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Modeling of contact problems by boundary element and finite element methods

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RÉSUMÉ. Cet article traite de la modélisation numérique des problèmes de contact avec frottement en élasticité linéaire. La méthode des éléments de frontière et la méthode des éléments finis sont présentées et comparées du point de vue algorithmique. L'analyse d'un problème de contact entre deux corps élastiques est effectuée afin de démontrer la validité des algorithmes développés et d'évaluer leurs performances.

ABSTRACT. This paper is concerned with the numerical modeling of contact problems in elastostatics with Coulomb friction. The boundary element method (BEM) and the finite element method (FEM) are presented and compared from an algorithmic point of view. Analysis of a contact problem between two elastic bodies is performed so as to demonstrate the developed algorithms and to highlight their performance.

MOTS-CLÉS : Contact, frottement, méthode des éléments de frontière, méthode des éléments finis

KEYWORDS: Contact, friction, finite element method, boundary element method

1. Introduction

Contact problems have received much attention as may be seen in the monographs by Johnson (Johnson, 1985), Wriggers (Wriggers, 2002) and Laursen (Laursen, 2002). Many numerical methods have been developed in the past twenty five years for solving such problems. These problems may be dynamic with nonlinear material behavior and geometrical nonlinearities (Feng *et al.*, 2003, Feng *et al.*, 2005). The finite element method (FEM) is no doubt the most popular one. However, modeling of frictional contact problems by the boundary element method (BEM) remains limited. Renaud *et al.* have solved frictionless and frictional contacts between an elastic body and a rigid body (Renaud *et al.*, 2003a, Renaud *et al.*, 2003b). The aim of the present paper is to briefly outline the BEM and FEM developed for contact modeling between two elastic bodies in elastostatics with Coulomb friction. Characteristics of each method are discussed. A test numerical example is performed in this study to show the validity of the developed algorithms.

2. Contact kinematics

For the sake of simplicity, let us consider contact between two bodies Ω_1 and Ω_2 . The displacements of the particles of Ω_1 and Ω_2 being respectively \mathbf{u}_1 and \mathbf{u}_2 , the relative displacement is : $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Let \mathbf{r} be the contact traction acting at P from Ω_2 onto Ω_1 . Let \mathbf{n} denote the normal unit vector at the projection point P' directed towards Ω_1 , and $\mathbf{T}(\mathbf{t}_1, \mathbf{t}_2)$ denotes the orthogonal plane to \mathbf{n} (Figure 1)

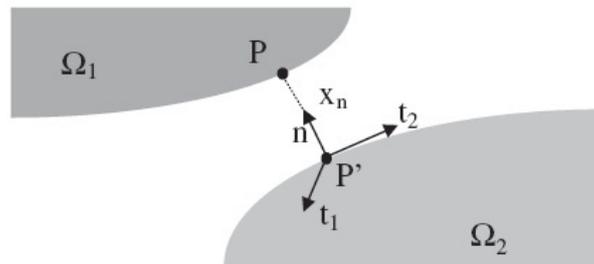


Figure 1. Contact projection and gap vector

The unilateral contact law is characterized by a geometric condition of non-penetration, a static condition of no-adhesion and a mechanical complementary condition. These three conditions are known as Signorini conditions expressed, for each contact point, in terms of the contact distance x_n and the normal contact force r_n by

$$x_n = u_n + g, \quad x_n \geq 0, \quad r_n \geq 0 \quad \text{and} \quad x_n r_n = 0 \quad [1]$$

where g is the initial gap. The classic Coulomb friction rule is used and defined by

$$\text{if } \|\dot{\mathbf{u}}_t\| = 0 \text{ then } \|\mathbf{r}_t\| \leq \mu r_n \text{ else } \mathbf{r}_t = -\mu r_n \frac{\dot{\mathbf{u}}_t}{\|\dot{\mathbf{u}}_t\|} \quad [2]$$

where \mathbf{r}_t is the tangential contact forces, $\dot{\mathbf{u}}_t$ is the relative tangential velocity and μ is the coefficient of friction.

3. The finite element method

The finite element method is often used in computational mechanics. Without going into all the details, we present here just the algorithm for contact modeling. After finite element discretization in the context of small displacements, the global set of equilibrium equations of two contacting elastic bodies can be written as

$$\mathbf{K}\mathbf{U} = \mathbf{F} + \mathbf{R} \quad [3]$$

where \mathbf{K} is the stiffness matrix, \mathbf{F} the vector of external loads, \mathbf{R} the vector of contact reaction forces and \mathbf{U} , the vector of nodal displacements. As \mathbf{U} and \mathbf{R} are both unknown, Eq.[3] cannot be directly solved. In the popular penalty method, \mathbf{K} is modified by introducing contact elements and the global set of equations is solved at each iteration. The resulting numerical algorithms are not very reliable. In this work, we adopt an alternative way. The key idea is to determine iteratively the contact reactions vector \mathbf{R} in a reduced system which only concerns the contact nodes. Then, vector \mathbf{U} can be computed in the whole structure, using contact reactions as external loading. The advantage is that the stiffness matrix is not changed as opposite to the penalty method or to the Lagrange multiplier method.

