A new discrete impulse response Gramian and its application to model reduction

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Abstract

Some fundamental properties of a new impulse response Gramian for linear time-invariant asymptotically stable discrete SISO systems are derived in this note. This Gramian is a system-invariant and can be found by solving a Lyapunov equation. The connection with standard controllability, observability and cross Gramians is proved. The significance of these results in model-order reduction is highlighted with an efficient procedure.

1. Introduction

Impulse Response Gramians (IRG) have been introduced by Sreeram and Agathoklis to derive Reduced Order Models (ROM) for linear time-invariant asymptotically stable continuous [1] or discrete [2] SISO systems. Usefulness of these Gramians has also been shown in system identification application.

An IRG contains elements which are inner products of functions given as successive derivatives or delays of the impulse response in continuous case and discrete case respectively. It can be obtained by solving the Lyapunov equation for the controllability canonical realization of the system.

In [1] the approach is based on matching the first q Markov parameters and $q \times q$ entries of the IRG. The procedure has been extended to discrete case in [3] where the relation to the q-Markov Cover method is discussed [4] [5] [6] . This usually yields good approximations at high frequencies, but a large error on the steady state behaviour is noticed. An improved low-frequency approximation is achieved for discrete systems in [10] by matching some initial time-moments and low-frequency power

moments. For continuous systems, this drawback has been overcome with a reciprocal transformation [7] [8] to preserve the first q time-moments and $q \times q$ entries of the Gram matrix [9] . Another ROM building procedure, both valid in the continuous case [11] and in the discrete case [2] , is based on the approximation of a energy criteria by a diagonalization and a direct truncation of the IRG. Use of the singular perturbation technique is suggested if a good approximation at low frequency is required. Note that the methods in [1] [2] [11] still apply if the IRG is weighted (WIRG).

The approach in [2] has been recently extended to MIMO systems with the definition of an Extended Impulse Response Gramians (EIRG) and a convergence property to balanced realization [13] [14] has been established.

Krajewski et al. have proposed a mixed use of the results in [1] and [8] to derive a ROM matching Markov parameters, time-moments and impulse response energies [15] . It is based on a generalized definition of the IRG of Sreeram and Agathoklis using successive derivatives and/or integrals of the impulse response. This method is efficient but applies only to continuous time systems.

The initial motivation for the present paper is the extension of this approach to discrete case. A Generalized Impulse Response Gramian (GIRG) composed with scalar products of successive differences and/or sums of the impulse response is introduced for linear time-invariant asymptotically stable discrete SISO systems. It is related to standard controllability, observability and cross Gramians, and is found to be the solution to the Lyapunov equation for a particular state space representation. It is also shown that the characteristic polynomial can be obtained using some impulse energies contained in the GIRG. Application of these properties to model reduction is then investigated and an efficient procedure is proposed. The ROM is elaborated in two major steps : a reduced characteristic polynomial is first computed and then some Markov parameters and/or time-moments are retained. The stability and minimality properties of this ROM are studied. A numerical example is proposed and a comparison with well known discrete model reduction techniques is carried out.

2. The discrete Generalized Impulse Response Gramian

In this section we first define the GIRG, and then describe properties of this Gramian.

Let (A, b, c) be an nth order minimal state space realization of a stable, linear, discrete SISO system with impulse response $h[k] = cA^{k-1}b$.

Definition 2.1 *The $(n+1)$th order Generalized Impulse Response Gramian is defined as follows :*

$$W_{q,n+1} \hat{=} [\ \langle w_{q+i-1}, w_{q+j-1} \rangle\ ]_{i,j=1,\ldots,n+1} \quad , \quad q = -n+1, \ldots, 0 \tag{1}$$

with $w_0[k] \hat{=} h[k]$ and

$$w_{l+1}[k] \hat{=} w_l[k+1] - w_l[k]\ ,\ l \in \mathbb{N}^+ \quad ;\ \ w_{l-1}[k] \hat{=} -\sum_{l'=k}^{\infty} w_l[l']\ ,\ l \in \mathbb{N}^- \tag{2}$$

and where $\langle f, g \rangle \hat{=} \sum_{k=1}^{\infty} f[k]g[k]$ denotes the inner product of two causal real functions $f[k]$, $g[k]$.

