Gevrey estimates of the resolvent and sub-exponential time-decay of solutions

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GEVREY ESTIMATES OF THE RESOLVENT AND SUB-EXPONENTIAL TIME-DECAY FOR THE HEAT AND SCHRÖDINGER SEMIGROUPS

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Abstract. In this article, we prove Gevrey estimates of the resolvent near threshold zero for a class of second order elliptic operators satisfying a weighted coercive condition. As application, we obtain large time expansions with sub-exponential time-decay estimates on the remainder for the heat and Schrödinger semigroups generated by non-selfadjoint Schrödinger operators. Our results cover the cases of threshold eigenvalue and positive resonances.

Résumé. Dans cet article, nous démontrons les estimations de Gevrey près du seul zéro pour la résolvante d'une classe d'opérateurs elliptiques du second ordre vérifiant une condition de coercivité à poids. Comme application, nous obtenons les développements en temps long pour les semi-groupes de la chaleur et de Schrödinger avec les estimations sous-exponentielles en temps sur le reste. Nos résultats couvrent les cas de valeur propre au seuil et de résonances positives.

Contents

1. Introduction

This work is concerned with sub-exponential time-decay of local energies for semigroups e^{-tH} and e^{-itH} as $t \to +\infty$ where $H = -\Delta + V(x)$ is a non-selfadjoint Schrödinger operator regarded as perturbation of some model operator $H_0 = -\Delta + V_0(x)$ satisfying a weighted coercive condition (see (2.3)). There is a large literature on large-time asymptotics of solutions.

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For selfadjoint Schrödinger operators $-\Delta + V(x)$ with a real-valued potential $V(x)$ verifying the decay estimate

$$|V(x)| \leq C\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

for some $\rho > 0$, where $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. We only mention [4, 13] for quickly decaying potentials ($\rho > 2$), [27] for critically decaying potentials ($\rho = 2$) under an assumption of Hardy inequality for the model operator and [26] in one-dimensional case when this Hardy condition is not satisfied. For slowly decreasing potentials ($0 < \rho < 2$), there are works of [8] when the potential is negative and [20, 31, 32] when it is globally positive and slowly decreasing. Non-selfadjoint Schrödinger operators appear naturally in many physical problems. See for example [6, 19, 15, 23, 24]. In this case, we only quote the works of [9, 28] on dispersive estimates. In [9], the absence of real resonances is assumed and in [28], only dissipative Schrödinger operators are considered. In the later case, positive resonances may exist but outgoing positive resonances (see Definition 2.5) are absent due to the dissipativity of the operator. Another related topic which partly motivated this work is return to equilibrium of Fokker-Planck operator with a sublinearly increasing potential such that its gradient decays slowly. This operator is non-selfadjoint and sub-elliptic. When the potential increases sublinearly, zero is an eigenvalue embedded in the essential spectrum of the Fokker-Planck operator. In [29], it is conjectured that the rate of return to equilibrium in this problem should be sub-exponential in time with a precise power in time explicitly determined by the rate of the potential. While polynomially decaying remainder estimate is established in [5, 7] by method of Markov processes, the sub-exponential remainder estimate is proved in a recent work of T. Li and Z. Zhang ([18]). Note that M. Klein and J. Rama ([17]) study Gevrey estimates in a different context to analyze large time evolution of quantum resonant states.

Most relevant to this work is selfadjoint Schrödinger operator with positive and slowly decreasing potential of the form $H_0 = -\Delta + V_0(x)$ where $V_0(x)$ satisfies the estimates

$$0 < c_1\langle x\rangle^{-2\mu} \leq V_0(x) \leq c_2\langle x\rangle^{-2\mu}, \quad x \in \mathbb{R}^n, \tag{1.2}$$

for some constants $\mu \in]0, 1[$ and $c_1, c_2 > 0$. In [31, 32], D. Yafaev studies the low-energy spectral properties and proves that in one dimensional case, if $V_0(x)$ is in addition analytic, then local energies of solutions decay sub-exponentially

$$\|e^{-itH_0}\|_{L^2_{\rm comp}\to L^2_{\rm loc}} = O(e^{-c|t|^\beta}), \quad |t| \to +\infty, \tag{1.3}$$

where $\beta = \frac{1-\mu}{1+\mu}$ and c is some positive constant. Making use of semiclassical method, S. Nakamura [20] proves, under a virial condition on the potential, the existence and the smoothness of boundary values of the resolvent

$$R_0(\lambda \pm i0) = \lim_{\epsilon\to 0_+} (H_0 - (\lambda \pm i\epsilon))^{-1}$$

in neighborhood of zero and the heat semigroup verifies for any dimension $n \geq 1$

$$\|e^{-tH_0}\|_{L^2_{\rm comp}\to L^2_{\rm loc}} = O(e^{-c|t|^\beta}), \quad |t| \to +\infty. \tag{1.4}$$

Higher dimensional analog of (1.3) remains unknown until now even for analytical potentials verifying (1.2).

In this paper, we consider non-selfadjoint Schrödinger operators H which are perturbation of some model operator of the form

$$H_0 = -\Delta + V_0(x), \quad V_0(x) = V_1(x) - iV_2(x),$$

V_1 and V_2 being real-valued, where H_0 satisfies a weighted coercive condition (2.3) with some index $\mu \in]0, 1[$. This condition can be compared with (1.2) if V_0 is real. Note that if H is a perturbation of H_0: $H = H_0 + O(\langle x\rangle^{-2\mu-\epsilon})$ for some $\epsilon > 0$, modifying V_0 outside a sufficiently

large compact if necessary, we can decompose $H = \widetilde{H}_0 + \widetilde{W}(x)$ where $\widetilde{H}_0$ still satisfies the weighted coercive condition (2.3) with the same index μ and $\widetilde{W}(x)$ is of compact support. In the following we only restrict ourselves to compactly supported perturbations of the model operator H_0:

$$H = -\Delta + V(x) = H_0 + W(x) \tag{1.5}$$

with $H_0 = -\Delta + V_0(x)$ satisfying (2.3) and $W(x) = V(x) - V_0(x)$ is a bounded, compactly supported measurable function.

One of the results proved in this work is the large-time expansion for the Schrödinger semigroup e^{-itH}, $t \geq 0$, with sub-exponential time-decay estimates on the remainder. Let the model potential V_0 be in the class of holomorphic potentials $\mathcal{A}$ introduced in Definition 2.4 (which implies in particular H_0 satisfied (2.3) for some $\mu \in]0,1[$). Let H be compactly supported perturbation of $H_0 = -\Delta + V_0(x)$. Then H has at most a finite number of complex eigenvalues in the region $\{z \in \mathbb{C}; 0 \leq \arg z \leq \frac{3\pi}{2}\}$ and a finite number of outgoing positive resonances. We use analytic distortion $H(\theta)$ of H outside some large compact set to define quantum resonances of H as poles of the meromorphic extension of the resolvent $R(z,\theta) = (H(\theta) - z)^{-1}$ from the infinity of the upper half complex plane $\mathbb{C}_+$. The real resonances of H are generalized eigenvalues defined through Definition 2.5. It will be proved that the quantum resonances in $\mathbb{C}_+$ are eigenvalues of H and those located in $\mathbb{R}_+$ are outgoing real resonances of H (see theorem 5.1). Moreover there exists a contour located in the lower half-plane touching the real axis only at the point 0 such that there is no quantum resonances of H between this curve and the real axis. We shall prove that the Schrödinger semigroup e^{-itH}, $t \geq 0$, admits a large-time expansion of the form

$$\|\chi \left(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \overline{\mathbb{C}}_+} e^{-itH}\Pi_\lambda - \Pi_0(t) - \sum_{\nu \in r_+(H)} e^{-it\nu} P_\nu(t) \right) \chi\|_{L^2 \to L^2} \leq C_\chi e^{-ct^{\frac{1-\mu}{1+\mu}}} \quad t > 1. \tag{1.6}$$

Here $\chi \in C_0^\infty(\mathbb{R}^n)$, $c > 0$ is independent of χ, $r_+(H)$ is the set of outgoing positive resonances of H (see Definition 2.5), Π_λ is the Riesz projection of H associated with discrete eigenvalue λ, $\Pi_0(t)$ and $P_\nu(t)$ are some operators of finite rank depending polynomially on t, arising respectively from threshold eigenvalue and positive resonance $\nu > 0$ of H. See Section 2 for more precise statement of conditions and results.

The proof of (1.6) combines several technics: threshold spectral analysis for non-selfadjoint operators, method of quantum resonances and Gevrey estimates of the resolvent at threshold zero for a class of second order elliptic operators (Theorem 2.1). To prove (1.6), we use both the technics of analytic dilation and analytic deformation to study $R(\theta, z)$. The analytic dilation is applied to the model operator H_0 to show that there exists a resonance-free sector below the positive half-axis in which a uniform bound holds for the dilated resolvent. Then we study the analytic deformation of H outside some sufficiently large ball in $\mathbb{R}^n$ and prove the existence of a curve Γ in the lower half-plane, intersecting the real axis only at point 0, such that above this curve, the meromorphic extension of cut-off resolvent $\chi R(z)\chi$ from $\mathbb{C}_+$ with $\operatorname{Im} z >> 1$ has at most a finite number of poles and those located in $]0, +\infty[$ are precisely outgoing positive resonances. In particular, we prove that if the model potential V_0 belongs to the class $\mathcal{A}$, zero is not an accumulation point of quantum resonances of H located in the region above the curve Γ and there is a uniform bound for the cut-off resolvent for z near zero and z above Γ. Under some assumption on the eigenfunctions associated with eigenvalue zero, we compute the resolvent expansion at threshold in presence of threshold eigenvalue and prove the Gevrey estimates on the remainder. Then (1.6) is deduced by representing $\chi e^{-itH}\chi$

as sum of some residue terms and a Cauchy integral of the cut-off resolvent on Γ. The sub-exponential time-decay estimate is obtained from the Gevrey estimates on the remainder of the resolvent expansion. The method of threshold spectral analysis for non-selfadjoint Schrödinger operators initiated in this work can be applied to other non-selfadjoint spectral problems (see Remark 5.4 and [1]).

Real resonances, called spectral singularities by J. Schwartz in [24] in more general framework and exceptional points by Y. Saito in [23], are the main obstacle to understand spectral properties of non-selfadjoint Schrödinger operators near positive half-axis. In general case, one only knows that real resonances form a bounded set with Lebesgue measure zero ([23]). In this work we prove that for potentials belonging to the class $\mathcal{A}$, outgoing positive resonances are at most finite. We also give some classes of analytic potentials for which real resonances are at most a countable set with zero as the only possible accumulation point. It will be shown that each outgoing positive resonance is a pole of the meromorphic extension of the resolvent from the upper half-plane, hence does contribute to large time asymptotics of solutions as $t \to +\infty$.

A key ingredient of the proof of (1.6) is Gevrey estimates on the remainder for the resolvent expansions at threshold zero which will be deduced from Gevrey estimates of the model resolvent at threshold zero. To establish Gevrey estimates on the resolvent of the model operator H_0 at the threshold, we first prove an energy estimate depending uniformly on powers of weight $s \in \mathbb{R}$. This kind of estimate for fixed s has already appeared in [32]. The uniformity on $s \in \mathbb{R}$ is crucial in present work, because it allows to control norms of the resolvent in weighted spaces with respect to some parameters (Theorem 3.4), from which we deduce Gevrey estimates on the model resolvent at threshold zero (Theorem 2.1). To estimate remainders in the asymptotic expansions of the full resolvent $R(z) = (H - z)^{-1}$ near $z = 0$, we make use of Theorem 2.1 for the model operator and operations for operator-valued functions in Gevrey classes. When threshold eigenvalue is absent, one can iterate the first resolvent equation and it is sufficient to use Gevrey estimates for the free resolvent at threshold and a polynomial bound of the full resolvent on some curves. When threshold eigenvalue is present, we need uniform Gevrey estimates on the remainders on these curves. This explains why the remainder estimate in Theorem 2.4 (a) is not as good as in other situations.

The organisation of this paper is as follows. In Section 2, we state the main results obtained in this work. Sections 3 and 4 are devoted to the model operator $H_0 = -\Delta + V_0(x)$ verifying the weighted coercive condition (2.3). In Section 3, we prove Gevrey estimates of the model resolvent at threshold zero (Theorem 2.1). We first establish a uniform energy estimate which allows to control the growth of powers of the resolvent at threshold in weighted spaces. Then Theorem 2.1 is deduced by an appropriate induction. In Section 4, we study spectral properties of H_0 in the right half-plane and establish resolvent bounds along certain curves located in the right or the lower half-planes where are used respectively for the heat or Schrödinger semigroups. As the first application of Gevrey estimates of the resolvent, we prove sub-exponential time-decay estimates for the semigroups e^{-tH_0} and e^{-itH_0} and a low-energy estimate on the spectral density in case when H_0 is selfadjoint. Compactly supported perturbations H of the model operator H_0 are studied in Sections 5. We first study properties of real resonances in Subsection 5.1 and show in Subsection 5.2 how to deduce from Gevrey estimates of the resolvent the large-time expansion for the semigroups e^{-tH} and e^{-itH} with sub-exponential time-decay estimates on the remainder (Theorem 2.2). In Subsection 5.3, we prove Theorem 2.3. Since the method of low-energy spectral analysis used in the proof of Theorem 2.3 is well known for selfadjoint operators, we only emphasize upon Gevrey estimates on the remainders. Finally in Subsection 5.4, we study more difficult case of threshold eigenvalue in non-selfadjoint

case and prove Theorem 2.4. Note that algebraic multiplicity and Riesz projection for embedded eigenvalues are not defined. Instead, we use Birmann-Schwinger method to construct an explicit representation for the Riesz projection of eigenvalue -1 of some compact operator. This allows to construct a Grushin problem and to obtain the resolvent expansion under some condition on the eigenfunction. The main attention here is payed to the calculation of leading terms in resolvent expansion. The Gevrey estimates on remainders can be proved as in Subsection 5.3 for seladjoint case and hence the details are omitted in Subsection 5.4.

Part of the results on the model operator H_0 are announced in [30].

Notation. We denote $H^{r,s}$, $r \geq 0, s \in \mathbb{R}$ the weighted Sobolev space of order r with the weight $\langle x\rangle^s$ on $\mathbb{R}^n$: $H^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^n); \|u\|_{r,s} = \|\langle x\rangle^s(1-\Delta)^{\frac{r}{2}}u\|_{L^2} < \infty\}$. For $r < 0$, $H^{r,s}$ is defined as the dual space of $H^{-r,-s}$ with dual product identified with the scalar product $\langle\cdot,\cdot\rangle$ of $L^2(\mathbb{R}^n)$. Set $H^{0,s} = L^{2,s}$. $\mathcal{B}(r,s;r',s')$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. If $(r,s) = (r',s')$, we denote $\mathcal{B}(r,s) = \mathcal{B}(r,s;r',s')$. Unless otherwise mentioned explicitly, $\|\cdot\|$ denotes norm in $L^2(\mathbb{R}^n)$ or in $\mathcal{B}(L^2)$ when no confusion is possible. $\mathbb{C}_\pm$ denote respectively the upper and the lower open half-plane and $\overline{\mathbb{C}}_\pm$ their closure. Set $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. For $\theta_1 < \theta_2$ and $r > 0$, $S(\theta_1,\theta_2)$ denotes the sector

$$S(\theta_1,\theta_2) = \{z \in \mathbb{C}^*; \theta_1 < \arg z < \theta_2\}$$

and $\Omega(r,\theta_1,\theta_2)$ is a part of $S(\theta_1,\theta_2)$ near zero :

$$\Omega(r,\theta_1,\theta_2) = \{z \in S(\theta_1,\theta_2); |z| < r\}.$$

In this work, the scalar product denoted as $\langle\cdot,\cdot\rangle$ is assumed to be linear with respect to the left variable.

