



**HAL**  
open science

## On Drinfel'd associators

G rard H. E. Duchamp, Ngoc Minh, K A Penson

► **To cite this version:**

| G rard H. E. Duchamp, Ngoc Minh, K A Penson. On Drinfel'd associators. 2017. hal-01509151

**HAL Id: hal-01509151**

**<https://hal.science/hal-01509151>**

Preprint submitted on 28 Apr 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destin e au d p t et   la diffusion de documents scientifiques de niveau recherche, publi s ou non,  manant des  tablissements d'enseignement et de recherche fran ais ou  trangers, des laboratoires publics ou priv s.

# On Drinfel'd associators

G. H. E. Duchamp<sup>#</sup>, V. Hoang Ngoc Minh<sup>◇</sup>, K. A. Penson<sup>♭</sup>

<sup>#</sup>Université Paris XIII, 99 Jean-Baptiste Clément, 93430 Villetaneuse, France.

<sup>◇</sup>Université Lille II, 1, Place Déliot, 59024 Lille, France.

<sup>♭</sup>Université Paris VI, 75252 Paris Cedex 05, France.

## 1 Knizhnik-Zamolodchikov differential equations and coefficients of Drinfel'd associators

In 1986 [6], in order to study the linear representations of the braid group  $B_n$  coming from the monodromy of the Knizhnik-Zamolodchikov differential equations, Drinfel'd introduced a class of formal power series  $\Phi$  on noncommutative variables over the finite alphabet  $X = \{x_0, x_1\}$ . Such a power series  $\Phi$  is called an *associator*. For  $n = 3$ , it leads to the following fuchsian noncommutative differential equation with three regular singularities in  $\{0, 1, +\infty\}$  :

$$(DE) \quad dG(z) = \left( x_0 \frac{dz}{z} + x_1 \frac{dz}{1-z} \right) G(z).$$

Solutions of  $(DE)$  are power series, with coefficients which are mono-valued functions on the simply connex domain  $\Omega = \mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$  and can be seen as multi-valued over<sup>1</sup>  $\mathbb{C} - \{0, 1\}$ , on noncommutative variables  $x_0$  and  $x_1$ . Drinfel'd proved that  $(DE)$  admits two particular mono-valued solutions on  $\Omega$ ,  $G_0(z) \underset{z \rightsquigarrow 0}{\sim} \exp[x_0 \log(z)]$  and  $G_1(z) \underset{z \rightsquigarrow 1}{\sim} \exp[-x_1 \log(1-z)]$  [7, 8]. and the existence of an associator  $\Phi_{KZ} \in \mathbb{R}\langle\langle X \rangle\rangle$  such that  $G_0 = G_1 \Phi_{KZ}$  [7, 8]. After that, via representations of the chord diagram algebras, Lê and Murakami [17] expressed the coefficients of  $\Phi_{KZ}$  as *linear* combinations of special values of several complex variables *zeta* functions,  $\{\zeta_r\}_{r \in \mathbb{N}_+}$ ,

$$\zeta_r : \mathcal{H}_r \rightarrow \mathbb{R}, \quad (s_1, \dots, s_r) \mapsto \sum_{n_1 > \dots > n_k > 0} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}, \quad (1)$$

where  $\mathcal{H}_r = \{(s_1, \dots, s_r) \in \mathbb{C}^r \mid \forall m = 1, \dots, r, \sum_{i=1}^m \Re(s_i) > m\}$ . For  $(s_1, \dots, s_r) \in \mathcal{H}_r$ , one has two ways of thinking  $\zeta_r(s_1, \dots, s_r)$  as limits, fulfilling identities [14, 13, 1]. Firstly, they are limits of *polylogarithms* and secondly, as truncated sums, they are limits of *harmonic sums*, for  $z \in \mathbb{C}, |z| < 1, N \in \mathbb{N}_+$  :

$$\text{Li}_{s_1, \dots, s_k}(z) = \sum_{n_1 > \dots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_k^{s_k}}, \quad \text{H}_{s_1, \dots, s_k}(N) = \sum_{n_1 > \dots > n_k > 0}^N \frac{1}{n_1^{s_1} \dots n_k^{s_k}}. \quad (2)$$

More precisely, if  $(s_1, \dots, s_r) \in \mathcal{H}_r$  then, after a theorem by Abel, one has

$$\lim_{z \rightarrow 1} \text{Li}_{s_1, \dots, s_k}(z) = \lim_{n \rightarrow \infty} \text{H}_{s_1, \dots, s_k}(n) =: \zeta_r(s_1, \dots, s_k) \quad (3)$$

---

<sup>1</sup>In fact, we have mappings from the connected covering  $\mathbb{C} - \widetilde{\{0, 1\}}$ .

else it does not hold, for  $(s_1, \dots, s_r) \notin \mathcal{H}_r$ , while  $\text{Li}_{s_1, \dots, s_k}$  is well defined over  $\{z \in \mathbb{C}, |z| < 1\}$  and so is  $\text{H}_{s_1, \dots, s_k}$ , as Taylor coefficients of the following function

$$(1-z)^{-1} \text{Li}_{s_1, \dots, s_k}(z) = \sum_{n \geq 1} \text{H}_{s_1, \dots, s_k}(n) z^n, \quad \text{for } z \in \mathbb{C}, |z| < 1. \quad (4)$$

Note also that, for  $r = 1$ ,  $\zeta_1$  is nothing else but the famous Riemann zeta function and, for  $r = 0$ , it is convenient to set  $\zeta_0$  to the constant function  $1_{\mathbb{R}}$ . In all the sequel, for simplification, we will adopt the notation  $\zeta$  for  $\zeta_r$ ,  $r \in \mathbb{N}$ .

In this work, we will describe the regularized solutions of  $(DE)$ .

For that, we are considering the alphabets  $X = \{x_0, x_1\}$  and  $Y_0 = \{y_s\}_{s \geq 0}$  equipped of the total ordering  $x_0 < x_1$  and  $y_0 > y_1 > y_2 > \dots$ , respectively. Let  $Y = Y_0 - \{y_0\}$ . The free monoid generated by  $X$  (resp.  $Y, Y_0$ ) is denoted by  $X^*$  (resp.  $Y^*, Y_0^*$ ) and admits  $1_{X^*}$  (resp.  $1_{Y^*}, 1_{Y_0^*}$ ) as unit.

