On Drinfel'd associators

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1 Knizhnik-Zamolodchikov differential equations and coefficients of Drinfel'd associators

In 1986 [6], in order to study the linear representations of the braid group B_n coming from the monodromy of the Knizhnik-Zamolodchikov differential equations, Drinfel'd introduced a class of formal power series Φ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such a power series Φ is called an *associator*. For $n = 3$, it leads to the following fuchsian noncommutative differential equation with three regular singularities in $\{0, 1, +\infty\}$:

$$(DE) \qquad dG(z) = \left(x_0 \frac{dz}{z} + x_1 \frac{dz}{1-z}\right) G(z).$$

Solutions of (DE) are power series, with coefficients which are mono-valued functions on the simply connex domain $\Omega = \mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$ and can be seen as multi-valued over[1] $\mathbb{C} - \{0, 1\}$, on noncommutative variables x_0 and x_1. Drinfel'd proved that (DE) admits two particular mono-valued solutions on Ω, $G_0(z) \underset{z \rightsquigarrow 0}{\sim} \exp[x_0 \log(z)]$ and $G_1(z) \underset{z \rightsquigarrow 1}{\sim} \exp[-x_1 \log(1-z)]$ [7, 8]. and the existence of an associator $\Phi_{KZ} \in \mathbb{R}\langle\langle X \rangle\rangle$ such that $G_0 = G_1 \Phi_{KZ}$ [7, 8]. After that, via representations of the chord diagram algebras, Lê and Murakami [17] expressed the coefficients of Φ_{KZ} as *linear* combinations of special values of several complex variables *zeta* functions, $\{\zeta_r\}_{r \in \mathbb{N}_+}$,

$$\zeta_r : \mathcal{H}_r \to \mathbb{R}, \qquad (s_1, \ldots, s_r) \mapsto \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{s_1} \ldots n_k^{s_r}}, \tag{1}$$

where $\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, .., r, \sum_{i=1}^{m} \Re(s_i) > m\}$. For $(s_1, \ldots, s_r) \in \mathcal{H}_r$, one has two ways of thinking $\zeta_r(s_1, \ldots, s_r)$ as limits, fulfilling identities [14, 13, 1]. Firstly, they are limits of *polylogarithms* and secondly, as truncated sums, they are limits of *harmonic sums*, for $z \in \mathbb{C}, |z| < 1, N \in \mathbb{N}_+$:

$$\mathrm{Li}_{s_1,\ldots,s_k}(z) = \sum_{n_1 > \ldots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_k^{s_k}}, \mathrm{H}_{s_1,\ldots,s_k}(N) = \sum_{n_1 > \ldots > n_k > 0}^{N} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}}. \tag{2}$$

More precisely, if $(s_1, \ldots, s_r) \in \mathcal{H}_r$ then, after a theorem by Abel, one has

$$\lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_k}(z) = \lim_{n \to \infty} \mathrm{H}_{s_1,\ldots,s_k}(n) =: \zeta_r(s_1, \ldots, s_k) \tag{3}$$

[1]In fact, we have mappings from the connected covering $\widetilde{\mathbb{C} - \{0, 1\}}$.

else it does not hold, for $(s_1,\dots,s_r)\notin\mathcal{H}_r$, while $\mathrm{Li}_{s_1,\dots,s_k}$ is well defined over $\{z\in\mathbb{C},|z|<1\}$ and so is $\mathrm{H}_{s_1,\dots,s_k}$, as Taylor coefficients of the following function

$$(1-z)^{-1}\mathrm{Li}_{s_1,\dots,s_k}(z)=\sum_{n\geq 1}\mathrm{H}_{s_1,\dots,s_k}(n)z^n,\quad\text{for}\quad z\in\mathbb{C},|z|<1. \tag{4}$$

Note also that, for $r=1$, ζ_1 is nothing else but the famous Riemann zeta function and, for $r=0$, it is convenient to set ζ_0 to the constant function $1_{\mathbb{R}}$. In all the sequel, for simplification, we will adopt the notation ζ for $\zeta_r, r\in\mathbb{N}$.

In this work, we will describe the regularized solutions of (DE).

For that, we are considering the alphabets $X=\{x_0,x_1\}$ and $Y_0=\{y_s\}_{s\geq 0}$ equipped of the total ordering $x_0<x_1$ and $y_0>y_1>y_2>\dots$, respectively. Let $Y=Y_0-\{y_0\}$. The free monoid generated by X (resp. Y, Y_0) is denoted by X^* (resp. Y^*, Y_0^*) and admits 1_{X^*} (resp. $1_{Y^*}, 1_{Y_0^*}$) as unit.

The sets of, respectively, polynomials and formal power series, with coefficients in a commutative $\mathbb{Q}$-algebra A, over X^* (resp. Y^*, Y_0^*) are denoted by $A\langle X\rangle$ (resp. $A\langle Y\rangle, A\langle Y_0\rangle$) and $A\langle\langle X\rangle\rangle$ (resp. $A\langle\langle Y\rangle\rangle, A\langle\langle Y_0\rangle\rangle$). The sets of polynomials are the A-modules and endowed with the associative concatenation, the associative commutative shuffle (resp. quasi-shuffle) product, over $A\langle X\rangle$ (resp. $A\langle Y\rangle, A\langle Y_0\rangle$). Their associated coproducts are denoted, respectively, $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\text{⧢}}$. The algebras $(A\langle X\rangle,\sqcup\!\sqcup,1_{X^*})$ and $(A\langle Y\rangle,\text{⧢},1_{Y^*})$ admit the sets of Lyndon words denoted, respectively, by $\mathcal{L}ynX$ and $\mathcal{L}ynY$, as transcendence bases [18] (resp. [15, 16]).

