



HAL
open science

Sticking motions of impact oscillators

C Toulemonde, C Gontier

► **To cite this version:**

C Toulemonde, C Gontier. Sticking motions of impact oscillators. *European Journal of Mechanics - A/Solids*, 1998, 17 (2), pp.339-366. 10.1016/S0997-7538(98)80090-9 . hal-01508925

HAL Id: hal-01508925

<https://hal.science/hal-01508925>

Submitted on 15 Apr 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sticking motions of impact oscillators

C. TOULEMONDE * and C. GONTIER **

ABSTRACT. – The present paper is devoted to the study of impact oscillators subjected to harmonic excitation. The interest is essentially centred on a family of behaviours which has never been systematically investigated before: the class of sticking periodic responses of impact oscillators. A formulation of this kind of motion is presented and a Poincaré application is built. A method is defined in order to produce its characteristics and an analytic differentiation of the responses is used to evaluate local stability. A methodology of analysis, based on a Predictor–Corrector method, is presented and applied to single and multiple degree of freedom systems: bifurcation diagrams and parameter space partitionings are developed.

1. Introduction

Among the various aspects of nonlinear vibrations, the problem of impacting structures deserves special attention in view of its scientific interest and its technical usefulness.

Indeed, the specific challenge of this problem comes from the strong nonlinearity resulting from the impact. Many researchers have noticed the tremendous variety of dynamic responses of these systems, even under a simple harmonic excitation (Thompson, 1986; Moon, 1990). As to its technical applications, the impact oscillator provides a realistic model for a very common situation in vibrations. The vibration of a structure with clearance, the response of an untightened bolted joint, the motion of a body travelling between guides are examples of such situations. In general, the dynamics of an impacting structure is so complex that most authors have restricted their analysis to the simplest model, *i.e.* the single degree of freedom (d.o.f.) oscillator. In this framework they were able to confirm the classic scenario of period doubling bifurcation (Holmes, 1983; Chow 1986). Some specific features such as grazing bifurcations have also been identified (Nordmark, 1991; Ivanov, 1994). A few studies were carried out using analytical approach but most studies proceeded from time integration of the dynamics (Paoli, 1993).

In spite of all this research there is a very common phenomenon which strangely enough has drawn very little interest. We will refer below to this phenomenon in terms of “sticking motion”. To describe roughly this phenomenon, let us assume that the impact is approximately governed by a restitution rule. Depending on the mechanical conditions and on the dynamic state of the body, a structure may undergo an infinite sequence of

* Électricité de France, Research and Development Division, Mechanics and Component Technology Division, Moret sur Loing, France.

** Laboratoire de Mécanique Sols, Structures et Matériaux, École Centrale de Paris, Chatenay-Malabry, France.

decreasing impacts, although finite in time, before it remains for a time in contact with the obstacle. In the case of multi-d.o.f. oscillators, this does not mean that the structure is motionless, but during a finite time contact is maintained with the stop, as if the body was stuck to it. Moreover, this kind of motion can converge asymptotically on a periodic solution including one or several sticking phases per period.

Apparently, this phenomenon has been the object of very few studies in the past. A particular case (Shaw, 1983) with a small or zero restitution ratio was analysed, and the domains of attraction of infinite impact periodic motions (Budd, 1994) have been numerically analysed. However, these studies were also restricted to the one d.o.f. oscillator. To our knowledge, a general investigation of this phenomenon is still missing, and is the subject of the present paper, whose aim, in general, is to define a suitable procedure for the analysis of these systems. First, a general methodology for handling the dynamics of multi-d.o.f. oscillators will be proposed. Then, by applying the procedure to the case of sticking motions, a method to determine the periodic solutions will be defined, and their stability will be calculated. Finally, the whole procedure will be applied to a few examples, that will serve as validation tests, and as a means of understanding some specific features of the motions.

2. Dynamic equations of an impacting structure

2.1. THE MECHANICAL SYSTEM

In this paper, the system under consideration is a multiple d.o.f. impact oscillator. This oscillator has a linear structure that can impact on a single rigid stop and it is subjected to harmonic excitation (*Fig. 1*). The type of nonlinear displacement constraint is determined by the position of the stop and by its orthogonal distance x_0 to the equilibrium position of the structure.

When the displacement constraint x_0 is positive the static system is always tensioned. For small excitation magnitudes this preload constrains one degree of freedom of the

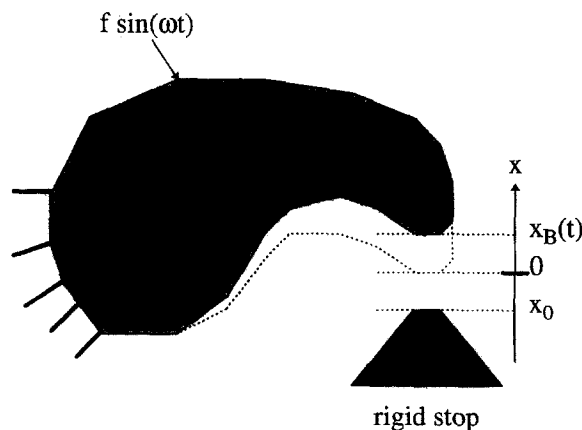


Fig. 1. – Multidimensional impacting structure.
The equilibrium state of the structure is outlined by the dashed curve.

structure onto the stop. In the case of a multi d.o.f. structure, linear harmonic motion continues while contact is maintained with the stop. When the displacement constraint x_0 is negative clearance exists in the system. In both cases, for small excitation magnitudes the structure behaves linearly and nonlinear behaviour does not exist. The dynamic behaviour changes when the stop is reached by the oscillator.

For such a dynamic system, a very large variety of asymptotic solutions can be encountered: finite impact periodic motions, infinite impact periodic motions, quasiperiodic motions, chaotic motions (Gontier, 1997). Infinite impact periodic motions and sticking motions in general are the subject of the present study.

Referring to Figure 1, let us note two major points of the system, A and B . Harmonic excitation is applied at point A in a constant direction. The stop is impacted by the structure at point B . As x_0 is small, it is assumed that the contact point B on the surface is constant and that the reaction of the stop has a constant direction orthogonal to the structure at B .

2.2. DESCRIPTION OF THE STICKING MOTIONS

Let us define more accurately the so-called sticking motions. For such dynamic responses, contact at point B is maintained during a finite time per period. This dynamic phenomenon may result from different situations. In the case of a purely plastic shock, *i.e.*, a shock involving no rebound, the post-impact velocity at point B is set to zero. In that case, if the incident acceleration is such that the structure impacts with the stop, *i.e.* if the acceleration is negative, the oscillator will be stuck on the stop. This phenomenon can be observed when the impact rule is a restitution rule whose coefficient is zero.

However, in most impacting systems, the post-impact velocity follows an elasto-plastic impact rule. Generally, this rule can be approached by a restitution rule with a non zero restitution coefficient. Even in this case, sticking periodic motions can be encountered, but now they are observed after an infinite sequence of decreasing impacts, although finite in time. Once the limit of the sequence is reached, it can be said that the structure sticks onto the stop. Such dynamic behaviour frequently occurs in impact oscillators, especially in the preload case, where it is encountered for rather small values of excitation magnitude.

2.3. THE FREE FLIGHT PHASE

During infinite impact periodic motions, the dynamics of the structure undergoes an infinite sequence of free flight phases. For the sake of simplicity, let us write the dynamics directly in the eigenbase of the linear structure. The equation of motion of a free flight phase is therefore

$$(1) \quad \ddot{X} + \Omega X = T_A f \sin(\omega t); \quad x_B(t) > x_0$$

In Eq. (1) t is time, $\left(\dot{}\right) = \frac{d}{dt}$ is the time derivative and X is the displacement field in the structure. Ω is the diagonal matrix of the squares of the eigenpulsations $(\omega_1^2, \dots, \omega_n^2)$. $x_B(t)$ is the resulting displacement at point B in the direction Bx . f and ω are respectively the magnitude and the pulsion of the harmonic excitation applied

at point A ; T_A is the projection of a unit force applied at A onto the eigenbase of the structure.

The vectors and the matrices of Eq. (1) have an arbitrary dimension n . Therefore, the theory is applicable to any multi-d.o.f. structure.

A part of the kinetic energy is lost when the structure impacts the stop, which happens when $x_B(t)$ is equal to x_0 . In the present study a restitution rule is applied in this case

$$(2) \quad \dot{x}_B(t^+) = -r\dot{x}_B(t^-); \quad \left\{ \begin{array}{l} x_B(t) = x_0 \\ \dot{x}_B(t) < 0 \end{array} \right\}$$

where r is the restitution coefficient, whose value lies between zero and unity. This formulation is equivalent to assuming that the impact duration is very small and that the reaction of the stop is an impulse. During sticking motions, and when the restitution coefficient is not zero, the sticking phenomenon will occur after an infinite sequence of impacts. theoretically, an infinite number of free flight phases will be encountered until the sticking conditions are reached.

2.4. THE STICKING PHASE

Sticking occurs when the position $x_B(t)$ of point B is equal to x_0 , its velocity $\dot{x}_B(t)$ is equal to 0 and the contact reaction of the stop is positive. This situation may be the consequence of a purely plastic impact, or of an infinite sequence of decreasing impacts. Sticking ends when the reaction of the stop is null and when it changes sign.

We denote the reaction of the stop as $R(t)$. The sticking phase motion equation is

$$(3) \quad \ddot{X} + \Omega X = T_A f \sin(\omega t) + T_B R(t); \quad \left\{ \begin{array}{l} x_B(t) = x_0 \\ \dot{x}_B(t) < 0 \end{array} \right\}$$

In Eq. (3) T_B is the projection of a unit force applied at B onto the eigenbase.