The sets of, respectively, polynomials and formal power series, with coefficients in a commutative  $\mathbb{Q}$ -algebra  $A$ , over  $X^*$  (resp.  $Y^*, Y_0^*$ ) are denoted by  $A\langle X \rangle$  (resp.  $A\langle Y \rangle, A\langle Y_0 \rangle$ ) and  $A\langle\langle X \rangle\rangle$  (resp.  $A\langle\langle Y \rangle\rangle, A\langle\langle Y_0 \rangle\rangle$ ). The sets of polynomials are the  $A$ -modules and endowed with the associative concatenation, the associative commutative shuffle (resp. quasi-shuffle) product, over  $A\langle X \rangle$  (resp.  $A\langle Y \rangle, A\langle Y_0 \rangle$ ). Their associated coproducts are denoted, respectively,  $\Delta_{\sqcup}$  and  $\Delta_{\boxplus}$ . The algebras  $(A\langle X \rangle, \sqcup, 1_{X^*})$  and  $(A\langle Y \rangle, \boxplus, 1_{Y^*})$  admit the sets of Lyndon words denoted, respectively, by  $\mathcal{Lyn}X$  and  $\mathcal{Lyn}Y$ , as transcendence bases [18] (resp. [15, 16]).

For  $Z = X$  or  $Y$ , denoting  $\mathcal{L}ie_A\langle Z \rangle$  and  $\mathcal{L}ie_A\langle\langle Z \rangle\rangle$  the sets of, respectively, Lie polynomials and Lie series, the enveloping algebra  $\mathcal{U}(\mathcal{L}ie_A\langle Z \rangle)$  is isomorphic to the Hopf algebra  $(A\langle X \rangle, \cdot, 1_{Z^*}, \Delta_{\sqcup}, \mathbf{e})$ . We get also  $\mathcal{H}_{\boxplus} := (A\langle Y \rangle, \cdot, 1_{Y^*}, \Delta_{\boxplus}, \mathbf{e}) \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\boxplus}))$ , where [15, 16]

$$\text{Prim}(\mathcal{H}_{\boxplus}) = \text{span}_A\{\pi_1(w) | w \in Y^*\}, \quad (5)$$

$$\pi_1(w) = \sum_{k=1}^{(w)} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w | u_1 \boxplus \dots \boxplus u_k \rangle u_1 \dots u_k. \quad (6)$$

## 2 Indexing polylogarithms and harmonic sums by words and their generating series

For any  $r \in \mathbb{N}$ , since any combinatorial composition  $(s_1, \dots, s_r) \in \mathbb{N}_+^r$  can be associated with words  $x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1 \in X^* x_1$  and  $y_{s_1} \dots y_{s_r} \in Y^*$ . Similarly, any multi-indices<sup>2</sup>  $(s_1, \dots, s_r) \in \mathbb{N}^r$  can be associated with words  $y_{s_1} \dots y_{s_r} \in Y_0^*$ . Then let  $\text{Li}_{x_0}(z) := (\log(z))^r / r!$ , and  $\text{Li}_{s_1, \dots, s_k}$  and  $\text{H}_{s_1, \dots, s_k}$  be indexed by words [14] :  $\text{Li}_{x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1} := \text{Li}_{s_1, \dots, s_r}$  and  $\text{H}_{y_{s_1} \dots y_{s_r}} := \text{H}_{s_1, \dots, s_r}$ . Similarly,  $\text{Li}_{-s_1, \dots, -s_k}$  and  $\text{H}_{-s_1, \dots, -s_k}$  be indexed by words<sup>3</sup> [4, 5] :  $\text{Li}_{y_{s_1} \dots y_{s_r}}^- :=$

<sup>2</sup>The weight of  $(s_1, \dots, s_r) \in \mathbb{N}_+^r$  (resp.  $\mathbb{N}^r$ ) is defined as the integer  $s_1 + \dots + s_r$  which corresponds to the weight, denoted  $(w)$ , of its associated word  $w \in Y^*$  (resp.  $Y_0^*$ ) and corresponds also to the length, denoted by  $|u|$ , of its associated word  $u \in X^*$ .

<sup>3</sup>Note that, all these  $\{\text{Li}_{\bar{w}}\}_{w \in Y_0^*}$  and  $\{\text{H}_{\bar{w}}\}_{w \in Y_0^*}$  are divergent at their singularities.

$\text{Li}_{-s_1, \dots, -s_r}^-$  and  $\text{H}_{y_{s_1} \dots y_{s_r}}^- := \text{H}_{-s_1, \dots, -s_r}^-$ . In particular,  $\text{H}_{y_0}^-(n) := \binom{n}{r} = (n)_r / r!$  and  $\text{Li}_{y_0}^-(z) := (z/(1-z))^r$ . There exists a law of algebra, denoted by  $\top$ , in  $\mathbb{Q}\langle\langle Y_0 \rangle\rangle$ , such that the following morphisms of algebras are surjective [4]

$$\mathbf{H}_\bullet^- : (\mathbb{Q}\langle Y_0 \rangle, \boxplus, 1_{Y_0^*}) \longrightarrow (\mathbb{Q}\{\text{H}_w^-\}_{w \in Y_0^*}, \times, 1), \quad w \longmapsto \text{H}_w^-, \quad (7)$$

$$\mathbf{Li}_\bullet^- : (\mathbb{Q}\langle Y_0 \rangle, \top, 1_{Y_0^*}) \longrightarrow (\mathbb{Q}\{\text{Li}_w^-\}_{w \in Y_0^*}, \times, 1), \quad w \longmapsto \text{Li}_w^-, \quad (8)$$

and  $\ker \mathbf{H}_\bullet^- = \ker \mathbf{Li}_\bullet^- = \mathbb{Q}\langle\{w - w \top 1_{Y_0^*} \mid w \in Y_0^*\}\rangle$  [4]. Moreover, the families  $\{\text{H}_{y_k}^-\}_{k \geq 0}$  and  $\{\text{Li}_{y_k}^-\}_{k \geq 0}$  are  $\mathbb{Q}$ -linearly independent.

On the other hand, the following morphisms of algebras are injective

$$\mathbf{H}_\bullet : (\mathbb{Q}\langle Y \rangle, \boxplus, 1_{Y^*}) \longrightarrow (\mathbb{Q}\{\text{H}_w\}_{w \in Y^*}, \times, 1), \quad w \longmapsto \text{H}_w, \quad (9)$$

$$\mathbf{Li}_\bullet : (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}) \longrightarrow (\mathbb{Q}\{\text{Li}_w\}_{w \in X^*}, \times, 1), \quad w \longmapsto \text{Li}_w \quad (10)$$

Moreover, the families  $\{\text{H}_w\}_{w \in Y^*}$  and  $\{\text{Li}_w\}_{w \in X^*}$  are  $\mathbb{Q}$ -linearly independent and the families  $\{\text{H}_l\}_{l \in \mathcal{L}yn Y}$  and  $\{\text{Li}_l\}_{l \in \mathcal{L}yn X}$  are  $\mathbb{Q}$ -algebraically independent. But at singularities of  $\{\text{Li}_w\}_{w \in X^*}$ ,  $\{\text{H}_w\}_{w \in Y^*}$ , the following convergent values

$$\forall u \in Y^* - y_1 Y^*, \zeta(u) := \text{H}_u(+\infty) \quad \text{and} \quad \forall v \in x_0 X^* x_1, \zeta(v) := \text{Li}_v(1) \quad (11)$$

are no longer linearly independent and the values  $\{\text{H}_l(+\infty)\}_{l \in \mathcal{L}yn Y - \{y_1\}}$  (resp.  $\{\text{Li}_l(1)\}_{l \in \mathcal{L}yn X - X}$ ) are no longer algebraically independent [14, 19].