For $Z=X$ or Y, denoting $\mathcal{L}ie_A\langle Z\rangle$ and $\mathcal{L}ie_A\langle\langle Z\rangle\rangle$ the sets of, respectively, Lie polynomials and Lie series, the enveloping algebra $\mathcal{U}(\mathcal{L}ie_A\langle Z\rangle)$ is isomorphic to the Hopf algebra $(A\langle X\rangle,.,1_{Z^*},\Delta_{\sqcup\!\sqcup},\mathtt{e})$. We get also $\mathcal{H}_{\text{⧢}}:=(A\langle Y\rangle,.,1_{Y^*},\Delta_{\text{⧢}},\mathtt{e})\cong\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\text{⧢}}))$, where [15, 16]

$$\mathrm{Prim}(\mathcal{H}_{\text{⧢}}) = \mathrm{span}_A\{\pi_1(w)|w\in Y^*\}, \tag{5}$$

$$\pi_1(w) = \sum_{k=1}^{(w)}\frac{(-1)^{k-1}}{k}\sum_{u_1,\dots,u_k\in Y^+}\langle w\mid u_1\text{⧢}\dots\text{⧢}u_k\rangle u_1\dots u_k. \tag{6}$$

2 Indexing polylogarithms and harmonic sums by words and their generating series

For any $r\in\mathbb{N}$, since any combinatorial composition $(s_1,\dots,s_r)\in\mathbb{N}_+^r$ can be associated with words $x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1\in X^*x_1$ and $y_{s_1}\dots y_{s_r}\in Y^*$. Similarly, any multi-indice[2] $(s_1,\dots,s_r)\in\mathbb{N}^r$ can be associated with words $y_{s_1}\dots y_{s_r}\in Y_0^*$. Then let $\mathrm{Li}_{x_0^r}(z):=(\log(z))^r/r!$, and $\mathrm{Li}_{s_1,\dots,s_k}$ and $\mathrm{H}_{s_1,\dots,s_k}$ be indexed by words [14] : $\mathrm{Li}_{x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1}:=\mathrm{Li}_{s_1,\dots,s_r}$ and $\mathrm{H}_{y_{s_1}\dots y_{s_r}}:=\mathrm{H}_{s_1,\dots,s_r}$. Similarly, $\mathrm{Li}_{-s_1,\dots,-s_k}$ and $\mathrm{H}_{-s_1,\dots,-s_k}$ be indexed by words[3] [4, 5] : $\mathrm{Li}^-_{y_{s_1}\dots y_{s_r}}:=$

[2]The weight of $(s_1,\dots,s_r)\in\mathbb{N}_+^r$ (resp. $\mathbb{N}^r$) is defined as the integer $s_1+\dots+s_r$ which corresponds to the weight, denoted (w), of its associated word $w\in Y^*$ (resp. Y_0^*) and corresponds also to the length, denoted by $|u|$, of its associated word $u\in X^*$.

[3]Note that, all these $\{\mathrm{Li}^-_w\}_{w\in Y_0^*}$ and $\{\mathrm{H}^-_w\}_{w\in Y_0^*}$ are divergent at their singularities.

$\mathrm{Li}_{-s_1,\ldots,-s_r}$ and $\mathrm{H}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{-s_1,\ldots,-s_r}$. In particular, $\mathrm{H}^-_{y_0^r}(n) := \binom{n}{r} = (n)_r/r!$ and $\mathrm{Li}^-_{y_0^r}(z) := (z/(1-z))^r$. There exists a law of algebra, denoted by $\top$, in $\mathbb{Q}\langle\langle Y_0\rangle\rangle$, such that he following morphisms of algebras are surjective [4]

$$\mathrm{H}^-_\bullet : (\mathbb{Q}\langle Y_0\rangle, \text{⧢}, 1_{Y_0^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}^-_w\}_{w\in Y_0^*}, \times, 1), \qquad w \longmapsto \mathrm{H}^-_w, \tag{7}$$

$$\mathrm{Li}^-_\bullet : (\mathbb{Q}\langle Y_0\rangle, \top, 1_{Y_0^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}^-_w\}_{w\in Y_0^*}, \times, 1), \qquad w \longmapsto \mathrm{Li}^-_w, \tag{8}$$

and $\ker \mathrm{H}^-_\bullet = \ker \mathrm{Li}^-_\bullet = \mathbb{Q}\langle\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}\rangle$ [4]. Moreover, the families $\{\mathrm{H}^-_{y_k}\}_{k\geq 0}$ and $\{\mathrm{Li}^-_{y_k}\}_{k\geq 0}$ are $\mathbb{Q}$-linearly independent.

On the other hand, the following morphisms of algebras are injective

$$\mathrm{H}_\bullet : (\mathbb{Q}\langle Y\rangle, \text{⧢}, 1_{Y^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, \times, 1), \qquad w \longmapsto \mathrm{H}_w, \tag{9}$$

$$\mathrm{Li}_\bullet : (\mathbb{Q}\langle X\rangle, ⧢, 1_{X^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1), \qquad w \longmapsto \mathrm{Li}_w \tag{10}$$

Moreover, the families $\{\mathrm{H}_w\}_{w\in Y^*}$ and $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathbb{Q}$-linearly independent and the families $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ and $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$are $\mathbb{Q}$-algebraically independent. But at singularities of $\{\mathrm{Li}_w\}_{w\in X^*}, \{\mathrm{H}_w\}_{w\in Y^*}$, the following convergent values

$$\forall u \in Y^* - y_1Y^*, \zeta(u) := \mathrm{H}_u(+\infty) \quad \text{and} \quad \forall v \in x_0X^*x_1, \zeta(v) := \mathrm{Li}_v(1) \tag{11}$$

are no longer linearly independent and the values $\{\mathrm{H}_l(+\infty)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (resp. $\{\mathrm{Li}_l(1)\}_{l\in\mathcal{L}ynX-X}$) are no longer algebraically independent [14, 19].