2. Statement of the results

The main tool of this work is Gevrey estimates of the resolvent at threshold zero for a class second order elliptic operators satisfying a weighted coercive condition. Let

$$H_0 = -\sum_{i,j=1}^{n} \partial_{x_i} a^{ij}(x)\partial_{x_j} + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + V_0(x), \tag{2.1}$$

where $a^{ij}(x)$, $b_j(x)$ and $V_0(x)$ are complex-valued measurable functions. Suppose that a^{ij}, b_j are of class C_b^1 on $\mathbb{R}^n$ (*i. e.*, bounded C^1 functions with bounded first order derivatives) and there exists some constant $c > 0$ such that

$$\mathrm{Re}\,(a^{ij}(x)) \geq cI_n, \quad \forall x \in \mathbb{R}^n. \tag{2.2}$$

Assume that V_0 is relatively bounded with respect to $-\Delta$ with relative bound zero, $\mathrm{Re}\, H_0 \geq 0$ and that there exists some constants $0 < \mu < 1$ and $c_0 > 0$ such that

$$|\langle H_0 u, u\rangle| \geq c_0(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu}u\|^2), \quad \forall\, u \in H^2, \tag{2.3}$$

$$\sup_x |\langle x\rangle^{\mu} b_j(x)| < \infty, \quad j = 1,\cdots,n. \tag{2.4}$$

Condition (2.3) is called weighted coercive condition.

Remark 2.1. *If $H_0 = -\Delta + V_0(x)$ with $V_0(x) = V_1(x) - iV_2(x)$ with real V_j. Assume that $-\alpha\Delta + V_1(x) \geq \tau(x) \geq 0$ for some $\alpha \in [0,1[$ in sense of selfadjoint operators. If $V_2 \geq 0$ is such that for some $c > 0$*

$$\tau(x) + V_2(x) \geq c\langle x\rangle^{-2\mu}, \quad x \in \mathbb{R}^n. \tag{2.5}$$

then the weighted coercive condition (2.3) is satisfied. If $V_1(x)$ is slowly decaying (i. e. $V_1(x) \geq c\langle x\rangle^{-2\mu}$ for some $\mu \in]0,1[$ and $c > 0$), then (2.3) is satisfied by $H_0 = -\Delta + V_1(x) - iV_2(x)$ for any real function V_2 which is $-\Delta$-bounded with relative bound zero.

Note that when we study Schrödinger operators $H_0 = -\Delta + V_0$ by technics of analytic dilation or analytic deformation, if H_0 verifies (2.3), the analytically dilated or distorted operators obtained from H_0 are of the form (2.1) and still satisfy (2.3) if the dilation or distortion parameter is small. The condition that $V_0(x)$ is $-\Delta$-bounded with relative bound zero allows to include a class of N-body potentials.

Under the assumptions 2.2, 2.3 and 2.4, one can show that H_0 is bijective from $D(H_0) = H^2(\mathbb{R}^n)$ onto $R(H_0)$ and $R(H_0)$ is dense in $L^2(\mathbb{R}^n)$. Let $G_0 : R(H_0) \to D(H_0)$ be the algebraic inverse of H_0. Denote $\mathcal{D} = \cap_{s\in\mathbb{R}} L^{2,s}$. Then $G_0(\mathcal{D}) \subset \mathcal{D}$ and G_0 is a densely defined, continuous from $R(H_0) \cap L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$ (see Lemma 3.3). To simplify notation, we still denote by G_0 its extension by density so that G_0 is regarded as a bounded operator from $L^{2,s}$ to $L^{2,s-2\mu}$. Consequently for any $N \in \mathbb{N}^*$, $G_0^N : L^{2,s} \to L^{2,s-2\mu N}$ is well defined for any $s \in \mathbb{R}$. Let $R_0(z) = (H_0 - z)^{-1}$ for $z \notin \sigma(H_0)$. Since $\operatorname{Re} H_0 \geq 0$ on L^2, $zR_0(z)$ is uniformly bounded for $z \in S(\delta) = \{z; \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta\}$ for each fixed $\delta > 0$. From the equation

$$R_0(z) = G_0 + zG_0R_0(z)$$

and an argument of density, we deduce that

$$\text{s-}\lim_{z\in S(\delta), z\to 0} \langle x\rangle^{-2\mu}(R_0(z) - G_0) = 0$$

in $L^2(\mathbb{R}^n)$. The same limit also holds in $\mathcal{B}(0, s; 0, s)$ for any $s \in \mathbb{R}$.

Theorem 2.1. *Assume the conditions (2.1)-(2.4). Then for any $a > 0$, there exist some constants $C_a, c_a > 0$ such that*

$$\|\langle x\rangle^{-\tau} e^{-a\langle x\rangle^{1-\mu}} G_0^N \langle x\rangle^{\tau}\| + \|\langle x\rangle^{\tau} G_0^N e^{-a\langle x\rangle^{1-\mu}} \langle x\rangle^{-\tau}\| \leq C_a c_a^{N+\tau} (N+\tau)^{\frac{\tau}{1-\mu}+\gamma N} \tag{2.6}$$

for all $N \in \mathbb{N}^*$ and $\tau \geq 0$. Here $\gamma = \frac{2\mu}{1-\mu}$.

Remark 2.2. *Since $\operatorname{Re} H_0 \geq 0$ and H_0 verifies (2.3), $H_0 - \lambda$ also verifies (2.3) uniformly in $\lambda \leq 0$. Repeating the proof of Theorem 2.1 with H_0 replaced by $H_0 - \lambda$, one can show that*

$$\|\langle x\rangle^{-\tau} e^{-a\langle x\rangle^{1-\mu}} R_0(\lambda)^N \langle x\rangle^{\tau}\| \leq C_a c_a^{N+\tau} (N+\tau)^{\frac{\tau}{1-\mu}+\gamma N} \tag{2.7}$$

for all $N \in \mathbb{N}^*$ and $\tau \geq 0$, uniformly in $\lambda < 0$. In Section 3, uniform Gevrey estimates for $R_0(z)$ will be proved for z in larger domains.

Since one has at least formally

$$\frac{d^N}{dz^N} R_0(z)|_{z=0} = N! G_0^{N+1},$$

Theorem 2.1 with $\tau = 0$ says that derivatives of the resolvent of H_0 at threshold zero satisfies the Gevrey estimates of order $\sigma = 1 + \gamma$ as operators from $L^2(\mathbb{R}^n; dx)$ to $L^2(\mathbb{R}^n; e^{-a\langle x\rangle^{1-\mu}} dx)$ for any $a > 0$. Estimate (2.6) with $\tau > 0$ will be used when we want to pass the weight $\langle x\rangle^{-\tau}$ across G_0^N. It follows from (2.6) with $\tau = 0$ that there exists some constant $C > 0$ (independent of cut-offs χ) such that $\forall \chi \in C_0^\infty(\mathbb{R}^n)$, there exists some constant $C_\chi > 0$ such that

$$\|\chi G_0^N\| + \|G_0^N \chi\| \leq C_\chi C^N N^{\gamma N}, \forall N \in \mathbb{N}^*. \tag{2.8}$$

Theorem 2.1 can be applied, for example, to the Laplace-Beltrami operator on some complete and non-compact Riemannian manifolds or to N-body Schrödinger operators with repulsive interactions. In this paper we use Theorem 2.1 to study non-selfadjoint Schrödinger operators $H = -\Delta + V(x)$ which are perturbation of a model operator $H_0 = -\Delta + V_0(x)$, with $V_0(x) = V_1(x) - iV_2(x)$ and V_1, V_2 real-valued, satisfying the conditions of Theorem 2.1. As explained in Introduction, we can assume without loss in some situation that H is a compactly supported perturbation of H_0:

$$H = H_0 + W(x), \quad W = V - V_0 \in L^\infty_{\rm comp}(\mathbb{R}^n). \tag{2.9}$$

Denote

$$H_1 = -\Delta + V_1(x) \tag{2.10}$$

the selfadjoint part of H_0. In order to study large time behavior of the semigroups e^{-tH} and e^{-itH}, $t \geq 0$, we introduce two classes of model potentials $\mathcal{V}$ and $\mathcal{A}$.

Definition 2.3. *Let $\mathcal{V}$ be the class of complex-valued potentials $V_0(x) = V_1(x) - iV_2(x)$ with V_1, V_2 real-valued such that*

$$V_0 \text{ is } -\Delta\text{-compact and (2.3) is satisfied for some constant } \mu \in]0,1[. \tag{2.11}$$

and

$$H_1 \geq -\alpha\Delta \text{ and } |V_2(x)| \leq C\langle x\rangle^{-2\mu_2} \tag{2.12}$$

for some constants $\alpha, \mu_2, C > 0$.

Results for the heat semigroup e^{-tH} will be proved for model potentials $V_0 \in \mathcal{V}$. To study the Schrödinger semigroup e^{-itH} we will use both technics of analytic dilation and analytic deformation, hence introduce a class of potentials with more restrictive conditions. Recall that $V_0(x)$ is sais to be dilation analytic if $V_0(e^\theta x)$ defined for real θ admits a holomorphic extension for θ in a complex neighborhood of zero and $V_0(e^\theta x)(-\Delta+1)^{-1}$ is a compact operator-valued holomorphic function for $\theta \in \mathbb{C}$ with $|\theta|$ small ([3]).

Definition 2.4. *Let $\mathcal{A}$ denote the class of complex-valued potentials $V_0(x) = V_1(x) - iV_2(x)$ for $x \in \mathbb{R}^n$ with $n \geq 2$ such that H_1 satisfies (2.3) for some constant $\mu \in]0,1[$. Assume in addition that V_1 and V_2 are dilation analytic and extend holomorphically into a complex region of the form*

$$\Omega = \{x \in \mathbb{C}^n; |x| > c^{-1}, |\mathrm{Im}\, x| < c|\ \mathrm{Re}\, x|\}$$

for some $c > 0$ and satisfy for some constants $c_1, c_2 > 0$ and $R \in [0, +\infty]$

$$|V_j(x)| \leq c_1\langle\ \mathrm{Re}\, x\rangle^{-2\mu}, x \in \Omega, \quad j = 1, 2, \tag{2.13}$$

$$V_2(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{2.14}$$

$$x \cdot \nabla V_1(x) \leq -c_2 \frac{x^2}{\langle x\rangle^{2\mu+2}}, \quad x \in \mathbb{R}^n \text{ with } |x| \geq R, \text{ and} \tag{2.15}$$

$$V_2(x) \geq c_2\langle x\rangle^{-2\mu}, \quad x \in \mathbb{R}^n \text{ with } |x| < R. \tag{2.16}$$

If Condition (2.16) is satisfied with $R = 0$, we assume in addition

$$0 < \mu < \tfrac{3}{4}, \text{ if } n = 2 \text{ and } 0 < \mu < 1, \text{ if } n \geq 3. \tag{2.17}$$

Remark that when $R = 0$, (2.15) is a global virial condition on V_1 and (2.16) is void; while if $R = +\infty$, no virial condition is needed on V_1, but (2.16) is required on the whole space which means that the dissipation is strong. Assume that $V_2(x)$ is non-negative and dilation analytic and extends holomorphically into Ω satisfying $|V_2(x)| \leq C\langle \mathrm{Re}\, x\rangle^{-2\mu}$ for $x \in \Omega$. Then potentials of the form

$$V_0(x) = \frac{c}{\langle x\rangle^{2\mu}} - iV_2(x), \tag{2.18}$$

satisfy conditions (2.13)-(2.16) with $R = 0$ if $c > 0$; and satisfy conditions (2.13)-(2.16) with $R = +\infty$ if $c = 0$ and $V_2(x) \geq c'\langle x\rangle^{-2\mu}$, $\forall x \in \mathbb{R}^n$, for some $c' > 0$. If $n = 3$, Coulomb-type potential $V_0 = \frac{a-ib}{|x|}$ belongs to the class $\mathcal{A}$ if $a, b \geq 0$ with $a + b > 0$.

For $V_0 \in \mathcal{A}$, one can study quantum resonances of $H_0 = -\Delta + V_0(x)$ by both analytic dilation or analytic deformation outside some large compact ([3, 12, 25]). Conditions (2.15), (2.16) and (2.17) are used to prove the absence of quantum resonances in a sector below the positive half-axis and the uniform boundedness of the cut-off resolvent there. See Lemma 4.6

Let $V_0 \in \mathcal{V}$ and $H_0 = -\Delta + V_0(x)$. Let $H = H_0 + W(x)$ be a compactly supported perturbation of H_0: $W \in L^\infty_{\rm comp} = \{u \in L^\infty(\mathbb{R}^n); \text{supp } u \text{ is compact }\}$. Let $\sigma_d(H)$ ($\sigma_p(H)$, resp.) denote the set of discrete eigenvalues of H (the set of eigenvalues of H, resp.). It will be proved in Section 5 that H has at most a finite number of discrete eigenvalues located on the left of a curve Γ of the form

$$\Gamma = \{z \in \mathbb{C}; \text{Re}\, z \geq 0, |\text{Im}\, z| = C(\text{Re}\, z)^{\mu'}\} \tag{2.19}$$

for some constants $C, \mu' > 0$ and that there exists a nice bound for the resolvent of H_0 on Γ. Complex eigenvalues of H may accumulate to zero from the right-hand side of Γ. Note that zero may be an embedded eigenvalue of H, but we shall see that it is never a resonance of H, *i. e.*, if $u \in L^2(\mathbb{R}^n; \langle x\rangle^{2s}dx) \cap H^2_{\rm loc}(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ such that $Hu = 0$, then $u \in H^2(\mathbb{R}^n)$ which means u is an eigenfunction of H.

More subtle is the role of positive resonances of non-selfadjoint Schrödinger operators. Recall that if V is of short-range: $V(x) = O(\langle x\rangle^{-1-\epsilon})$ for some $\epsilon > 0$, $\lambda > 0$ is called real resonance of $H = -\Delta + V(x)$ if the equation $Hu = \lambda u$ admits a non-trivial solution $u \in H^2_{\rm loc}(\mathbb{R}^n)$ satisfying one of Sommerfeld radiation conditions:

$$u(x) = \frac{e^{\pm i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}}(a_\pm(\omega) + o(1)), \quad |x| \to \infty, \tag{2.20}$$

for some $a_\pm \in L^2(\mathbb{S}^{n-1}), a_\pm \neq 0$. λ is called an outgoing (resp., incoming) positive resonance of H if u verifies (2.20) with sign + (resp. with sign −). It is known that if $V(x)$ is real, then positives resonances are absent ([2]) and if $\text{Im}\, V \leq 0$, outgoing resonances are absent ([21]). In this paper, we use a slight different definition for real resonances, because our potentials $V(x)$ may have a complex long-range tail. Let U_0 be a complex valued potential such that $(x\cdot\nabla_x)^j U_0$, $j = 0, 1, 2$, are $-\Delta$-compact. Assume $\text{Im}\, U_0 \leq 0$. Then for any $\lambda > 0$ the boundary value of the resolvent

$$(-\Delta + U_0 - (\lambda + i0))^{-1} = \lim_{z\to\lambda, \text{Im}\, z>0}(-\Delta + U_0 - z)^{-1} \tag{2.21}$$

exists in $\mathcal{B}(0, s; 0, -s)$ for any $s > \frac{1}{2}$ and is Hölder-continuous for $\lambda > 0$. See [22].

Definition 2.5. *Let $U(x)$ be a Lebesgue measurable function such that $U(x)-U_0(x)$ is bounded and of short-range on $\mathbb{R}^n$. $\lambda > 0$ is called outgoing resonance of $H = -\Delta + U$ if -1 is an eigenvalue of the compact operator $(-\Delta + U_0 - (\lambda + i0))^{-1}(U - U_0)$ in $L^{2,-s}$ for $s > \frac{1}{2}$ and sufficiently close to $\frac{1}{2}$. Denote $r_+(H)$ the set of outgoing resonances of H. For $\lambda \in r_+(H)$, define $m_+(\lambda)$ as the algebraic multiplicity of eigenvalue -1 of $(-\Delta+U_0-(\lambda+i0))^{-1}(U-U_0)$. Similarly if $\text{Im}\, U_0 \geq 0$, one can define the set of incoming positive resonances $r_-(H)$ and the algebraic multiplicity $m_-(\lambda)$ for $\lambda \in r_-(H)$.*

If U is of short-range, our definition coincides with the usual one given in [24]. See also [23] where complex short-range perturbations of real long-range electro-magnetic potentials are considered. In fact if U is of short-range, then U_0 is also of short-range. In this case, the

equation

$$1+(-\Delta-z)^{-1}U=\big(1+(-\Delta-z)^{-1}U_0\big)\big(1+(-\Delta+U_0-z)^{-1}(U-U_0)\big)$$

valid for $z\notin\sigma(-\Delta+U_0)$ can be extended up to $z=\lambda+i0$ in $\mathcal{B}(0,-s)$, $s>\frac{1}{2}$. It follows that -1 is an eigenvalue of the compact operator $(-\Delta+U_0-(\lambda+i0))^{-1}(U-U_0)$ if and only if -1 is an eigenvalue of $(-\Delta-(\lambda+i0))^{-1}U$. The latter is equivalent with the condition that the equation $(-\Delta+U(x)-\lambda)u=0$ admits a solution satisfying the outgoing Sommerfeld radiation condition.

