approximate solutions are compared to a benchmark obtained from a mesh with 12800 triangles and  $\Delta t = 0.05$ .

We solve the variational inequality using the time-step Uzawa algorithm. In a given time step, we stop the Uzawa iteration when subsequent iterates either have a relative difference of less than  $10^{-12}$  or if the  $\ell_\infty$ -norm is less than  $10^{-10}$ .

Figure 9 shows the solution  $u_{\Delta t, h}$  to the punch problem and its Lagrange multiplier  $\lambda_{\Delta t, h}$ . Contact is observed for most of the considered time interval.

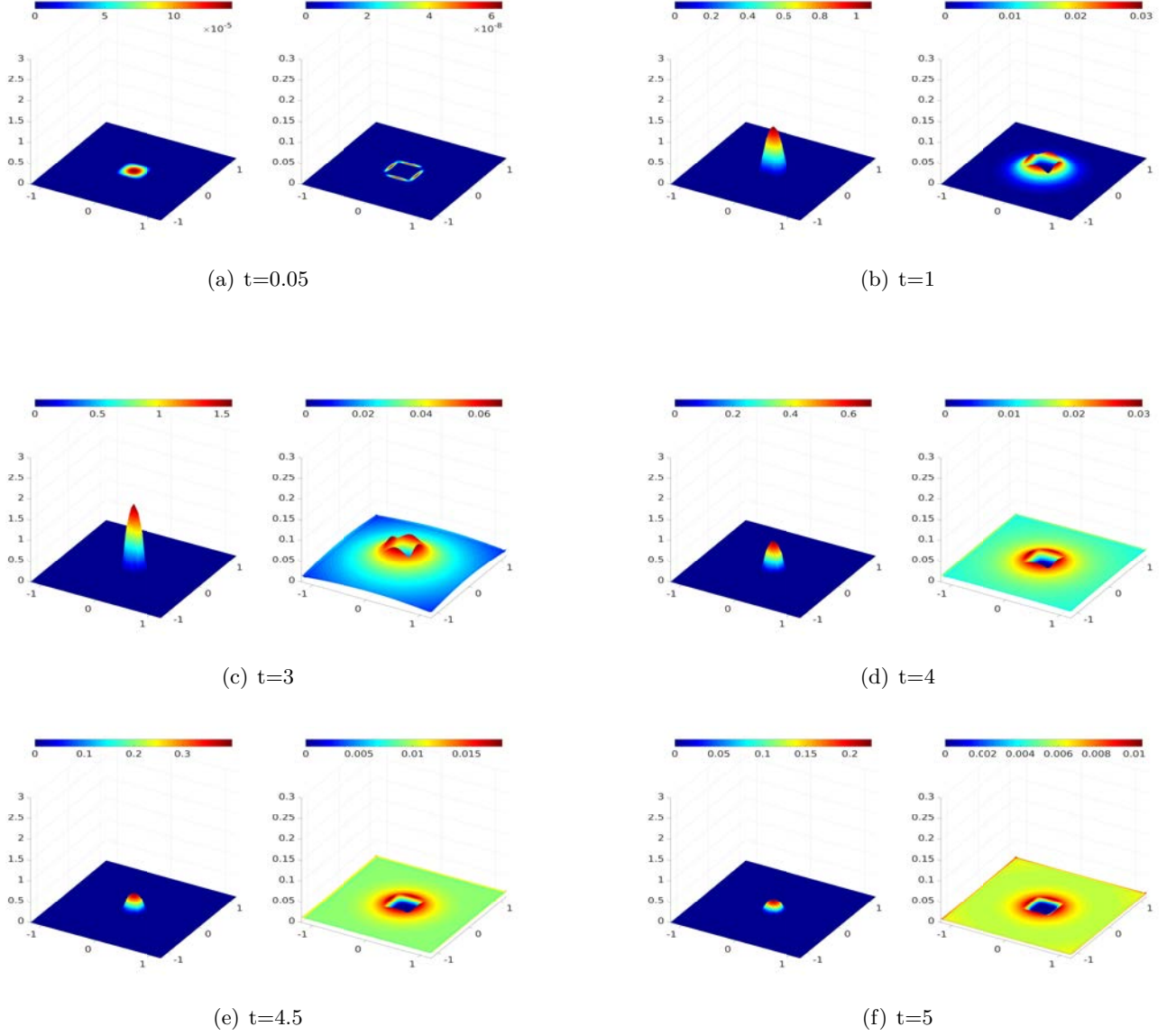


Figure 9: Evolution of  $u$  and  $\lambda$  for the punch problem with  $G = [-1.2, 1.2]^2 \times \{0\}$ , Example 27.