Successive differences and sums of the impulse response defined in (2) have been previously used as candidates for constructing a set of approximating functions in [16] . An interesting property of such operators is that they preserve the original poles in the z-domain.

Some key properties of the GIRG are now considered in the following Theorem.

Theorem 2.1

i) The nth order GIRG $W_{q,n}$ can be written as

$$W_{q,n} = \mathcal{C}_q^T W_o \mathcal{C}_q \quad , \quad W_{q,n} = \mathcal{O}_q W_c \mathcal{O}_q^T \quad , \quad W_{q,n} = \mathcal{O}_q W_{co} \mathcal{C}_q \tag{3}$$

where W_c, W_o, W_{co} denote respectively the standard controllability, observability and cross Gramian for any minimal realization (A, b, c).

The matrices $\{\mathcal{C}_q,\ \mathcal{O}_q\}$ used in the above factorizations are given by

$$\mathcal{C}_q = \left[(A-I)^q\, b, \ldots, (A-I)^{q+n-1}\, b\right] \ \text{and}\ \mathcal{O}_q^T = \left[\left[c\,(A-I)^q\right]^T, \ldots, \left[c\,(A-I)^{q+n-1}\right]^T\right] \tag{4}$$

ii) $W_{q,n}$ is the solution to the Lyapunov equation

$$W_{q,n} - \hat{A}^T W_{q,n} \hat{A} = \hat{c}_q^T \hat{c}_q \tag{5}$$

where $(\hat{A}, \hat{b}_q, \hat{c}_q)$ is derived from (A, b, c) by the coordinate transformation $\mathcal{C}_q$.

iii) The realization $(\hat{A}, \hat{b}_q, \hat{c}_q)$ has the following structure (in accordance with the proposed values

for q, see eqn. (1)) :

$$\hat{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\bar{a}_n \\ 1 & 1 & \ddots & \vdots & -\bar{a}_{n-1} \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & -\bar{a}_2 \\ 0 & \cdots & 0 & 1 & -\bar{a}_1 + 1 \end{bmatrix} \tag{6}$$

$$\hat{b}_q^T = \left[\underbrace{0, \ldots, 0}_{-q}, 1, \underbrace{0, \ldots, 0}_{q+n-1} \right] \quad , \quad \hat{c}_q = \left[\underbrace{\ldots, -t_2, -t_1}_{-q}, \underbrace{m_1', m_2', \ldots}_{q+n} \right] \tag{7}$$

where $\{t_i\}_{i=1,2,\ldots}$ are the time-moments of the system and $\left\{ m_i' = c\,(A-I)^{i-1}\,b \right\}_{i=1,2,\ldots}$ are given as linear combinations of the Markov parameters $\{m_i\}_{i=1,2,\ldots}$:

$$t_i = c\,(I-A)^{-i}\,b \quad , \quad m_i = cA^{i-1}b \quad , \quad m_i' = \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} m_{i-j} \tag{8}$$

and where $\{\bar{a}_i\}_{i=1,\ldots,n}$ denote the characteristic polynomial coefficients for $(A-I)$.

Proof.

i) Starting with $w_0[k] = h[k]$, it is easily shown that for any $l \in \mathbb{N}$ the function $w_l[k]$ derived using one of the transformations in (2) can be expressed as $w_l[k] = c\,(A-I)^l\,A^{k-1}b = cA^{k-1}\,(A-I)^l\,b$.

Writing each inner product $\langle w_{q+i-1}, w_{q+j-1} \rangle$ appearing in (1) in terms of the (A, b, c) matrices then yields directly the relations in (3). As A is assumed to be asymptotically stable ($\|A\| < 1$), the matrix $(A-I)$ is nonsingular [19] . Thus, the existence of $\{\mathcal{C}_q,\ \mathcal{O}_q\}$ is ensured and because $\{A, b\}$ is controllable, $\mathcal{C}_q$ is nonsingular.

ii) The observability Gramian for $\left\{ \hat{A}, \hat{c}_q \right\}$ is given by $\mathcal{C}_q^T W_o \mathcal{C}_q$, which is seen to be the nth order GIRG for $h[k]$ in view of (3).

iii) Let $q = 0$ and $\bar{p}(z) = \sum_{i=0}^{n} \bar{a}_i z^{n-i}$ be the characteristic polynomial for $(A-I)$. It is well known that a similarity transformation using the standard controllability matrix yields a state matrix under companion form :

$$\mathcal{C}^{-1} A \mathcal{C} = \hat{A}' = \left[\begin{array}{c|c} \mathbf{0} & -\mathbf{a} \\ \hline I_{n-1} & \end{array} \right] \quad , \quad \mathcal{C} = \begin{bmatrix} b & Ab & \ldots & A^{n-1}b \end{bmatrix}$$

where $\mathbf{a}^T = [a_n, ..., a_1]$ and $p(z) = \sum_{i=0}^{n} a_i z^{n-i}$ is the characteristic polynomial of A.