The graphs of the isomorphisms of algebras, $\mathrm{Li}_\bullet$ and $\mathrm{H}_\bullet$, as generating series, read then [2, 14]

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w\, w = \prod^{\searrow}_{l\in\mathcal{L}ynX} e^{\mathrm{Li}_{S_l} P_l}, \qquad \mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w = \prod^{\searrow}_{l\in\mathcal{L}ynY} e^{\mathrm{H}_{\Sigma_l}\Pi_l}, \tag{12}$$

where the PBW basis $\{P_w\}_{w\in X^*}$ (resp. $\{\Pi_w\}_{w\in Y^*}$) is expanded over the basis of $\mathcal{L}ie_A\langle X\rangle$ (resp. $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\text{⧢}}))$, $\{P_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$), and $\{S_w\}_{w\in X^*}$ (resp. $\{\Sigma_w\}_{w\in Y^*}$) is the basis of $(\mathbb{Q}\langle Y\rangle, ⧢, 1_{X^*})$ (resp. $(\mathbb{Q}\langle Y\rangle, \text{⧢}, 1_{Y^*})$) containing the transcendence basis $\{S_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$).

By termwise differentiation, L satisfies the noncommutative differential equation (DE) with the boundary condition $\mathrm{L}(z) \underset{z\to 0^+}{\sim} e^{x_0\log(z)}$. It is immediate that the power series H and L are group-like, for $\Delta_{\text{⧢}}$ and $\Delta_{⧢}$, respectively. Hence, the following noncommutative generating series are well defined and are group-like, for $\Delta_{\text{⧢}}$ and $\Delta_{⧢}$, respectively [14, 15, 16] :

$$Z_{\text{⧢}} := \prod^{\searrow}_{l\in\mathcal{L}ynY-\{y_1\}} e^{\mathrm{H}_{\Sigma_l}(+\infty)\Pi_l} \quad \text{and} \quad Z_{⧢} := \prod^{\searrow}_{l\in\mathcal{L}ynX-X} e^{\mathrm{Li}_{S_l}(1)P_l}. \tag{13}$$

Definitions (3) and (11) lead then to the following surjective poly-morphism

$$\zeta : \begin{array}{c}(\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1, ⧢, 1_{X^*}) \\ (\mathbb{Q}1_{Y^*} \oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle, \text{⧢}, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, \times, 1), \tag{14}$$

$$\begin{array}{c} x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1} \\ y_{s_1}\ldots y_{s_k}\end{array} \longmapsto \sum_{n_1>\ldots>n_k>0} n_1^{-s_1}\ldots n_k^{-s_k}, \tag{15}$$

where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra generated by $\{\zeta(l)\}_{l\in\mathcal{L}ynX-X}$ (resp. $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$), or equivalently, generated by $\{\zeta(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (resp. $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$).

Now, let $t_i\in\mathbb{C},|\,t_i\,|<1,i\in\mathbb{N}$. For $z\in\mathbb{C},|\,z\,|<1$, we have [11]

$$\sum_{n\geq 0}\mathrm{Li}_{x_0^n}(z)\;t_0^n=z^{t_0}\quad\text{and}\quad\sum_{n\geq 0}\mathrm{Li}_{x_1^n}(z)\;t_1^n=(1-z)^{-t_1}. \tag{16}$$

These suggest to extend the morphism $\mathrm{Li}_\bullet$ over $(\mathrm{Dom}(\mathrm{Li}_\bullet),\sqcup,1_{X^*})$, via *Lazard's elimination*, as follows (subjected to be convergent)

$$\mathrm{Li}_S(z)=\sum_{n\geq 0}\langle S\mid x_0^n\rangle\frac{\log^n(z)}{n!}+\sum_{k\geq 1}\sum_{w\in(x_0^*x_1)^kx_0^*}\langle S\mid w\rangle\,\mathrm{Li}_w(z) \tag{17}$$

with $\mathbb{C}\langle X\rangle\sqcup\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\sqcup\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle\subset\mathrm{Dom}(\mathrm{Li}_\bullet)\subset\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ denotes the closure, of $\mathbb{C}\langle X\rangle$ in $\mathbb{C}\langle\langle X\rangle\rangle$, by $\{+,.,^*\}$. For example [11, 12],

1. For any $i,j\in\mathbb{N}_+$ and $x\in X$, since $(t_0x_0+t_1x_1)^*=(t_0x_0)^*\sqcup(t_1x_1)^*$ and $(x^*)^{\sqcup i}=(ix)^*$ then $\mathrm{Li}_{(x_0^*)^{\sqcup i}\sqcup(x_1^*)^{\sqcup j}}(z)=z^i(1-z)^{-j}$.