In the case zero is not an eigenvalue of non-selfadjoint Schrödinger operator $H=H_0+W(x)$, we prove the following

Theorem 2.2. *Assume that zero is not an eigenvalue of H.*

(a). Let $V_0\in\mathcal{V}$. For any $a>0$ there exist $c_a,C_a>0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}}\left(e^{-tH}-\sum_{\lambda\in\sigma_d(H),\mathrm{Re}\,\lambda\le 0}e^{-tH}\Pi_\lambda\right)\|\le C_ae^{-c_at^\beta}\quad t>0,\tag{2.22}$$

where

$$\beta=\frac{1-\mu}{1+\mu}.\tag{2.23}$$

(b). Let $V_0\in\mathcal{A}$. Then the set of outgoing resonances $r_+(H)$ of H is at most finite. There exists some constant $c>0$ such that for any $\chi\in C_0^\infty(\mathbb{R}^n)$ one has

$$\|\chi\left(e^{-itH}-\sum_{\lambda\in\sigma_d(H)\cap\overline{\mathbb{C}}_+}e^{-itH}\Pi_\lambda-\sum_{\nu\in r_+(H)}e^{-it\nu}P_\nu(t)\right)\chi\|\le C_\chi e^{-c\,t^\beta}\quad t>0,\tag{2.24}$$

Here Π_λ denotes the Riesz projection associated with the discrete eigenvalue λ of H and $P_\nu(t)$ is an operator depending polynomially on t with coefficients of rank not exceeding $m_+(\nu)$.

Remark that for $\lambda\in\sigma_d(H)\cap\overline{\mathbb{C}}_+$,

$$e^{-itH}\Pi_\lambda=e^{-it\lambda}Q_\lambda(t)$$

where $Q_\lambda(t)$ is polynomial in t with rank less than or equal to the algebraic multiplicity of eigenvalue λ. If in addition $\mathrm{Im}\,V\le 0$, then the set of positive resonances $r_+(H)$ is empty. Recall that for any $\lambda>0$, one can construct a complex-valued potential $V\in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{Im}\,V\ge 0$ such that λ is an outgoing positive resonance of H. See [28] for example of incoming positive resonance with $V\in C_0^\infty$ and $\mathrm{Im}\,V\le 0$. The result (2.24) for $n\ge 2$ is new even for the selfadjoint model operator $H=H_0$. See [31] for a result in one dimensional case.

Consider now the case when zero is an eigenvalue of H. If H is selfadjoint, then H has only a finite number of negative eigenvalues and both positive eigenvalues and positive resonances of H are absent. We can apply the known method in threshold spectral analysis for selfadjoint operators ([13]) to compute low-energy expansion of the resolvent. Theorem 2.1 allows to estimate the remainder in Gevrey spaces and to prove the following

Theorem 2.3. *Assume that zero is an eigenvalue of H and that both H and H_0 are selfadjoint.*

(a). If $V_0 \in \mathcal{V}$, then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}}\left(e^{-tH} - \sum_{\lambda\in\sigma_p(H)} e^{-t\lambda}\Pi_\lambda\right)\| \leq C_a e^{-c_a t^\beta} \quad t > 0, \tag{2.25}$$

(b). Let $V_0 \in \mathcal{A}$. Then there exists some constant $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has

$$\|\chi\left(e^{-itH} - \sum_{\lambda\in\sigma_p(H)} e^{-it\lambda}\Pi_\lambda\right)\chi\| \leq C_\chi e^{-c\,t^\beta} \quad t > 0, \tag{2.26}$$

where $\beta = \frac{1-\mu}{1+\mu}$ and Π_λ is the orthogonal eigenprojection of H associated with eigenvalue λ.

The case of threshold eigenvalue in non-selfadjoint case is more difficult, because in this case the associated algebraic multiplicity and Riesz projection are not defined. There does not yet exist general method to treat resolvent expansion near threshold eigenvalue. In [9], threshold eigenvalue of H is studied under several conditions on subspaces Ker (H^j) in $L^2(\mathbb{R}^n)$, $j \in \mathbb{N}^*$. In this work, we generalize the usual approach in threshold spectral analysis known in selfadjoint case ([13, 27]) to non-selfadjoint problems. Under the conditions of Theorem 2.1, G_0W is a compact operator on $L^2(\mathbb{R}^n)$. One can show that zero is an eigenvalue of H if and only if -1 is an eigenvalue of compact operator G_0W. Consequently, threshold eigenvalue of H, if it does exist, is of finite geometrical multiplicity. Although the algebraic multiplicity of eigenvalue zero of H is not defined, that of eigenvalue -1 of compact operator G_0W is well defined. Let m denote the algebraic multiplicity of eigenvalue -1 of operator G_0W. Then we shall show that there exists some numerical Gevrey function $\omega(z)$ such that for z near zero, $z \in \sigma_d(H)$ if and only if $\omega(z) = 0$. (See Proposition 5.12). In addition, $\omega(z)$ admits an asymptotic expansion of any order in powers of z: there exist some constants $\omega_j \in \mathbb{C}$, $j \in \mathbb{N}$, such that

$$\omega(z) = \omega_1 z + \cdots + \omega_N z^N + O(|z|^{N+1}), \tag{2.27}$$

for z near 0 and $\mathrm{Re}\, z < 0$ and for any $N \in \mathbb{N}^*$.

Theorem 2.4. *Assume that zero is an eigenvalue of H and that there exists some constant $\omega_k \neq 0$ such that*

$$\omega(z) = \omega_k z^k + O(|z|^{k+1}), \tag{2.28}$$

for z near 0 and $\mathrm{Re}\, z < 0$. Then the following results hold.

(a). If $V_0 \in \mathcal{V}$, then for any $a > 0$ there exist some constants $c_a, C_a > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}}\left(e^{-tH} - \sum_{\lambda\in\sigma_d(H),\mathrm{Re}\,\lambda\leq 0} e^{-tH}\Pi_\lambda - \Pi_0(t)\right)\| \leq C_a e^{-c_a t^{\beta'}} \quad t > 0, \tag{2.29}$$

where $\beta' = \frac{1-\mu}{1+\kappa\mu}$ for some integer $\kappa \geq 1$ given by Corollary 4.2.

(b). Let $V_0 \in \mathcal{A}$. Then the set of outgoing resonances $r_+(H)$ of H is at most finite and there exists $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has

$$\|\chi\left(e^{-itH} - \sum_{\lambda\in\sigma_d(H)\cap\overline{\mathbb{C}}_+} e^{-itH}\Pi_\lambda - \Pi_0(t) - \sum_{\nu\in r_+(H)} e^{-it\nu}P_\nu(t)\right)\chi\| \leq C_\chi e^{-c\,t^\beta} \quad t > 0, \tag{2.30}$$

Here β, Π_λ and $P_\nu(t)$ have the same meaning as in Theorem 2.2 (b) and $\Pi_0(t)$ is polynomial in t of the form

$$\Pi_0(t) = \sum_{j=0}^{k-1} t^j \Pi_{0,j} \tag{2.31}$$

where $\Pi_{0,j}$, $0 \leq j \leq k-1$ is an operator of rank not exceeding m, m being the algebraic multiplicity of -1 as eigenvalue of G_0W.

Remarks 2.6. *1. If H_0 is selfadjoint or if $\mathrm{Re}\, H_0$ already satisfies (2.3) for some $\mu \in]0,1[$ and $\mu_2 \geq \mu$, one can take $\kappa = 1$ in Theorem 2.4 (a) and recovers $\beta' = \beta$. See Remark 4.1.*

2. If threshold zero is an eigenvalue of selfadjoint Schrödinger operator H, the condition (2.28) is always satisfied and $\Pi_0(t) = \Pi_0$ is the eigenprojection associated with the eigenvalue zero of H. In non-selfadjoint case, under the (2.28), the existence of a resolvent expansion near threshold can be deduced by standard method of threshold spectral analysis. Now non-trivial problems are to check when the condition (2.28) is satisfied and to calculate more explicitly the term $\Pi_0(t)$. In fact a large part of Subsection 5.4 is devoted to proving the following more explicit results:

- *Assume that threshold eigenvalue of H is geometrically simple.*
  - *If (2.28) is satisfied, then $\Pi_{0,k-1}$ is of rank one and is given by*

$$\Pi_{0,k-1} = \langle \cdot, J\psi_0 \rangle \psi_0 \tag{2.32}$$

  for some eigenfunction ψ_0 associated with threshold eigenvalue of H. Here J is the complex conjugation $J : f(x) \to \overline{f(x)}$;

  - *If there exists an associated eigenfunction φ_0 such that*

$$\int_{\mathbb{R}^n} (\varphi_0(x))^2 dx = 1, \tag{2.33}$$

  then Condition (2.28) is satisfied with $k = 1$ and one has

$$\Pi_0(t) = \Pi_0 = \langle \cdot, J\varphi_0 \rangle \varphi_0. \tag{2.34}$$

  Note that although the Riesz projection is not defined for threshold eigenvalue in non-selfadjoint case, Π_0 is a projection from $L^2(\mathbb{R}^n)$ onto the eigenspace associated with eigenvalue zero of H.
- *Assume that eigenvalue -1 of G_0W is semi-simple (i.e. its algebraic and geometrical multiplicities are equal) with multiplicity m and that condition (5.83) is satisfied. Then (2.28) is valid with $k = m$ and one has $\Pi_{0,j} = 0$, $j = 1, \cdots, m-1$ and $\Pi_{0,0}$ is given by*

$$\Pi_{0,0} = \sum_{j=1}^{m} \langle \cdot, J\psi_j \rangle \varphi_j \tag{2.35}$$

  where $\{\varphi_j; j = 1, \cdots, m\}$ and $\{\psi_j; j = 1, \cdots, m\}$ are two basis of the eigenspace of H associated with eigenvalue zero.

The method developed in the proof of Theorem 2.4 is general and applies to some other situations. See Remarks 5.2 and 5.4. At the end of this paper, we give an example such that threshold eigenvalue of H is geometrically simple and (2.33) is satisfied.

3. Gevrey resolvent estimates for the model operator

The starting point of our Gevrey estimates of the resolvent of H_0 is a uniform *a priori* energy estimate for the model operator H_0. In the sequel, we need to apply this kind of energy estimates to the Schrödinger operator $-\Delta + V_1(x) - iV_2(x)$ and to its analytically

deformed versions as well. For this purpose, we begin with a class of second order elliptic differential operators of the form

$$H_0 = -\sum_{i,j=1}^{n} \partial_{x_i} a^{ij}(x)\partial_{x_j} + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + V_0(x) \tag{3.1}$$

satisfying conditions (2.2), (2.3) and (2.4).

Denote $b = (b_1, \cdots, b_n)$ and

$$|a|_\infty = \max_{1\le i,j\le n} \sup_{x\in\mathbb{R}^n} |a^{ij}(x)|, \quad |b|_{\mu,\infty} = \max_{1\le j\le n} \sup_{x\in\mathbb{R}^n} |\langle x\rangle^\mu b_j(x)|. \tag{3.2}$$

For $s \in \mathbb{R}$, denote

$$\varphi_s(x) = \left(1 + \frac{|x|^2}{R_s^2}\right)^s, \tag{3.3}$$

where $R_s = M\langle s\rangle^{\frac{1}{1-\mu}}$ with $M = M(c_0, |a|_\infty, |b|_\infty) > 1$ large enough, but independent of $s \in \mathbb{R}$. For each s, φ_s is equivalent to a weight of order s in x. Rescaling x by the s-dependent function $R_s = M\langle s\rangle^{\frac{1}{1-\mu}}$ in φ_s is crucial in this work, because it allows to prove an energy estimate uniformly in $s \in \mathbb{R}$ (Lemma 3.1) and to control the size of constants appeared in the inductive proof of Gevrey estimates of the resolvent (see (3.26)). A key estimate for φ_s which is used in the proof of the following Lemma 3.1 is

$$|\nabla\varphi_s(x)|^2 \le \frac{c\varphi_s(x)^2}{M^{2(1-\mu)}\langle x\rangle^{2\mu}}, \quad x \in \mathbb{R}^n, \tag{3.4}$$

for some constant $c > 0$ independent of M and s.

3.1. **A uniform energy estimate.**

Lemma 3.1. *Let H_0 be given by (3.1). Under the conditions (2.2), (2.3) and (2.4) with $0 < \mu < 1$, there exist some constants $C, M > 0$ depending only on $|a|_\infty$, $|b|_{\mu,\infty}$ and c_0 given in (2.3) such that*

$$\|\langle x\rangle^{-\mu}\varphi_s(x)u\| + \|\nabla(\varphi_s(x)u)\| \le C\|\langle x\rangle^\mu \varphi_s(x)H_0u\| \tag{3.5}$$

for any $s \in \mathbb{R}$ and $u \in H^2_{\rm loc}(\mathbb{R}^n)$ with $\langle x\rangle^{s+\mu}H_0u \in L^2$.

Proof. We calculate $\langle u, \varphi_s^2 H_0 u\rangle$ for $u \in C_0^\infty$:

$$\begin{aligned}
&\langle u, \varphi_s^2 H_0 u\rangle \\
&= \langle \varphi_s u, H_0(\varphi_s u)\rangle + \langle \varphi_s u, \left[\sum_{i,j=1}^{n} \partial_{x_i} a^{ij}\partial_{x_j}, \varphi_s\right] u\rangle - \langle \varphi_s u, (b\cdot\nabla\varphi_s)u)\rangle \\
&= I + II + III,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
I &= \langle \varphi_s u, H_0(\varphi_s u)\rangle \\
II &= \langle \varphi_s u, \sum_{i,j=1}^{n} \big((\partial_{x_i}\varphi_s)a^{ij}\partial_{x_j}u + \partial_{x_i}(a^{ij}(\partial_{x_j}\varphi_s)u)\big)\rangle \\
III &= -\langle \varphi_s u, (b\cdot\nabla\varphi_s)u\rangle.
\end{aligned}$$

Since $\varphi_s\partial_{x_j}u=\partial_{x_j}(\varphi_s u)-(\partial_{x_j}\varphi_s)u$, one has

$$\begin{aligned}
&|\langle\varphi_s u,(\partial_{x_i}\varphi_s)a^{ij}\partial_{x_j}u+\partial_{x_i}\left(a^{ij}(\partial_{x_j}\varphi_s)u\right)\rangle|\\
&= |\langle(\partial_{x_i}\varphi_s)u,a^{ij}\left(\partial_{x_j}(\varphi_s u)-(\partial_{x_j}\varphi_s)u\right)\rangle+\langle\varphi_s u,\partial_{x_i}(a^{ij}(\partial_{x_j}\varphi_s)u)\rangle|\\
&= |\langle(\partial_{x_i}\varphi_s)u,a^{ij}\left(\partial_{x_j}(\varphi_s u)-(\partial_{x_j}\varphi_s)u\right)\rangle-\langle\partial_{x_i}(\varphi_s u),a^{ij}(\partial_{x_j}\varphi_s)u\rangle|\\
&\leq |a|_\infty\left(\|(\partial_{x_i}\varphi_s)u\|(\|\partial_{x_j}(\varphi_s u)\|+\|(\partial_{x_j}\varphi_s)u\|)+\|\partial_{x_i}(\varphi_s u)\|\|(\partial_{x_j}\varphi_s)u\|\right)
\end{aligned}$$