Then it follows that

$$\mathcal{C}_0^{-1}(A-I)\mathcal{C}_0 = \widehat{(A-I)}' = \left[\begin{array}{c|c} \mathbf{0} & \\ \hline I_{n-1} & -\bar{\mathbf{a}} \end{array}\right]$$

where $\mathcal{C}_0 = \left[\begin{array}{cccc} b & (A-I)b & \dots & (A-I)^{n-1}b \end{array}\right]$ and $\bar{\mathbf{a}}^T = [\bar{a}_n, ..., \bar{a}_1]$.

Then we get (6). As the differences and sums in (2) preserve the original poles, the state matrix is the same for $q = -n+1, \dots, 0$. The proof of (7) is straightforward and is omitted. ■

Remark 1 *As* $\left\{\hat{A}, \hat{c}_q\right\}$ *is observable, it follows from (5) that* $W_{q,n}$ *is positive definite and the* l*th order GIRG* $W_{q,l}$ *is positive definite for any* $l \leqslant n$.

The following theorem shows that the characteristic polynomial can be extracted from the GIRG.

Theorem 2.2 *Let* $p(z) = \sum_{i=0}^{n} a_i z^{n-i}$ $(a_0 = 1)$ *be the characteristic polynomial for any minimal realization* (A, b, c) *of the system. Let the corresponding* $(n+1)$*th order GIRG be partitionned as*

$$W_{q,n+1} = \left[\begin{array}{c|c} W_{q,n} & \mathbf{w}_{q,n+1} \\ \hline \mathbf{w}_{q,n+1}^T & w_{q,n+1} \end{array}\right]$$

with $W_{q,n}$ *the* n*th order GIRG,* $\mathbf{w}_{q,n+1} \in \mathbb{R}^{n\times 1}$ *and* $w_{q,n+1} \in \mathbb{R}$. *Then the following equation holds:*

$$\bar{\mathbf{a}} = -\left(W_{q,n}\right)^{-1} \mathbf{w}_{q,n+1} \tag{9}$$

where $\bar{\mathbf{a}}^T = [\bar{a}_n, \dots, \bar{a}_1]$ *and* $\sum_{i=0}^{n} \bar{a}_i z^{n-i} = p(z+1)$.

Proof. By definition $\mathbf{w}_{q,n+1}$ is given as

$$\mathbf{w}_{q,n+1} = \sum_{k=1}^{\infty} \left[w_q[k], ..., w_{q+n-1}[k]\right]^T w_{q+n}[k] \quad \text{with} \quad w_{q+n}[k] = cA^{k-1}\left(A-I\right)^{q+n} b \tag{10}$$

Let $\bar{p}(z) = \sum_{i=0}^{n} \bar{a}_i z^{n-i}$ be the characteristic polynomial for $(A-I)$. Then from the Cayley-Hamilton theorem we get

$$\left(A-I\right)^{q+n} = -\sum_{i=1}^{n} \bar{a}_i \left(A-I\right)^{q+n-i} \quad \text{and} \quad w_{q+n}[k] = -\sum_{i=1}^{n} \bar{a}_i w_{q+n-i}[k] \tag{11}$$

Finally, substituting (11) into (10) yields the relation (9). ■

Usefulness of these results in model order reduction is shown in the next section.

3. Model Order Reduction

Let us consider an nth order original model described by the stable proper transfer function $H(z) = \frac{N(z)}{D(z)}$ with a minimal realization $\{x[k+1] = Ax[k] + bu[k]\ ,\ y[k] = cx[k]\}$.