During the sticking phase, the contact reaction $R(t)$ can be easily calculated. The position and the velocity at point B remain constant during this period, therefore acceleration at point B is zero and the calculation of reaction $R(t)$ results directly from the expression $\ddot{x}_B(t) = 0$. Let us first assume relation between scalar displacement $x_B(t)$ and displacement field $X(t)$:

$$(4) \quad x_B(t) = X_B^t X(t)$$

In Eq. (4) T_B is a vector and $(^t)$ is the transpose operator. In the same manner, the equation of motion (3) can be projected and the reaction of the stop can be extracted:

$$(5) \quad R(t) = \frac{X_B^t \Omega X - X_B^t T_A \sin(\omega t)}{X_B^t T_B}; \quad R(t) \geq 0$$

The result (5) will be used to modify the equation of motion (3) and to determine the end of the sticking phase when the reaction is zero.

The eigenbase of the structure being linear, vector X_B can be chosen such that $X_B^t = [1 \cdots 1]$, *i.e.* such that $x_B(t) = \sum_{i=1}^n X_i(t)$.

2.5. RESTRICTED EQUATION OF MOTION OF THE STICKING PHASE

During the sticking phase, one of the degrees of freedom of the structure is constrained by the stop. The dynamics of the constrained system is ruled by a $(n - 1)$ -sized dynamic equation. First, by using Eq. (3) and Eq. (5) it is possible to give a new n -sized formulation of the dynamics:

$$(6) \quad \ddot{X} + \left(\Omega - \frac{T_B[\omega_1^2 \cdots \omega_n^2]}{\sum_{i=1}^n T_{B_i}} \right) X = \left(\begin{array}{c} T_A - \frac{\sum_{i=1}^n T_{A_i}}{n} T_B \\ \sum_{i=1}^n T_{B_i} \end{array} \right) f \sin(\omega t); \quad \left\{ \begin{array}{l} x_B(t) = x_0 \\ \dot{x}_B(t) = 0 \end{array} \right\}$$

Eq. (6) only holds when the reaction of the stop is positive. Once the dynamics described by Eq. (6) are integrated, the reaction can be calculated with the help of expression (5). As shown by this relation, the reaction of the stop is a linear function of the displacement coordinates X . Therefore, in this example the stiffness matrix is modified by the expression of the contact reaction and its eigenvalues are the square of the eigenpulsations of the linear structure constrained at point B .

In order to achieve the complete transformation of the dynamic equation a number of other conditions must be added. During the sticking phase, only $n - 1$ components of the n -sized vector X are free. The last coordinate of X can be expressed as a sum of the first $n - 1$ components:

$$(7) \quad x_B(t) = x_0 \Leftrightarrow X(n) = x_0 - \sum_{i=1}^{n-1} X(i)$$

We denote the vector of the first $n - 1$ coordinates of vector X as Y . By using this new vector, the n -sized equation of motion (6) can be reduced to the following $(n - 1)$ -sized equation

$$(8) \quad \ddot{Y} + \left(\begin{array}{ccc} \left[\begin{array}{ccc} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_{n-1}^2 \end{array} \right] & - \frac{\left[\begin{array}{c} T_{B_1} \\ \vdots \\ T_{B_{n-1}} \end{array} \right] [\omega_1^2 - \omega_n^2 \cdots \omega_{n-1}^2 - \omega_n^2]}{\sum_{i=1}^n T_{B_i}} & \end{array} \right) Y \\ = \left(\begin{array}{c} \left[\begin{array}{c} T_{A_1} \\ \vdots \\ T_{A_{n-1}} \end{array} \right] - \frac{\sum_{i=1}^n T_{A_i}}{\sum_{i=1}^n T_{B_i}} \left[\begin{array}{c} T_{B_1} \\ \vdots \\ T_{B_{n-1}} \end{array} \right] \\ \sum_{i=1}^n T_{B_i} \end{array} \right) f \sin(\omega t) + \frac{\omega_n^2}{\sum_{i=1}^n T_{B_i}} \left[\begin{array}{c} T_{B_1} \\ \vdots \\ T_{B_{n-1}} \end{array} \right] x_0; \\ \left\{ \begin{array}{l} x_B(t) = x_0 \\ \dot{x}_B(t) = 0 \end{array} \right\}$$

The reaction of the stop can also be calculated using the restricted displacement field Y :

$$(9) \quad R(t) = \frac{[1 \cdots 1] \begin{bmatrix} \omega_1^2 - \omega_n^2 & & 0 \\ & \ddots & \\ 0 & & \omega_{n-1}^2 - \omega_n^2 \end{bmatrix} Y + \omega_n^2 x_0 - \sum_{i=1}^n T_{A_i} f \sin(\omega t)}{\sum_{i=1}^n T_{B_i}}$$

In the case of a one dimensional impact oscillator, the restricted displacement field Y cannot be defined and Eq. (7) does not exist. In this case, the entire structure is stationary during the sticking phases, and the expression of the reaction reduced to the simple relation $R(t) = \frac{\omega_1^2 x_0 - T_{A_1} f \sin(\omega t)}{T_{B_1}}$.

2.6. DYNAMIC IMPACT EQUATION

The last two subsections were devoted respectively to the two equations of motion, Eq. (1) and Eq. (3). Now the impact rule (2) will also be expressed in detail. In the case of a multidimensional structure the restitution rule has to be applied carefully and relation (2) cannot be translated directly using the velocity field \dot{X} .

The stop is rigid, therefore the shock stiffness is very high and the impact duration may be very short, as described in Eq. (2). The reaction can be approximated by an impulse, and the relation between the pre-impact and post-impact velocity fields can be deduced from the time integration, in the sense of distributions, of the acceleration field during the shock:

$$(10) \quad \dot{X}(t^+) = \dot{X}(t^-) + p \cdot T_B$$

In Eq. (10) p is the magnitude of the impulse to which the structure is subjected, which can be calculated by taking the impact rule (2), Eq. (10) and relation (4) into account:

$$(11) \quad p = -(1+r) \frac{[1 \cdots 1] \dot{X}(t^-)}{\sum_{i=1}^n T_{B_i}}$$

Then, the relation between the two velocity fields results from Eq. (10) and Eq. (11).

The multidimensional restitution rule can now be rewritten as

$$(12) \quad \dot{X}(t^+) = \left[I - (1+r) \frac{T_B [1 \cdots 1]}{\sum_{i=1}^n T_{B_i}} \right] \dot{X}(t^-)$$

3. General method for the determination of sticking periodic motions

3.1. COMBINING THE TWO KINDS OF DYNAMICS

This paper is mainly focused on the study of sticking periodic motions. Depending on the value of the restitution coefficient, two different situations can be encountered. First, when the restitution coefficient is zero, *i.e.* when the impact rule is a plastic impact rule, each free flight phase is followed by a sticking phase (*Fig. 2*). The motion is an alternate sequence of free flight phases and sticking phases.

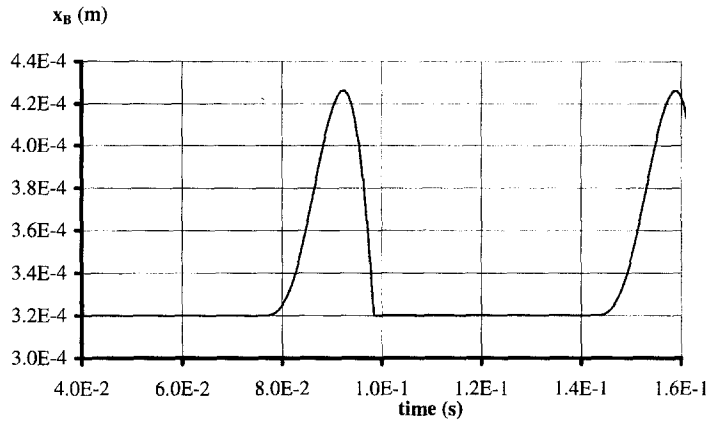


Fig. 2. – Plastic impact rule, sticking periodic motion (two d.o.f. impact oscillator with preload).

When the restitution coefficient is not null, *i.e.* when the impact rule is elasto-plastic, the motion encounters an infinite number of impacts before any sticking phase (*Fig. 3*). From a practical point of view, it is impossible to calculate this infinite amount of impacts, but it is possible to approach it as the limit of a geometrical series.

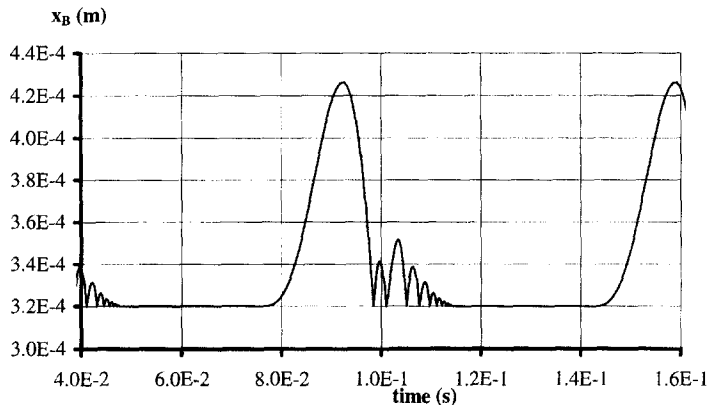


Fig. 3. – Elastic impact rule, infinite impact periodic motion (two d.o.f. impact oscillator with preload).