Because of the potentially low spatial regularity of the solution, which a priori only belongs to a Sobolev space with negative exponent, we do not consider the error of the numerical solutions in  $L^2([0, T] \times \Gamma)$ . As a weaker measure, we consider an approximate energy norm. An approximation for the energy is computed via  $E(u) = \frac{1}{2}u^\top V u - u^\top h$ .

Figure 10 shows the convergence of the numerical solutions in energy. The energy similarly shows a convergence rate of  $\alpha = 1$ .

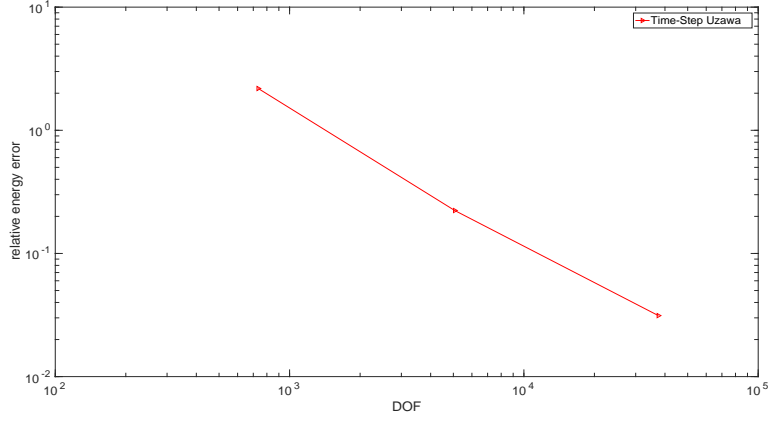


Figure 10: Relative energy error for the punch problem for fixed  $\frac{\Delta t}{h}$ , Example 27.