The objective of model reduction is to find a state space realization $\{\tilde{x}[k+1] = A_r\tilde{x}[k] + b_r u[k]\ \ ,\ \tilde{y}[k] = c_r\tilde{x}[k]\}$ with $\tilde{x}[k] \in \mathbb{R}^{r\times 1}$ and $r < n$, such that $\tilde{y}$ approximates y as close as possible for all admissible inputs.

Let $W_{q,r+1}$ and $\tilde{W}_{q,r+1}$ be the $(r+1)$th order GIRG for the original and reduced-order model respectively, with the q parameter choosen in the set $\{-r+1, ..., 0\}$.

An efficient GIRG-based model reduction technique will be proposed in the following. This technique can be seen as an extension to discrete systems of the approach considered in [15] . It consists in an approximation of some impulse response energies by first finding a reduced-order characteristic polynomial $\tilde{p}_q(z)$ and then matching some Markov parameters and/or time-moments.

Let the Gramians $W_{q,r+1}$ and $\tilde{W}_{q,r+1}$ be partitionned as

$$W_{q,r+1} = \left[\begin{array}{c|c} W_{q,r} & \mathbf{w}_{q,r+1} \\ \hline \mathbf{w}_{q,r+1}^T & w_{q,r+1} \end{array}\right] \quad , \quad \tilde{W}_{q,r+1} = \left[\begin{array}{c|c} \tilde{W}_{q,r} & \tilde{\mathbf{w}}_{q,r+1} \\ \hline \tilde{\mathbf{w}}_{q,r+1}^T & \tilde{w}_{q,r+1} \end{array}\right] \tag{12}$$

Suppose that $\tilde{W}_{q,r}$ matches the original rth order GIRG: $\tilde{W}_{q,r} = W_{q,r}$.

We will now calculate a rth degree polynomial $\tilde{p}_q(z)$ such that $\|\tilde{\mathbf{w}}_{q,r+1} - \mathbf{w}_{q,r+1}\|_2^2$ is minimized : From Theorem 2.2 $\tilde{\mathbf{w}}_{q,r+1} = -W_{q,r}\overline{\tilde{\mathbf{a}}}_q$, where $\overline{\tilde{\mathbf{a}}}_q^T = \left[\overline{\tilde{a}}_r, \ldots, \overline{\tilde{a}}_1\right]$ and $\overline{\tilde{p}}_q(z) = \sum_{i=0}^{r} \overline{\tilde{a}}_i z^{r-i} = \tilde{p}_q(z+1)$. Hence, $\tilde{\mathbf{w}}_{q,r+1}$ matches $\mathbf{w}_{q,r+1}$ if

$$\overline{\tilde{\mathbf{a}}}_q = -\left(W_{q,r}\right)^{-1}\mathbf{w}_{q,r+1}. \tag{13}$$

As $W_{q,r}$ is positive definite, its nonsingularity is ensured (Remark 1). Once $\overline{\tilde{\mathbf{a}}}_q$ has been computed, a reduced-order state matrix $\widehat{\tilde{A}}_q$ with a form like (6) is readily obtained; it remains to choose input and output vectors $(\widehat{\tilde{b}}_q, \widehat{\tilde{c}}_q)$ to get our ROM.

From Theorem 2.1 we know that the rth order GIRG for the ROM solves the Lyapunov equation

$$\tilde{W}_{q,r} - \widehat{\tilde{A}}_q^T \tilde{W}_{q,r} \widehat{\tilde{A}}_q = \widehat{\tilde{c}}_q^T \widehat{\tilde{c}}_q \tag{14}$$

where $\left\{\widehat{\tilde{b}}_q, \widehat{\tilde{c}}_q\right\}$ have a form like (7).

The matrices $\left\{\widehat{\tilde{A}}_q, \widehat{\tilde{b}}_q\right\}$ are known assuming that a characteristic polynomial for the ROM has been

computed using (13). Then, (14) suggests that $\widehat{\tilde{c}}_q$ may be choosen so that some of the time-moments and/or Markov-type parameters (8) of the original model are matched.

The main steps involved in our reduction procedure are then summarized as:

Step 1. Given an nth order original model (A, b, c), choose any $q \in \{-r+1, ..., 0\}$ with $r < n$ and then determine the particular realization $(\hat{A}, \hat{b}_q, \hat{c}_q)$ using the similarity transformation $\mathcal{C}_q$ in (4).

Step 2. Solve the Lyapunov equation (5) to determine the nth order GIRG $W_{q,n}$.