3.2. DISCRETISATION OF THE FREE FLIGHT DYNAMICS

The equation of motion (1) of free flight motion is deterministic, thus its continuous dynamics can be completely defined by the initial conditions. Expressed in the structure

eigenbase, the whole displacement and velocity fields between two successive impacts (impact number j and impact number $j + 1$) can be written as the sum

$$(13) \quad \left. \begin{cases} X(t) = U(t) + \sum_{i=1}^n [a_{i,j} \cos(\omega_i(t - t_j)) + b_{i,j} \sin(\omega_i(t - t_j))] X_i \\ \dot{X}(t) = \dot{U}(t) + \sum_{i=1}^n \omega_i [-a_{i,j} \sin(\omega_i(t - t_j)) + b_{i,j} \cos(\omega_i(t - t_j))] X_i \end{cases} \right\};$$

$$\forall t \in [t_j; t_{j+1}[$$

In Eq. (13) $U(t)$ is the forced harmonic displacement field and t_j is the time of impact numbered j . Set $(X_i)_{i \in [1,n]}$ is the set of eigenvectors of the free structure.

The motion defined in Eq. (13) depends only on t_j and on the set of $2n$ variables $(a_{i,j}, b_{i,j})_{i \in [1,n]}$. But t_j is the time of impact j and it must satisfy the relation $x_B(t_j) = X_B^t X(t_j) = x_0$ and thus depends on the variables $(a_{i,j}, b_{i,j})_{i \in [1,n]}$. Therefore, only $2n$ variables of the $2n + 1$ variables describing the dynamics in Eq. (13) are free. The dimension of the free flight dynamics is then $2n$. Let us define V_j as the $2n$ dimensional degree of freedom vector describing the free flight phase numbered j :

$$(14) \quad V_j = \begin{bmatrix} a_{1,j} \\ b_{1,j} \\ \vdots \\ a_{n,j} \\ b_{n,j} \end{bmatrix}$$

The free flight continuous motion dynamics can then be reduced to a discrete dynamics relationship, equivalent to a recurrence relation between vector V_j and vector V_{j+1} . V_{j+1} can then be expressed explicitly using V_j and the two successive impact times t_j and t_{j+1} :

$$(15) \quad V_{j+1} = \begin{pmatrix} \ddots & & & & 0 \\ & \cos(\omega_i(t_{j+1} - t_j)) & \sin(\omega_i(t_{j+1} - t_j)) & & \\ & -\sin(\omega_i(t_{j+1} - t_j)) & \cos(\omega_i(t_{j+1} - t_j)) & & \\ & & & \ddots & \\ 0 & & & & \ddots \end{pmatrix} V_j$$

$$- \frac{(1+r)\dot{x}_B(t_{j+1}^-)}{\sum_{j=1}^n T_{B,j}} \begin{bmatrix} 0 \\ T_{B,1} \\ \vdots \\ 0 \\ T_{B,n} \end{bmatrix}$$

But time t_j and t_{j+1} are determined by the implicit relations $x_B(t_j) = x_0$ and $x_B(t_{j+1}) = x_0$. They are therefore two implicit functions of the degree of freedom vector V_j . Consequently, V_{j+1} can be defined as an implicit function of V_j .

3.3. DISCRETISATION OF THE STICKING PHASE

In the same manner, the dynamics of the sticking phase can be discretised. The equation of motion (8) of the sticking system is also deterministic and its dynamics can also be completely defined by the discrete initial conditions. We denote the set of the $n - 1$ eigenvectors of the linear homogeneous dynamic system of Eq. (8) as $(Y_i)_{i \in [1, n-1]}$; we also denote the set of eigenpulsations of this system as $(\omega'_i)_{i \in [1, n-1]}$. During the sticking phase, the restricted displacement and velocity fields in the structure can be written as the sum

$$(16) \quad \left\{ \begin{array}{l} Y(t) = Y_0 + U'(t) + \sum_{i=1}^{n-1} [c_i \cos(\omega'_i(t - t'_0)) + d_i \sin(\omega'_i(t - t'_0))] X_i \\ \dot{Y}(t) = \dot{U}'(t) + \sum_{i=1}^{n-1} \omega'_i [-c_i \sin(\omega'_i(t - t'_0)) + d_i \cos(\omega'_i(t - t'_0))] X_i \end{array} \right\};$$

$$\forall t \in [t'_0; t'_1[$$

In Eq. (16) $U'(t)$ is the forced displacement field of the blocked structure induced by the harmonic excitation; Y_0 is the constant response associated with the constant excitation term of Eq. (8); t'_0 and t'_1 are respectively the initial and final time of the sticking phase. Thus the motion defined in Eq. (16) depends only on t'_0 and on the set of $2n - 2$ variables $(c_i, d_i)_{i \in [1, n-1]}$. We now define W_0 as the $2n - 1$ dimensional vector describing the beginning of the sticking phase:

$$(17) \quad W_0 = \begin{bmatrix} t'_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ d_1 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

At time t'_1 the restricted displacement and velocity fields in the structure can also be expressed with the help of the eigenbase of the constrained structure

$$(18) \quad \left\{ \begin{array}{l} Y(t'_1) = Y_0 + U'(t'_1) + \sum_{i=1}^{n-1} e_i Y_i \\ \dot{Y}(t'_1) = \dot{U}'(t'_1) + \sum_{i=1}^{n-1} \omega'_i f_i Y_i \end{array} \right\}$$

As shown by relation (9) the reaction of the stop depends only on the time and on the restricted displacement field Y . Time t'_1 and restricted degree of freedom vector $Y(t'_1)$ are then linked by the relation $R(t'_1) = 0$. By taking Eq. (18) into account, it can be expressed

the degrees of freedom at the end of sticking phase W_1 and the degrees of freedom of the first free flight phase V_1 :

$$(23) \quad \left\{ \begin{array}{l} H_1 : \mathfrak{R}^{2n-2} \rightarrow \mathfrak{R}^{2n} \\ W_1 \mapsto V_1 \end{array} \right\}$$

We also define the relation between the degrees of freedom at the end of last free flight V_∞ and the degrees of freedom at the beginning of the next sticking phase

$$(24) \quad \left\{ \begin{array}{l} H_2 : \mathfrak{R}^{2n} \rightarrow \mathfrak{R}^{2n-1} \\ V_\infty \mapsto W_0 \end{array} \right\}$$

The complete sequence can be defined by linking the last four described functions. Let $W_{1,j}$ be the vector describing the state of the structure at the end of the sticking phase numbered j . The dynamics between the beginning and the end of two successive sticking phases, *i.e.* between $W_{1,j}$ and $W_{1,j+1}$, can be described by a implicit function

$$(25) \quad \left\{ \begin{array}{l} P : \mathfrak{R}^{2n-2} \rightarrow \mathfrak{R}^{2n-2} \\ W_{1,j} \mapsto W_{1,j+1} = (G \circ H_2 \circ F^\infty \circ H_1)(W_{1,j}) \end{array} \right\}$$

In Eq. (25) application P appears as a composed function, and defines a Poincaré map associated with sticking motions. Every continuous orbit of this kind can be made equivalent to the series $(W_{1,j})_{j \in \mathbb{N}}$ of restricted degree of freedom vectors. As defined above, a Poincaré map P is a very convenient representation of the dynamics, and is well adapted to the analysis of the asymptotic behaviour of the system. For instance, an infinite impact periodic motion with only one sticking phase per period can be defined by the following fixed point equation:

$$(26) \quad P(W_1) = W_1$$

where W_1 defines the restricted degree of freedom vector at the end of each sticking phase. In the case of a one dimensional impact oscillator Eq. (25) and Eq. (26) cannot be defined. As mentioned in subsection 3.3, it is possible to discretise the sticking phase, *i.e.* to calculate $H_1 \circ G$. The Poincaré application P can then be replaced by $H_2 \circ F^\infty \circ H_1 \circ G$.

3.5. RESOLUTION METHOD

The simplest method to determine stable periodic motions is based on iterative integration of the dynamics until stabilisation is found. It is well known that the efficiency of these methods depends heavily on the stability of the motions and the choice of the initial conditions. Although sticking periodic motions are generally stable, the method is in any case expensive and often not very accurate. A more direct method based on analytic resolution is proposed here. Using the fixed point formalism introduced in the last subsection, the problem of periodic motions reduces to finding the roots of a Boundary Value Problem

$$(27) \quad \phi(W_1) = P(W_1) - W_1 = 0$$

This equation is easily solved by using an algorithm based on the Newton method requiring local determination of the gradient $D_w\phi$ of ϕ .

3.6. ANALYTIC APPROACH TO INFINITE SERIES OF IMPACTS

In Eq. (25), the Poincaré map appears as a chain of several functions. In the case of an infinite impact periodic motion the theory implies that application F will be iterated an infinite number of times. Pratically, since this calculation cannot be carried out exactly, the problem must be simplified in order for it to be solved.

Qualitatively, in the case of infinite impact periodic motions, it can be shown that the sequence of impacts converges towards a geometric series. The ratio of this series is equal to the restitution coefficient r . Let $(t_j)_{j \in [1, +\infty[}$ denote the series of impact times in this infinite sequence of impacts. The series of impact times and the series of impact velocities satisfy the following relations:

$$(28) \quad \left\{ \begin{array}{l} \frac{t_{i+1} - t_i}{t_i - t_{i-1}} \xrightarrow{i \rightarrow +\infty} r \\ \frac{\dot{x}_B(t_{i+1}^-)}{\dot{x}_B(t_i^-)} \xrightarrow{i \rightarrow +\infty} r \end{array} \right\}$$

Series $(V_j)_{j \in [1, +\infty[}$ of the degrees of freedom satisfies relation (15). Using this recurrence relation and the asymptotic geometric behaviour described in Eq. (28) the limit V_∞ of series $(V_j)_{j \in [1, +\infty[}$ can be approached by

$$(29) \quad \left(\begin{array}{ccc} \ddots & & 0 \\ \cos\left(\omega_j \frac{t_{k+1} - t_k}{1 - r}\right) & \sin\left(\omega_j \frac{t_{k+1} - t_k}{1 - r}\right) & \\ -\sin\left(\omega_j \frac{t_{k+1} - t_k}{1 - r}\right) & \cos\left(\omega_j \frac{t_{k+1} - t_k}{1 - r}\right) & \\ 0 & & \ddots \end{array} \right) V_k$$

$$- \frac{(1+r)\dot{x}_B(t_{k+1}^-)}{(1-r)\sum_{j=1}^n T_{B,j}} \begin{bmatrix} 0 \\ T_{B,1} \\ \vdots \\ 0 \\ T_{B,n} \end{bmatrix} \xrightarrow{k \rightarrow +\infty} V_\infty$$

Let L denote the application defined by Eq. (29). A good approximation of the complete sequence of impacts F^∞ can be obtained with the help of the previous expression

$$(30) \quad L \circ F^{k-1} \xrightarrow{k \rightarrow +\infty} F^\infty$$

The new application defined by equation (30) will be used in place of the original function.