The term II in (3.6) can be bounded by

$$\begin{aligned}
|II| &\leq |a|_\infty(\sum_{i=1}^n\|(\partial_{x_i}\varphi_s)u\|)\sum_{j=1}^n(2\|\partial_{x_j}(\varphi_s u)\|+\|(\partial_{x_j}\varphi_s)u\|)\\
&\leq n^2|a|_\infty\|(\nabla\varphi_s)u\|(2\|\nabla(\varphi_s u)\|+\|(\nabla\varphi_s)u\|))\\
&\leq n^2|a|_\infty\left(\epsilon\|\nabla(\varphi_s u)\|^2+(1+\frac{1}{\epsilon})\|(\nabla\varphi_s)u\|^2\right)
\end{aligned}$$

for any $\epsilon>0$. Clearly, III verifies

$$|III|\leq|b|_{\mu,\infty}\|\langle x\rangle^{-\mu}\varphi_s u\|\|(\nabla\varphi_s)u\|\leq|b|_{\mu,\infty}(\epsilon\|\langle x\rangle^{-\mu}\varphi_s u\|^2+\frac{1}{4\epsilon}\|(\nabla\varphi_s)u\|^2)\tag{3.7}$$

Taking $\epsilon=\epsilon(c_0,|a|_\infty,|b|_{\mu,\infty})>0$ appropriately small where $c_0>0$ is given by (2.3), it follows from (2.3) that

$$\begin{aligned}
|\langle u,\varphi_s^2H_0u\rangle| &\geq |I|-|II|-|III| \qquad (3.8)\\
&\geq \frac{c_0}{2}(\|\nabla(\varphi_s u)\|^2+\|\langle x\rangle^{-\mu}\varphi_s(x)u\|^2)-\langle u,W_s u\rangle
\end{aligned}$$

where $W_s(x)=c_1|\nabla\varphi_s|^2$ with $c_1>0$ some constant depending only on c_0, $|a|_\infty$ and $|b|_{\mu,\infty}$. One can check that

$$\begin{aligned}
|\nabla\varphi_s|^2 &= \frac{4s^2x^2}{R_s^4(1+\frac{x^2}{R_s^2})^2}(1+\frac{x^2}{R_s^2})^{2s}\\
&\leq \frac{4s^2x^2}{(R_s^2+x^2)^2}\varphi_s^2\leq\frac{4s^2}{R_s^2+x^2}\varphi_s^2
\end{aligned}$$

Since $R_s^2+x^2\geq 2^{-2\mu}R_s^{2(1-\mu)}\langle x\rangle^{2\mu}$ and $R_s=M\langle s\rangle^{\frac{1}{1-\mu}}$, $W_s(x)$ is bounded by

$$0\leq W_s(x)\leq\frac{4c_1\langle s\rangle^2}{R_s^2+x^2}\varphi_s^2\leq\frac{2^{2\mu}4c_1}{M^{2(1-\mu)}\langle x\rangle^{2\mu}}\varphi_s^2.\tag{3.9}$$

Noticing that $0<\mu<1$, one can choose $M=M(c_0,|a|_\infty,|b|_{\mu,\infty})>1$ large enough so that $\frac{2^{2\mu}4c_1}{M^{2(1-\mu)}}<\frac{c_0}{4}$. Consequently, the above estimate combined with (3.8) gives

$$|\langle u,\varphi_s^2H_0u\rangle|\geq\frac{c_0}{4}(\|\nabla(\varphi_s u)\|^2+\|\langle x\rangle^{-\mu}\varphi_s u\|^2).\tag{3.10}$$

Remark that

$$|\langle u,\varphi_s^2H_0u\rangle|\leq\|\langle x\rangle^{-\mu}\varphi_s u\|\|\langle x\rangle^{\mu}\varphi_s H_0u\|\leq\frac{c_0}{8}\|\langle x\rangle^{-\mu}\varphi_s u\|^2+\frac{2}{c_0}\|\langle x\rangle^{\mu}\varphi_s H_0u\|^2.$$

It follows from (3.10) that

$$\|\langle x\rangle^{\mu}\varphi_s H_0u\|^2\geq\frac{c_0^2}{16}(\|\langle x\rangle^{-\mu}\varphi_s u\|^2+\|\nabla(\varphi_s u)\|^2),\quad u\in C_0^\infty(\mathbb{R}^n).\tag{3.11}$$

By an argument of density, one obtains (3.5) with some constant $C>0$ independent of $s\in\mathbb{R}$. ■

The same proof as that for Lemma 3.1 shows that

$$\|\langle x\rangle^{-\mu}\varphi_s(x)u\|+\|\nabla(\varphi_s(x)u)\|\leq C\|\langle x\rangle^{\mu}\varphi_s(x)(H_0-1)u\|\tag{3.12}$$

uniformly in $s \in \mathbb{R}$, $\lambda \leq 0$ and and $u \in H^2_{\rm loc}(\mathbb{R}^n)$ with $\langle x\rangle^{s+\mu} H_0 u \in L^2$.

Corollary 3.2. *Under the conditions of Lemma 3.1, there exists some constant $C > 0$ such that for any $r \in \mathbb{R}$, $f \in L^{2,r}$ and $u \in H^2_{\rm loc}$ such that $H_0 u = f$, one has: $u \in L^{2,r-2\mu}$, $\nabla u \in L^{2,r-\mu}$ and*

$$\|\langle x\rangle^{r-\mu}\nabla u\| + \|\langle x\rangle^{r-2\mu} u\| \leq C\|\langle x\rangle^r f\|. \tag{3.13}$$

Proof. It follows from Lemma 3.1 with $s = \frac{r-\mu}{2}$. ■

Lemma 3.1 shows that $H_0 : D(H_0) \to R(H_0) := \text{Range}(H_0) \subset L^2(\mathbb{R}^n)$ is bijective. Let G_0 denote its algebraic inverse with domain $D(G_0) = R(H_0)$. Then one has

$$H_0 G_0 = 1 \text{ on } R(H_0), \quad G_0 H_0 = 1 \text{ on } D(H_0) \tag{3.14}$$

Lemma 3.3. *Let $\mathcal{D} = \cap_{s\in\mathbb{R}} L^{2,s}$. Then one has*

(a). $\mathcal{D} \subset D(G_0)$. G_0 maps $\mathcal{D}$ into $\mathcal{D}$ and is a densely defined closed operator on $L^2(\mathbb{R}^n)$. If H_0 is selfadjoint (resp., maximally dissipative), then $-G_0$ is also selfadjoint (resp., maximally dissipative).

(b). There exists some constant $C > 0$ such that

$$\|\nabla(\varphi_s G_0 \varphi_{-s}\langle x\rangle^{-\mu} w)\| + \|\langle x\rangle^{-\mu}\varphi_s G_0 \varphi_{-s}\langle x\rangle^{-\mu} w)\| \leq C\|w\| \tag{3.15}$$

for all $w \in \mathcal{D}$ and $s \in \mathbb{R}$.

Proof. We first show that $\mathcal{D} \subset D(G_0)$. Remark that $\text{Re}\, H_0 \geq 0$. Let $f \in \mathcal{D}$ and $u_\epsilon = (H_0 + \epsilon)^{-1} f$, $\epsilon > 0$. Since $\text{Re}\, H_0 \geq 0$ and H_0 verifies the weighted coercive condition (2.3), $H_0 + \epsilon$ satisfies also (2.3) with the same constant $c_0 > 0$ independent of $\epsilon > 0$. Following the proof of Lemma 3.1 with H_0 replaced by $H_0 + \epsilon$, one has that for any $s > 0$

$$\|\langle x\rangle^{s-\mu}\nabla u_\epsilon\| + \|\langle x\rangle^{s-2\mu} u_\epsilon\| \leq C_s\|\langle x\rangle^s f\| \tag{3.16}$$

uniformly in $\epsilon > 0$. For $s > 2\mu$, this estimate implies that the sequence $\{u_\epsilon; \epsilon \in]0,1]\}$ is relatively compact in L^2. Therefore there exists a subsequence $\{u_{\epsilon_k}; k \in \mathbb{N}\}$ and $u \in L^2$ such that $\epsilon_k \to 0$ and $u_{\epsilon_k} \to u$ in L^2 as $k \to +\infty$. It follows that $H_0 u = f$ in the sense of distributions. The ellipticity of H_0 implies that $u \in H^2(\mathbb{R}^n)$. Therefore $f \in R(H_0) = D(G_0)$. This shows that $\mathcal{D} \subset D(G_0)$. In particular $D(G_0)$ is dense in $L^{2,r}$ for any $r \in \mathbb{R}$. The upper-bound (3.16) uniform in $\epsilon > 0$ implies that $u \in L^{2,s}$ for any s. Since H_0 is injective by assumption (2.3), one has $G_0 f = u$ and G_0 maps $\mathcal{D}$ into $\mathcal{D}$. The closeness of G_0 follows from that of H_0. The other assertions of Part (a) can be easily checked.

The argument used above shows that for any $w \in \mathcal{D}$, one can find $u \in D(H_0)$ such that $H_0 u = \varphi_{-s}\langle x\rangle^{-\mu} w$. (3.15) follows from (3.5). ■

Lemma 3.3 shows that for any s, $\langle x\rangle^{-\mu}\varphi_s G_0 \varphi_{-s}\langle x\rangle^{-\mu}$ defined on $\mathcal{D}$ can be uniquely extended to a bounded operator on $L^2(\mathbb{R}^n)$, or in other words, for any $s \in \mathbb{R}$, G_0 is bounded from $D(G_0) \cap L^{2,s}$ to $L^{2,s-2\mu}$:

$$\|\langle x\rangle^{-\mu}\varphi_s G_0 u\| \leq C\|\varphi_s\langle x\rangle^{\mu} u\| \tag{3.17}$$

uniformly in $u \in D(G_0) \cap L^{2,s}$ and $s \in \mathbb{R}$. This implies that G_0 maps $\mathcal{D}$ into $\mathcal{D}$ and G_0 extends to a continuous operator from $L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$. By an induction, one can check that G_0^N extends to a bounded operator from $L^{2,s}$ to $L^{2,s-2N\mu}$ for any $s \in \mathbb{R}$. To simplify notation, we still denote G_0 (resp., G_0^N) its continuous extension by density as operator from $L^{2,s}$ to $L^{2,s-2\mu}$ (resp., from $L^{2,s}$ to $L^{2,s-2N\mu}$). $\mathcal{D}$ is a core of G_0^N for any $N \geq 1$.

3.2. **Resolvent estimates at threshold for the model operator.**

Theorem 3.4. *Let $M > 1$ be given in Lemma 3.1. Denote*

$$x_{N,r} = \frac{x}{R_{N,r}} \text{ with } R_{N,r} = R_{(2N-1+r)\mu} = M\langle(2N-1+r)\mu\rangle^{\frac{1}{1-\mu}} \tag{3.18}$$

where $N \in \mathbb{N}^*$ and $r \in \mathbb{R}_+$ and $M > 0$ is a constant given by Lemma 3.1. Set $\langle x_{N,r}\rangle = (1+|x_{N,r}|^2)^{\frac{1}{2}}$. Then there exists some constant $C > 0$ such that

$$\|\langle x_{N,r}\rangle^{-(2N+r)\mu} G_0^N \langle x_{N,r}\rangle^{r\mu}\| \leq C^N \langle(2N-1+r)\mu\rangle^{\gamma N}, \tag{3.19}$$

for any integer $N \geq 1$ and any $r \geq 0$. Here

$$\gamma = \frac{2\mu}{1-\mu}. \tag{3.20}$$

Proof. Making use of Lemma 3.1, one can check that operator

$$I_N = \langle x_{N,r}\rangle^{-2N\mu - r\mu} G_0^N \langle x_{N,r}\rangle^{r\mu} \tag{3.21}$$

is well defined on $\mathcal{D}$ and extends to a bounded operator on L^2. To show the estimate (3.19), we use an induction on N. Since $\langle x\rangle \leq R\langle\frac{x}{R}\rangle$ for $R \geq 1$, it follows from (3.15) that

$$\|\langle\frac{x}{R_s}\rangle^{-s-\mu} G_0 \langle\frac{x}{R_s}\rangle^{s-\mu}\| \leq C' R_s^{2\mu} \leq C_1\langle s\rangle^{\gamma} \tag{3.22}$$

uniformly in s, where $R_s = M\langle s\rangle^{\frac{1}{1-\mu}}$. In particular, when $s = (1+r)\mu$, one has $R_s = M\langle(1+r)\mu\rangle^{\frac{1}{1-\mu}} = R_{1,r}$ and

$$\|I_1\| \leq C_1\langle(1+r)\mu\rangle^{\gamma} \tag{3.23}$$

for all $r \geq 0$, which proves (3.19) when $N = 1$. Assume now that $N \geq 2$ and that one has proved for some $C > 0$ independent of N and $r \geq 0$ that

$$\|I_{N-1}\| \leq C^{N-1}\langle(2N-3+r)\mu\rangle^{\gamma(N-1)}. \tag{3.24}$$

Write I_N as

$$I_N = \langle x_{N,r}\rangle^{-(2N+r)\mu} G_0 \langle x_{N-1,r}\rangle^{(2N-2+r)\mu} \cdot I_{N-1} \cdot \langle x_{N-1,r}\rangle^{-r\mu} \langle x_{N,r}\rangle^{r\mu}$$

Notice that

$$\langle x_{N,r}\rangle \leq \langle x_{N-1,r}\rangle \leq \frac{R_{N,r}}{R_{N-1,r}}\langle x_{N,r}\rangle$$

for any $N \geq 2$. Applying (3.22) with $s = (2N-1+r)\mu$, one obtains

$$\|\langle x_{N,r}\rangle^{-(2N+r)\mu} G_0 \langle x_{N,r}\rangle^{(2N-2+r)\mu}\| \leq C_1\langle(2N-1+r)\mu\rangle^{\gamma}.$$

Making use of the induction hypothesis (3.24), one can estimate I_N as follows:

$$\begin{aligned}
\|I_N\| &\leq \|\langle x_{N,r}\rangle^{-(2N+r)\mu} G_0 \langle x_{N-1,r}\rangle^{(2N-2+r)\mu}\| \cdot \|I_{N-1}\| \\
&\leq \|\langle x_{N,r}\rangle^{-(2N+r)\mu} G_0 \langle x_{N,r}\rangle^{(2N-2+r)\mu}\| \cdot \|(\frac{\langle x_{N-1,r}\rangle}{\langle x_{N,r}\rangle})^{(2N-2+r)\mu}\|_{L^\infty} \cdot \|I_{N-1}\| \\
&\leq C_1\langle(2N-1+r)\mu\rangle^{\gamma} \cdot \left(\frac{\langle(2N-1+r)\mu\rangle}{\langle(2N-3+r)\mu\rangle}\right)^{\gamma(2N-2+r)} \cdot C^{N-1}\langle(2N-3+r)\mu\rangle^{\gamma(N-1)} \\
&\leq C_1\left(\frac{2N-1+r}{2N-3+r}\right)^{\gamma(2N-2+r)} C^{N-1}\langle(2N-1+r)\mu\rangle^{\gamma N}.
\end{aligned} \tag{3.25}$$

The sequence $\{\left(\frac{2m-1+r}{2m-3+r}\right)^{\gamma(2m-2+r)}; m \geq 2\}$ is uniformly bounded in $r \geq 0$. Hence there exists some constant $C_2 > 0$ independent of m such that

$$C_1\left(\frac{2m-1+r}{2m-3+r}\right)^{\gamma(2m-2+r)} \leq C_2 \tag{3.26}$$

for all $m \geq 2$ and $r \geq 0$. Increasing the constant C if necessary, one can suppose without loss that $C_2 \leq C$ and obtains from (3.26) that

$$\|I_N\| \leq C^N \langle (2N-1+r)\mu \rangle^{N\gamma} \tag{3.27}$$

Theorem 3.4 is proved by an induction on N. ■

(3.19) with $r = 0$ shows that there exists some contant $C > 0$ such that

$$\|\langle x \rangle^{-2N\mu} G_0^N\| + \|G_0^N \langle x \rangle^{-2N\mu}\| \leq C^N N^{\gamma N} \tag{3.28}$$

for all $N \geq 1$. Let $R_0(z) = (H_0 - z)^{-1}$ denote the resolvent of H_0 and

$$\Omega_-(\delta) \equiv \Omega\left(\delta, \frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta\right) = \left\{ z \in \mathbb{C}^*; |z| < \delta, \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta \right\},$$

with $\delta > 0$. Since $\operatorname{Re} H_0 \geq 0$, there exists some constant $C_1 > 0$ such that

$$\|R_0(z)\| \leq \frac{C_1}{|z|}, \quad z \in \Omega_-(\delta).$$

From the equation $R_0(z) = G_0 + zG_0R_0(z)$, it follows that as operators from $L^{2,s}$ to $L^{2,s-2\mu}$, $s \in \mathbb{R}$, $R_0(z)$ is uniformly bounded for $z \in \Omega_-(\delta)$ and one has

$$\text{s-}\lim_{z \in \Omega_-(\delta), z \to 0} R_0(z) = G_0. \tag{3.29}$$

Similarly one can check that for any $N \in \mathbb{N}^*$, one has

$$\text{s-}\lim_{z \in \Omega_-(\delta), z \to 0} R_0(z)^N = G_0^N. \tag{3.30}$$

as operators from $L^{2,s}$ to $L^{2,s-2N\mu}$. By an abuse of notation, we set $R_0(0) = G_0$. Thus $R_0(z)$ is defined for z in $\Omega_-(\delta) \cup \{0\}$.