Step 3. Partition the $(r+1)$th order original GIRG as (12) and solve (13) to obtain a rth degree characteristic polynomial .

Step 4. Form the reduced realization $(\widehat{\tilde{A}}_q, \widehat{\tilde{b}}_q, \widehat{\tilde{c}}_q)$: the state matrix $\widehat{\tilde{A}}_q$ follows from step 3 with a structure as in eqn. (6) and $\left\{\widehat{\tilde{b}}_q, \widehat{\tilde{c}}_q\right\}$ matches the first r entries of $\left\{\hat{b}_q, \hat{c}_q\right\}$.

The condition to preserve the initial stability is given by the next theorem:

Theorem 3.1 *Let $(\widehat{\tilde{A}}_q, \widehat{\tilde{b}}_q, \widehat{\tilde{c}}_q)$ be the rth order ROM of any asymptotically stable initial system (A, b, c) derived using our GIRG-based algorithm : $\tilde{p}_q(z)$ cannot have any zeros outside the unit circle. Furthermore, provided $\left\{\widehat{\tilde{A}}_q, \widehat{\tilde{c}}_q\right\}$ is observable, the ROM is asymptotically stable.*

Proof. From Theorem 2.1 it is known that the nth order original GIRG $W_{q,n}$ is the solution to the Lyapunov equation $W_{q,n} - \hat{A}^T W_{q,n} \hat{A} = \hat{Q}_q$ with $\hat{Q}_q = \hat{c}_q^T \hat{c}_q$, and where $(\hat{A}, \hat{b}_q, \hat{c}_q)$ is obtained from any realization (A, b, c) using the similarity transformation $\mathcal{C}_q$ defined in (4). It is seen that the rth order original GIRG solves the following equation :

$$W_{q,r} - \widehat{\tilde{A}}_q^T W_{q,r} \widehat{\tilde{A}}_q = \hat{Q}_q(1:r, 1:r) + \widehat{\tilde{Q}}_q^+ \tag{15}$$

$$\text{with} \quad \widehat{\tilde{Q}}_q^+ = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha \end{bmatrix} \quad , \quad \alpha = w_{q,r+1} - W_{q,r+1}(r+1:r+1, 1:r)\overline{\overline{\mathbf{a}}}_q \tag{16}$$

and where the subscripting notation $M\left(i:j, i^{'}:j^{'}\right)$ stands for the submatrix with rows $i..j$ and columns $i^{'}..j^{'}$ of matrix M.

$\hat{Q}_q$ is a positive semidefinite matrix. Therefore, $\hat{Q}_q(1:r, 1:r)$ is positive semidefinite.

To get the required result we shall now show that $\widehat{\tilde{Q}}_q^+$ is also a positive semidefinite matrix.

It is seen from (13) that the following equation holds :

$$W_{q,r+1}(r+1:r+1,1:r)\overline{\tilde{\mathbf{a}}}_q = \langle \tilde{w}_{q+r}, w_{q+r} \rangle \tag{17}$$

where $\tilde{w}_{q+r}[k]$ is the l_2-optimal approximation of the function $w_{q+r}[k]$ with the set $\{w_{q+i-1}[k]\}$:

$$\tilde{w}_{q+r}[k] = -\sum_{i=1}^{r} \overline{\tilde{a}}_i w_{q+r-i}[k] \tag{18}$$

Hence we have the following expression for the last diagonal entry of $\widehat{\tilde{Q}}_q^+$:

$$\alpha = \langle w_{q+r}, w_{q+r} \rangle - \langle \tilde{w}_{q+r}, w_{q+r} \rangle \tag{19}$$

From the *orthogonality principle* we know that the error $e[k] \hat{=} w_{q+r}[k] - \tilde{w}_{q+r}[k]$ is orthogonal to the approximating functions: $\langle e, w_{q+r-i} \rangle = 0 \ , \ \ i = 1, \ldots, r$. This implies that the second scalar product in (19) is the energy of the approximate function:

$$\langle \tilde{w}_{q+r}, w_{q+r} \rangle = \langle \tilde{w}_{q+r}, \tilde{w}_{q+r} \rangle \tag{20}$$

A known property derived from the orthogonality principle is that the following inequality holds :

$$\langle \tilde{w}_{q+r}, \tilde{w}_{q+r} \rangle \ < \ \langle w_{q+r}, w_{q+r} \rangle \tag{21}$$