Another way to solve this problem is to suppose a purely plastic impact rule as soon as the impact velocity is smaller than a given value.

3.7. FIRST ORDER DIFFERENTIATION TERMS

As mentioned in section 3.4, equation (25) can be reduced to a root finding problem, (26). The roots of this system are found by applying a Newton algorithm. Although the whole nonlinear application is quite heavy to handle it is possible to calculate its first order derivatives, *i.e.* its gradient, in an analytic manner. The first order derivative of ϕ is composed of five different linear applications, DH_1 , DH_2 , DF , DG and DL .

The five implicit functions have been analytically differentiated with respect to their inputs. In the case of DH_1 and DH_2 the calculation is very simple. The expression of function F is given by Eq. (15) and its differentiation requires the calculation of a number of first order terms, such as dt_j and dt_{j+1} , that can be performed by using the relation

$$(31) \quad x_B(t_j) = X_B^t U(t_j) + [1 \quad 0 \quad \cdots \quad 1 \quad 0] V_j = x_0$$

By differentiating Eq. (31) the first order term dt_j appears as a linear function of dV_j . DG and DL are calculated in the same manner.

3.8. ANALYTIC AND EXPLICIT RESOLUTION IN A SIMPLE CASE

In the simple case of a one degree of freedom impact oscillator, when the restitution coefficient r is null, the characteristics of the periodic sticking motions can be easily calculated. For the sake of simplicity, we consider a non dimensional impact oscillator, *i.e.* in the case where x_0 and the first eigenpulsation are equal to 1.

By definition time t'_1 is the time at the end of the sticking phase, but it is also the time at the beginning of the next free flight phase. t'_1 thus satisfies the following conditions:

$$(32) \quad \left\{ \begin{array}{l} x_B(t'_1) = 1 \\ \dot{x}_B(t'_1) = 0 \\ \ddot{x}_B(t'_1) = 0 \\ \dddot{x}_B(t'_1) > 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \sin(\omega t'_1) = \frac{1}{f} \\ \cos(\omega t'_1) = \frac{\sqrt{f^2 - 1}}{f} \end{array} \right\}$$

With the help of the previous relation it is easy to predict that no periodic response with sticking phases exist for values of amplitude f less than 1 because in that case the trigonometric equations would not have any real solutions.

Taking these initial conditions into account, the free flight phase motion can be explicitly integrated. Time t'_0 then satisfies the condition

$$(33) \quad x_B(t'_0) = -\frac{\omega^2}{1 - \omega^2} \cos(t'_0 - t'_1) - \frac{\omega \sqrt{f^2 - 1}}{1 - \omega^2} \sin(t'_0 - t'_1) + \frac{f}{1 - \omega^2} \sin(\omega t'_0) = 1$$

A new sticking phase at time t'_0 . As a consequence, t'_0 must also satisfy the relation $\ddot{x}(t'_0) < 0$. In the two following limit cases, by developing Eq. (33), the root of Eq. (33) can be calculated explicitly:

$$(34) \quad \left\{ \begin{array}{l} t'_0 \xrightarrow{\omega \rightarrow 0} \frac{\pi}{\omega} - t'_1 \\ t'_0 \xrightarrow{f \rightarrow 1} t'_1 + \frac{4\sqrt{f^2 - 1}}{\omega} \end{array} \right\}$$

In general Eq. (33) can be used to determine the zone of existence in parameter space of a large class of sticking periodic motions. In the case of a nondimensional single degree of freedom impact oscillator with zero restitution coefficient, subharmonic periodic responses can be encountered. These responses occur when solution t'_0 of Eq. (33) is such that $t'_0 - t'_1$ is larger than the period $\frac{2\pi}{\omega}$ (Fig. 4). In that case, a subharmonic coefficient q of the subharmonic sticking periodic responses is introduced.

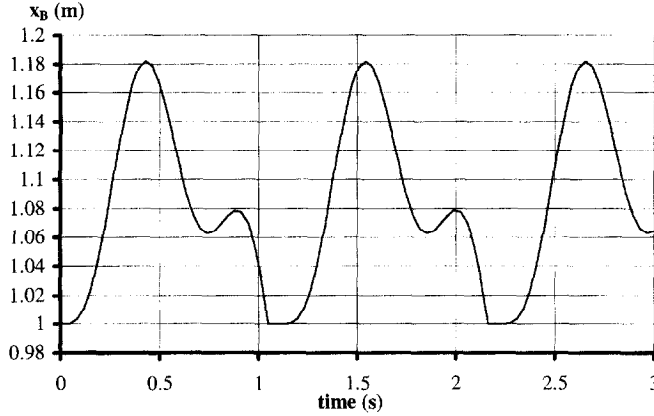


Fig. 4. – Subharmonic periodic response with sticking phases; the subharmonic coefficient is equal to 2. (r, ω, f) = (0, 10, 6).

In the following equation we denote the smaller amplitude for which these subharmonic orbits exist as f_{sticking} . The responses corresponding to this limit are such that

$$(35) \quad \left\{ \begin{array}{l} x_B(t'_0) = 1 \\ \ddot{x}_B(t'_0) = 0 \\ \ddot{\ddot{x}}_B(t'_0) < 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \sin(\omega t'_1) = \frac{1}{f_{\text{sticking}}} \\ \cos(\omega t'_1) = -\frac{\sqrt{f_{\text{sticking}}^2 - 1}}{f_{\text{sticking}}} \end{array} \right\}$$

Taking condition (33) and equation (35) into account, the critical amplitude f_{sticking} can be implicitly calculated:

$$(36) \quad \frac{\sqrt{f_{\text{sticking}}^2 - 1}}{\omega} = \frac{1 - \cos\left(\frac{(2q-1)\pi}{\omega} - \frac{2}{\omega} \arcsin\left(\frac{1}{f_{\text{sticking}}}\right)\right)}{\sin\left(\frac{(2q-1)\pi}{\omega} - \frac{2}{\omega} \arcsin\left(\frac{1}{f_{\text{sticking}}}\right)\right)}$$

When the subharmonic coefficient q is equal to 1 the value of f_{sticking} is constant and equal to 1. The limit determined by this expression has asymptotes in the parametric plane defined by the two parameters (ω, f) :

$$(37) \quad \left\{ \begin{array}{l} \omega \xrightarrow{f_{\text{sticking}} \rightarrow +\infty} 2q - 1 \\ f_{\text{sticking}} \xrightarrow{\omega \rightarrow +\infty} f \text{ such that } 4\sqrt{f^2 - 1} = (2q - 1)\pi - 2 \arcsin\left(\frac{1}{f}\right) \end{array} \right\}$$

In the same manner, the larger amplitude for which these subharmonic responses exist can be calculated analytically. We assign a value $f_{\text{free-flight}}$ to this amplitude, which is determined by introducing relation $t'_0 - t'_1 = \frac{2q\pi}{\omega}$ in equation (33):

$$(38) \quad f_{\text{free-flight}} = \sqrt{1 + \omega^2 \left[\frac{1 - \cos\left(\frac{2q\pi}{\omega}\right)}{\sin\left(\frac{2q\pi}{\omega}\right)} \right]^2}$$

This expression also has two limits :

$$(39) \quad \left\{ \begin{array}{l} \omega \xrightarrow{f_{\text{free-flight}} \rightarrow +\infty} 2q \\ f_{\text{free-flight}} \xrightarrow{\omega \rightarrow +\infty} \sqrt{1 + q^2\pi^2} \end{array} \right\}$$

When the restitution coefficient r is not null but small, the conclusions about the existence of the subharmonic responses are always valid, as the duration of the infinite series of impacts is very small. When the restitution coefficient is larger, and when the subharmonic coefficients is greater than one, the conditions required by striking periodic motions are rarely fulfilled.