Corollary 3.5. *For any $a > 0$, there exists some constant $C_a > 0$ such that one has*

$$\|e^{-a\langle x \rangle^{1-\mu}} R_0(z)^N\| + \|R_0(z)^N e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a^{N+1} N^{\gamma N} \tag{3.31}$$

for all $N \geq 1$ and $z \in \Omega_-(\delta) \cup \{0\}$. Here $\gamma = \frac{2\mu}{1-\mu}$.

Proof. Since $\operatorname{Re} H_0 \geq 0$, one has

$$\|R_0(z)\| \leq \frac{1}{|\operatorname{Re} z|}$$

for $\operatorname{Re} z < 0$. Hence $\|zR_0(z)\|$ is uniformly bounded for z in $\Omega_-(\delta) \cup \{0\}$ ($\delta > 0$ is fixed). Iterating the first resolvent equation, one has

$$R_0(z)^N = G_0^N (1 + zR_0(z))^N.$$

According to Theorem 3.4 with $r = 0$, one has for some constant $C > 0$ depending on δ

$$\|\langle x_{N,0} \rangle^{-2N\mu} R_0(z)^N\| \leq \|\langle x_{N,0} \rangle^{-2N+\mu} G_0^N\| (1 + \|zR_0(z)\|^N) \leq C^N N^{\gamma N}, \tag{3.32}$$

for any integer $N \geq 1$ and $z \in \Omega_-(\delta) \cup \{0\}$. Let $a > 0$. Then

$$\|e^{-a\langle x \rangle^{1-\mu}} R_0(z)^N\| \leq \|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty} C^N N^{\gamma N}.$$

To evaluate the norm $\|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty}$, consider the function

$$f(r) = e^{-ar^{1-\mu}} \langle \frac{r}{R_N} \rangle^{2N\mu},$$

where $r = |x|$ and $R_N = R_{N,0} = M\langle (2N-1)\mu \rangle^{\frac{1}{1-\mu}}$. One calculates:

$$f'(r) = \frac{f(r)}{r^\mu (R_N^2 + r^2)} \left(-2a(1-\mu)(R_N^2 + r^2) + 2N\mu r^{1+\mu} \right), r \geq 1.$$

Let $A \geq 1$. Since $R_N \sim c' N^{\frac{1}{1-\mu}}$ for some constant $c' > 0$ as $N \to \infty$, one can check that $Nr^{1+\mu} \leq \frac{c}{A^{1-\mu}} r^2$ if $r \geq AR_N$ for some constant $c > 0$ independent of A, r and N. Therefore, if $A = A(\mu, a) > 1$ is chosen sufficiently large, one has

$$f'(r) < 0, \quad r > AR_N,$$

thus $f(r)$ is decreasing in $[AR_N, +\infty[$. It is now clear that

$$\|e^{-a\langle x\rangle^{1-\mu}} \langle x_{N,0}\rangle^{2N\mu}\|_{L^\infty} \leq \sup_{0\leq r\leq AR_N} f(r) \leq \langle A\rangle^{2N\mu}$$

Corollary 3.5 is proved for some appropriate constant C_a. ■

Proof of Theorem 2.1. Theorem 2.1 for $\tau = 0$ is a particular case of Corollary 3.5. In the general case $\tau \geq 0$, we apply Theorem 3.4 with $\tau = r\mu$ and remark that

$$\frac{1}{R_{N,r}} \langle x\rangle \leq \langle x_{N,r}\rangle \leq \langle x\rangle.$$

As in the proof of Corollary 3.5, one can show that

$$\begin{aligned} &\|\langle x\rangle^{-r\mu} e^{-a\langle x\rangle^{1-\mu}} G_0^N \langle x\rangle^{r\mu}\| \\ &\quad \leq \quad B_1 B_2^N R_{N,r}^{r\mu} \|\langle x_{N,r}\rangle^{-(2N+r)\mu} G_0^N \langle x_{N,r}\rangle^{r\mu}\| \\ &\quad \leq \quad D_1 D_2^{N+r\mu} (N+r)^{\frac{r\mu}{1-\mu}+\gamma N} \end{aligned} \tag{3.33}$$

for some constants $B_j, D_j > 0$, uniformly in $r \geq 0$ and $N \in \mathbb{N}^*$. The proof of Theorem 2.2 is complete for all $\tau = r\mu \geq 0$ and $N \in \mathbb{N}^*$. ■

4. Heat and Schrödinger semigroups of the model operator

As the first application of Theorem 2.1, we show in this Section how to deduce sub-exponential time-decay estimates for the heat and Schrödinger equations associated with a model operator of the form $H_0 = -\Delta + V_0(x)$ with $V_0(x) = V_1(x) - iV_2(x)$, $V_1(x)$ and $V_2(x)$ being real and satisfying Condition (2.3).

4.1. Sub-exponential time-decay of the heat semigroup.

To study the heat semigroup e^{-tH_0}, $t \geq 0$, we use Cauchy integral formula for semigroups and need an upper-bound of the resolvent on a contour in the right half-plane passing through the origin.

Proposition 4.1. *Assume that* $\operatorname{Re} H_0 \geq -a\Delta$ *for some* $a > 0$ *and that the imaginary part of the potential* $V_0(x)$ *verifies the estimate*

$$|V_2(x)| \leq C\langle x\rangle^{-2\mu_2}, \quad \forall x \in \mathbb{R}^n, \tag{4.1}$$

for some $\mu_2 > 0$. *Let* μ' *such that* $0 < \mu' \leq \min\{\mu_2, 1\}$ *and* $0 < \mu' < \frac{n}{2}$. *Then there exists some constant* $C_0 > 0$ *such that the numerical range* $N(H_0)$ *of* H_0 *is contained in a region of the form* $\{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| \leq C_0 (\operatorname{Re} z)^{\mu'}\}$. *Consequently, for* $\delta > 0$ *small enough there exists some constant* M_0 *such that*

$$\|R_0(z)\| \leq \frac{M_0}{|z|^{\frac{1}{\mu'}}} \tag{4.2}$$

for $z \in O(\delta)$ *where*

$$O(\delta) = \{z \in \mathbb{C}^*; |z| < \delta, \operatorname{Re} z < \delta |\operatorname{Im} z|^{\frac{1}{\mu'}}\}. \tag{4.3}$$

Proof. For $z = \langle u, H_0 u\rangle \in N(H_0)$ where $u \in D(H_0)$ and $\|u\| = 1$, one has

$$\begin{aligned} \mathrm{Re}\, z &= \mathrm{Re}\, \langle u, H_0 u\rangle \geq a\|\nabla u\|^2 \\ |\mathrm{Im}\, z| &\leq \langle u, |V_2| u\rangle \leq C\|\langle x\rangle^{-\mu_2} u\|^2. \end{aligned}$$

According to the generalized Hardy inequality ([11]), for $0 < \mu' < \frac{n}{2}$ and $0 < \mu' \leq \mu_2$ there exists some constant $C_{\mu'}$ such that

$$\|\langle x\rangle^{-\mu_2} u\|^2 \leq \||x|^{-\mu'} u\|^2 \leq C_{\mu'} \||\nabla|^{\mu'} u\|^2. \tag{4.4}$$

Let $\hat{u}$ denote the Fourier transform of u normalized such that $\|\hat{u}\| = \|u\|$ and $\tau = \|\nabla u\|$. Then for $0 < \mu' \leq 1$ noticing that $\|u\| = 1$, one has

$$\begin{aligned} \||\nabla|^{\mu'} u\|^2 &= \||\xi|^{\mu'} \hat{u}\|^2 = \||\xi|^{\mu'} \hat{u}\|^2_{L^2(|\xi|\geq\tau)} + \||\xi|^{\mu'} \hat{u}\|^2_{L^2(|\xi|<\tau)} \\ &\leq \tau^{2(\mu'-1)} \||\xi| \hat{u}\|^2_{L^2(|\xi|\geq\tau)} + \tau^{2\mu'} \|\hat{u}\|^2_{L^2(|\xi|<\tau)} \\ &\leq 2\tau^{2\mu'} = 2\|\nabla u\|^{2\mu'}. \end{aligned}$$

This proves that $\mathrm{Re}\, z \geq 0$ and $|\mathrm{Im}\, z| \leq C_0 (\mathrm{Re}\, z)^{\mu'}$ when $z \in N(H_0)$. The other assertions of Proposition are immediate, since $\sigma(H_0) \subset \overline{N(H_0)}$ and

$$\|R_0(z)\| \leq \frac{1}{\mathrm{dist}(z, N(H_0))}.$$

■

Remark 4.1. *Let $H_1 = -\Delta + V_1(x)$ be the real part of H_0. From the proof of Proposition 4.1, one sees that if $V_1 \in \mathcal{V}$ for some $\mu \in]0,1[$ and if $\mu_2 \geq \mu$, then the numerical range of H_0 is contained in the sector $\{z; \mathrm{Re}\, z \geq 0, |\mathrm{Im}\, z| \leq C \mathrm{Re}\, z\}$ for some $C > 0$. Therefore the results of Proposition 4.1 hold with $\mu' = 1$. In particular, this is the case if $V_2 = 0$ and $V_1 \in \mathcal{V}$. Note that Proposition 4.1 can be applied to operators of the form $H_0 = -\Delta - i\frac{1}{\langle x\rangle^{2\mu}}$, $\mu \in]0,1[$.*

Making use of the equation

$$R_0(z) = \sum_{j=0}^{\kappa-1} z^j G_0^{j+1} + z^\kappa G_0^\kappa R_0(z) \tag{4.5}$$

we deduce from Theorem 3.4 (with $r = 0$ and $N = \kappa$) and Proposition 4.1 that for $\kappa \in \mathbb{N}^*$ and $\kappa \geq \frac{1}{\mu'}$, one has

$$\|\langle x\rangle^{-2\kappa\mu} R_0(z)\| \leq C \tag{4.6}$$

uniformly in $z \in \Omega$ and z near 0. It follows that

$$G_0 = \text{s-}\lim_{z\in\Omega, z\to 0} R_0(z)$$

as operator from $L^{2,s}$ to $L^{2,s-2\kappa\mu}$. As before we denote $R_0(0) = G_0$.

Notice that under the conditions of Proposition 4.1, one can not exclude possible accumulation of complex eigenvalues towards zero. Making use of Proposition 4.1, one can prove the following uniform Gevrey estimates in a domain located in the right half-plane.

Corollary 4.2. *Under the conditions of Proposition 4.1, let $\kappa \in \mathbb{N}^*$ be the smallest integer such that $\kappa \geq \frac{1}{\mu'}$. For any $a > 0$, there exist $c, C > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} \frac{d^{N-1}}{dz^{N-1}} R_0(z)\| \leq C c^N N^{(1+\kappa\gamma)N}, \quad \forall N \geq 1, \tag{4.7}$$

uniformly in $z \in O_0(\delta) \equiv O(\delta) \cup \{0\}$, where $O(\delta)$ is defined by (4.3).

Proof. For $z \in \Omega$, decompose $R_0(z)$ as

$$R_0(z) = A(z) + G_0^\kappa B(z)$$

with $A(z) = \sum_{j=0}^{\kappa-1} z^j G_0^{j+1}$ and $B(z) = z^\kappa R_0(z)$. By Proposition 4.1, $\|B(z)\|$ is uniformly bounded for $z \in \Omega$. Theorem 3.4 shows that for some constant C_1

$$\begin{aligned} \|\langle x_{\kappa,r}\rangle^{-(2\kappa+r)\mu} G_0^\kappa \langle x_{\kappa,r}\rangle^{r\mu}\| &\leq C_1 \langle (2\kappa+r)\mu\rangle^{\gamma\kappa}, \quad (4.8) \\ \|\langle x_{\kappa,r}\rangle^{-(2\kappa+r)\mu} A(z) \langle x_{\kappa,r}\rangle^{r\mu}\| &\leq C_1 \langle (2\kappa+r)\mu\rangle^{\gamma\kappa} \quad (4.9) \end{aligned}$$

for any $r \geq 0$ and $|z| \leq 1$. Making use of the relation

$$R_0(z)^N = A(z) R_0(z)^{N-1} + G_0^\kappa R_0(z)^{N-1} B(z)$$

one can show by an induction on N that there exists some constant $C > 0$ such that

$$\|\langle x_{\kappa N,0}\rangle^{-2\kappa N\mu} R_0(z)^N\| \leq C^N N^{N\gamma\kappa} \quad (4.10)$$

for any $N \geq 1$ and $z \in \Omega$. In fact, the case $N = 1$ follows from (4.6). If (4.10) is proved with N replaced by $N-1$ for some $N \geq 2$, noticing that $x_{\kappa N,0} = x_{\kappa,2\kappa(N-1)}$, (4.8) and (4.9) with $r = 2\kappa(N-1)$ show that

$$\begin{aligned} &\|\langle x_{\kappa N,0}\rangle^{-2\kappa N\mu} R_0(z)^N\| \\ &\leq C_1 \langle (2\kappa N-1)\mu\rangle^{\gamma\kappa} (\|\langle x_{\kappa(N-1),0}\rangle^{-2\kappa(N-1)\mu} R_0(z)^{N-1}\| \\ &\quad + \|\langle x_{\kappa(N-1),0}\rangle^{-2\kappa(N-1)\mu} R_0(z)^{N-1}\| \|B(z)\|) \\ &\leq C_2 C^{N-1} N^{N\gamma\kappa} \end{aligned}$$

for some constant C_2 independent of N. Increasing the constant C if necessary, one obtains (4.10) for all $N \geq 1$ by induction. (4.7) is deduced from (4.10) as in the proof of Corollary 3.5. ∎

By Remark 4.1, if $V_1 \in \mathcal{V}$ for some $\mu \in]0,1[$ (which implies in particular that $H_1 = -\Delta + V_1(x)$ satisfies (2.3)) and $V_2 = O(\langle x\rangle^{-2\mu})$, then one can take $\kappa = 1$ in Corollary 4.2 and the order of Gevrey estimates in Corollary 4.2 is the same as in Corollary 3.5. As another consequence of Proposition 4.1, we establish the following estimate on resolvent expansion at threshold.

Corollary 4.3. *Under the conditions of Proposition 4.1, assume in addition (2.3) with $\mu \in]0,1[$. Then for any $a > 0$, there exist some constants $C, c > 0$ such that for any $z \in O(\delta)$, one has for some N (depending on z) such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} (R_0(z) - \sum_{j=0}^{N} z^j G_0^{j+1})\| \leq C e^{-c|z|^{-\frac{1}{\gamma}}}. \quad (4.11)$$

Proof. It follows from (4.5) and Theorem 2.1 with $\tau = 2\kappa\mu$ that for some $C_a, c_a > 0$

$$\|e^{-a\langle x\rangle^{1-\mu}} (R_0(z) - \sum_{j=0}^{N} z^j G_0^{j+1})\| \leq C_a c_a^N N^{\gamma N} |z|^{N+1}, \quad (4.12)$$

for all $z \in O(\delta)$. The remainder estimate can be minimized by choosing an appropriate N in terms of $|z|$. For fixed $M' > 1$ and $z \neq 0$, take $N = [\frac{1}{(c_a M'|z|)^{\frac{1}{\gamma}}}]$. Then

$$c_a^N N^{\gamma N} |z|^{N+1} \leq e^{-c|z|^{-\frac{1}{\gamma}}}$$

for $z \in O(\delta)$, where $c > 0$ is independent of z and N. ∎

Theorem 4.4. *Let $H_0 = -\Delta + V_0(x)$ with $V_0 \in \mathcal{V}$. Then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} e^{-tH_0}\| + \|e^{-tH_0} e^{-a\langle x\rangle^{1-\mu}}\| \leq C_a e^{-c_a t^\beta}, \quad t > 0, \tag{4.13}$$

with β given by (2.23).