Hence, it is seen from (16) that the only non-zero entry of $\widehat{\tilde{Q}}_q^+$ is positive, which implies that the right-hand side of the Lyapunov equation (15) is a positive semidefinite matrix. Then the characteristic polynomial of $\widehat{\tilde{A}}_q$ cannot have any zeros outside the unit circle and moreover, if $\left\{\widehat{\tilde{A}}_q, \widehat{\tilde{c}}_q\right\}$ is observable, $\widehat{\tilde{A}}_q$ is asymptotically stable. ■

Remark 2 *Proof of Theorem 3.1 reveals an important difference between the discrete case and the continuous case considered in* [15] *, where Gramians are defined using operators* $\int_\infty^t$ *and/or* $\frac{d}{dt}$*. Here, the right hand side of the Lyapunov equation (15) that solves the* r*th order principal submatrice of the original GIRG is not only composed from the parameters* t_i *and/or* $m_i^{'}$ *of the original realization in (7), due to the additional term* $\widehat{\tilde{Q}}_q^+$*.*

The controllability of the obtained ROM is now established in the following theorem.

Theorem 3.2 *Provided that* $\tilde{p}_q(1) \neq 0$*, the* r*th order ROM* $(\widehat{\tilde{A}}_q, \widehat{\tilde{b}}_q, \widehat{\tilde{c}}_q)$ *is controllable.*

Proof. Let $\tilde{p}_q(z)$ be the characteristic polynomial for the state matrix $\widehat{\tilde{A}}_q$ and $\overline{\tilde{p}}_q(z) = \sum_{i=0}^{r} \overline{\tilde{a}}_i z^{r-i} =$

$\tilde{p}_q(z+1)$. Since $\widehat{\tilde{A}}_q$ has the same structure as in eqn. (6), it is clear that $\left(\widehat{\tilde{A}}_q - I\right)$ is a companion matrix. Now it is seen that

$$\det\left[\mathcal{C}\left\{\left(\widehat{\tilde{A}}_q - I\right), \widehat{\tilde{b}}_q\right\}\right] = \begin{cases} \overline{\tilde{a}}_r^{-q} & \text{if } r \text{ odd} \\ (-1)^{-q+r+1}\overline{\tilde{a}}_r^{-q} & \text{if } r \text{ even} \end{cases}$$

where $\mathcal{C}\{.\}$ denotes the standard controllability matrix. Since $\mathcal{C}\left\{\left(\widehat{\tilde{A}}_q - I\right), \widehat{\tilde{b}}_q\right\}$ and $\mathcal{C}\left\{\widehat{\tilde{A}}_q, \widehat{\tilde{b}}_q\right\}$ have the same rank, the controllability matrix for our ROM realization is of full rank provided that $\overline{\tilde{a}}_r \neq 0$. Noting that $\overline{\tilde{a}}_r = \tilde{p}_q(1)$ achieves the proof.

Finally, assuming $\left\{\widehat{\tilde{A}}_q, \widehat{\tilde{c}}_q\right\}$ is observable yields the asymptotic stability (see Theorem 3.1), which implies $\tilde{p}_q(1) \neq 0$ and therefore controllability; in this case the ROM is minimal. ■

Remarks:

1) As the ROM matches the ith Markov-type parameters m'_i in (8), it also matches the ith Markov parameters.

2) It is well known that the high frequency behaviour is related to Markov parameters while the low frequency one is related to time-moments. Therefore, a GIRG with $q \sim 0$ is expected to give a better approximation at high frequencies than a GIRG with $q \sim -r+1$. Note that the original DC-gain is preserved for $q \neq 0$.

3) The present algorithm is very easy to implement using a standard numerical software (e.g. the *Matlab* script is ten lines long and is available upon request to the authors).

4. Example

Numerous examples have been studied in [17] to verify the validity of previous results.

Consider now the 7th order transfer function of a supersonic jet engine inlet proposed by Lalonde in [18] :

$$h(z) = \frac{2.0434z^6 - 4.9825z^5 + 6.57z^4 - 5.8189z^3 + 3.636z^2 - 1.4105z + 0.2997}{z^7 - 2.46z^6 + 3.433z^5 - 3.333z^4 + 2.5460z^3 - 1.584z^2 + 0.7478z - 0.2520}$$

With this model, the characteristics of any order reduction technique is clearly highlighted by comparing the ROM frequency response with the original response which is characterized by peaks at distinct frequencies.