2. For $a\in\mathbb{C},x\in X,i\in\mathbb{N}_+$, since $(ax)^{*i}=(ax)^*\sqcup(1+ax)^{i-1}$ then

$$\mathrm{Li}_{(ax_0)^{*i}}(z) = z^a\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{(a\log(z))^k}{k!}, \tag{18}$$

$$\mathrm{Li}_{(ax_1)^{*i}}(z) = \frac{1}{(1-z)^a}\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{(a\log((1-z)^{-1})^k}{k!}. \tag{19}$$

3. Let $V=(t_1x_0)^{*s_1}x_0^{s_1-1}x_1\ldots(t_rx_0)^{*s_r}x_0^{s_r-1}x_1$, for $(s_1,\ldots,s_r)\in\mathbb{N}_+^r$. Then

$$\mathrm{Li}_V(z)=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}. \tag{20}$$

   In particular, for $s_1=\ldots=s_r=1$, then one has

$$\begin{aligned}\mathrm{Li}_V(z) &= \sum_{n_1,\ldots,n_r>0}\mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}(z)\;t_0^{n_1-1}\ldots t_r^{n_r-1}\\ &= \sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{(n_1-t_1)\ldots(n_r-t_r)}.\end{aligned} \tag{21}$$

4. From the previous points, one has

$$\begin{aligned}\{\mathrm{Li}_S\}_{S\in\mathbb{C}\langle X\rangle\sqcup\mathbb{C}[x_0^*]\sqcup\mathbb{C}[(-x_0^*)]\sqcup\mathbb{C}[x_1^*]} &= \mathrm{span}_{\mathbb{C}}\left\{\frac{z^a}{(1-z)^b}\,\mathrm{Li}_w(z)\right\}_{w\in X^*}^{a\in\mathbb{Z},b\in\mathbb{N}}\\ &\subset \mathrm{span}_{\mathbb{C}}\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{Z}^r}\\ &\quad\oplus\mathrm{span}_{\mathbb{C}}\{z^a|a\in\mathbb{Z}\},\end{aligned} \tag{22}$$

$$\begin{aligned}\{\mathrm{Li}_S\}_{S\in\mathbb{C}\langle X\rangle\sqcup\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\sqcup\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle} &= \mathrm{span}_{\mathbb{C}}\left\{\frac{z^a}{(1-z)^b}\,\mathrm{Li}_w(z)\right\}_{w\in X^*}^{a,b\in\mathbb{C}}\\ &\subset \mathrm{span}_{\mathbb{C}}\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{C}^r}\\ &\quad\oplus\mathrm{span}_{\mathbb{C}}\{z^a|a\in\mathbb{C}\}.\end{aligned} \tag{23}$$

3 Noncommutative evolution equations

As we said previously Drinfel'd proved that (DE) admits two particular solutions on Ω. These new tools and results can be considered as pertaining to the domain of *noncommutative evolution equations*. We will, here, only mention what is relevant for our needs.

Even for one sided [4] differential equations, in order to cope with limit initial conditions (see applications below), one needs the two sided version.

Let then $\Omega \subset \mathbb{C}$ be simply connected and open and $\mathcal{H}(\Omega)$ denote the algebra of holomorphic functions on Ω. We suppose given two series (called *multipliers*) $M_1, M_2 \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$ (X is an alphabet and the subscript indicates that the series have no constant term). Let then

$$(DE_2) \qquad \mathbf{d}S = M_1 S + S M_2.$$

be our equation.

3.1 The main theorem

Theorem 1. *Let*

$$(DE_2) \qquad \mathbf{d}S = M_1 S + S M_2. \tag{24}$$

(i) Solutions of (DE_2) form a $\mathbb{C}$-vector space.

(ii) Solutions of (DE_2) have their constant term (as coefficient of 1_{X^*}) which are constant functions (on Ω); there exists solutions with constant coefficient 1_Ω (hence invertible).

(iii) If two solutions coincide at one point $z_0 \in \overline{\Omega}$, they coincide everywhere.

(iv) Let be the following one-sided equations

$$(DE^{(1)}) \quad \mathbf{d}S = M_1 S \qquad (DE^{(2)}) \quad \mathbf{d}S = S M_2. \tag{25}$$

and let S_i, $i = 1, 2$ a solution of $(DE^{(i)})$, then $S_1 S_2$ is a solution of (DE_2). Conversely, every solution of (DE_2) can be constructed so.

(v) If M_i, $i = 1, 2$ are primitive and if S, a solution of (DE_2), is group-like at one point, (or, even at one limit point) it is globally group-like.

Proof. Omitted. □

Remark 1.

- *Every holomorphic series $S(z) \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ which is group-like ($\Delta(S) = S \otimes S$ and $\langle S \mid 1_{X^*}\rangle$) is a solution of a left-sided dynamics with primitive multiplier (take $M_1 = \mathbf{d}(S)S^{-1}$ and $M_2 = 0$).*
- *Invertible solutions of an equation of type $S' = M_1 S$ are on the same orbit by multiplication on the right by invertible constant series i.e. let S_i, $i = 1, 2$ be invertible solutions of $(DE^{(1)})$, then there exists an unique invertible $T \in \mathbb{C}\langle\langle X\rangle\rangle$ such that $S_2 = S_1.T$. From this and point (iv) of the theorem, one can parametrize the set of invertible solutions of (DE_2).*

[4] As the left (DE) for instance.