4. Stability of infinite impact periodic motions

4.1. PERTURBATION OF THE DYNAMICS

The motion of the impact oscillators defined by Eq. (1) and Eq. (3) is highly nonlinear, and so is their dynamics. Therefore, systems may have more than a single response to a given periodic excitation. This phenomenon is called coexistence (Guckenheimer, 1983). For a given set of external parameters, infinite impact periodic motions have been proved exist together with other finite impact motions (*Fig. 5*) $(r, \omega, f, x_0) = (0.75, 10, 1.1, 1)$. Depending on the value of the control parameters the existence, as well as the stability of the response, may change. This section is devoted to an evaluation of the stability of infinite impact periodic responses. Using a Poincaré formalism, the problem of local stability of periodic motion reduces to a fixed point stability problem. Local stability

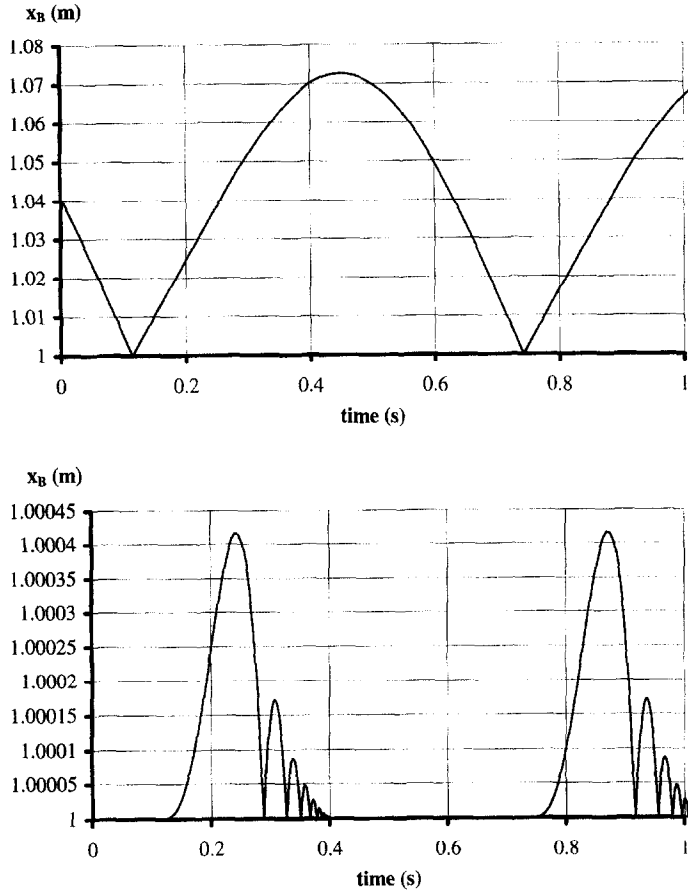


Fig. 5. – a) Coexistence: single impact period one motion, single d.o.f. impact oscillator, $(r, \omega, f) = (0.75, 10, 1.1)$. b) Coexistence: infinite impact period one motion, single d.o.f. impact oscillator, $(r, \omega, f) = (0.75, 10, 1.1)$.

evaluation can be carried out by using the first order perturbation of impacting dynamics. The relation between the end of sticking phase j and the end of sticking phase $j + 1$, *i.e.* between $W_{1,j}$ and $W_{1,j+1}$ is defined by a Poincaré application P .

Local stability can be calculated by using a perturbation method. The first order perturbation term $dW_{1,j+1}$, which is the image of first order perturbation term $dW_{1,j}$, is calculated by differentiation of the dynamic, *i.e.* through differentiation of P . This first order perturbation is equivalent to calculating the Jacobian matrix DP of the Poincaré application. This Jacobian matrix can be determined with the help of the first order terms described in subsection 3.7:

$$(40) \quad \left\{ \begin{array}{l} DP : \mathbb{R}^{2n-2} \rightarrow \mathbb{R}^{2n-2} \\ dW_{1,j} \mapsto dW_{1,j+1} = (DG \circ DH_2 \circ DL \circ DF^k \circ DH_1)(dW_{1,j}) \end{array} \right\}$$

When the impact oscillator is one dimensional, the Jacobian DP cannot be defined. In that case t'_1 is uniquely defined, and it is easy to prove that these motions are always stable.

4.2. LOSS OF LOCAL STABILITY, LOCAL BIFURCATION

Once the Jacobian is calculated, the discussion regarding local stability depends on the eigenvalues of matrix DP . This $(2n - 2)$ square matrix is real but not symmetric, so some of its $2n - 2$ eigenvalues may be complex and conjugate. When all of them are inscribed in the unit circle of the complex plane the motion is stable. Conversely if only one of the eigenvalues lies outside this circle, it is unstable. If at least one eigenvalue lies on the unit circle, the dynamics of the system may bifurcate (Gumovski, 1980).

In the latter case, depending on the phase of eigenvalue λ , three kinds of bifurcation may be encountered. When $\lambda = -1$ the dynamics crosses over a flip or subharmonic bifurcation. In that case the previous motion still exists but becomes unstable and the dynamics of the system stabilises on a response whose period is twice the previous one (Guckenheimer, 1983). When $\lambda = 1$ the periodic solution disappears into a saddle node bifurcation (Guckenheimer, 1983). When $\lambda = e^{i\varphi} \notin \mathbb{R}$ and $\varphi \neq \pm\frac{\pi}{3}(2\pi)$ (Gumovski, 1980), the motion bifurcates into a Hopf bifurcation and the periodic motion turns into a quasiperiodic one.

In the case of a one dimensional impact oscillator, sticking periodic motions are always stable, and it can be proved that such local bifurcations cannot be encountered in their dynamics.

4.3. "RISING-BIFURCATION"

Infinite impact periodic responses may lose their stability through a new kind of dynamic phenomenon in which the sticking phase can be interrupted. The sticking phase of the motion ends when the contact reaction is zero and when it changes sign. This reaction is given in Eq. (9). The contact reaction is a function of time but it is also a function of the restricted displacement field Y . Therefore, in the case of a multidimensional impact oscillator, *i.e.* when Y can be defined, the end of the sticking phase is not uniquely defined as in the one dimensional case.

When varying a control parameter, the characteristics of a sticking response can change suddenly if a new zero contact reaction appears between t'_0 and t'_1 . We refer to this phenomenon as a "rising bifurcation" (Fig. 6) $(r, \omega, f, x_0) = (0.75, 148, 28.5, 3.2 \cdot 10^{-4})$. This bifurcation is qualitatively similar to a 'grazing bifurcation' (Nordmark, 1991) which can be encountered in finite impact periodic dynamics.

When a rising bifurcation is encountered, the original sticking periodic dynamics does not exist any more. This kind of bifurcation may lead to other kinds of dynamics, *e.g.* other kinds of sticking periodic dynamics (Fig. 7), or sticking chaotic dynamics. For example periodic motions with two or three sticking phases per period may result from these bifurcations. Similar to grazing bifurcations, rising bifurcations would merit specific investigation, but will be no longer analysed in the present paper.

5. Methods of investigation

5.1. ANALYTIC PREDICTOR-CORRECTOR METHOD

The method and results detailed in the two sections provide information regarding the existence and stability of sticking responses of multi-d.o.f. impact oscillators, but

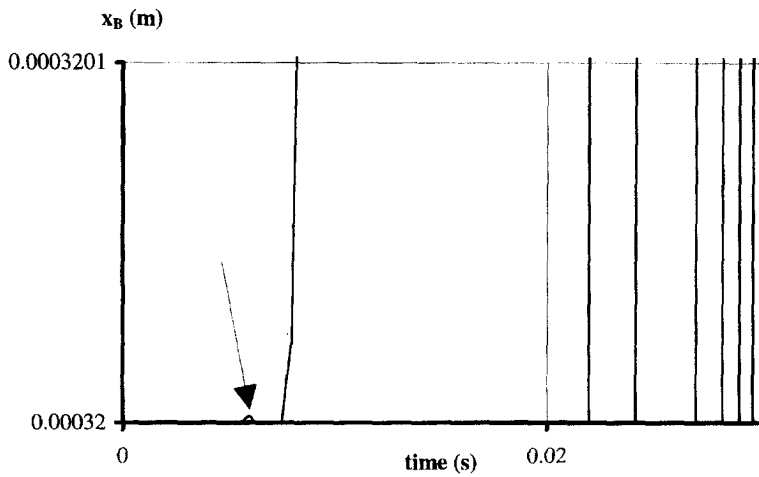
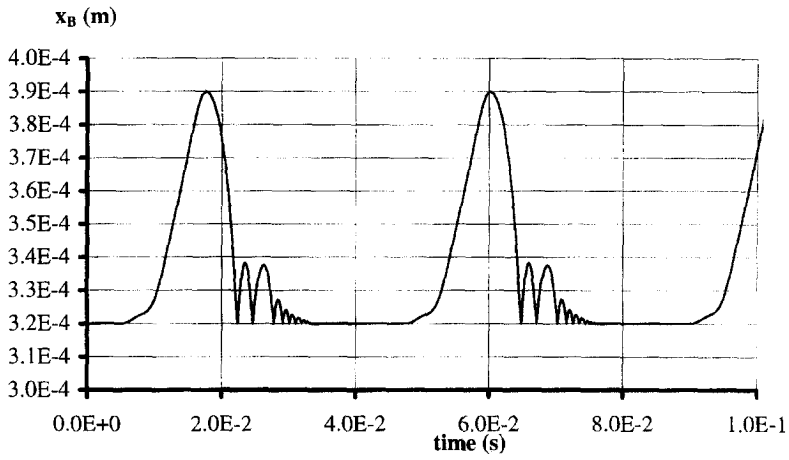
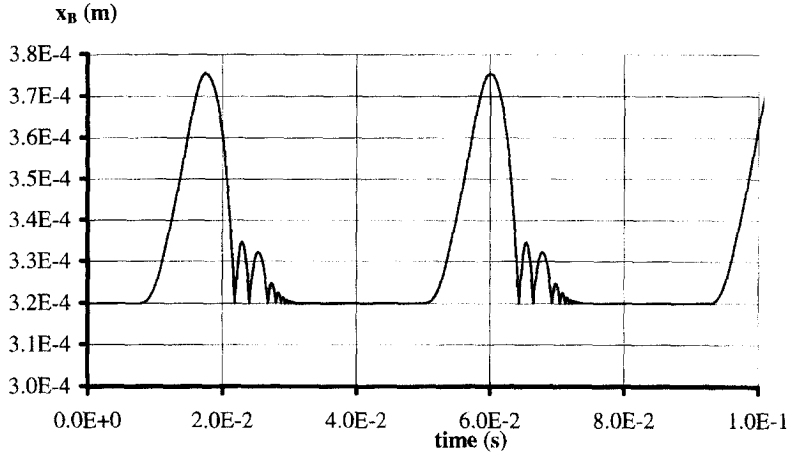


Fig. 6. – a) Infinite impact periodic reaction before a rising bifurcation. b) Infinite impact reaction after a rising bifurcation. c) Detail of a sticking motion during a rising bifurcation.