Using our GIRG-based technique of the previous section with $q = -1$ we get the following five-order

model (GIRG5)

$$\tilde{h}(z) = \frac{2.0434z^4 - 3.0842z^3 + 2.1696z^2 - 1.4130z + 0.7100}{z^5 - 1.5310z^4 + 1.2594z^3 - 0.9770z^2 + 0.6962z - 0.3241}$$

To measure the approximations, consider the error criteria $\{Q = \|e\|_2^2 \ , \ e[k] = y[k] - \tilde{y}[k]\}$ where $y[k]$ and $\tilde{y}[k]$ are the responses of the original and reduced-order model, respectively. For impulse responses, the criteria is usually normalized: $Q' = Q/\|h\|_2^2$.

Table 1 compares GIRG5 with models derived through balanced realization (BR5, see [13]), Weighted Impulse Response Gramian (WIRG5, see [2]) and least-squares with scaling (LS5S5, see [18]):

Models	Impulse Error (%)	Step Error	DC-Gain Error (%)
GIRG5	1.27	0.1139	0
BR5	1.19	∞	6.25
WIRG5	1.13	∞	12.69
LS5S5	2.36	0.0691	0

Table 1. Errors for the reduced order models

Models BR5 and WIRG5 give the best approximations from the point of view of the impulse response, but their step responses are not acceptable. Model LS5S5 provides a reasonable impulse response and a close approximation of the original step response. Model derived by GIRG exhibits a very good behaviour on both impulse and step responses (the DC-gain is retained). The Bode plots of the original and reduced-order models (WIRG5, LS5S5, GIRG5) are shown in Figs 1 and 2. Very small reduction errors are obtained with model GIRG5 at high frequencies and low frequencies, as well as middle frequencies.

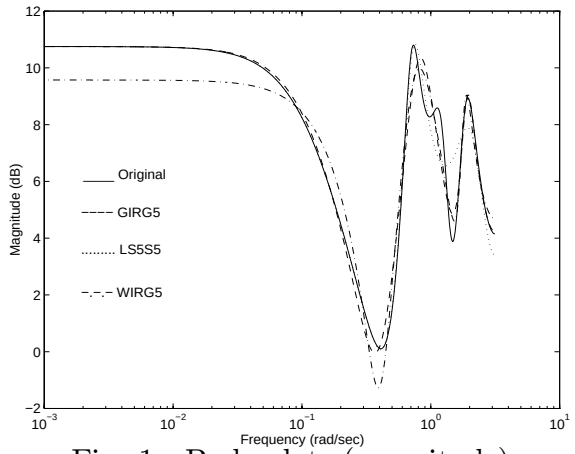

Fig. 1 - Bode plots (magnitude)

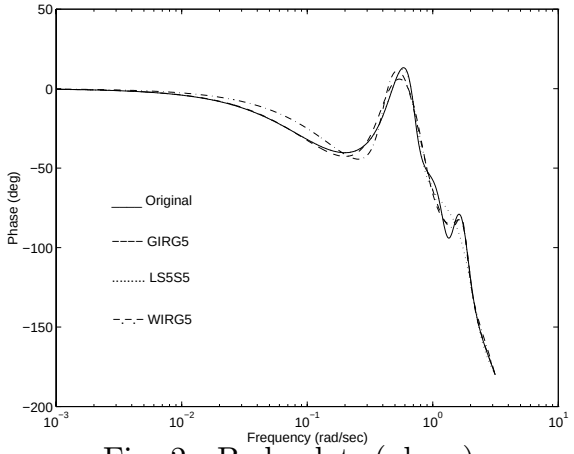

Fig. 2 - Bode plots (phase)

5. Conclusion

A new impulse response Gramian has been introduced for linear time-invariant asymptotically stable discrete SISO systems. It is easily obtained by solving a Lyapunov equation for a particular realization and is connected to standard Gramians. It has been further shown that it contains information about the characteristic polynomial. A model reduction method based on these properties has been proposed. The rth order ROM is choosen in a set of r solutions: the poles are first computed through a minimization of a l_2 error criteria and then we match some Markov parameters and/or time-moments. This ROM cannot have any poles outside the unit circle and is asymptotically stable and minimal provided it is observable. This method can ensure a close approximation for a given frequency range. As shown by the numerical example, the proposed solution compares well with those obtained with other techniques.