3.2 Application: Unicity of solutions with asymptotic conditions.

In a previous work [3], we proved that asymptotic group-likeness, for a series, implies[5] that the series in question is group-like everywhere. The process above (theorem (1), Picard's process) can be performed, under certain conditions with improper integrals we then construct the series L recursively as

$$\langle \mathrm{L} \mid w\rangle = \begin{cases} \dfrac{\log^n(z)}{n!} & \text{if} \quad w = x_0^n \\ \int_0^z \Big((\frac{x_1}{1-z})\langle \mathrm{L} \mid u\rangle\Big)[s]\,ds & \text{if} \quad w = x_1 u \\ \int_0^z \Big((\frac{x_0}{z})\langle \mathrm{L} \mid u x_1 x_0^n\rangle\Big)[s]\,ds & \text{if} \quad w = x_0 u x_1 x_0^n. \end{cases}$$

one can check that

- this process is well defined at each step and computes the series L as below.
- L is solution of (DE), is exactly G_0 and is group-like

We here only prove that G_0 is unique using the theorem above. Consider the series $T = Le^{-x_0\log(z)}$. Then T is solution of an equation of the type (DE_2)

$$T' = (\frac{x_0}{z} + \frac{x_1}{1-z})T + T(\frac{x_0}{z}) \tag{26}$$

but $\lim_{z\to z_0} G_0 e^{-x_0\log(z)} = 1$ so, by the point (iii) of theorem (1) one has $G_0 e^{-x_0\log(z)} = Le^{-x_0\log(z)}$ and then $G_0 = L$.

A similar (and symmetric) argument can be performed for G_1 and then, in this interpretation and context, Φ_{KZ} is unique.

4 Double global regularization of associators

Global singularities analysis leads to to the following global renormalization [2]

$$\begin{aligned} \lim_{z\to 1}\exp\left(-y_1\log\frac{1}{1-z}\right)\pi_Y(\mathrm{L}(z)) &= \lim_{n\to\infty}\exp\left(\sum_{k\geq 1}\mathrm{H}_{y_k}(n)\frac{(-y_1)^k}{k}\right)\mathrm{H}(n) \\ &= \pi_Y(Z_{\sqcup\!\sqcup}). \end{aligned} \tag{27}$$

Thus, the coefficients $\{\langle Z_{\sqcup\!\sqcup}|u\rangle\}_{u\in X^*}$ (*i.e.* $\{\zeta_{\sqcup\!\sqcup}(u)\}_{u\in X^*}$) and $\{\langle Z_{\text{ш}}|v\rangle\}_{v\in Y^*}$ (*i.e.* $\{\zeta_{\text{ш}}(v)\}_{v\in Y^*}$) represent the finite part of the asymptotic expansions, in $\{(1-z)^{-a}\log^b(1-z)\}_{a,b\in\mathbb{N}}$ (resp. $\{n^{-a}\mathrm{H}_1^b(n)\}_{a,b\in\mathbb{N}}$) of $\{\mathrm{Li}_w\}_{u\in X^*}$ (resp. $\{\mathrm{H}_w\}_{v\in Y^*}$). On the other way, by a transfer theorem [10], let $\{\gamma_w\}_{v\in Y^*}$ be the finite parts of $\{\mathrm{H}_w\}_{v\in Y^*}$, in $\{n^{-a}\log^b(n)\}_{a,b\in\mathbb{N}}$, and let Z_γ be their noncommutative generating series. The map $\gamma_\bullet : (\mathbb{Q}\langle Y\rangle, \text{ш}, 1_{Y^*}) \to (\mathcal{Z}, \times, 1)$, mapping w

[5] Under the condition that the multiplier be primitive, result extended as point (v) of the theorem above.

to γ_w, is then a character and Z_γ is group-like, for $\Delta_{\text{ш}}$. Moreover [15, 16],

$$Z_\gamma = \exp(\gamma y_1) \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} \exp(\zeta(\Sigma_l)\Pi_l) = \exp(\gamma y_1) Z_{\text{ш}}. \tag{28}$$

The asymptotic behavior leads to the bridge[6] equation [2, 15, 16]

$$Z_\gamma = B(y_1)\pi_Y(Z_{\sqcup\!\sqcup}) \quad \text{or equivalently} \quad Z_{\sqcup\!\sqcup} = B'(y_1)\pi_Y(Z_{\sqcup\!\sqcup}) \tag{29}$$

where $B(y_1) = \exp(\gamma y_1 - \sum_{k\geq 2}(-y_1)^k\zeta(k)/k)$ and $B'(y_1) = \exp(-\gamma y_1)B(y_1)$.

Similarly, there is $C_w^- \in \mathbb{Q}$ and $B_w^- \in \mathbb{N}$, such that $\mathrm{H}_w^-(N) \widetilde{_{N\to+\infty}} N^{(w)+|w|}C_w^-$ and $\mathrm{Li}_w^-(z) \widetilde{_{z\to 1}} (1-z)^{-(w)-|w|}B_w^-$ [4]. Moreover,

$$C_w^- = \prod_{w=uv,v\neq 1_{Y_0^*}} ((v)+|v|)^{-1} \quad \text{and} \quad B_w^- = ((w)+|w|)!C_w^-. \tag{30}$$

Now, one can then consider the following noncommutative generating series :

$$\mathrm{L}^- := \sum_{w\in Y_0^*} \mathrm{Li}_w^- w, \quad \mathrm{H}^- := \sum_{w\in Y_0^*} \mathrm{H}_w^- w, \quad C^- := \sum_{w\in Y_0^*} C_w^- w. \tag{31}$$

Then H^- and C^- are group-like for, respectively, $\Delta_{\text{ш}}$ and $\Delta_{\sqcup\!\sqcup}$ and [4]

$$\lim_{z\to 1} h^{\odot -1}((1-z)^{-1}) \odot \mathrm{L}^-(z) = \lim_{N\to+\infty} g^{\odot -1}(N) \odot \mathrm{H}^-(N) = C^-, \tag{32}$$

$$h(t) = \sum_{w\in Y_0^*} ((w)+|w|)!t^{(w)+|w|}w \quad \text{and} \quad g(t) = \left(\sum_{y\in Y_0} t^{(y)+1}y\right)^*. \tag{33}$$