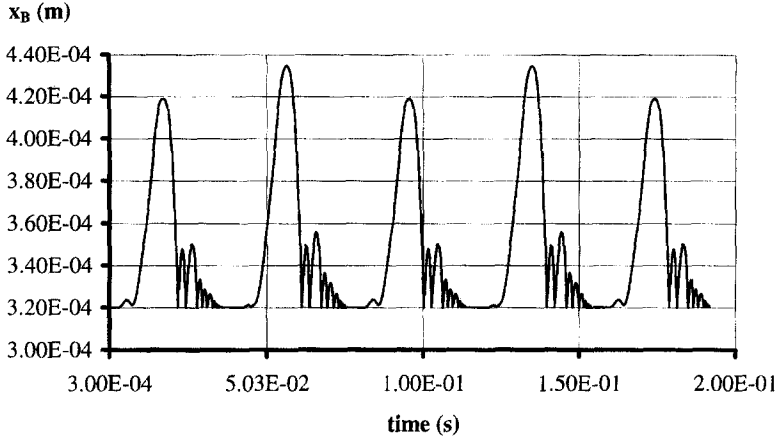


Fig. 7. – Infinite impact periodic two motions.

these results correspond to a given value (r, ω, f, x_0) of the control parameters. The purpose of constructing diagrams is to extend these results to a wider area inside the parameter space. Many numerical studies have been carried out, including bifurcation diagrams (Thompson, 1982). These calculations were limited to single d.o.f. impact oscillators. At the same time, analytic studies (Shaw, 1989) were made, but they were restricted to single impact periodic motions of one dimensional oscillators. The authors (Gontier, 1997), (Toulemonde, 1997) have carried out an analytic study of finite impact periodic motions of these kinds of oscillators.

It is useful to remark that the displacement constraint x_0 and magnitude f are redundant parameters, where $\mu = (r, \omega, f)$ denotes the reduced parameter vector of the system.

The analytic method used to build bifurcation diagrams is based on the continuation method and results from the differentiation of the Poincaré application defined by Eq. (25) (Allgower, 1990). The aim of the present investigation is to explore the phase parameter space from two points of view: by determining bifurcation diagrams; and constructing a partitioning of the parameter space. The first order differentiation of the periodic motions in Eq. (26) is the main tool for this investigation, and the procedure defined is a continuation method.

Essentially, there are two different types of continuation methods (Allgower, 1990). The Predictor Corrector (P.C.) methods and the Piecewise Linear (P.L.) methods. The idea of P.C. methods is to trace implicitly defined curves numerically by generating a sequence of points along the curve while satisfying a chosen tolerance criterion. In P.L. methods, the curve is linearised, piece by piece, then fitted to the system equations. Essentially, the continuation method can be used as a convex homotopic method applied to the solution of the nonlinear Eq. (27).

Equation (27) can be rewritten showing the relation between the parameter vector μ defined above and the unknown vector

$$(41) \quad \phi(\mu, W_1) = 0$$

When vector μ follows a curve in the parameter space, Eq. (41) implicitly defines a curve $W_1(\mu)$ in the space of unknowns. We assume the existence of a given starting point $(\mu_0, W_{1,0})$ satisfying Eq. (41). The aim of this method is to generate a sequence of points $(\mu_j, W_{1,j})_{j \in \mathbb{N}}$ satisfying Eq. (41) with a chosen tolerance criterion.

Assume that $(\mu_j, W_{1,j})$ is a root of the multidimensional function ϕ . A new point $(\mu_{j+1}, W_{1,j+1})$ along the implicitly defined curve is obtained by first making a Predictor Step, generally an Euler Predictor. This step is based on the first order differentiation of Eq. (41):

$$(42) \quad \left\{ \begin{array}{l} (\mu'_{j+1}, W'_{1,j+1}) = (\mu_j, W_{1,j}) + h(d\mu, dW_1) \\ \frac{\partial \phi}{\partial \mu}(\mu_j, W_{1,j})d\mu + \frac{\partial \phi}{\partial W_1}(\mu_j, W_{1,j})dW_1 = 0 \end{array} \right\}$$

In Eq. (42) h represents the “step-size”. The manner in which h is to be chosen depends on the application of the P.C. method; for example, it may be controlled by an arclength method. However, solution $(\mu'_{j+1}, W'_{1,j+1})$ is only a first order prediction and it must be corrected $(\mu_{j+1}, W_{1,j+1})$ is then obtained by applying a Corrector step to $(\mu'_{j+1}, W'_{1,j+1})$. An obvious way to obtain this correction is to use a Newton-like method. Defining the correction by $(\mu_{j+1}, W_{1,j+1}) = (\mu'_{j+1}, W'_{1,j+1}) + (d\mu, dW_1)$, the first Corrector step is such that

$$(43) \quad \frac{\partial \phi}{\partial \mu}(\mu'_{j+1}, W'_{1,j+1})d\mu + \frac{\partial \phi}{\partial W_1}(\mu'_{j+1}, W'_{1,j+1})dW_1 = -\phi(\mu'_{j+1}, W'_{1,j+1})$$

The solution of Eq. (43) is not unique, so that another relation between $d\mu$ and dW_1 must be looked for. This relation defines a correction strategy, which depends on the application of the continuation method. In view of constructing a bifurcation diagram, choosing $d\mu = 0$ was found to be sufficient. However, when partitioning parameter space, a strategy at μ_1 or μ_2 constant can be deficient because of the shape of the curves, especially if they have asymptotic branches. So another equation is added to Eq. (43), which defines the corrector step. This equation determines $d\mu$ and then dW_1 is derived from Eq. (43).

5.2. ADVANTAGES OF THE P.C. METHOD

Many authors have obtained bifurcation diagrams by time integration of the dynamics. Actually, numerical approaches usually produce rather noisy diagrams, whose inaccuracies are inherent in the weak stability of the dynamics and the integration process. There are two reasons why more accurate results are obtained here. First, in some cases, as in the present situation, the dynamics of infinite impact periodic motions is totally described in an analytic manner. This represents a dramatic improvement in accuracy over a step-by-step numerical integration.

The second reason derives from the considerable loss of accuracy of numerical methods when approaching bifurcation or when motions are not stable. In these zones, periodic solutions are considerably less attractive (Thompson, 1982), and numerical methods

hardly converge. Conversely, both the determination of infinite impact periodic motions described in section 3 and the continuation method described in this section are not sensitive to local instability, as the points in the bifurcation diagram remain defined by an analytic method.

Another advantage of the continuation method is that it requires less CPU time than direct integration methods. The computation of a neighbouring point requires only the calculation of the two differential elements of Eq. (41). A very good approximation of the neighbouring point is obtained with only one Predictor step and less than two Corrector steps, which take very little time to perform on a standard 486 PC.

Moreover the continuation method is characteristically efficient and accurate, especially when applied to analytic functions. For a given set of control parameters the method investigates the finite impact periodic responses and it also able to compute the Floquet multipliers analytically. These results are very useful in bifurcation diagrams, as they are the key to an accurate prediction of local bifurcations and subsequently of new branches.

5.3. BIFURCATION DIAGRAMS

The method described in the last two subsections makes it possible to slide along a branch of a bifurcation diagram. Thus the diagrams presented here will stop at the point at which the motions bifurcate in a non-periodic way-quasiperiodicity occurs after Hopf bifurcation and chaos may appear after rising bifurcation. In the bifurcation diagrams presented in this article the sticking proportion is projected onto the vertical axis and the magnitude f of the harmonic excitation is used as a control parameter.

5.4. PARTITIONING THE PARAMETER SPACE

The P.C. method can also be applied to the partitioning of a parameter space, for instance the (ω, f) plane. In the parameter plane, curves are drawn representing the motions with a constant sticking proportion. The partitioning lines correspond to constant sticking proportion motions, defined as a ratio of $\frac{t'_1 - t'_0}{T}$. Thus starting from a given point (ω_0, f_0) in the plane associated with a given proportion, a curve can be drawn by a continuation process at constant proportion, representing a set of equivalent sticking orbits.

The algorithm used to calculate these curves is described in subsection 5.1 where a new constant proportion condition is added to Eq. (42) during the Predictor step. In the same fashion the Corrector Step must take the error in the calculation of the proportion into account.

6. Computational results

6.1. SYSTEMS WITH ONLY ONE D.O.F.

In this subsection the above methods are applied to the one d.o.f. oscillator. As stated above, if a sticking periodic response exists in the dynamics of a one d.o.f. system, it can

be easily proved that this motion is locally stable. Local bifurcation such as subharmonic bifurcation or Hopf bifurcation cannot be encountered in this kind of dynamics.

As a consequence, the cascades of flip bifurcations observed in the dynamics of finite impact motions and referenced in the literature (Shaw, 1989; Gontier, 1997) do not exist in the sticking dynamics of single degree of freedom impact oscillators.

Depending on the sign of displacement constraint x_0 , two classes of impact oscillators and therefore two classes of dynamic behaviour can be encountered. Sticking periodic motions have thus been proved to exist in both cases, depending on whether the systems have clearance or are preloaded.

(i) In the case of a preloaded system, *i.e.* when $x_0 > 0$, sticking periodic motions exist for every value of the the excitation pulsation. They exist for excitation magnitude values larger than the value of the preload force induced by x_0 ; below this force, sticking motions cannot be encountered. These responses exist either with a plastic impact or elastic restitution rule (*Fig. 5b*).