The graphs of the isomorphisms of algebras,  $\mathbf{Li}_\bullet$  and  $\mathbf{H}_\bullet$ , as generating series, read then [2, 14]

$$\mathbf{L} := \sum_{w \in X^*} \text{Li}_w w = \prod_{l \in \mathcal{L}yn X}^{\searrow} e^{\text{Li}_{s_l} P_l}, \quad \mathbf{H} := \sum_{w \in Y^*} \text{H}_w w = \prod_{l \in \mathcal{L}yn Y}^{\searrow} e^{\text{H}_{\Sigma_l} \Pi_l} \quad (12)$$

where the PBW basis  $\{P_w\}_{w \in X^*}$  (resp.  $\{\Pi_w\}_{w \in Y^*}$ ) is expanded over the basis of  $\mathcal{L}ie_A\langle X \rangle$  (resp.  $\mathcal{U}(\text{Prim}(\mathcal{H}_{\boxplus}))$ ),  $\{P_l\}_{l \in \mathcal{L}yn X}$  (resp.  $\{\Pi_l\}_{l \in \mathcal{L}yn Y}$ ), and  $\{S_w\}_{w \in X^*}$  (resp.  $\{\Sigma_w\}_{w \in Y^*}$ ) is the basis of  $(\mathbb{Q}\langle Y \rangle, \sqcup, 1_{X^*})$  (resp.  $(\mathbb{Q}\langle Y \rangle, \boxplus, 1_{Y^*})$ ) containing the transcendence basis  $\{S_l\}_{l \in \mathcal{L}yn X}$  (resp.  $\{\Sigma_l\}_{l \in \mathcal{L}yn Y}$ ).

By termwise differentiation,  $\mathbf{L}$  satisfies the noncommutative differential equation (DE) with the boundary condition  $\mathbf{L}(z) \underset{z \rightarrow 0^+}{\sim} e^{x_0 \log(z)}$ . It is immediate that the power series  $\mathbf{H}$  and  $\mathbf{L}$  are group-like, for  $\Delta_{\boxplus}$  and  $\Delta_{\sqcup}$ , respectively. Hence, the following noncommutative generating series are well defined and are group-like, for  $\Delta_{\boxplus}$  and  $\Delta_{\sqcup}$ , respectively [14, 15, 16] :

$$\mathbf{Z}_{\boxplus} := \prod_{l \in \mathcal{L}yn Y - \{y_1\}}^{\searrow} e^{\text{H}_{\Sigma_l} (+\infty) \Pi_l} \quad \text{and} \quad \mathbf{Z}_{\sqcup} := \prod_{l \in \mathcal{L}yn X - X}^{\searrow} e^{\text{Li}_{s_l} (1) P_l}. \quad (13)$$

Definitions (3) and (11) lead then to the following surjective poly-morphism

$$\zeta : (\mathbb{Q}1_{X^*} \oplus x_0 \mathbb{Q}\langle X \rangle x_1, \sqcup, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1), \quad (14)$$

$$\begin{aligned} & (\mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\}) \mathbb{Q}\langle Y \rangle, \boxplus, 1_{Y^*}) \\ & \quad x_0 x_1^{r_1-1} \dots x_0 x_1^{r_k-1} \\ & \quad \quad y_{s_1} \dots y_{s_k} \end{aligned} \longmapsto \sum_{n_1 > \dots > n_k > 0} n_1^{-s_1} \dots n_k^{-s_k} \quad (15)$$

where  $\mathcal{Z}$  is the  $\mathbb{Q}$ -algebra generated by  $\{\zeta(l)\}_{l \in \mathcal{L}ynX-X}$  (resp.  $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX-X}$ ), or equivalently, generated by  $\{\zeta(l)\}_{l \in \mathcal{L}ynY-\{y_1\}}$  (resp.  $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY-\{y_1\}}$ ).

Now, let  $t_i \in \mathbb{C}, |t_i| < 1, i \in \mathbb{N}$ . For  $z \in \mathbb{C}, |z| < 1$ , we have [11]

$$\sum_{n \geq 0} \text{Li}_{x_0^n}(z) t_0^n = z^{t_0} \quad \text{and} \quad \sum_{n \geq 0} \text{Li}_{x_1^n}(z) t_1^n = (1-z)^{-t_1}. \quad (16)$$

These suggest to extend the morphism  $\text{Li}_\bullet$  over  $(\text{Dom}(\text{Li}_\bullet), \sqcup, 1_{X^*})$ , via *Lazard's elimination*, as follows (subjected to be convergent)

$$\text{Li}_S(z) = \sum_{n \geq 0} \langle S \mid x_0^n \rangle \frac{\log^n(z)}{n!} + \sum_{k \geq 1} \sum_{w \in (x_0^* x_1)^k x_0^*} \langle S \mid w \rangle \text{Li}_w(z) \quad (17)$$

with  $\mathbb{C}\langle X \rangle \sqcup \mathbb{C}^{\text{rat}}\langle\langle x_0 \rangle\rangle \sqcup \mathbb{C}^{\text{rat}}\langle\langle x_1 \rangle\rangle \subset \text{Dom}(\text{Li}_\bullet) \subset \mathbb{C}^{\text{rat}}\langle\langle X \rangle\rangle$  and  $\mathbb{C}^{\text{rat}}\langle\langle X \rangle\rangle$  denotes the closure, of  $\mathbb{C}\langle X \rangle$  in  $\mathbb{C}\langle\langle X \rangle\rangle$ , by  $\{+, \cdot, *\}$ . For example [11, 12],

1. For any  $i, j \in \mathbb{N}_+$  and  $x \in X$ , since  $(t_0 x_0 + t_1 x_1)^* = (t_0 x_0)^* \sqcup (t_1 x_1)^*$  and  $(x^*)^{\sqcup i} = (ix)^*$  then  $\text{Li}_{(x_0^*)^{\sqcup i} \sqcup (x_1^*)^{\sqcup j}}(z) = z^i (1-z)^{-j}$ .
2. For  $a \in \mathbb{C}, x \in X, i \in \mathbb{N}_+$ , since  $(ax)^{*i} = (ax)^* \sqcup (1+ax)^{i-1}$  then

$$\text{Li}_{(ax_0)^{*i}}(z) = z^a \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a \log(z))^k}{k!}, \quad (18)$$

$$\text{Li}_{(ax_1)^{*i}}(z) = \frac{1}{(1-z)^a} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a \log((1-z)^{-1}))^k}{k!}. \quad (19)$$