Next, for any $w \in Y_0^*$, there exists then a unique polynomial $p \in (\mathbb{Z}[t], \times, 1)$ of degree $(w)+|w|$ such that [4]

$$\mathrm{Li}_w^-(z) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{(1-z)^k} = \sum_{k=0}^{(w)+|w|} p_k e^{-k\log(1-z)} \in (\mathbb{Z}[(1-z)^{-1}], \times, 1), \tag{34}$$

$$\mathrm{H}_w^-(n) = \sum_{k=0}^{(w)+|w|} p_k \binom{n+k-1}{k-1} = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!}(n)_k \in (\mathbb{Q}[(n)_\bullet], \times, 1), \tag{35}$$

where[7] where $(n)_\bullet : \mathbb{N} \longrightarrow \mathbb{Q}$ mapping i to $(n)_i = n(n-1)\dots(n-i+1)$. In other terms, for any $w \in Y_0^*, k \in \mathbb{N}, 0 \leq k \leq (w)+|w|$, one has $\langle \mathrm{Li}_w^- \mid (1-z)^{-k}\rangle = k!\langle \mathrm{H}_w^- \mid (n)_k\rangle$.

[6] This equation is different from Jean Écalle's one [9].

[7] Here, it is also convenient to denote $\mathbb{Q}[(n)_\bullet]$ the set of "polynomials" expanded as follows

$$\forall p \in, \mathbb{Q}[(n)_\bullet], \quad p = \sum_{k=0}^{d} p_k(n)_k, \quad \deg(p) = d.$$

Hence, denoting $\tilde{p}$ the exponential transformed of the polynomial p, one has $\mathrm{Li}^-_w(z) = p((1-z)^{-1})$ and $\mathrm{H}^-_w(n) = \tilde{p}((n)_\bullet)$ with

$$p(t) = \sum_{k=0}^{(w)+|w|} p_k t^k \in (\mathbb{Z}[t], \times, 1) \quad \text{and} \quad \tilde{p}(t) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} t^k \in (\mathbb{Q}[t], \times, 1). \tag{36}$$

Let us then associate p and $\tilde{p}$ with the polynomial $\check{p}$ obtained as follows

$$\check{p}(t) = \sum_{k=0}^{(w)+|w|} k! p_k t^k = \sum_{k=0}^{(w)+|w|} p_k t^{\sqcup\!\sqcup\, k} \in (\mathbb{Z}[t], \sqcup\!\sqcup, 1). \tag{37}$$

Let us recall also that, for any $c \in \mathbb{C}$, one has $(n)_{c\widetilde{\ }_{n\to+\infty}} n^c = e^{c\log(n)}$ and, with the respective scales of comparison, one has the following finite parts

$$\mathrm{f.p.}_{z\to 1} c\log(1-z) = 0, \qquad \{(1-z)^a \log^b((1-z)^{-1})\}_{a\in\mathbb{Z}, b\in\mathbb{N}}, \tag{38}$$

$$\mathrm{f.p.}_{n\to+\infty} c\log n = 0, \qquad \{n^a \log^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}. \tag{39}$$

Hence, using the notations given in (34) and (35), one can see, from (38) and (39), that the values $p(1)$ and $\tilde{p}(1)$ obtained in (36) represent

$$\mathrm{f.p.}_{z\to 1} \mathrm{Li}^-_w(z) = \mathrm{f.p.}_{z\to 1} \mathrm{Li}_{R_w}(z) \quad = \quad p(1) \in \mathbb{Z}, \tag{40}$$

$$\mathrm{f.p.}_{n\to+\infty} \mathrm{H}^-_w(n) = \mathrm{f.p.}_{n\to+\infty} \mathrm{H}_{\pi_Y(R_w)}(n) \quad = \quad \tilde{p}(1) \in \mathbb{Q}. \tag{41}$$

One can use then these values $p(1)$ and $\tilde{p}(1)$, instead of the values B^-_w and C^-_w, to regularize, respectively, $\zeta_{\sqcup\!\sqcup}(R_w)$ and $\zeta_\gamma(\pi_Y(R_w))$ as showed Theorem 2 bellow because, essentially, $B^-_\bullet$ and $C^-_\bullet$ do not realize characters for, respectively, $(\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \mathtt{e})$ and $(\mathbb{Q}\langle Y\rangle, \text{⧢}, 1_{Y^*}, \Delta_{\text{⧢}}, \mathtt{e})$ [4].

Now, in virtue of the extension of $\mathrm{Li}_\bullet$, defined as in (16) and (17), and of the Taylor coefficients, the previous polynomials $p, \tilde{p}$ and $\check{p}$ given in (36)–(37) can be determined explicitly thanks to

Proposition 1. *1. The following morphisms of algebras are bijective*

$$\begin{array}{rcccc} \lambda : (\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*}) & \longrightarrow & (\mathbb{Z}[(1-z)^{-1}], \times, 1), & R & \longmapsto & \mathrm{Li}_R, \\ \eta : (\mathbb{Q}[y_1^*], \text{⧢}, 1_{Y^*}) & \longrightarrow & (\mathbb{Q}[(n)_\bullet], \times, 1), & S & \longmapsto & \mathrm{H}_S. \end{array}$$