For computation of contact forces \mathbf{r} in the local coordinate frame $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{n})$, we adopt the bi-potential method proposed by de Saxcé and Feng (de Saxcé *et al.*, 1998). They have shown that the contact law is equivalent to the following differential inclusion :

$$\left(-\dot{\mathbf{u}}_t - (\dot{u}_n + \mu \|\dot{\mathbf{u}}_t\|) \mathbf{n} \right) \in \partial \Psi_{K_\mu}(\mathbf{r}) \quad [4]$$

where $\Psi_{K_\mu}(\mathbf{r})$ denotes the so-called indicator function of the convex Coulomb cone K_μ . In order to avoid non-differentiable potentials that occur in nonlinear mechanics, such as in contact problems, it is convenient to use the Augmented Lagrangian Method (de Saxcé *et al.*, 1998) which leads to an iterative process involving one predictor-corrector step :

$$\begin{aligned} \text{Predictor } \boldsymbol{\tau}^{i+1} &= \mathbf{r}^i + \varrho^i \left(-\dot{\mathbf{u}}_t^i - (\dot{u}_n^i + \mu \|\dot{\mathbf{u}}_t^i\|) \mathbf{n} \right), \\ \text{Corrector } \mathbf{r}^{i+1} &= \text{proj}(\boldsymbol{\tau}^{i+1}, K_\mu). \end{aligned} \quad [5]$$

It is worth noting that, in this algorithm, the unilateral contact and the friction are coupled.

4. The boundary element method

The BEM resolution is based on the approach (Renaud *et al.*, 2003a) which consists in determining contact zone sizes (slip and stick areas) at the same time as

the usual unknowns (displacements and surfaces forces in BEM). This method has already been tested on some classical contact problems (Renaud *et al.*, 2003a, Renaud *et al.*, 2003b) with or without friction. The position of the contact zones is defined through geometrical parameters. Introduction of these extra unknowns requires additional equations in the same number. Under the assumption of regular bodies boundaries and loads in the vicinity of the contact zone, additional equations are given by limit contact and limit slip conditions which allows to reduce the unilateral contact problems to a nonlinear equations system. The contact zones are defined by P contact parameters $\rho_q, q = 1$ to P and adhesion areas by P_a contact parameters $\bar{\rho}_r, r = 1$ to P_a , then overabundant conditions will be given by the contact limit conditions :

$$\sigma_n(M_q) = 0 \quad q = 1 \text{ to } P \quad [6]$$

where M_q are boundary points between contact zones and no contact ones and by the limit sliding conditions :

$$u \cdot t^{(1)}(I_r) + u \cdot t^{(2)}(I_r) = 0 \quad r = 1 \text{ to } P_a \quad [7]$$

where I_r are the boundary points between sliding zones and adhesion ones. So the contact and sliding areas positions are defined by $P' = P + P_a$ contact parameters. In the contact between two elastic bodies, a node to node scheme is considered. The action and reaction principle leads to following equations : $\sigma n^{(1)} = \sigma n^{(2)}$. Moreover in the example (Figure 2), the normal displacement is the same for both bodies such as : $u^{(1)}n^{(1)} = -u^{(2)}n^{(2)}$. The boundary $\partial\Omega^b$ of the domain $\Omega^b (b = 1, 2)$ is divided into four parts : Γ_u^b where displacements are given, Γ_F^b where forces are imposed, Γ_a and Γ_s are respectively the adhesion subareas and the sliding subareas of the contact zone. The formulation of the unilateral frictional contact problem whose contact state can be described by the P' contact parameters reads :

$$\begin{aligned} & \text{div } \sigma(u^b) = 0 \quad \text{in } \Omega^b \\ & u = \bar{u}^b \quad \text{on } \Gamma_u^b \\ & \sigma n^b = \bar{F}^b \quad \text{on } \Gamma_F^b \\ & \text{body 1 } \left\{ \begin{array}{l} u t^{(1)} = -u t^{(2)} \\ (\sigma n^{(1)}) t^{(1)} = (\sigma n^{(2)}) t^{(2)} \end{array} \right\} \quad \text{on } \Gamma_a - \{I_r\} \\ & \text{body 2 } \left\{ \begin{array}{l} u n^{(2)} = -u n^{(1)} \\ (\sigma n^{(2)}) n^{(2)} = (\sigma n^{(1)}) n^{(1)} \end{array} \right\} \\ & \text{body 1 } \left\{ \begin{array}{l} u t^{(1)} + u t^{(2)} = -e^\alpha \sigma n^{(1)} t^{(1)} \\ u n^{(1)} = -u n^{(2)} \end{array} \right\} \quad \text{on } \Gamma_s - \{M_q, I_r\} \\ & \text{body 2 } \left\{ \begin{array}{l} \sigma n^{(2)} = \sigma n^{(1)} \\ u t^{(1)} + u t^{(2)} = 0 \\ u n^{(1)} = -u n^{(2)} \\ \sigma_t^2 = \mu^2 \sigma_n^2 \end{array} \right\} \quad \text{at } \{I_r\} \\ & \left. \begin{array}{l} u n^{(1)} = -u n^{(2)} \\ \sigma_n^{(1)} = 0 \end{array} \right\} \quad \text{at } \{M_q\} \end{aligned} \quad [8]$$