3. Let  $V = (t_1 x_0)^{*s_1} x_0^{s_1-1} x_1 \dots (t_r x_0)^{*s_r} x_0^{s_r-1} x_1$ , for  $(s_1, \dots, s_r) \in \mathbb{N}_+^r$ . Then

$$\text{Li}_V(z) = \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}. \quad (20)$$

In particular, for  $s_1 = \dots = s_r = 1$ , then one has

$$\begin{aligned} \text{Li}_V(z) &= \sum_{n_1, \dots, n_r > 0} \text{Li}_{x_0^{n_1-1} x_1 \dots x_0^{n_r-1} x_1}(z) t_0^{n_1-1} \dots t_r^{n_r-1} \\ &= \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{(n_1 - t_1) \dots (n_r - t_r)}. \end{aligned} \quad (21)$$

4. From the previous points, one has

$$\begin{aligned} \{\text{Li}_S\}_{S \in \mathbb{C}\langle X \rangle \sqcup \mathbb{C}\langle x_0^* \rangle \sqcup \mathbb{C}\langle (-x_0^*) \rangle \sqcup \mathbb{C}\langle x_1^* \rangle} &= \text{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1-z)^b} \text{Li}_w(z) \right\}_{\substack{a \in \mathbb{Z}, b \in \mathbb{N} \\ w \in X^*}} \\ &\subset \text{span}_{\mathbb{C}} \{ \text{Li}_{s_1, \dots, s_r} \}_{s_1, \dots, s_r \in \mathbb{Z}^r} \\ &\quad \oplus \text{span}_{\mathbb{C}} \{ z^a \mid a \in \mathbb{Z} \}, \end{aligned} \quad (22)$$

$$\begin{aligned} \{\text{Li}_S\}_{S \in \mathbb{C}\langle X \rangle \sqcup \mathbb{C}^{\text{rat}}\langle\langle x_0 \rangle\rangle \sqcup \mathbb{C}^{\text{rat}}\langle\langle x_1 \rangle\rangle} &= \text{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1-z)^b} \text{Li}_w(z) \right\}_{\substack{a, b \in \mathbb{C} \\ w \in X^*}} \\ &\subset \text{span}_{\mathbb{C}} \{ \text{Li}_{s_1, \dots, s_r} \}_{s_1, \dots, s_r \in \mathbb{C}^r} \\ &\quad \oplus \text{span}_{\mathbb{C}} \{ z^a \mid a \in \mathbb{C} \}. \end{aligned} \quad (23)$$

### 3 Noncommutative evolution equations

As we said previously Drinfel'd proved that  $(DE)$  admits two particular solutions on  $\Omega$ . These new tools and results can be considered as pertaining to the domain of *noncommutative evolution equations*. We will, here, only mention what is relevant for our needs.

Even for one sided <sup>4</sup> differential equations, in order to cope with limit initial conditions (see applications below), one needs the two sided version.

Let then  $\Omega \subset \mathbb{C}$  be simply connected and open and  $\mathcal{H}(\Omega)$  denote the algebra of holomorphic functions on  $\Omega$ . We suppose given two series (called *multipliers*)  $M_1, M_2 \in \mathcal{H}(\Omega)_+ \langle\langle X \rangle\rangle$  ( $X$  is an alphabet and the subscript indicates that the series have no constant term). Let then

$$(DE_2) \quad \mathbf{d}S = M_1S + SM_2.$$

be our equation.

#### 3.1 The main theorem

**Theorem 1.** *Let*

$$(DE_2) \quad \mathbf{d}S = M_1S + SM_2. \quad (24)$$

- (i) *Solutions of  $(DE_2)$  form a  $\mathbb{C}$ -vector space.*
- (ii) *Solutions of  $(DE_2)$  have their constant term (as coefficient of  $1_{X^*}$ ) which are constant functions (on  $\Omega$ ); there exists solutions with constant coefficient  $1_\Omega$  (hence invertible).*
- (iii) *If two solutions coincide at one point  $z_0 \in \overline{\Omega}$ , they coincide everywhere.*
- (iv) *Let be the following one-sided equations*

$$(DE^{(1)}) \quad \mathbf{d}S = M_1S \quad (DE^{(2)}) \quad \mathbf{d}S = SM_2. \quad (25)$$

*and let  $S_i$ ,  $i = 1, 2$  a solution of  $(DE^{(i)})$ , then  $S_1S_2$  is a solution of  $(DE_2)$ . Conversely, every solution of  $(DE_2)$  can be constructed so.*

- (v) *If  $M_i$ ,  $i = 1, 2$  are primitive and if  $S$ , a solution of  $(DE_2)$ , is group-like at one point, (or, even at one limit point) it is globally group-like.*

*Proof.* Omitted. □

**Remark 1.** • *Every holomorphic series  $S(z) \in \mathcal{H}(\Omega) \langle\langle X \rangle\rangle$  which is group-like ( $\Delta(S) = S \otimes S$  and  $\langle S | 1_{X^*} \rangle$ ) is a solution of a left-sided dynamics with primitive multiplier (take  $M_1 = \mathbf{d}(S)S^{-1}$  and  $M_2 = 0$ ).*

- *Invertible solutions of an equation of type  $S' = M_1S$  are on the same orbit by multiplication on the right by invertible constant series i.e. let  $S_i$ ,  $i = 1, 2$  be invertible solutions of  $(DE^{(1)})$ , then there exists an unique invertible  $T \in \mathbb{C} \langle\langle X \rangle\rangle$  such that  $S_2 = S_1.T$ . From this and point (iv) of the theorem, one can parametrize the set of invertible solutions of  $(DE_2)$ .*

---

<sup>4</sup>As the left  $(DE)$  for instance.

### 3.2 Application: Unicity of solutions with asymptotic conditions.

In a previous work [3], we proved that asymptotic group-likeness, for a series, implies<sup>5</sup> that the series in question is group-like everywhere. The process above (theorem (1), Picard's process) can be performed, under certain conditions with improper integrals we then construct the series  $L$  recursively as

$$\langle L | w \rangle = \begin{cases} \frac{\log^n(z)}{n!} & \text{if } w = x_0^n \\ \int_0^z \left( \left( \frac{x_1}{1-z} \right) \langle L | u \rangle \right) [s] ds & \text{if } w = x_1 u \\ \int_0^z \left( \left( \frac{x_0}{z} \right) \langle L | u x_1 x_0^n \rangle \right) [s] ds & \text{if } w = x_0 u x_1 x_0^n. \end{cases}$$

one can check that

- this process is well defined at each step and computes the series  $L$  as below.
- $L$  is solution of  $(DE)$ , is exactly  $G_0$  and is group-like

We here only prove that  $G_0$  is unique using the theorem above. Consider the series  $T = L e^{-x_0 \log(z)}$ . Then  $T$  is solution of an equation of the type  $(DE_2)$

$$T' = \left( \frac{x_0}{z} + \frac{x_1}{1-z} \right) T + T \left( \frac{x_0}{z} \right) \quad (26)$$

but  $\lim_{z \rightarrow z_0} G_0 e^{-x_0 \log(z)} = 1$  so, by the point (iii) of theorem (1) one has  $G_0 e^{-x_0 \log(z)} = L e^{-x_0 \log(z)}$  and then  $G_0 = L$ .