2. For any $w = y_{s_1}, \ldots y_{s_r} \in Y_0^*$, there exists a unique polynomial R_w belonging to $(\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*})$ of degree $(w)+|w|$, such that

$$\begin{array}{rcll} \mathrm{Li}_{R_w}(z) = \mathrm{Li}^-_w(z) = & p((1-z)^{-1}) & \in (\mathbb{Z}[(1-z)^{-1}], \times, 1), \\ \mathrm{H}_{\pi_Y(R_w)}(n) = \mathrm{H}^-_w(n) = & \tilde{p}((n)_\bullet) & \in (\mathbb{Q}[(n)_\bullet], \times, 1). \end{array}$$

In particular, via the extension, by linearity, of $R_\bullet$ over $\mathbb{Q}\langle Y_0\rangle$ and via the linear independent family $\{\mathrm{Li}^-_{y_k}\}_{k\geq 0}$ in $\mathbb{Q}\{\mathrm{Li}^-_w\}_{w\in Y_0^*}$, one has

$$\forall k, l \in \mathbb{N}, \qquad \mathrm{Li}_{R_{y_k} \sqcup\!\sqcup R_{y_l}} = \mathrm{Li}_{R_{y_k}} \mathrm{Li}_{R_{y_l}} = \mathrm{Li}^-_{y_k} \mathrm{Li}^-_{y_l} = \mathrm{Li}^-_{y_k \top y_l} = \mathrm{Li}_{R_{y_k \top y_l}}.$$

3. *For any w, one has $\check{p}(x_1^*) = R_w$.*

4. *More explicitly, for any $w = y_{s_1}, \ldots y_{s_r} \in Y_0^*$, there exists a unique polynomial R_w belonging to $(\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*})$ of degree $(w)+ |w|$, given by*

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1} \sum_{k_2=0}^{s_1+s_2-k_1} \cdots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots \binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}\rho_{k_1}\sqcup\!\sqcup\ldots\sqcup\!\sqcup\rho_{k_r},$$

where, for any $i = 1, \ldots, r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else, for $k_i > 0$, denoting the Stirling numbers of second kind by $S_2(k, j)$'s, one has

$$\rho_{k_i} = \sum_{j=1}^{k_i} S_2(k_i, j)(j!)^2 \sum_{l=0}^{j} \frac{(-1)^l}{l!}\frac{(x_1^*)^{\sqcup\!\sqcup(j-l+1)}}{(j-l)!}.$$

Proposition 2 ([2, 15, 16])**.** *With notations of* (14)*, similar to the character $\gamma_\bullet$, the poly-morphism ζ can be extended as follows*

$$\zeta_{\sqcup\!\sqcup} : (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1), \qquad \zeta_{\text{ш}} : (\mathbb{Q}\langle Y\rangle, \text{ш}, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1)$$

satisfying, for any $l \in \mathcal{L}ynY - \{y_1\}$, $\zeta_{\sqcup\!\sqcup}(\pi_X(l)) = \zeta_{\text{ш}}(l) = \gamma_l = \zeta(l)$ and, for the generators of length (resp. weight) one, for X^* (resp. Y^*), $\gamma_{y_1} = \gamma$ and $\zeta_{\sqcup\!\sqcup}(x_0) = \zeta_{\sqcup\!\sqcup}(x_1) = \zeta_{\text{ш}}(y_1) = 0$.

Now, to regularize $\{\zeta(s_1, \ldots, s_r)\}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}$, we use

Lemma 1 ([4])**.** 1. *The power series x_0^* and x_1^* are transcendent over $\mathbb{C}\langle X\rangle$.*

2. *The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$.*

3. *The module $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is $\mathbb{C}\langle X\rangle$-free and the family $\{(x_0^*)^{\sqcup\!\sqcup k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ forms a $\mathbb{C}\langle X\rangle$-basis of it.*

   Hence, $\{w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup l}\}_{w\in X^*}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}$-basis of it.

4. *One has, for any $x_i \in X$, $\mathbb{C}^{\mathrm{rat}}\langle\langle x_i\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(tx_i)^* \sqcup\!\sqcup \mathbb{C}\langle x_i\rangle | t \in \mathbb{C}\}$.*

Since, for any $t \in \mathbb{C}, |t| < 1$, one has $\mathrm{Li}_{(tx_1)^*}(z) = (1-z)^{-t}$ and

$$\mathrm{H}_{\pi_Y(tx_1)^*} = \sum_{k\geq 0} \mathrm{H}_{y_1^k} t^k = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_k}\frac{(-t)^k}{k}\right) \tag{42}$$

then, with the notations of Proposition 2, we extend extend the characters $\zeta_{\sqcup\!\sqcup}$ and $\gamma_\bullet$, defined in Proposition 2, over $\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}[x_1^*]$ and $\mathbb{C}\langle Y\rangle$ ш $\mathbb{C}[y_1^*]$, respectively, as follows

Proposition 3 ([4]). *The characters $\zeta_{\sqcup\!\sqcup}$ and $\gamma_\bullet$ can be extended as follows*

$$\begin{array}{rcl}
\zeta_{\sqcup\!\sqcup} : (\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}[x_1^*], \sqcup\!\sqcup, 1_{X^*}) & \longrightarrow & (\mathbb{C}, \times, 1_{\mathbb{C}}),\\
\forall t \in \mathbb{C}, |t|<1, \quad (tx_1)^* & \longmapsto & 1_{\mathbb{C}}.\\
\gamma_\bullet : (\mathbb{C}\langle Y\rangle \text{ш} \mathbb{C}[y_1^*], \text{ш}, 1_{Y^*}) & \longrightarrow & (\mathbb{C}, \times, 1_{\mathbb{C}}),\\
\forall t \in \mathbb{C}, |t|<1, \quad (ty_1)^* & \longmapsto & \exp\Big(\gamma t - \sum_{n\geq 2} \zeta(n)\frac{(-t)^n}{n}\Big) = \frac{1}{\Gamma(1+t)}.
\end{array}$$