Proof. Let Γ be the contour defined by $\Gamma = \{z; \mathrm{Re}\, z \geq 0, |\mathrm{Im}\, z| = C(\mathrm{Re}\, z)^{\mu'}\}$ oriented in anti-clockwise sense, where $C > 0$ is sufficiently large. Here $\mu' > 0$ is appropriately small such that both conditions (2.12) and (4.1) are satisfied. By Proposition 4.1, the numerical range of H_0 is located on the right-hand side of Γ and one has

$$e^{-tH_0} = \frac{i}{2\pi} \int_\Gamma e^{-tz} R_0(z) dz. \tag{4.14}$$

Decompose Γ as $\Gamma = \Gamma_0 + \Gamma_1$ where Γ_0 is the part of Γ with $0 \leq \mathrm{Re}\, z \leq \delta$ while Γ_1 is the part of Γ with $\mathrm{Re}\, z > \delta$ where $\delta > 0$ is sufficiently small. Clearly, the integral on Γ_1 is exponentially decreasing as $t \to \infty$

$$\| \int_{\Gamma_1} e^{-tz} R_0(z) dz\| \leq C e^{-\delta t}, t > 0,$$

for some constant $C > 0$. For $z \in \Gamma_0$, denote $f_N(z) = R_0(z) - \sum_{j=0}^N z^j G_0^{j+1}$. Then

$$f_N(z) = z^{N+1} G_0^{N+1} R_0(z).$$

Then Theorem 2.1 with $\tau = 2\kappa\mu$ shows that for any $a > 0$ there exist some constants $C, C_1 > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}} f_N(z)\| \leq C_1 C^N |z|^{N+1} N^{\gamma N} \tag{4.15}$$

for $z \in \Gamma_0$. It follows that

$$\begin{aligned}
&\| \int_{\Gamma_0} e^{-tz} e^{-a\langle x\rangle^{1-\mu}} R_0(z) dz\| \\
&\leq \sum_{j=0}^N \|e^{-a\langle x\rangle^{1-\mu}} G_0^{j+1}\| | \int_{\Gamma_0} e^{-tz} z^j dz| + \| \int_{\Gamma_0} e^{-tz} e^{-a\langle x\rangle^{1-\mu}} f_N(z) dz\| \\
&\leq C_2 \sum_{j=0}^N C^j j^{\gamma j} e^{-\delta t} + C_2 C^N N^{\gamma N} \int_{\Gamma_0} |e^{-tz}| |z|^{N+1} |dz|
\end{aligned}$$

for some $C_2 > 0$ and for all $t > 0$ and $N \geq 1$. Parameterizing Γ_0 by $z = \lambda \pm ic\lambda^{\frac{1}{\mu'}}$ with $\lambda \in]0, \delta]$, one can evaluate the last integral as follows:

$$\begin{aligned}
\int_{\Gamma_0} |e^{-tz}| |z|^{N+1} |dz| &\leq C_3^N \int_0^\delta e^{-t\lambda} \lambda^{N+1} d\lambda \\
&= C_3^N t^{-N-2} \int_0^{\delta t} e^{-\tau} \tau^{N+1} d\tau \\
&\leq C_4^N t^{-N-2} N^N
\end{aligned}$$

for some $C_3, C_4 > 0$. This proves that there exist some constants B_0 and $B_1 > 0$ such that

$$\| \int_{\Gamma_0} e^{-tz} e^{-a\langle x\rangle^{1-\mu}} R_0(z) dz\| \leq B_0 B_1^N N^{\gamma N} (N e^{-\delta t} + N^N t^{-N-2}) \tag{4.16}$$

for any $t > 0$ and $N \geq 1$. To minimize the remainder, we choose N in terms of t such that $N \simeq (\frac{t}{M_1 B_1})^{\frac{1}{1+\gamma}}$ as $t \to +\infty$ for some fixed appropriately chosen constant M_1. One obtains

$$\| \int_{\Gamma_0} e^{-tz} e^{-a\langle x\rangle^{1-\mu}} R_0(z) dz\| \leq C e^{-\delta_0 t^{\frac{1}{1+\gamma}}} \tag{4.17}$$

for some $C, \delta_0 > 0$. This proves that there exist some constants $C, c > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}} e^{-tH_0}\| \leq Ce^{-c\,t^\beta}, \quad t > 0, \tag{4.18}$$

with $\beta = \frac{1+\mu}{1-\mu}$. ■

As a consequence of Theorem 4.4, one obtains that there exists some constant $c > 0$ such that

$$\|e^{-tH_0} f\| \leq C_R e^{-c\,t^\beta} \|f\|, \quad t > 0, \tag{4.19}$$

for all $f \in L^2(\mathbb{R}^n)$ with support contained in $\{|x| \leq R\}$, $R > 0$. In selfadjoint case, this result is already proved by S. Nakamura in [20] by different method.

4.2. **Sub-exponential time-decay of the Schrödinger semigroup.** To obtain sub-exponential time-decay of solutions to the Schrödinger equation associated with H_0, we take $V_0 \in \mathcal{A}$ and use both techniques of analytic dilation and analytic deformation to study quantum resonances of H_0. It is well known that these different techniques give rise to the same set of quantum resonances ([3, 10, 12, 25]). The conditions for potentials V_0 in the class $\mathcal{A}$ are used, among others, to show that quantum resonances of H_0 do not accumulate to threshold zero. The main task of this subsection is to prove that the resolvent of the analytically distorted Hamiltonian $H_0(\theta)$ of H_0 verifies Gevrey estimates along some curve located in $\mathbb{C}_-$.

Firstly, we use the analytic dilation method to prove that if $V_0 \in \mathcal{A}$, then quantum resonances of $H_0 = -\Delta + V_0(x)$ can not accumulate to zero. Denote $\widetilde{H}_0(\theta)$ the operator obtained from H_0 by analytic dilation:

$$\widetilde{H}_0(\theta) = -(1+\theta)^{-2}\Delta + V_0((1+\theta)x)$$

for $\theta \in \mathbb{C}$ and θ near 0. Set $\widetilde{R}_0(z,\theta) = (\widetilde{H}_0(\theta) - z)^{-1}$. For θ real, $\widetilde{R}_0(z,\theta)$ is holomorphic in $\mathbb{C}_+$ and meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$. Since $V_0 \in \mathcal{A}$, there exists some constant $\delta > 0$ such that $\{\widetilde{H}_0(\theta); \theta \in \mathbb{C}, |\theta| < \delta\}$ is a holomorphic family of type A. One has

$$\sigma_{\rm ess}(\widetilde{H}_0(\theta)) = \{\frac{r}{(1+\theta)^2}; r \geq 0\}. \tag{4.20}$$

Consequently for $\mathrm{Im}\,\theta > 0$ and $|\theta|$ small enough, the resolvent $\widetilde{R}_0(z,\theta)$ defined for $z \in \mathbb{C}_+$ with $\mathrm{Im}\, z >> 1$ can be meromorphically extended across the positive real half-axis $\mathbb{R}_+$ into the sector $\{z; \arg z > -\mathrm{Im}\,\theta\}$. The poles of this meromorphic extension of $\widetilde{R}_0(z,\theta)$ are by definition quantum resonances of H_0, which are independent of θ ([3]).

We begin with the following elementary Hardy type inequality.

Lemma 4.5. *Let $n \geq 2$ and $0 < s < n-1$. One has*

$$\|\langle x\rangle^{-1-\frac{s}{2}} u\|^2 \leq \frac{1}{2\sqrt{(n-1)(n-1-s)}} (\|\nabla u\|^2 + \|\langle x\rangle^{-s} u\|^2) \tag{4.21}$$

for all $u \in H^1(\mathbb{R}^n)$.

Proof. Let $x = r\omega$, $r \geq 0$ and $\omega \in \mathbb{S}^{n-1}$. For $u \in \mathcal{S}(\mathbb{R}^n)$, denote

$$F(r,\omega) = \frac{|u(r\omega)|^2 r^{n-1}}{\langle r\rangle^s}.$$

Then one has

$$F'_r(r,\omega) = \frac{((n-1)(1+r^2) - sr^2)|u(r\omega)|^2 r^{n-2}}{\langle r\rangle^{s+2}} + 2\frac{r^{n-1}}{\langle r\rangle^s} \mathrm{Re}\,(u'_r(r\omega)\overline{u(r\omega)}).$$

Here $F'_r(r,\omega)$ is the derivative of $F(r,\omega)$ with respect to r. Since $n \geq 2$ and $u \in \mathcal{S}$ one has $\int\int_{\mathbb{R}_+\times\mathbb{S}^{n-1}} F'(r,\omega)\, dr d\omega = 0$. It follows that

$$\int\int_{\mathbb{R}^n} \frac{(n-1)+(n-1-s)r^2}{r\langle r\rangle^{s+2}}|u(x)|^2\, dx = -2\int\int_{\mathbb{R}^n} \langle x\rangle^{-s}\mathrm{Re}\,(u'_r\overline{u})\, dx \tag{4.22}$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$. Inequality (4.21) follows from the trivial bounds

$$2\sqrt{(n-1)(n-1-s)} \leq \frac{(n-1)+(n-1-s)r^2}{r}$$

and

$$-2\int\int_{\mathbb{R}^n} \langle x\rangle^{-s}\mathrm{Re}\,(u'_r\overline{u})\, dx \leq 2\|u'_r\|\|\langle x\rangle^{-s}u\| \leq \|\nabla u\|^2 + \|\langle x\rangle^{-s}u\|^2$$

together with an argument of density. ■

Lemma 4.6. *Let $V_0 \in \mathcal{A}$. Then there exist some constants $c_0 > 0$ and $\gamma_0 \in]\pi, \frac{3\pi}{2}[$ such that for $\theta \in \mathbb{C}$ with $|\theta|$ sufficiently small and $\mathrm{Im}\,\theta > 0$, one has*

$$\sigma(\widetilde{H}_0(\theta)) \cap S(-c_0\theta, \gamma_0) = \emptyset \tag{4.23}$$

and

$$\|\langle x\rangle^{-2\mu}\widetilde{R}_0(z,\theta)\| \leq \frac{1}{c_0\mathrm{Im}\,\theta\langle z\rangle} \tag{4.24}$$

for $z \in S(-c_0\theta, \gamma_0)$. Here

$$S(-c_0\theta, \gamma_0) = \{z \in \mathbb{C}^*; -c_0\mathrm{Im}\,\theta < \arg z < \gamma_0\} \tag{4.25}$$

Proof. We only consider the case $\theta = i\tau$ with $\tau > 0$ small enough. Since $V_0 = V_1 - iV_2 \in \mathcal{A}$, one has

$$\begin{aligned} V_0((1+i\tau)x) &= V_1(x) + \tau x\cdot\nabla V_2(x) \\ &\quad -i(V_2(x) - \tau x\cdot\nabla V_1(x) + O(\tau^2\langle x\rangle^{-2\mu}) \end{aligned}$$

for $\tau > 0$ sufficiently small. Let $z = \langle u, \widetilde{H}_0(\theta)u\rangle$, $u \in H^2$ with $\|u\| = 1$. Then,

$$\mathrm{Re}\, z = \frac{1-\tau^2}{(1+\tau^2)^2}\|\nabla u\|^2 + \langle u, (V_1(x) + O(\tau\langle x\rangle^{-2\mu}))u\rangle, \tag{4.26}$$

$$\begin{aligned} \mathrm{Im}\, z &= -\frac{2\tau}{(1+\tau^2)^2}\|\nabla u\|^2 - \langle u, (V_2(x) - \tau x\cdot\nabla V_1(x))u\rangle \\ &\quad +\langle u, O(\tau^2\langle x\rangle^{-2\mu}))u\rangle. \end{aligned} \tag{4.27}$$

Since $\mathrm{Re}\, H_0 \geq 0$ and satisfies (2.3), there exists $c > 0$ such that

$$\mathrm{Re}\, z \geq c(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu}u\|^2) \tag{4.28}$$

for $\tau > 0$ sufficiently small. If $R \in]0,\infty]$, one has for some $c' > 0$

$$V_2(x) - \tau x\cdot\nabla V_1(x) \geq c'\tau\langle x\rangle^{-2\mu}, \forall x \in \mathbb{R}^n,$$

which gives that

$$\mathrm{Im}\, z \leq -c''\tau(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu}u\|^2) \tag{4.29}$$

for some $c'' > 0$. This shows that $\mathrm{Im}\, z \leq -C\tau\mathrm{Re}\, z$ ($C = c''c^{-1}$) if $R \in]0,+\infty]$.

If $R = 0$, one has $V_2(x) \geq 0$ for all $x \in \mathbb{R}^n$ and

$$V_2(x) - \tau x\cdot\nabla V_1(x) \geq c_3\tau\frac{x^2}{\langle x\rangle^{2\mu+2}}, \quad \forall x \in \mathbb{R}^n,$$

for some $c_3 > 0$. In this case, one has

$$\mathrm{Im}\, z \leq -C\tau(\|\nabla u\|^2 + \langle u, \frac{x^2}{\langle x\rangle^{2\mu+2}} u\rangle) + C\tau^2 \|\langle x\rangle^{-\mu} u\|^2. \tag{4.30}$$

Lemma 4.5 with $s = \mu$ shows

$$\frac{1}{\langle x\rangle^{2\mu+2}} \leq \frac{1}{2\sqrt{(n-1)(n-1-\mu)}}(-\Delta + \frac{1}{\langle x\rangle^{2\mu}})$$

in the sense of selfadjoint operators. For $0 < \mu < \frac{3}{4}$ when $n = 2$ and $\mu \in]0, 1[$ if $n \geq 3$, one has

$$\alpha \equiv \frac{1}{2\sqrt{(n-1)(n-1-\mu)}} < 1.$$

This proves that

$$\begin{aligned} \|\nabla u\|^2 + \langle u, \frac{x^2}{\langle x\rangle^{2\mu+2}} u\rangle &= \|\nabla u\|^2 + \langle u, (\frac{1}{\langle x\rangle^{2\mu}} - \frac{1}{\langle x\rangle^{2\mu+2}})u\rangle \\ &\geq (1-\alpha)(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu} u\|^2). \end{aligned}$$

Consequently, one obtains

$$\mathrm{Im}\, z \leq -C(1-\alpha)\tau(\|\nabla u\|^2 + \|\langle x\rangle^{-\mu} u\|^2) + C\tau^2 \|\langle x\rangle^{-\mu} u\|^2 \leq -C_1 \tau \mathrm{Re}\, z \tag{4.31}$$

for some $C_1 > 0$ if $\tau > 0$ is small enough. This proves that the numerical range $N(\widetilde{H}_0(\theta))$ of $\widetilde{H}_0(\theta)$ is contained in Σ_θ where

$$\Sigma_\theta = \{z; \mathrm{Re}\, z \geq 0, \mathrm{Im}\, z \leq -C_1 \mathrm{Im}\, \theta \mathrm{Re}\, z\}, \quad \mathrm{Im}\, \theta > 0, \tag{4.32}$$

for some constant $C_1 > 0$. Since $\sigma(\widetilde{H}_0(\theta)) \subset \Sigma_\theta$ and $\|\widetilde{R}_0(z,\theta)\| \leq dist(z, \Sigma_\theta)^{-1}$, one has

$$\|\widetilde{R}_0(z,\theta)\| \leq \frac{1}{c_0 \mathrm{Im}\, \theta\, |z|} \tag{4.33}$$

for $z \in \mathbb{C}\backslash\{0\}$ with $-c_0 \mathrm{Im}\, \theta < \arg z < \frac{3\pi}{2} - c_0$ for some $c_0 > 0$ small enough. For $V_0 \in \mathcal{A}$, $\widetilde{H}_0(\theta)$ verifies Conditions (2.1) − (2.4) uniformly in θ with $\mathrm{Im}\, \theta \geq 0$ and $|\theta| < \delta$. $\widetilde{G}_0(\theta) = \widetilde{H}_0(\theta)^{-1}$ is well defined as in the case of $\theta = 0$ and Theorem 3.4 holds for $\widetilde{G}_0(\theta)$. (4.24) follows from the equation

$$\widetilde{R}_0(z,\theta) = \widetilde{G}_0(\theta) + z\widetilde{G}_0(\theta)\widetilde{R}_0(z,\theta)$$

and Theorem 3.4 applied to $\widetilde{G}_0(\theta)$. ■

In order to obtain sub-exponential time-decay estimates for $\chi e^{-itH_0}\chi$, $\chi \in C_0^\infty(\mathbb{R}^n)$, we use analytic distortion of H_0 outside the support of χ. Let $R_0 > 1$ and $\rho \in C^\infty(\mathbb{R})$ with $0 \leq \rho \leq 1$ and $\rho(r) = 0$ if $r \leq 1$ and $\rho(r) = 1$ if $r \geq 2$. Set

$$F_\theta(x) = x\left(1 + \theta\rho(\frac{|x|}{R_0})\right), \quad x \in \mathbb{R}^n. \tag{4.34}$$