A similar (and symmetric) argument can be performed for  $G_1$  and then, in this interpretation and context,  $\Phi_{KZ}$  is unique.

## 4 Double global regularization of associators

Global singularities analysis leads to to the following global renormalization [2]

$$\begin{aligned} \lim_{z \rightarrow 1} \exp\left(-y_1 \log \frac{1}{1-z}\right) \pi_Y(L(z)) &= \lim_{n \rightarrow \infty} \exp\left(\sum_{k \geq 1} H_{y_k}(n) \frac{(-y_1)^k}{k}\right) H(n) \\ &= \pi_Y(Z_{\llbracket \cdot \rrbracket}). \end{aligned} \quad (27)$$

Thus, the coefficients  $\{\langle Z_{\llbracket \cdot \rrbracket} | u \rangle\}_{u \in X^*}$  (i.e.  $\{\zeta_{\llbracket \cdot \rrbracket}(u)\}_{u \in X^*}$ ) and  $\{\langle Z_{\llbracket \cdot \rrbracket} | v \rangle\}_{v \in Y^*}$  (i.e.  $\{\zeta_{\llbracket \cdot \rrbracket}(v)\}_{v \in Y^*}$ ) represent the finite part of the asymptotic expansions, in  $\{(1-z)^{-a} \log^b(1-z)\}_{a,b \in \mathbb{N}}$  (resp.  $\{n^{-a} H_1^b(n)\}_{a,b \in \mathbb{N}}$ ) of  $\{Li_w\}_{u \in X^*}$  (resp.  $\{H_w\}_{v \in Y^*}$ ). On the other way, by a transfer theorem [10], let  $\{\gamma_w\}_{v \in Y^*}$  be the finite parts of  $\{H_w\}_{v \in Y^*}$ , in  $\{n^{-a} \log^b(n)\}_{a,b \in \mathbb{N}}$ , and let  $Z_\gamma$  be their noncommutative generating series. The map  $\gamma_\bullet : (\mathbb{Q}\langle Y \rangle, \llbracket \cdot \rrbracket, 1_{Y^*}) \rightarrow (\mathcal{Z}, \times, 1)$ , mapping  $w$

<sup>5</sup>Under the condition that the multiplier be primitive, result extended as point (v) of the theorem above.

to  $\gamma_w$ , is then a character and  $Z_\gamma$  is group-like, for  $\Delta_{\mathbf{L}}$ . Moreover [15, 16],

$$Z_\gamma = \exp(\gamma y_1) \prod_{l \in \mathcal{L}y_1Y - \{y_1\}} \exp(\zeta(\Sigma_l)\Pi_l) = \exp(\gamma y_1)Z_{\mathbf{L}}. \quad (28)$$

The asymptotic behavior leads to the bridge<sup>6</sup> equation [2, 15, 16]

$$Z_\gamma = B(y_1)\pi_Y(Z_{\mathbf{L}}) \quad \text{or equivalently} \quad Z_{\mathbf{L}} = B'(y_1)\pi_Y(Z_{\mathbf{L}}) \quad (29)$$

where  $B(y_1) = \exp(\gamma y_1 - \sum_{k \geq 2} (-y_1)^k \zeta(k)/k)$  and  $B'(y_1) = \exp(-\gamma y_1)B(y_1)$ .

Similarly, there is  $C_w^- \in \mathbb{Q}$  and  $B_w^- \in \mathbb{N}$ , such that  $H_w^-(N) \underset{N \rightarrow +\infty}{\sim} N^{(w)+|w|} C_w^-$  and  $\text{Li}_w^-(z) \underset{z \rightarrow 1}{\sim} (1-z)^{-(w)-|w|} B_w^-$  [4]. Moreover,

$$C_w^- = \prod_{w=uv, v \neq 1_{Y_0^*}} ((v)+|v|)^{-1} \quad \text{and} \quad B_w^- = ((w)+|w|)! C_w^-. \quad (30)$$

Now, one can then consider the following noncommutative generating series :

$$L^- := \sum_{w \in Y_0^*} \text{Li}_w^- w, \quad H^- := \sum_{w \in Y_0^*} H_w^- w, \quad C^- := \sum_{w \in Y_0^*} C_w^- w. \quad (31)$$

Then  $H^-$  and  $C^-$  are group-like for, respectively,  $\Delta_{\mathbf{L}}$  and  $\Delta_{\mathbf{L}}$  and [4]

$$\lim_{z \rightarrow 1} h^{\odot -1}((1-z)^{-1}) \odot L^-(z) = \lim_{N \rightarrow +\infty} g^{\odot -1}(N) \odot H^-(N) = C^-, \quad (32)$$

$$h(t) = \sum_{w \in Y_0^*} ((w)+|w|)! t^{(w)+|w|} w \quad \text{and} \quad g(t) = \left( \sum_{y \in Y_0} t^{(y)+1} y \right)^*. \quad (33)$$

Next, for any  $w \in Y_0^*$ , there exists then a unique polynomial  $p \in (\mathbb{Z}[t], \times, 1)$  of degree  $(w)+|w|$  such that [4]

$$\text{Li}_w^-(z) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{(1-z)^k} = \sum_{k=0}^{(w)+|w|} p_k e^{-k \log(1-z)} \in (\mathbb{Z}[(1-z)^{-1}], \times, 1), \quad (34)$$

$$H_w^-(n) = \sum_{k=0}^{(w)+|w|} p_k \binom{n+k-1}{k-1} = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} (n)_k \in (\mathbb{Q}[(n)_\bullet], \times, 1), \quad (35)$$

where<sup>7</sup> where  $(n)_\bullet : \mathbb{N} \rightarrow \mathbb{Q}$  mapping  $i$  to  $(n)_i = n(n-1) \dots (n-i+1)$ . In other terms, for any  $w \in Y_0^*$ ,  $k \in \mathbb{N}$ ,  $0 \leq k \leq (w)+|w|$ , one has  $\langle \text{Li}_w^- | (1-z)^{-k} \rangle = k! \langle H_w^- | (n)_k \rangle$ .

<sup>6</sup>This equation is different from Jean Écalle's one [9].