Therefore, in virtue of Propositions 1 and 3, we obtain then

Theorem 2. *1. For any $(s_1,\ldots,s_r) \in \mathbb{N}_+^r$ associated with $w \in Y^*$, there exists a unique polynomial $p \in \mathbb{Z}[t]$ of valuation 1 and of degree $(w)+|w|$ such that*

$$\begin{array}{rcll}
\check{p}(x_1^*) & = & R_w & \in (\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*})\\
p((1-z)^{-1}) & = & \mathrm{Li}_{R_w}(z) & \in (\mathbb{Z}[(1-z)^{-1}], \times, 1),\\
\tilde{p}((n)_\bullet) & = & \mathrm{H}_{\pi_Y(R_w)}(n) & \in (\mathbb{Q}[(n)_\bullet], \times, 1),\\
\zeta_{\sqcup\!\sqcup}(-s_1,\ldots,-s_r) = p(1) & = & \zeta_{\sqcup\!\sqcup}(R_w) & \in (\mathbb{Z}, \times, 1),\\
\gamma_{-s_1,\ldots,-s_r} = \tilde{p}(1) & = & \gamma_{\pi_Y(R_w)} & \in (\mathbb{Q}, \times, 1).
\end{array}$$

2. Let $\Upsilon(n) \in \mathbb{Q}[(n)_\bullet]\langle\langle Y\rangle\rangle$ and $\Lambda(z) \in \mathbb{Q}[(1-z)^{-1}][\log(z)]\langle\langle X\rangle\rangle$ be the noncommutative generating series of $\{\mathrm{H}_{\pi_Y(R_w)}\}_{w\in Y^*}$ and $\{\mathrm{Li}_{R_{\pi_Y(w)}}\}_{w\in X^*}$:

$$\Upsilon := \sum_{w\in Y^*} \mathrm{H}_{\pi_Y(R_w)} w \text{ and } \Lambda := \sum_{w\in X^*} \mathrm{Li}_{R_{\pi_Y(w)}}\, w, \text{ with } \langle \Lambda(z) \mid x_0\rangle = \log(z).$$

Then Υ and Λ are group-like, for respectively $\Delta_{\text{ш}}$ and $\Delta_{\sqcup\!\sqcup}$, and :

$$\Upsilon = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\pi_Y(R_{\Sigma_l})}\Pi_l} \quad \text{and} \quad \Lambda = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\pi_Y(S_l)}}P_l}.$$

3. Let $Z_\gamma^- \in \mathbb{Q}\langle\langle Y\rangle\rangle$ and $Z_{\sqcup\!\sqcup}^- \in \mathbb{Z}\langle\langle X\rangle\rangle$ be the noncommutative generating series of $\{\gamma_{\pi_Y(R_w)}\}_{w\in Y^*}$ and[8] $\{\zeta_{\sqcup\!\sqcup}(R_{\pi_Y(w)})\}_{w\in X^*}$, respectively :

$$Z_\gamma^- := \sum_{w\in Y^*} \gamma_{\pi_Y(R_w)} w \quad \text{and} \quad Z_{\sqcup\!\sqcup}^- := \sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(R_{\pi_Y(w)}) w.$$

Then Z_γ^- and $Z_{\sqcup\!\sqcup}^-$ are group-like, for respectively $\Delta_{\text{ш}}$ and $\Delta_{\sqcup\!\sqcup}$, and :

$$Z_\gamma^- = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\pi_Y(R_{\Sigma_l})}\Pi_l} \quad \text{and} \quad Z_{\sqcup\!\sqcup}^- = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\zeta_{\sqcup\!\sqcup}(\pi_Y(S_l))P_l}.$$

Moreover, $\mathrm{F.P.}_{n\to+\infty}\Upsilon(n) = Z_\gamma^-$ and $\mathrm{F.P.}_{z\to 1}\Lambda(z) = Z_{\sqcup\!\sqcup}^-$ meaning that, for any $v \in Y^*$ and $u \in X^*$, one has

$$\mathrm{f.p.}_{n\to+\infty}\langle\Upsilon(n) \mid v\rangle = \langle Z_\gamma^- \mid v\rangle \quad \text{and} \quad \mathrm{f.p.}_{z\to 1}\langle\Lambda(z) \mid u\rangle = \langle Z_{\sqcup\!\sqcup}^- \mid u\rangle. \quad (43)$$

[8] On the one hand, by Proposition 2, one has $\langle Z_{\sqcup\!\sqcup}^- \mid x_0\rangle = \zeta_{\sqcup\!\sqcup}(x_0) = 0$.

On the other hand, since $R_{y_1} = (2x_1)^* - x_1^*$ then $\mathrm{Li}_{R_{y_1}}(z) = (1-z)^{-2} - (1-z)^{-1}$ and $\mathrm{H}_{\pi_Y(R_{y_1})}(n) = \binom{n}{2} - \binom{n}{1}$. Hence, one also has $\langle Z_{\sqcup\!\sqcup}^- \mid x_1\rangle = \zeta_{\sqcup\!\sqcup}(R_{\pi_Y(y_1)}) = 0$ and $\langle Z_\gamma^- \mid x_1\rangle = \gamma_{\pi_Y(R_{y_1})} = -1/2$.

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