When $\theta \in \mathbb{R}$ with $|\theta|$ sufficiently small, $x \to F_\theta(x)$ is a global diffeomorphism on $\mathbb{R}^n$. Set

$$U_\theta f(x) = |DF_\theta(x)|^{\frac{1}{2}} f(F_\theta(x)), \quad f \in L^2(\mathbb{R}^n), \tag{4.35}$$

where $DF_\theta(x)$ is the Jacobi matrix and $|DF_\theta(x)|$ the Jacobian of the change of variables: $x \to F_\theta(x)$. One has

$$|DF_\theta(x)| = \begin{cases} 1, & |x| < R_0; \\ (1+\theta)^n, & |x| > 2R_0 \end{cases} \tag{4.36}$$

U_θ is unitary in $L^2(\mathbb{R}^n)$ for θ real with $|\theta|$ sufficiently small. Define the distorted operator $H_0(\theta)$ by

$$H_0(\theta) = U_\theta H_0 U_\theta^{-1}. \tag{4.37}$$

One can calculate that

$$H_0(\theta) = -\Delta_\theta + V(F_\theta(x)) \tag{4.38}$$

where $-\Delta_\theta =^t \nabla_\theta \cdot \nabla_\theta$ with

$$\nabla_\theta = ({}^tDF_\theta)^{-1} \cdot \nabla - \frac{1}{|DF_\theta|^2}({}^tDF_\theta)^{-1} \cdot (\nabla |DF_\theta|) \tag{4.39}$$

In particular, $\nabla_\theta f = (1+\theta)^{-1}\nabla f$ if f is supported outside the ball $B(0, 2R_0)$. If $V_0 \in \mathcal{A}$, $H_0(\theta)$ can be extended to a holomorphic family of type A for θ in a small complex neighborhood of zero. $H_0(\theta)$ and $\widetilde{H}_0(\theta)$ coincide outside the ball $B(0, 2R_0)$ and they have the same essential spectra. In addition their discrete eigenvalues are also the same so long as they are uncovered by the essential spectra ([3, 12, 10]). Since $\widetilde{R}_0(z, \theta)$ is holomorphic for z in $S(-c_0\theta, \gamma_0)$, so is $R_0(z, \theta) = (H_0(\theta) - z)^{-1}$. We want to establish an upper bound on $R_0(z, \theta)$ when z in $S(-c_0\theta, \gamma_0)$ which will imply $\langle x\rangle^{-2\mu}R_0(z, \theta)$ is uniformly bounded for z near zero and z in $S(-c_0\theta, \gamma_0)$.

Remark that if $V_0 \in \mathcal{A}$, the distorted operator $H_0(\theta)$ satisfies the conditions (2.3) and (2.4) with some constant $c_0 > 0$ independent of $R_0 > 1$ and $\theta \in \mathbb{C}$ with $|\theta|$ small. Therefore Lemma 3.1 can be applied to $H_0(\theta)$. One can define $G_0(\theta)$ by

$$G_0(\theta) = s - \lim_{\mathrm{Re}\, z<0, z\to 0} R_0(z, \theta)$$

as operators from $L^{2,s}$ to $L^{2,s-2\mu}$ and Theorem 3.4 holds for $G_0(\theta)$ uniformly in θ when $|\theta|$ is small. To simplify statement, denote

$$R_0(0, \theta) = G_0(\theta).$$

In the following θ is fixed with $|\theta|$ small and $\mathrm{Im}\,\theta > 0$. Although $R_0(z, \theta)$ and $\widetilde{R}_0(z, \theta)$ have same poles, their norms may be rather different. In the following Proposition we give an argument to deduce an estimate on $R_0(z, \theta)$ for z near 0 from those on $\widetilde{R}_0(z, \theta)$ and $R_0(0, \theta)$.

Proposition 4.7. *Assume the conditions of Lemma 4.6. Then one has* $\sigma(H_0(\theta)) \cap S(-c_0\theta, \gamma_0) = \emptyset$ *and there exists some constant* $C > 0$ *such that*

$$\|\langle x\rangle^{-2\mu}R_0(z, \theta)\| \le \frac{C}{\langle z\rangle}, \quad z \in S(-c_0\theta, \gamma_0). \tag{4.40}$$

Proof. For $z \in S(-c_0\theta, \gamma_0)$ and $|z|$ large, (4.40) follows from Lemma 4.6 by an argument of perturbation. For $z \in S(-c_0\theta, \gamma_0)$ with $|z|$ bounded, we compare $R_0(z, \theta)$ with $\widetilde{R}_0(z, \theta)$ for $|x|$ large and with $R_0(0, \theta)$ for $|z|$ small.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ if $|x| \le 2R_0$. Take $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ such that $\widetilde{\chi}\chi = \chi$. On the support of $1 - \chi$, $H_0(\theta) = \widetilde{H}_0(\theta)$. For $z \in S(-c_0\theta, \gamma_0)$ and $|z|$ small, one has

$$\begin{aligned} R_0(z, \theta) &= R_0(0, \theta) + zR_0(0, \theta)R_0(z, \theta) \\ &= R_0(0, \theta) + zR_0(0, \theta)(\widetilde{\chi} + (1 - \widetilde{\chi}))R_0(z, \theta) \\ &= R_0(0, \theta) + zR_0(0, \theta)\widetilde{\chi}R_0(z, \theta) \\ &\quad + zR_0(0, \theta)(1 - \widetilde{\chi})\widetilde{R}_0(z, \theta)(1 - \chi) \\ &\quad + zR_0(0, \theta)(1 - \widetilde{\chi})\widetilde{R}_0(z, \theta)[(1 + \theta)^{-2}\Delta, \chi]R_0(z, \theta). \end{aligned}$$

Recall that for $\mathrm{Im}\,\theta > 0$, there exists some constant $C > 0$ such that

$$\|\widetilde{R}_0(z, \theta)\langle x\rangle^{-2\mu}\| \le C, \text{ for } z \in S(-c_0\theta, \gamma_0).$$

By the ellipticity of the operator and the first resolvent equation

$$\widetilde{R}_0(z, \theta) = \widetilde{R}_0(-1, \theta) + (z - 1)\widetilde{R}_0(z, \theta)\widetilde{R}_0(-1, \theta)$$

we derive that

$$\|\widetilde{R}_0(z,\theta)(1-\Delta)\langle x\rangle^{-2\mu}\| \leq C, \tag{4.41}$$

for $z \in S(-c_0\theta, \gamma_0)$ and $|z| \leq 1$. Therefore there exists some constant C_1 such that

$$\|\langle x\rangle^{-2\mu} R_0(z,\theta)\langle x\rangle^{-2\mu}\| \leq C_1 + C_1|z|\|\langle x\rangle^{-2\mu} R_0(z,\theta)\langle x\rangle^{-2\mu}\| \tag{4.42}$$

for $z \in S(-c_0\theta, \gamma_0)$ and $|z| \leq 1$. This shows that $\|\langle x\rangle^{-2\mu} R_0(z,\theta)\langle x\rangle^{-2\mu}\|$ is uniformly bounded for $z \in S(-c_0\theta, \gamma_0)$ and $|z|$ sufficiently small. (4.40) now follows from Lemme 4.6 and the second resolvent equation

$$R_0(z,\theta) = \widetilde{R}_0(z,\theta) + R_0(z,\theta)(\widetilde{H}_0(\theta) - H_0(\theta))\widetilde{R}_0(z,\theta) \tag{4.43}$$

since $\widetilde{H}_0(\theta) - H_0(\theta)$ is a second order differential operator with compactly supported coefficients. ■

Let $\mathrm{Im}\,\theta > 0$ and $\delta > 0$ be small. Set

$$\Omega(\delta,\theta) = \{z \in \mathbb{C}^*; |z| < \delta, -\delta \mathrm{Im}\,\theta < \arg z < \frac{3\pi}{2} - \delta\} \text{ and } \Omega_0(\delta,\theta) = \Omega(\delta,\theta) \cup \{0\}. \tag{4.44}$$

Corollary 4.8. *With the convention $R_0(0,\theta) = G_0(\theta)$, for any $a > 0$, there exist some constants $c, C > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} R_0(z,\theta)^N\| \leq C c^N N^{\gamma N}, \quad N \in \mathbb{N}^* \tag{4.45}$$

for all $z \in \Omega_0(\delta,\theta)$.

Proof. In the proof of Lemma 4.6, it is shown that $\|\widetilde{R}_0(z,\theta)\| \leq C_{\delta,\theta}|z|^{-1}$ for $z \in \Omega(\delta,\theta)$ if $\delta > 0$ is small enough. Since $H_0(\theta)$ and $\widetilde{H}_0(\theta)$ differ only in a compact set, we deduce from Proposition 4.7 and (4.43) that

$$\|R_0(z,\theta)\| \leq C_\theta |z|^{-1}$$

for $z \in \Omega(\delta,\theta)$. With the above bound, (4.45) follows from Theorem 2.1 applied to $G_0(\theta)$ and the equation

$$R_0(z,\theta)^N = G_0(\theta)^N (1 + zR_0(z,\theta))^N, \quad z \in \Omega(\delta,\theta).$$

■

Theorem 4.9. *Let $V_0 \in \mathcal{A}$. There exists some constant $c > 0$ such that for any function $\chi \in C_0^\infty(\mathbb{R}^n)$ there exists some constant $C_\chi > 0$ such that*

$$\|\chi e^{-itH_0}\chi\| \leq C_\chi e^{-c|t|^\beta}, \quad t > 0. \tag{4.46}$$

Proof. Let $R_1 > 0$ such that $\mathrm{supp}\chi \subset B(0,R_1)$. Let $U(\theta)$ be defined as before with $R_0 > R_1$ and $H_0(\theta) = U(\theta)^{-1} H_0 U(\theta)$. Then one has

$$\chi R_0(z)\chi = \chi R_0(z,\theta)\chi \quad \text{and} \quad \chi e^{-itH_0}\chi = \chi e^{-itH_0(\theta)}\chi$$

for $\theta \in \mathbb{R}$ with $|\theta|$ small. Since $V_0 \in \mathcal{A}$, the right hand sides of the above equations can be extended holomorphically in z to a complex neighborhood of 0. For $\theta \in \mathbb{C}$ with θ near zero and $\mathrm{Im}\,\theta > 0$, $H_0(\theta)$ is strictly sectorial and the resolvent $R_0(z,\theta)$ is holomorphic in $z \in \mathbb{C}$ with $-c\mathrm{Im}\,\theta < \arg z < \pi + c$ for some $c > 0$. Making use of Proposition 4.7, one can check that

$$\chi e^{-itH_0}\chi = \frac{i}{2\pi}\int_{\Gamma'} e^{-itz}\chi R_0(z,\theta)\chi dz \tag{4.47}$$

where

$$\Gamma' = \{z = re^{-i\eta}; r \geq 0\} \cup \{z = -re^{i\eta}, r \geq 0\}$$

for some $\eta = \eta(\theta) > 0$ small enough. Γ' is oriented in anti-clockwise sense.

The remaining part of the proof of (4.46) is the same as in Theorem 4.4 and will not be repeated here. We just indicate that

$$R_0(z,\theta) = \sum_{j=0}^{N} z^j G_0(\theta)^j + z^{N+1} G_0(\theta)^N R_0(z,\theta)$$

for $z \in \Gamma'$ and z near 0. Theorem 2.1 and Proposition 4.7 show that for $a > 0$

$$\begin{aligned} &\|\chi G_0(\theta)^N R_0(z,\theta)\chi\| \qquad (4.48)\\ &\leq \quad C_\chi \|e^{-a\langle x\rangle^{1-\mu}} G_0(\theta)^N R_0(z,\theta)\langle x\rangle^{-2\mu}\| \leq C_{a,\chi,\mathrm{Im}\,\theta} C_a^N N^{\gamma N} \end{aligned}$$

with some constant C_a independent of χ. By choosing appropriately N in terms of t as in the proof of Theorem 4.4, one obtains (4.46) with some constant $c > 0$ independent of χ. ■

Theorem 4.9 generalizes a result of D. Yafaev [31] for one-dimensional selfadjoint Schrödinger operators to higher dimensions $n \geq 2$.

Example 4.2. *If $n = 3$, Coulomb-type potential $V_0(x) = \frac{a-ib}{|x|}$ with $a, b \geq 0$ and $a + b > 0$ belongs to the class $\mathcal{A}$ with $\mu = \frac{1}{2}$. Consequently, the results of this section can be applied to $H_0 = -\Delta + \frac{a-ib}{|x|}$. In particular, for any $\chi \in C_0^\infty(\mathbb{R}^3)$, making use of analytic deformation outside the support of χ, one has $\chi R_0(z)\chi = \chi R_0(z,\theta)\chi$. Corollary 4.8 shows that $\chi R_0(z)\chi$ satisfies Gevrey estimates of order 3 for $z \in \Omega_0(\delta,\theta)$ and (4.46) holds with $\beta = \frac{1}{3}$.*

4.3. **A low-energy estimate on the spectral density.** For the selafdjoint Schrödinger operator H_0 with slowly decreasing potential V_0, it is proved by S. Nakamura ([20]) that under some additional conditions the spectral density $E_0'(\lambda)$ of H_0 satisfies the estimate that for any $N > 0$

$$E_0'(\lambda) = O_N(\lambda^N), \qquad (4.49)$$

in appropriately weighted spaces, as $\lambda \to 0_+$. The Gevery estimates of the resolvent at threshold allow to improve this result.

Lemma 4.10. *Let $V_0(x) = V_1(x) - iV_2(x)$ with $V_1(x), V_2(x)$ real. Assume that V_1 is of class C^2 on $\mathbb{R}^n$ and that there exists $\mu \in]0,1[$ and some constants $c_j > 0$, $j = 1,2,3$, such that*

$$\begin{aligned} c_1\langle x\rangle^{-2\mu} &\leq V_1(x) \leq c_2\langle x\rangle^{-2\mu}, &(4.50)\\ |(x\cdot\nabla)^j V_1(x)| &\leq c_2\langle x\rangle^{-2\mu}, \quad j = 1,2 &(4.51)\\ x\cdot\nabla V_1(x) &\leq -c_3\langle x\rangle^{-2\mu}, \quad |x| > R \text{ for some } R > 0, &(4.52)\\ |V_2(x)| &\leq c_2\langle x\rangle^{-1-\mu-\epsilon_0}, \quad \epsilon_0 > 0. &(4.53) \end{aligned}$$

Then the eigenvalues of H_0 are absent in a neighborhood of zero and the boundary values of the resolvent $R_0(\lambda \pm i0) = \lim_{z\to\lambda,\pm\mathrm{Im}\,z>0}(H_0 - z)^{-1}$ exist for $\lambda \in [0,\delta]$ for some $\delta > 0$ and are Hölder continuous as operators in $\mathcal{B}(0,s;0-s)$, $s > \frac{1+\mu}{2}$.