By applying the method described in section 5, bifurcation diagrams can be constructed. For a given value of excitation pulsation, the excitation magnitude f can be regarded as a control parameter of the dynamic system. It can be proved that Sticking periodic motions can be proved to be non existent for values of f smaller than the preload induced by x_0 . But they begin to appear when f is slightly larger. These first responses give rise to a time ratio of sticking dynamics slightly less than 1. It is then observed that the sticking proportion decreases as a function of amplitude (*Fig. 8*). For values of f larger than a given level, sticking motions disappear and finite impact responses occur (*cf.* Eq. (38)). This result shows that for a given value of excitation pulsation, sticking period one responses exist only for a finite segment of excitation magnitude.

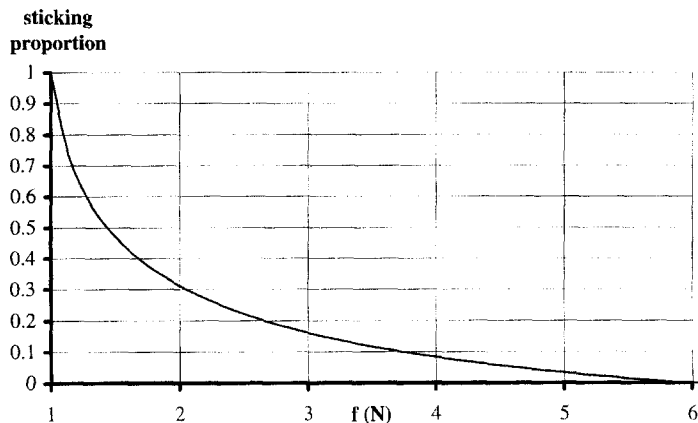


Fig. 8. – Bifurcation diagram, one dimensional system with preload. $(r, \omega) = (0.75, 2.8)$.

Infinite impact responses of impact oscillators with preload can also be represented in a two dimensional (ω, f) parameter space, where sets of sticking motions with the same sticking ratio can be easily represented (*Fig. 9*). It must be noted that these curves are very smooth and that they are not interrupted by any kind of bifurcation.

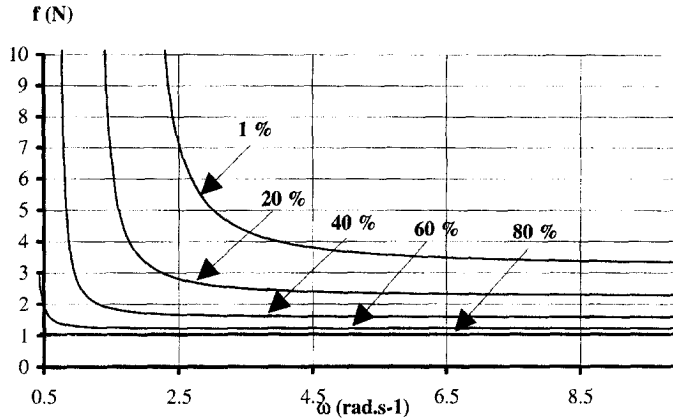


Fig. 9. – Parameter space partitioning, one dimensional system with preload, $r = 0.75$.

For large values of excitation pulsation, the constant sticking ratio curves are parallel to the pulsation axis. This means the qualitative behaviour of these motions does not depend on the pulsation when it is sufficiently large. For small values of excitation pulsation and for large values of excitation magnitude, the constant sticking ratio curves are parallel to the magnitude axis. In case the qualitative aspect of the impact response does not depend on the excitation magnitude. Partitioning (Fig. 9) clearly shows that sticking motions exist in a restricted area of the parameter space and that they will not be observed in the upper right quarter of the (ω, f) plane. It should be noted that the limit provided by these numerical calculations have been predicted by the purely analytic investigation of section 3.8.

The conclusions of section 3.8 prove that subharmonic sticking periodic responses exist and that the responses predicted by these analytic calculations have only one free-flight phase per period. When the restitution coefficient is null, some other kinds of orbits exist, some of which have two sticking phases per period (Fig. 10).

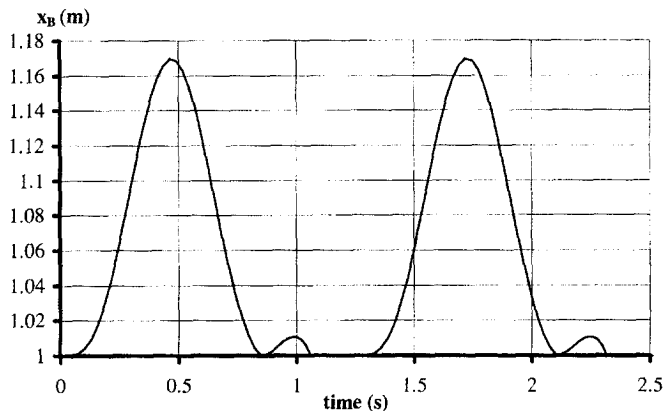


Fig. 10. – Periodic response with two free-flight phases per period, purely plastic impact rule, $(r, \omega, f) = (0, 10, 4.8)$.

(ii) In the case of a mechanical system with clearance, *i.e.* when $x_0 < 0$, sticking periodic responses have been proved to exist either with a purely plastic impact rule or with an elasto-plastic impact rule (*Fig. 11*), if the restitution coefficient is too large, sticking periodic responses are much more difficult to discover.

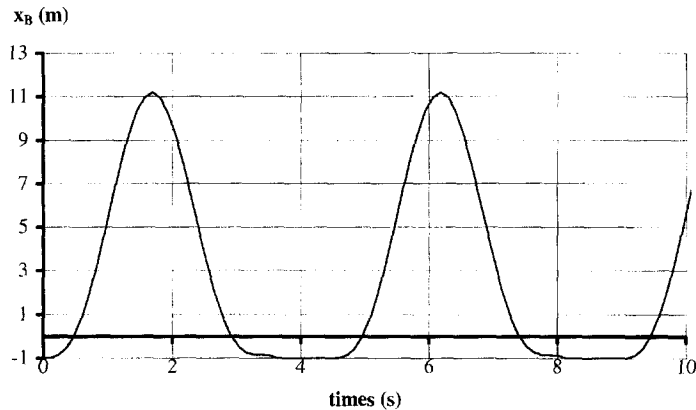


Fig. 11. – Sticking periodic motion, plastic impact rule, one dimensional system with clearance, $(r, \omega, f) = (0, 1.4, 10)$.

Bifurcation diagrams have also been constructed, and for a given value of the excitation pulsation, the field of excitation magnitudes have been explored. The least value of the amplitude for which sticking motions exist here corresponds to a very low sticking proportion. Although local bifurcation cannot be encountered in these dynamics, grazing bifurcations have been found. Here, the single-periodic sticking responses can disappear, and lead to two-periodic sticking responses (*Fig. 12*). This diagram shows that such motions can be observed for large values of excitation magnitude.

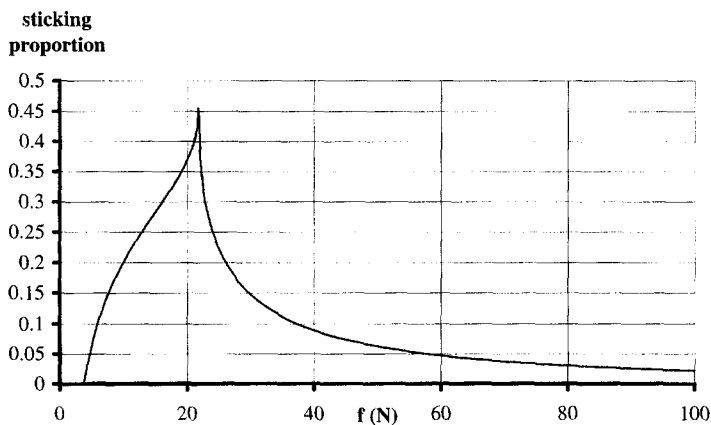


Fig. 12. – Bifurcation diagram, one dimensional system with clearance, $(r, \omega) = (0, 1.4)$.

A parameter space partitioning has also been constructed for such systems (*Fig. 13*), where the curves are much more complex than in the preload case, and different classes

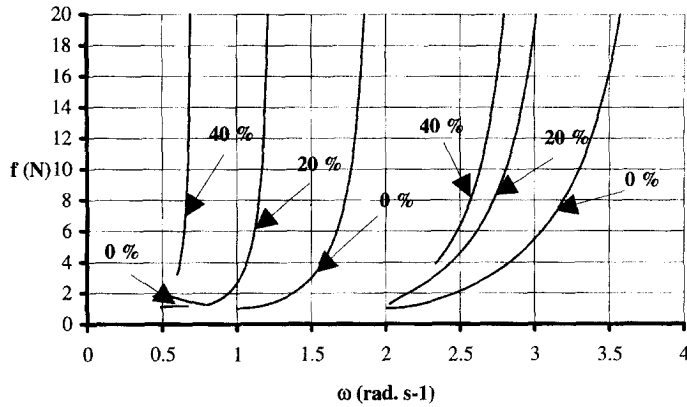


Fig. 13. – Parameter space partitioning, one dimensional system with clearance, $r = 0$.

of curves can be observed, which seem to be asymptotically parallel to the magnitude axis. It is easy to prove here that the reaction cannot be maintained at a positive value for half the period, and consequently that the sticking ratio is always smaller than 50%.

6.2. SYSTEMS WITH SEVERAL D.O.F.

The methods described above have been extended to some structures with several d.o.f. There are no major differences in shape between multidimensional responses and motions in the one d.o.f. case. Sticking periodic motions either exist with a purely plastic or with an elasto-plastic impact rule (Fig. 2-3).

In the sticking dynamics of multi-d.o.f. systems, periodic motions can lose their stability in different ways. For instance, local bifurcation can be encountered, as in the case of a three d.o.f. system a Hopf bifurcation is encountered and a sticking periodic motion is turned into a sticking quasiperiodic response (Fig. 14). Such dynamic behaviour could not have been observed in the single d.o.f. sticking dynamics.