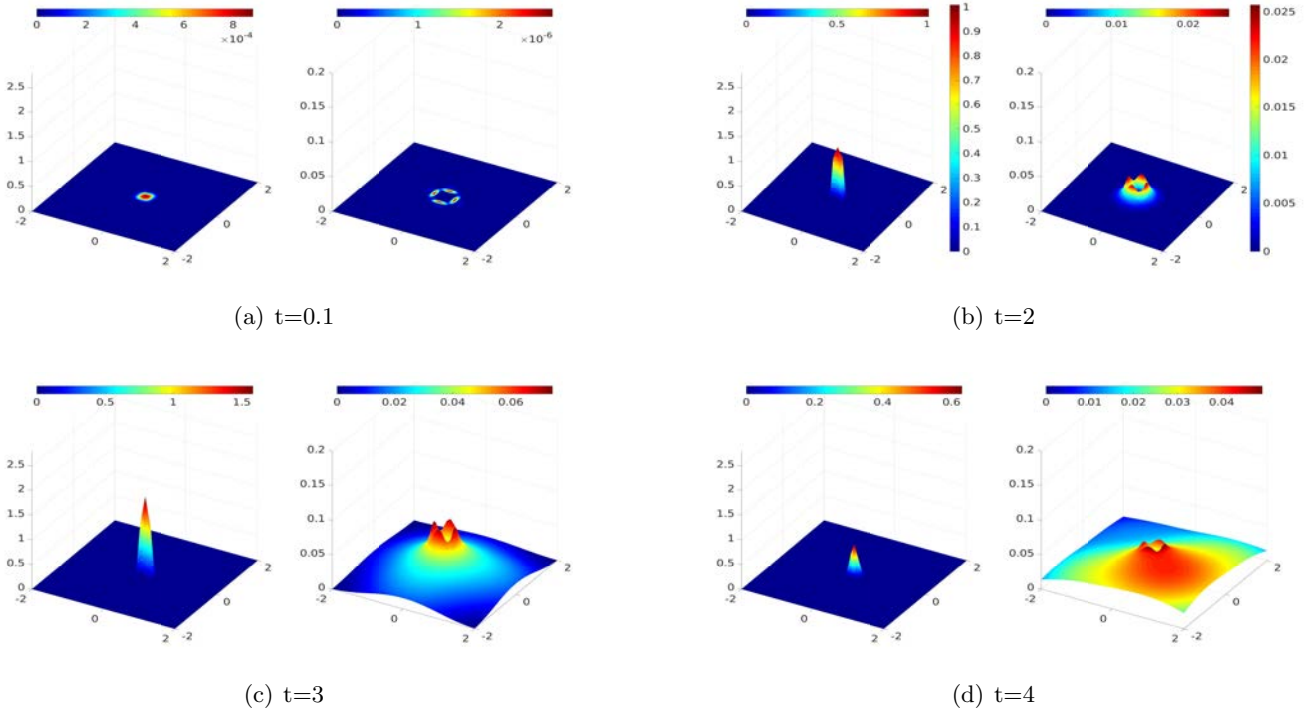


Figure 11: Evolution of the density and the Lagrange multiplier for the punch problem, Example 28.

We finally consider the punch problem with the entire surface of the cube as contact area.

**Example 28.** For the punch problem (28) we choose  $G = \Gamma$  to be the surface of  $[-2, 2]^3$ . We set  $T = 4$  and keep the CFL-ratio fixed at  $\frac{\Delta t}{h} \approx 0.7$ . As right hand side we consider the same

function as in Example 26,

$$h(t, x) = e^{-2t} t^4 \cos(2\pi x) \cos(2\pi y) \chi_{[-0.25, 0.25]}(x) \chi_{[-0.25, 0.25]}(y) ,$$

centered in the midpoint of the top, front and right face. We approximate the reference solution on a mesh with 19200 triangles and  $\Delta t = 0.1$ .

The variational inequality is solved using the time-step Uzawa algorithm as in Example 27. Figure 11 shows the solution  $u_{\Delta t, h}$  to the punch problem and its Lagrange multiplier  $\lambda_{\Delta t, h}$  on the surface of the cube. Again contact is observed for most times.

As in the previous example, in Figure 12 we show the convergence in energy. We here compute a convergence rate of  $\alpha = 0.5$ .

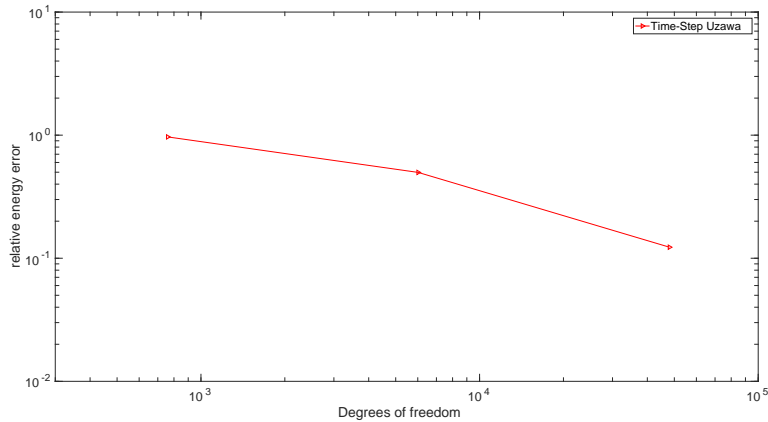


Figure 12: Relative error of the energy for the solution to the punch problem for fixed CFL  $\approx 0.7$ .

## 10 Conclusions

In this work we propose and analyze a Galerkin boundary element method to solve dynamic contact problems for the wave equation. Boundary elements provide a natural and efficient formulation, as the contact takes place at the interface between two materials.

Analytically, we obtain a first a priori error analysis for a variational inequality involving the Dirichlet-to-Neumann operator, as well as a similar analysis for a mixed formulation. The analysis and the stability of the method are crucially based on the weak coercivity of the formulation, and for the mixed method an inf-sup condition in space-time. The proof requires a flat contact area, as only in this case the existence of solutions to the continuous problem is known. Also a variational inequality and a mixed formulation for the single layer operator are considered, which do not require the approximation of the operator.

Numerical experiments demonstrate the efficiency and convergence of the proposed mixed method. A time-stepping Uzawa method for the solution of the variational inequality proves

more efficient, but also potentially less stable than a similar solver for the space-time system. The latter is shown to be provably convergent. As a key point, the numerical experiments indicate stability and convergence beyond flat geometries.

The current work provides a first, rigorous step towards efficient boundary elements for dynamic contact. Future work will focus on the a posteriori error analysis, which is essential for adaptive mesh refinements to resolve the singularities of the solution in space and time [19], as well as on stabilized mixed space-time formulations [8, 11]. For applications to traffic noise [9], also the nonsmooth variational inequalities for frictional contact will be of interest.

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