<sup>7</sup>Here, it is also convenient to denote  $\mathbb{Q}[(n)_\bullet]$  the set of "polynomials" expanded as follows

$$\forall p \in \mathbb{Q}[(n)_\bullet], \quad p = \sum_{k=0}^d p_k (n)_k, \quad \deg(p) = d.$$



Hence, denoting  $\tilde{p}$  the exponential transformed of the polynomial  $p$ , one has  $\text{Li}_w^-(z) = p((1-z)^{-1})$  and  $\text{H}_w^-(n) = \tilde{p}((n)_\bullet)$  with

$$p(t) = \sum_{k=0}^{(w)+|w|} p_k t^k \in (\mathbb{Z}[t], \times, 1) \quad \text{and} \quad \tilde{p}(t) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} t^k \in (\mathbb{Q}[t], \times, 1). \quad (36)$$

Let us then associate  $p$  and  $\tilde{p}$  with the polynomial  $\check{p}$  obtained as follows

$$\check{p}(t) = \sum_{k=0}^{(w)+|w|} k! p_k t^k = \sum_{k=0}^{(w)+|w|} p_k t^{\sqcup k} \in (\mathbb{Z}[t], \sqcup, 1). \quad (37)$$

Let us recall also that, for any  $c \in \mathbb{C}$ , one has  $(n)_{c, n \rightarrow +\infty} n^c = e^{c \log(n)}$  and, with the respective scales of comparison, one has the following finite parts

$$\text{f.p.}_{z \rightarrow 1} c \log(1-z) = 0, \quad \{(1-z)^a \log^b((1-z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \quad (38)$$

$$\text{f.p.}_{n \rightarrow +\infty} c \log n = 0, \quad \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}. \quad (39)$$

Hence, using the notations given in (34) and (35), one can see, from (38) and (39), that the values  $p(1)$  and  $\tilde{p}(1)$  obtained in (36) represent

$$\text{f.p.}_{z \rightarrow 1} \text{Li}_w^-(z) = \text{f.p.}_{z \rightarrow 1} \text{Li}_{R_w}(z) = p(1) \in \mathbb{Z}, \quad (40)$$

$$\text{f.p.}_{n \rightarrow +\infty} \text{H}_w^-(n) = \text{f.p.}_{n \rightarrow +\infty} \text{H}_{\pi_Y(R_w)}(n) = \tilde{p}(1) \in \mathbb{Q}. \quad (41)$$

One can use then these values  $p(1)$  and  $\tilde{p}(1)$ , instead of the values  $B_w^-$  and  $C_w^-$ , to regularize, respectively,  $\zeta_{\sqcup}(R_w)$  and  $\zeta_\gamma(\pi_Y(R_w))$  as showed Theorem 2 below because, essentially,  $B_\bullet^-$  and  $C_\bullet^-$  do not realize characters for, respectively,  $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}, \Delta_{\sqcup}, \mathbf{e})$  and  $(\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*}, \Delta_{\sqcup}, \mathbf{e})$  [4].

Now, in virtue of the extension of  $\text{Li}_\bullet$ , defined as in (16) and (17), and of the Taylor coefficients, the previous polynomials  $p, \tilde{p}$  and  $\check{p}$  given in (36)–(37) can be determined explicitly thanks to

**Proposition 1.** 1. *The following morphisms of algebras are bijjective*

$$\begin{aligned} \lambda : (\mathbb{Z}[x_1^*], \sqcup, 1_{X^*}) &\longrightarrow (\mathbb{Z}[(1-z)^{-1}], \times, 1), & R &\longmapsto \text{Li}_R, \\ \eta : (\mathbb{Q}[y_1^*], \sqcup, 1_{Y^*}) &\longrightarrow (\mathbb{Q}[(n)_\bullet], \times, 1), & S &\longmapsto \text{H}_S. \end{aligned}$$

2. *For any  $w = y_{s_1} \dots y_{s_r} \in Y_0^*$ , there exists a unique polynomial  $R_w$  belonging to  $(\mathbb{Z}[x_1^*], \sqcup, 1_{X^*})$  of degree  $(w) + |w|$ , such that*

$$\begin{aligned} \text{Li}_{R_w}(z) = \text{Li}_w^-(z) &= p((1-z)^{-1}) \in (\mathbb{Z}[(1-z)^{-1}], \times, 1), \\ \text{H}_{\pi_Y(R_w)}(n) = \text{H}_w^-(n) &= \tilde{p}((n)_\bullet) \in (\mathbb{Q}[(n)_\bullet], \times, 1). \end{aligned}$$

*In particular, via the extension, by linearity, of  $R_\bullet$  over  $\mathbb{Q}\langle Y_0 \rangle$  and via the linear independent family  $\{\text{Li}_{y_k}^-\}_{k \geq 0}$  in  $\mathbb{Q}\{\text{Li}_w^-\}_{w \in Y_0^*}$ , one has*

$$\forall k, l \in \mathbb{N}, \quad \text{Li}_{R_{y_k \sqcup y_l}} = \text{Li}_{R_{y_k}} \text{Li}_{R_{y_l}} = \text{Li}_{y_k}^- \text{Li}_{y_l}^- = \text{Li}_{y_k \sqcup y_l}^- = \text{Li}_{R_{y_k \sqcup y_l}}.$$

3. For any  $w$ , one has  $\check{p}(x_1^*) = R_w$ .

4. More explicitly, for any  $w = y_{s_1}, \dots, y_{s_r} \in Y_0^*$ , there exists a unique polynomial  $R_w$  belonging to  $(\mathbb{Z}[x_1^*], \sqcup, 1_{X^*})$  of degree  $(w) + |w|$ , given by

$$R_{y_{s_1} \dots y_{s_r}} = \sum_{k_1=0}^{s_1} \sum_{k_2=0}^{s_1+s_2-k_1} \dots \sum_{k_r=0}^{\binom{s_1+\dots+s_r}{k_1+\dots+k_{r-1}}} \binom{s_1}{k_1} \binom{s_1+s_2-k_1}{k_2} \dots \binom{s_1+\dots+s_r-k_1-\dots-k_{r-1}}{k_r} \rho_{k_1} \sqcup \dots \sqcup \rho_{k_r},$$

where, for any  $i = 1, \dots, r$ , if  $k_i = 0$  then  $\rho_{k_i} = x_1^* - 1_{X^*}$  else, for  $k_i > 0$ , denoting the Stirling numbers of second kind by  $S_2(k, j)$ 's, one has

$$\rho_{k_i} = \sum_{j=1}^{k_i} S_2(k_i, j) (j!)^2 \sum_{l=0}^j \frac{(-1)^l}{l!} \frac{(x_1^*)^{\sqcup(j-l+1)}}{(j-l)!}.$$

**Proposition 2** ([2, 15, 16]). *With notations of (14), similar to the character  $\gamma_\bullet$ , the poly-morphism  $\zeta$  can be extended as follows*

$$\zeta_{\sqcup} : (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1), \quad \zeta_{\sqcup} : (\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1)$$

satisfying, for any  $l \in \mathcal{L}yn Y - \{y_1\}$ ,  $\zeta_{\sqcup}(\pi_X(l)) = \zeta_{\sqcup}(l) = \gamma_l = \zeta(l)$  and, for the generators of length (resp. weight) one, for  $X^*$  (resp.  $Y^*$ ),  $\gamma_{y_1} = \gamma$  and  $\zeta_{\sqcup}(x_0) = \zeta_{\sqcup}(x_1) = \zeta_{\sqcup}(y_1) = 0$ .