Subharmonic bifurcations have not been encountered in these sticking dynamics. This is a major difference with finite impact periodic dynamics where they have been very

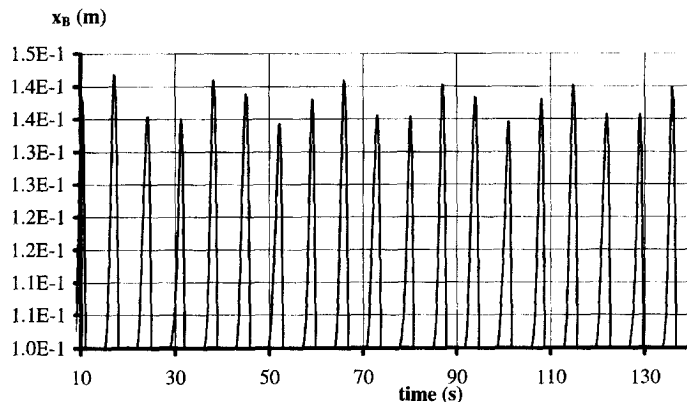


Fig. 14. – Quasiperiodic sticking response, three d.o.f. impact oscillator.

frequently observed (Gontier, 1997; Shaw, 1989) either in one or two dimensional systems (Toulemonde, 1997).

The new kind of bifurcation, called “rising bifurcation” and predicted in subsection 4.3, which cannot be predicted with the help of eigenvalues, can be found in a system with at least two d.o.f. Although it has no major influence on the shape of the motions (Fig. 6) it has a very strong influence on the shape of bifurcation diagrams whose smoothness can be broken by this kind of bifurcation. The sticking proportion decreases very sharply following rising bifurcations (Fig. 15) and some non-periodic motions may result from them. This result can be compared to the existence of chaotic motions after grazing bifurcations (Nordmark, 1991), as immediately afterwards a succession of chaotic (Fig. 16) and periodic windows can be observed.

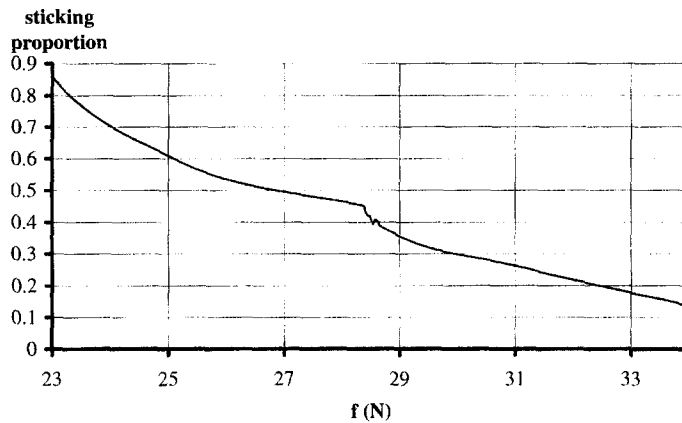


Fig. 15. – Bifurcation diagram, two dimensional system with preload. A rising bifurcation is encountered in this diagram.

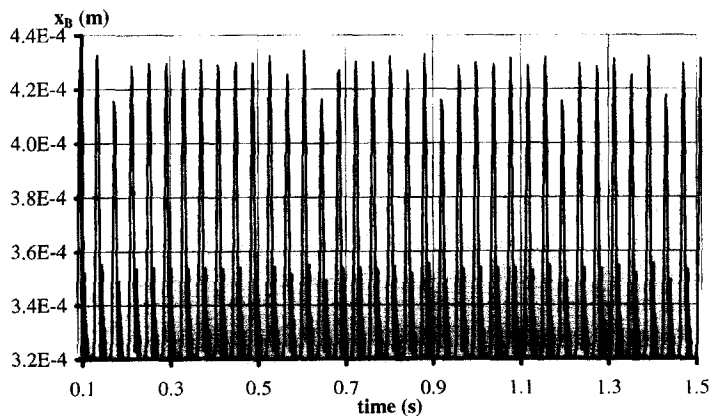


Fig. 16. – Chaotic response with sticking phases, two d.o.f. impact oscillator.

The phenomena cited above are typical of the sticking behaviours of multiple degree of freedom impact oscillators, which cannot be encountered in the sticking dynamics of

single degree of freedom impacting systems. And it has been proved in section 4.3 that rising bifurcations (Fig. 15) cannot be observed in the dynamics of a one dimensional system. Also, the chaotic behaviour (Fig. 16) can be observed only in multidimensional cases because it has been proved that the sticking motions of the single degree of freedom impact oscillators are always periodic and stable.

A parameter space partitioning has also been constructed for a multiple d.o.f. case, where constant sticking proportion curves are interrupted by the existence of rising bifurcations (Fig. 17). These bifurcations can turn periodic responses into chaotic ones so that the periodic branch cannot be continued. Thus it can be noted that the lower the sticking ratio, the more often the curve is interrupted.

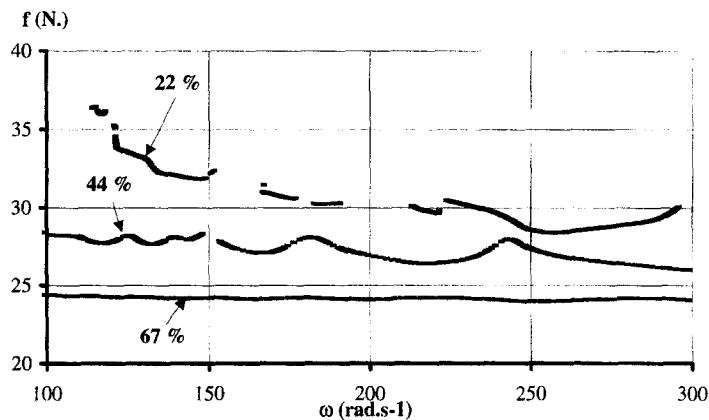


Fig. 17. – Parameter space partitioning, two dimensional system. The curves are interrupted by rising bifurcation.

7. Conclusions

The phenomenon of sticking periodic motions of impact oscillators has been analysed and a general methodology of approach suitable for multi-d.o.f. oscillators as well as one d.o.f. oscillators has been defined; bifurcation diagrams and parameter space partitionings have been determined. In the case of multi-d.o.f. oscillators some specific features such as quasiperiodic sticking motions and a new kind of bifurcation called “rising bifurcation” have been identified. All these examples have indicated the efficiency of the method used to investigate these sorts of problems. At the present time, the tool developed here should be used more extensively to provide a better understanding of the sticking solutions of impact oscillators. Rising bifurcations, transitions from unstable solutions to stable solutions, the limit of chaotic responses are some questions among many which would deserve a more thorough investigation. From a general point of view, multiple d.o.f. impact oscillator behaviour has to a large extent been unexplored up to now.

Acknowledgements. – The authors would like to thank Michel Panet and Alain Leger at E.D.F., for their assistance.

REFERENCES

- ALLGOWER E. L., GEORG K., 1990, *Numerical Continuation Method, An introduction*, Springer-Verlag, New York, U.S.A.
- BUD C., FELIX D., 1994, Chattering and related behaviour in impact oscillators, *Philosophical Transactions of the Royal Society of London, Series A*, **347**, 365-389.
- CHOW S. N., SHAW S. W., 1986, Bifurcation of Subharmonics, *Journal of Differential Equations*, **65**, 304-320.
- GONTIER C., TOULEMONDE C., 1997, Approach to the Periodic and Chaotic Behaviour of the Impact Oscillator by a Continuation Method, *European Journal of Mechanics A/Solid*, **16**, No. 1, 141-163.
- GUCKENHEIMER J., HOLMES P. J., 1983, *Non-linear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, U.S.A.
- GUMOVSKI L., MIRA C., 1980, *Dynamique chaotique : transformations ponctuelles, transitions ordre-désordre*, Cepadues, Paris, France.
- HOLMES P. J., MOON F. C., 1983, Strange Attractors and Chaos in Nonlinear Mechanics, *ASME Journal of Applied Mechanics*, **50**, 1021-1032.
- IVANOV A. P., 1994, Impact Oscillations: Linear Theory of Stability and Bifurcations, *Journal of Sound and Vibration*, **178**, 361-378.
- MOON F. C., LI G. X., 1990, Experimental Study of Chaotic Vibrations in a Pin-jointed Space Truss Structure, *AIAA Journal*, **28**, 915-921.
- NORDMARK A. B., 1991, Non-Periodic Motions Caused by Grazing Incidence in an Impact Oscillator, *Journal of Sound and Vibrations*, **145**, 279-298.
- PAOLI L., 1993, *Analyse numérique de vibrations avec contraintes unilatérales*, PhD, Université Claude-Bernard, Lyon-I, Laboratoire d'Analyse Numérique.
- SHAW S. W., HOLMES P. J., 1983, Periodically forced linear oscillator with impacts: chaos and long-period motions, *Physical Review Letters*, **51**, No. 8, 623-626.
- SHAW S. W., RAND R. H., 1989, The Transition to Chaos in a Simple Mechanical System, *International Journal of Non-Linear Mechanics*, **24**, 41-56.
- THOMPSON J. M. T., GHAFFARI R., 1982, Chaos After Period-Doubling Bifurcations in the Resonance of an Impact Oscillator, *Physics Letters*, **91 A**, 5-8.
- THOMPSON J. M. T., STEWART. 1986, *Nonlinear Dynamics and Chaos*, John Wiley and Sons, New York, U.S.A.
- TOULEMONDE C., GONTIER C., 1997, Multiple Degree of Freedom Impact Oscillator, *European Journal of Mechanics A/Solid*, **16**, No. 5, 879-904.