The above equation is reduced to an algebraic form using the approximation of the boundary $\partial\Omega^{(1)}$ (resp. $\partial\Omega^{(2)}$) by a set of $N^{(1)}$ (resp. $N^{(2)}$) constant elements. The final form of the nonlinear system solved by the Newton Raphson method is then :

$$\begin{aligned}
\sum_{m=1}^{N^{(1)}} \sum_{j=1}^2 A_{jmk}^{(1)}(\rho, \bar{\rho}) p_{jm}^{(1)} - \sum_{m=1}^{N^{(1)}} \sum_{j=1}^2 B_{ijmk}^{(1)}(\rho, \bar{\rho}) u_{jm}^{(1)} &= 0 \\
\sum_{m=1}^{N^{(2)}} \sum_{j=1}^2 A_{jmk}^{(2)}(\rho, \bar{\rho}) p_{jm}^{(2)} - \sum_{m=1}^{N^{(2)}} \sum_{j=1}^2 B_{ijmk}^{(2)}(\rho, \bar{\rho}) u_{jm}^{(2)} &= 0 \\
\sigma_{n^{(1)}}(M_q) &= 0 \\
u t^{(1)}(I_r) - u t^{(2)}(I_r) &= 0 \\
\sigma_t^2 - \mu^2 \sigma_n^2 &= 0 \quad \text{on each sliding node of body 1}
\end{aligned} \tag{9}$$

where the coefficients $A_{ijmk}^{(b)}$ and $B_{ijmk}^{(b)}$ depend on contact parameters. More details are given in (Renaud *et al.*, 2003a). In a more general form, this new discretized formulation of unilateral contact problems with friction leads to the following nonlinear equations

$$F_i(X) = 0 \quad i = 1, \dots, 2N^{(1)} + 2N^{(2)} + P' + N_s \tag{10}$$

where X is a $2N^{(1)} + 2N^{(2)} + P' + N_s$ vector composed of P' contact parameters, $2N^{(1)} + 2N^{(2)}$ classical unknowns and N_s values of α when sliding is considered. N_s is the number of sliding nodes and the parameter α comes from a rewriting of Coulomb law (Renaud *et al.*, 2003a).

5. Numerical results

A test example with a linearly elastic law is considered in this study. The example concerns the contact between two elastic plates $\Omega_1(10 \times 10 \times 1 \text{ mm})$ and $\Omega_2(20 \times 24 \times 1 \text{ mm})$ under the assumption of plane stress. The problem is displayed in Figure 2. The upper surface of Ω_1 is given a vertical rigid motion of 0.5 mm . The lower surface comes into contact with the upper surface of Ω_2 . The characteristics of this example are : Young's modulus : $E = 100 \text{ MPa}$, Poisson's ratio : $\nu = 0.3$, Friction coefficient : $\mu = 0.1$ and $\mu = 0.2$. Symmetry conditions are imposed on two lateral sides of the plates as shown in Figure 2. The analysis was performed by means of both FEM and BEM developed above. The finite element discretization includes 320 eight-node isoparametric elements and 1062 nodes. Each element has nine integration points. The boundary element discretization includes 162 nodes. Figure 3 shows the contact pressure distribution. The results obtained by BEM and FEM are in good agreement.

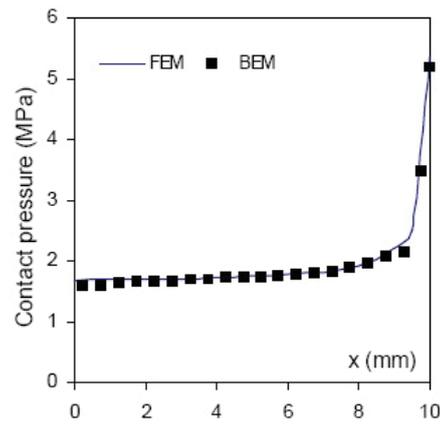
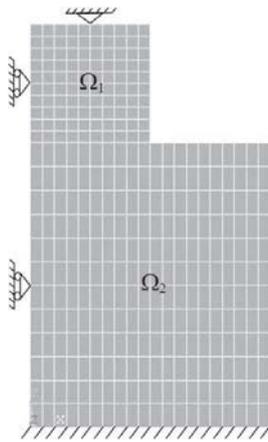


Figure 2. Contact between elastic plates **Figure 3.** Contact pressure

6. Conclusions

In this work, contact problems with friction have been solved by the finite element method and the boundary element method. The BEM consists of solving a nonlinear system of equations involving displacements, contact tractions and geometrical parameters on the boundary. The FEM developed solves the contact problem iteratively in a reduced linear system and computes the displacements in the whole structure, using contact reactions as external loading. The numerical test shows that both methods yield accurate results.

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