Now, to regularize  $\{\zeta(s_1, \dots, s_r)\}_{(s_1, \dots, s_r) \in \mathbb{C}^r}$ , we use

**Lemma 1** ([4]). 1. *The power series  $x_0^*$  and  $x_1^*$  are transcendent over  $\mathbb{C}\langle X \rangle$ .*

2. *The family  $\{x_0^*, x_1^*\}$  is algebraically independent over  $(\mathbb{C}\langle X \rangle, \sqcup, 1_{X^*})$  within  $(\mathbb{C}\langle\langle X \rangle\rangle, \sqcup, 1_{X^*})$ .*

3. *The module  $(\mathbb{C}\langle X \rangle, \sqcup, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$  is  $\mathbb{C}\langle X \rangle$ -free and the family  $\{(x_0^*)^{\sqcup k} \sqcup (x_1^*)^{\sqcup l}\}_{(k,l) \in \mathbb{Z} \times \mathbb{N}}$  forms a  $\mathbb{C}\langle X \rangle$ -basis of it.*

*Hence,  $\{w \sqcup (x_0^*)^{\sqcup k} \sqcup (x_1^*)^{\sqcup l}\}_{w \in X^*}^{(k,l) \in \mathbb{Z} \times \mathbb{N}}$  is a  $\mathbb{C}$ -basis of it.*

4. *One has, for any  $x_i \in X$ ,  $\mathbb{C}^{\text{rat}}\langle\langle x_i \rangle\rangle = \text{span}_{\mathbb{C}}\{(tx_i)^* \sqcup \mathbb{C}\langle x_i \rangle \mid t \in \mathbb{C}\}$ .*

Since, for any  $t \in \mathbb{C}$ ,  $|t| < 1$ , one has  $\text{Li}_{(tx_1)^*}(z) = (1-z)^{-t}$  and

$$H_{\pi_Y((tx_1)^*)} = \sum_{k \geq 0} H_{y_1^k} t^k = \exp\left(-\sum_{k \geq 1} H_{y_k} \frac{(-t)^k}{k}\right) \quad (42)$$

then, with the notations of Proposition 2, we extend extend the characters  $\zeta_{\sqcup}$  and  $\gamma_\bullet$ , defined in Proposition 2, over  $\mathbb{C}\langle X \rangle \sqcup \mathbb{C}[x_1^*]$  and  $\mathbb{C}\langle Y \rangle \sqcup \mathbb{C}[y_1^*]$ , respectively, as follows

**Proposition 3** ([4]). *The characters  $\zeta_{\sqcup}$  and  $\gamma_{\bullet}$  can be extended as follows*

$$\begin{aligned} \zeta_{\sqcup} : (\mathbb{C}\langle X \rangle \sqcup \mathbb{C}[x_1^*], \sqcup, 1_{X^*}) &\longrightarrow (\mathbb{C}, \times, 1_{\mathbb{C}}), \\ \forall t \in \mathbb{C}, |t| < 1, \quad (tx_1)^* &\longmapsto 1_{\mathbb{C}}. \\ \gamma_{\bullet} : (\mathbb{C}\langle Y \rangle \boxplus \mathbb{C}[y_1^*], \boxplus, 1_{Y^*}) &\longrightarrow (\mathbb{C}, \times, 1_{\mathbb{C}}), \\ \forall t \in \mathbb{C}, |t| < 1, \quad (ty_1)^* &\longmapsto \exp\left(\gamma t - \sum_{n \geq 2} \zeta(n) \frac{(-t)^n}{n}\right) = \frac{1}{\Gamma(1+t)}. \end{aligned}$$

Therefore, in virtue of Propositions 1 and 3, we obtain then

**Theorem 2.** *1. For any  $(s_1, \dots, s_r) \in \mathbb{N}_+^r$  associated with  $w \in Y^*$ , there exists a unique polynomial  $p \in \mathbb{Z}[t]$  of valuation 1 and of degree  $(w) + |w|$  such that*

$$\begin{aligned} \check{p}(x_1^*) &= R_w && \in (\mathbb{Z}[x_1^*], \sqcup, 1_{X^*}) \\ p((1-z)^{-1}) &= \text{Li}_{R_w}(z) && \in (\mathbb{Z}[(1-z)^{-1}], \times, 1), \\ \tilde{p}((n)_{\bullet}) &= \text{H}_{\pi_Y(R_w)}(n) && \in (\mathbb{Q}[(n)_{\bullet}], \times, 1), \\ \zeta_{\sqcup}(-s_1, \dots, -s_r) = p(1) &= \zeta_{\sqcup}(R_w) && \in (\mathbb{Z}, \times, 1), \\ \gamma_{-s_1, \dots, -s_r} = \tilde{p}(1) &= \gamma_{\pi_Y(R_w)} && \in (\mathbb{Q}, \times, 1). \end{aligned}$$

*2. Let  $\Upsilon(n) \in \mathbb{Q}[(n)_{\bullet}]\langle Y \rangle$  and  $\Lambda(z) \in \mathbb{Q}[(1-z)^{-1}][\log(z)]\langle X \rangle$  be the noncommutative generating series of  $\{\text{H}_{\pi_Y(R_w)}\}_{w \in Y^*}$  and  $\{\text{Li}_{R_{\pi_Y(w)}}\}_{w \in X^*}$  :*

$$\Upsilon := \sum_{w \in Y^*} \text{H}_{\pi_Y(R_w)} w \quad \text{and} \quad \Lambda := \sum_{w \in X^*} \text{Li}_{R_{\pi_Y(w)}} w, \quad \text{with } \langle \Lambda(z) \mid x_0 \rangle = \log(z).$$

*Then  $\Upsilon$  and  $\Lambda$  are group-like, for respectively  $\Delta_{\boxplus}$  and  $\Delta_{\sqcup}$ , and :*

$$\Upsilon = \prod_{l \in \mathcal{L}ynY} \overset{\curvearrowright}{\prod} e^{\text{H}_{\pi_Y(R_{\Sigma_l})} \Pi_l} \quad \text{and} \quad \Lambda = \prod_{l \in \mathcal{L}ynX} \overset{\curvearrowright}{\prod} e^{\text{Li}_{R_{\pi_Y(S_l)}} P_l}.$$