References

[1] P. Agathoklis and V. Sreeram, "Identification and model reduction from impulse response data," *Int. J. Systems Sci.*, vol. 21, no. 8, pp. 1541-1552, 1990.

[2] V. Sreeram and P. Agathoklis, "Model reduction of linear discrete systems via weighted impulse response Gramians," *Int. J. Contr.*, vol. 53, no. 1, pp. 129-144, 1991.

[3] V. Sreeram and P. Agathoklis, "Discrete system reduction via impulse response Gramians and its relation to q-Markov covers," *IEEE Trans. Automat. Contr.*, vol. 37, no. 5, pp. 653-658, 1992.

[4] D. A. Wagie and R. E. Skelton, "A projection approach to covariance equivalent realization," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 1114-1120, 1986.

[5] B. D. O. Anderson and R. E. Skelton, "The generation of all $q-$Markov covariance equivalent realizations," *IEEE Trans. Circuits Syst.*, vol. CAS-35, pp. 375-384, Apr. 1988.

[6] R. E. Skelton and B. D. O. Anderson, "Weighted $q-$Markov covariance equivalent realizations," *Int. J. Contr.*, vol. 49, no. 5, pp. 1755-1771, 1989.

[7] V. Sreeram and P. Agathoklis, "On the properties of Gram matrix," *IEEE Trans. Circuits Syst.*, vol. 41, no. 3, pp. 234-237, Mar. 1994.

[8] V. Sreeram and P. Agathoklis, "On the computation of the Gram matrix in time domain and its application," *IEEE Trans. Automat. Contr.*, vol. 38, no. 10, pp. 1516-1520, Oct., 1993.

[9] V. K. Jain and R. D. Gupta, "Identification of linear systems through a Gramian technique," *Int. J. Contr.*, vol. 12, pp. 421-431, 1970.

[10] V. Sreeram and P. Agathoklis, "The discrete time q-Markov cover models with improved low-frequency approximation," *IEEE Trans. Automat. Contr.*, vol. AC-39, no. 5, pp. 1102-1105, 1994.

[11] V. Sreeram and P. Agathoklis, "Model reduction of linear continuous systems using weighted impulse response Gramians," *IEE Proc.* D, vol. 140, no. 5, pp. 345-352, Sept., 1993.

[12] E. J. Ang, V. Sreeram, and W. Q. Liu, "Identification/Reduction to a balanced realization via the extended impulse response Gramian," *IEEE Trans. Automat. Contr.*, vol. 40, no. 12, pp. 2153-2158, Dec., 1995.

[13] L. Pernebo and L. M. Silverman, "Model reduction via balanced state space representation," *IEEE Trans. Automat. Contr.*, vol. AC-27, no. 2, pp. 382-387, Apr., 1982.

[14] S. Azou, P. Vilbé, and L. C. Calvez, "Balanced realization using an orthogonalization procedure and modular polynomial arithmetic," *J. Franklin Inst.*, vol. 335, no. 8, pp. 1507-1518, 1998.

[15] W. Krajewski, A. Lepschy, and U. Viaro, "Model reduction by matching Markov parameters, time-moments, and impulse-response energies," *IEEE Trans. Automat. Contr.*, vol. AC-40, no. 5, pp. 949-953, May, 1995.

[16] P. Bréhonnet, A. Derrien, P. Vilbé, and L. C. Calvez, "Quasi-optimal model reduction of discrete-time systems," *Electron. Lett.*, vol. 32, no. 16, pp. 1521-1522, Aug., 1996.

[17] S. Azou, "Balanced systems realization via orthogonalization of input maps - Grammians and approximation," *Thesis*, Univ. Bretagne Occidentale, Brest, France, Dec. 1997.

[18] R. J. Lalonde, T. T. Hartley, and J. A. De Abreu-Garcia, "Least-squares model order reduction enhancements," *IEEE Trans. Ind. Elect.*, vol. 40, no. 6, pp. 533-541, 1993.

[19] G. H. Golub and C. F. Van Loan, "Matrix computations," *The Johns Hopkins Univ. Press*, 1989.