*3. Let  $Z_{\gamma}^- \in \mathbb{Q}\langle Y \rangle$  and  $Z_{\sqcup}^- \in \mathbb{Z}\langle X \rangle$  be the noncommutative generating series of  $\{\gamma_{\pi_Y(R_w)}\}_{w \in Y^*}$  and<sup>8</sup>  $\{\zeta_{\sqcup}(R_{\pi_Y(w)})\}_{w \in X^*}$ , respectively :*

$$Z_{\gamma}^- := \sum_{w \in Y^*} \gamma_{\pi_Y(R_w)} w \quad \text{and} \quad Z_{\sqcup}^- := \sum_{w \in X^*} \zeta_{\sqcup}(R_{\pi_Y(w)}) w.$$

*Then  $Z_{\gamma}^-$  and  $Z_{\sqcup}^-$  are group-like, for respectively  $\Delta_{\boxplus}$  and  $\Delta_{\sqcup}$ , and :*

$$Z_{\gamma}^- = \prod_{l \in \mathcal{L}ynY} \overset{\curvearrowright}{\prod} e^{\gamma_{\pi_Y(R_{\Sigma_l})} \Pi_l} \quad \text{and} \quad Z_{\sqcup}^- = \prod_{l \in \mathcal{L}ynX} \overset{\curvearrowright}{\prod} e^{\zeta_{\sqcup}(\pi_Y(S_l)) P_l}.$$

Moreover,  $\text{F.P.}_{n \rightarrow +\infty} \Upsilon(n) = Z_{\gamma}^-$  and  $\text{F.P.}_{z \rightarrow 1} \Lambda(z) = Z_{\sqcup}^-$  meaning that, for any  $v \in Y^*$  and  $u \in X^*$ , one has

$$\underline{\text{f.p.}_{n \rightarrow +\infty} \langle \Upsilon(n) \mid v \rangle} = \langle Z_{\gamma}^- \mid v \rangle \quad \text{and} \quad \text{f.p.}_{z \rightarrow 1} \langle \Lambda(z) \mid u \rangle = \langle Z_{\sqcup}^- \mid u \rangle. \quad (43)$$

<sup>8</sup>On the one hand, by Proposition 2, one has  $\langle Z_{\sqcup}^- \mid x_0 \rangle = \zeta_{\sqcup}(x_0) = 0$ .

On the other hand, since  $R_{y_1} = (2x_1)^* - x_1^*$  then  $\text{Li}_{R_{y_1}}(z) = (1-z)^{-2} - (1-z)^{-1}$  and  $\text{H}_{\pi_Y(R_{y_1})}(n) = \binom{n}{2} - \binom{n}{1}$ . Hence, one also has  $\langle Z_{\sqcup}^- \mid x_1 \rangle = \zeta_{\sqcup}(R_{\pi_Y(y_1)}) = 0$  and  $\langle Z_{\gamma}^- \mid x_1 \rangle = \gamma_{\pi_Y(R_{y_1})} = -1/2$ .

## References

- [1] V.C. Bui, G.H.E. Duchamp, Hoang Ngoc Minh.– *Structure of Polyzetes and Explicit Representation on Transcendence Bases of Shuffle and Stuffle Algebras*, J. of Sym. Comp. (2016).
- [2] Costermans C., Hoang Ngoc Minh.– *Noncommutative algebra, multiple harmonic sums and applications in discrete probability*, J. of Sym. Comp. (2009), 801-817.
- [3] M. Deneufchâtel, G.H.E. Duchamp, Hoang Ngoc Minh, A.I. Solomon.– *Independence of hyperlogarithms over function fields via algebraic combinatorics*, dans Lecture Notes in Computer Science (2011), Volume 6742/2011, 127-139.
- [4] G.H.E. Duchamp, Hoang Ngoc Minh, Q.H. Ngo– *Harmonic sums and polylogarithms at negative multi-indices*, J. of Sym. Comp. (2016).
- [5] G.H.E. Duchamp, Hoang Ngoc Minh, Q.H. Ngo– *Double regularization of polyzetes at negative multiindices and rational extensions*, en préparation.
- [6] V. Drinfel'd– *Quantum group*, Proc. Int. Cong. Math., Berkeley, 1986.
- [7] V. Drinfel'd– *Quasi-Hopf Algebras*, Len. Math. J., 1, 1419-1457, 1990.
- [8] V. Drinfel'd– *On quasitriangular quasi-hopf algebra and a group closely connected with  $\text{gal}(\bar{q}/q)$* , Len. Math. J., 4, 829-860, 1991.
- [9] Jean Écalle–*L'équation du pont et la classification analytique des objets locaux*, in *Les fonctions résurgentes*, **3**, Publications de l'Université de Paris-Sud, Département de Mathématique (1985).
- [10] Philippe Flajolet & Andrew Odlyzko– *Singularity Analysis of Generating Functions*, SIAM J. Discrete Math., 3(2), pp. 216-240, 1982.
- [11] Hoang Ngoc Minh– *Summations of Polylogarithms via Evaluation Transform*, Math. & Computers in Simulations, 1336, pp 707-728, 1996.
- [12] Hoang Ngoc Minh & Jacob G.– *Symbolic Integration of meromorphic differential equation via Dirichlet functions*, Disc. Math. 210, pp. 87-116, 2000.
- [13] Hoang Ngoc Minh, Jacob G., N.E. Oussous, M. Petitot– *De l'algèbre des  $\zeta$  de Riemann multivariées à l'algèbre des  $\zeta$  de Hurwitz multivariées*, journal électronique du Séminaire Lotharingien de Combinatoire, 44, (2001).
- [14] Hoang Ngoc Minh & M. Petitot– *Lyndon words, polylogarithmic functions and the Riemann  $\zeta$  function*, Disc. Math., 217, 2000, pp. 273-292.
- [15] Hoang Ngoc Minh– *On a conjecture by Pierre Cartier about a group of associators*, Acta Math. Vietnamica (2013), 38, Issue 3, 339-398.

- [16] Hoang Ngoc Minh– *Structure of polyzetas and Lyndon words*, Vietnamese Math. J. (2013), 41, Issue 4, 409-450.
- [17] T.Q.T. Lê & J. Murakami– *Kontsevich’s integral for Kauffman polynomial*, Nagoya Math., pp 39-65, 1996.
- [18] Reutenauer C.– *Free Lie Algebras*, London Math. Soc. Monographs (1993).
- [19] D. Zagier– *Values of zeta functions and their applications*, in “First European Congress of Mathematics”, vol. 2, Birkhäuser, pp. 497-512, 1994.