Proof. Let $H_1 = -\Delta + V_1(x)$ be the selfadjoint part of H_0 and $R_1(z) = (H_1 - z)^{-1}$. Then one knows from [20] that under the condition of this Lemma, $R_1(\lambda \pm i0)$ exists for $\lambda \in [0,\delta]$ for some $\delta > 0$ and are Hölder continuous as operators in $\mathcal{B}(0,s;0,-s)$, $s > \frac{1+\mu}{2}$. Note that the smoothness assumption on the potential used in [20] is only needed for higher order resolvent estimates.

One knows that $G_{0,1} = \lim_{z\to 0, z\notin\mathbb{R}_+} R_1(z)$ exists and that $G_{0,1}V_2$ is a compact operator in $L^{2,-s}$ for $\frac{1+\mu}{2} < s < \frac{1+\mu+\epsilon_0}{2}$. Therefore the kernel of $1 + iG_{0,1}V_2$ is of finite dimension. From

Lemma 3.3 applied to G_{01}, one deduces that this kernel is contained in $L^{2,r}$ for any $r > 0$. Since $(1 + iG_{0,1}V_2)u = 0$ if and only if $H_0u = 0$, Lemma 3.1 implies that $\mathrm{Ker}(1 + iG_{0,1}V_2)$ in $L^{2,-s}$ is trivial. Therefore $(1+iG_{0,1}V_2)^{-1}$ is bounded in $L^{2,-s}$. By the continuity of $R_1(z)$ for z near 0 and $z \notin \mathbb{R}_+$, one deduces that $1+iR_1(z)V_2$ is invertible in $L^{2,-s}$ and its inverse is Hölder continuous in $\mathcal{B}(L^{2,-s})$ for z near 0 and $z \notin \mathbb{R}_+$. This implies in particular that the eigenvalues of H_0 are absent in a neighborhood of zero and the limits $R_0(\lambda \pm i0) = \lim_{z\to\lambda,\pm\mathrm{Im}\, z>0}(H_0-z)^{-1}$ exist for $\lambda \geq 0$ and small enough and are Hölder continuous in $\lambda \in [0, \delta]$ for some $\delta > 0$. ■

Corollary 4.11. *Under the conditions of Lemma 4.10, assume in addition that H_0 is self-adjoint ($V_2 = 0$). Denote by $E_0(\lambda)$ the spectral projection of H_0 associated with the interval $]-\infty, \lambda]$. Let $s > \frac{1+\mu}{2}$. Then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} E_0'(\lambda)\langle x\rangle^{-s}\| \leq C_a e^{-c_a|\lambda|^{-\frac{1}{\gamma}}}, \quad 0 < \lambda \leq \delta. \tag{4.54}$$

Proof. Since $E_0'(\lambda) = \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0))$, $\|\langle x\rangle^{-s}E_0'(\lambda)\langle x\rangle^{-s}\|$ is uniformly bounded for $\lambda > 0$ and λ near 0, if $s > \frac{1+\mu}{2}$ (see [20]). Iterating the first resolvent equation, one obtains for any $N \in \mathbb{N}^*$

$$E_0'(\lambda) = \lambda^N G_0^N E_0'(\lambda), \quad 0 < \lambda \leq \delta. \tag{4.55}$$

Applying Theorem 2.1 with $\tau = s$, one deduces that for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}} E_0'(\lambda)\langle x\rangle^{-s}\| \leq C_a c_a^N N^{\gamma N}\lambda^N \tag{4.56}$$

for all $N \in \mathbb{N}^*$ and $\lambda \in]0, \delta]$. It remains to minimize the right-hand side by choose N in terms of $\lambda > 0$ such that $N \approxeq c\lambda^{-\frac{1}{\gamma}}$ as $\lambda \to 0_+$ for some appropriate $c > 0$. Then

$$c_a^N N^{\gamma N}\lambda^N \leq C' e^{-c'\lambda^{-\frac{1}{\gamma}}}, \quad 0 < \lambda \leq \delta,$$

for some constants $c', C' > 0$. (4.54) is proved. ■

5. Heat et Schrödinger semigroups for perturbed operators

Consider non-selfadjoint Schrödinger operator H of the form

$$H = H_0 + W(x). \tag{5.1}$$

where $H_0 = -\Delta + V_0(x)$ with $V_0 \in \mathcal{V}$ and $W \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. Then the essential spectrum of H is equal to $[0, +\infty[$ and the possible accumulation points of complex eigenvalues of H are contained in $\mathbb{R}_+$. We begin with the analysis of positive resonances for non-selfadjoint Schrödinger operator $H = -\Delta + V(x)$ with $V(x)$ holomorphic outside some compact set. Since we are interested in behavior of solutions as $t \to +\infty$, the main attention is paid to outgoing positive resonances, because incoming positive resonances are invisible in the limit $t \to +\infty$.

5.1. Positive resonances of non-selfadjoint Schrödinger operators.

To begin with, consider a class of non-selfadjoint Schrödinger operators $H = -\Delta + V(x)$ which are compactly supported perturbations of $H_0 = -\Delta + V_0$ with $\mathrm{Im}\, V_0(x) \leq 0$ and $V_0(x)$ extends to a holomorphic function in a region of the form $\{x \in \mathbb{C}^n; |x| > R, |\mathrm{Im}\, x| < \delta|\mathrm{Re}\, x|\}$ and satisfies there

$$|V_0(x)| \leq C\langle \mathrm{Re}\, x\rangle^{-\rho} \tag{5.2}$$

for some constants $R, \delta, C, \rho > 0$. Suppose in addition $(x\cdot\nabla_x)^j V_0$, $j = 0, 1, 2$, are $-\Delta$-compact. Then the set $r_+(H)$ of outgoing positive resonances of H is well defined by Definition 2.5. Since H_0 is dissipative, one has $r_+(H_0) = \emptyset$ and the boundary value of the resolvent

$$R_0(\lambda + i0) = \lim_{z\to\lambda, \mathrm{Im}\, z>0} (H_0 - z)^{-1} \tag{5.3}$$

exists in $\mathcal{B}(-1, s; 1, -s)$, $s > \frac{1}{2}$, for $\lambda > 0$ and is Hölder-continuous in $\lambda > 0$ ([22]). Let U_θ be the analytic distortion defined by (4.35) with R_0 sufficiently large. Then H_θ

$$H_\theta = U_\theta H U_\theta^{-1}$$

defined for θ real can be extended to a holomorphic family of type A for θ in a complex neighborhood of zero. For $\mathrm{Im}\,\theta > 0$, spectrum of H_θ in $\{z, \arg z > -c\mathrm{Im}\,\theta\}$ is discrete for some constant $c > 0$ and is independent of the function ρ used in the distortion ([12]).

Theorem 5.1. *Let $H = H_0 + W(x)$ be a compactly supported perturbation of $H_0 = -\Delta + V_0(x)$. Assume (5.2) is satisfied. Then there exists some constant $\theta_0 > 0$ such that for $\mathrm{Im}\,\theta > 0$ and $|\theta| < \theta_0$ one has*

$$\sigma_d(H(\theta)) \cap \mathbb{R}_+ = r_+(H) \qquad \text{and} \tag{5.4}$$

$$\sigma_d(H(\theta)) \cap \mathbb{C}_+ = \sigma_d(H) \cap \mathbb{C}_+. \tag{5.5}$$

In particular, outgoing positive resonances of H are at most countable with zero as the only possible accumulation point.

Proof. Let $\lambda_0 > 0$. Note that positive eigenvalues of H are absent ([14]). If $\lambda_0 \notin r_+(H)$, then -1 is not an eigenvalue of the compact operator $R_0(\lambda_0 + i0)W$ in $L^{2,-s}$ for $\frac{1}{2} < s < \rho - \frac{1}{2}$. Therefore operator $1 + R_0(\lambda_0 + i0)W$ is invertible in $L^{2,-s}$. Since $\lambda \to R_0(\lambda + i0)$ is continuous as operator from $L^{2,s}$ to $L^{2,-s}$, we deduce that -1 is not an eigenvalue of $R_0(\lambda + i0)W$ for $\lambda \in \mathbb{R}$ and λ sufficiently near λ_0. It follows that the boundary value of the resolvent $R(\lambda + i0) = \lim_{z\in\mathbb{C}_+, z\to\lambda} R(z)$ exists and

$$\|\langle x\rangle^{-s} R(\lambda + i0)\langle x\rangle^{-s}\| \le C \tag{5.6}$$

for λ near λ_0 and $s > \frac{1}{2}$. This proves that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi R(z)\chi$ is bounded for $z \in \mathbb{C}_+$ and z near λ_0. Therefore the meromorphic extension of $\chi R(z)\chi$ from the upper half-plane is in fact holomorphic in complex neighborhood of λ_0. It follows that $\lambda_0 \notin \sigma_d(H_\theta)$. This proves the inclusion $\sigma_d(H_\theta) \cap \mathbb{R}_+ \subset r_+(H)$.

Conversely if $\lambda_0 \in r_+(H)$, then there exists a non-zero function $u \in L^{2,-s}$ for any $s > \frac{1}{2}$ such that

$$u = -R_0(\lambda_0 + i0)Wu.$$

Let $R_0(z, \theta) = U_\theta R_0(z) U_\theta^{-1}$ and $u_\theta = U_\theta u$, for $\theta \in \mathbb{R}$. Then one has

$$u_\theta = -R_0(\lambda_0 + i0, \theta)Wu \tag{5.7}$$

if the analytic distorsion is made outside the support of W. Since outgoing resonances of the dissipative operator H_0 are absent, $R_0(z, \theta)$ defined for $\mathrm{Im}\, z > 0$ and θ real can be holomorphically extended for $\theta \in \mathbb{C}$ with $\mathrm{Im}\,\theta > 0$ and $|\theta| < \theta_0$ for some $\theta_0 > 0$ depending on domain of the analyticity of V_0. After this extension in θ, $R_0(z, \theta)$ is holomorphic for z near λ_0 and $\mathrm{Im}\, z > -c\mathrm{Im}\,\theta\mathrm{Re}\, z$ for some $c > 0$. By (5.7), u_θ can be extended in θ for $\mathrm{Im}\,\theta > 0$. $u_\theta \in L^2$ because $R_0(\lambda_0 + i0, \theta)$ is bounded on L^2 when $\mathrm{Im}\,\theta > 0$. It is clear that $u_\theta \neq 0$ when $|\theta|$ is small, because

$$\|\langle x\rangle^{-s'}(u_\theta - u)\| \le \|\langle x\rangle^{-s'}(R_0(\lambda_0 + i0, \theta) - R_0(\lambda_0 + i0))Wu\| \le C|\theta|^\eta \tag{5.8}$$

for some $\eta > 0$ if $s' > \frac{1}{2}$, because $(-\Delta - e^{2\theta}(\lambda + i0))^{-1}$ is Hölder continuous in $\theta \in D_+(0, \theta_0)$ as operator-valued function from $L^{2,s}$ to $L^{2,-s}$. This proves λ_0 is an eigenvalue of H_θ with u_θ

as an eigenfunction when $\mathrm{Im}\,\theta > 0$ and $|\theta|$ is small enough. Therefore $r_+(H) \subset \sigma_d(H_\theta)$ which completes the proof of (5.4).

To prove (5.5), notice that the resolvent $R(z) = (H-z)^{-1}$ is meromorphic in $\mathbb{C}_+$ with poles at $\sigma_d(H) \cap \mathbb{C}_+$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ on supp W such that the analytic distortion of H is made outside the support of χ. For $\theta \in \mathbb{R}$ and $|\theta| < \theta_0$, one has

$$\chi R(z)\chi = \chi R(z,\theta)\chi \tag{5.9}$$

for $z \in \mathbb{C}_+$ with $\mathrm{Im}\, z > 1$ large enough. Since $\theta \to H(\theta)$ is a holomorphic family of type A ([16]), the above equality still holds for $\mathrm{Im}\,\theta > 0$ and $|\theta| < \theta_0$ and for $z \in \mathbb{C}_+$ with $\mathrm{Im}\, z > 1$ large enough. The uniqueness of meromorphic extensions in $\mathbb{C}_+$ implies that the above equality holds as meromorphic functions in $\mathbb{C}_+$, therefore the poles of the two cut-off resolvents are the same. Since the poles of $R(z,\theta)$ are independent of the analytic distortion used ([10]), the support of the cut-off χ can be arbitrarily large. Noticing that

$$\sigma_d(H(\theta)) \cap \mathbb{C}_+ = \cup_{\chi \in C_0^\infty} \{ \text{ poles of } \chi R(z,\theta)\chi \text{ in } \mathbb{C}_+ \}$$

and the same property for H, we obtain $\sigma_d(H(\theta)) \cap \mathbb{C}_+ = \sigma_d(H) \cap \mathbb{C}_+$. ■

In Theorem 5.1, the condition $\mathrm{Im}\, V_0 \leq 0$ is used to study the outgoing real resonances. Similarly if $\mathrm{Im}\, V_0 \geq 0$ one can prove

$$\sigma_d(H_\theta) \cap \mathbb{R}_+ = r_-(H) \tag{5.10}$$

for $\mathrm{Im}\,\theta < 0$ and $|\theta| < \theta_0$ and $r_-(H)$ is at most a countable set. Consequently we obtain the following

Corollary 5.2. *Under the conditions of Theorem 5.1, suppose in addition that $V_0(x)$ is real-valued for $x \in \mathbb{R}^n$. Then, real resonances of $H = H_0 + W(x)$ are at most countable with zero as the only possible accumulation point.*

The condition on $\mathrm{Im}\, V_0$ is not necessary if V_0 is of short-range (consequently $V = V_0 + W$ is of short-range). For simplicity we only give a result in dilation analytic case. Let $H = -\Delta + V(x)$ with V a dilation analytic short-range potential: $V_\theta(x) = V(e^\theta x)$ defined for θ real extends to a holomorphic function for θ in a complex neighborhood of zero:

$$|V_\theta(x)| \leq C\langle x\rangle^{-\rho}, \tag{5.11}$$

for $x \in \mathbb{R}^n$ and $|\theta| < \theta_0$ for some $C, \theta_0 > 0$ and $\rho > 1$. Then $H_\theta = -e^{-2\theta}\Delta + V_\theta$ is a holomorphic family of type A for $\theta \in \mathbb{C}$, $|\theta| < \theta_0$

Theorem 5.3. *Let $H = -\Delta + V(x)$ where $V(x)$ is a short-range dilation analytic potential verifying (5.11) for some $\rho > 1$. One has*

$$\sigma_d(H_\theta) \cap \mathbb{R}_+ = r_\pm(H) \tag{5.12}$$

for $\pm\mathrm{Im}\,\theta > 0$ and $|\theta| < \theta_0$. In particular, positive resonances of H form an at most countable set with zero as the only possible accumulation point.

Proof. The inclusion $\sigma_d(H_\theta) \cap \mathbb{R}_+ \subset r_+(H)$ for $\mathrm{Im}\,\theta > 0$ can be proved in the same way as the first part of the proof of Theorem 5.1. It remains to show that $r_+(H) \subset \sigma_d(H_\theta), \mathrm{Im}\,\theta > 0$. Let $\lambda \in r_+(H)$. Then -1 is an eigenvalue of the compact operator $K = (-\Delta - \lambda - i0)^{-1}V$ on $L^{2,-s}$, $\frac{1}{2} < s < \frac{\rho}{2}$. Let

$$K_\theta = (-e^{-2\theta}\Delta - \lambda - i0)^{-1}V_\theta = e^{2\theta}(-\Delta - e^{2\theta}(\lambda + i0))^{-1}V_\theta,$$

for $\mathrm{Im}\,\theta \geq 0$, $|\theta| < \theta_0$. K_θ is a family of compact operators on $L^{2,-s}$ continuous with respect to θ in the half disk

$$D_+(0,\theta_0) = \{\theta \in \mathbb{C}; \mathrm{Im}\,\theta \geq 0, |\theta| < \theta_0\}$$

and holomorphic for θ in its interior. In addition,

$$\|\langle x\rangle^{-s}(K_\theta - K)\langle x\rangle^{s}\| \leq C|\theta|^{\eta} \tag{5.13}$$

for $\frac{1}{2} < s < \frac{\rho}{2}$ and for some $\eta > 0$. It follows that in any small neighborhood of -1, K_θ has at least one eigenvalue z_θ for $\theta \in D_+(0,\delta)$ if $\delta > 0$ is sufficiently small. Since K_θ and $K_{\theta'}$ are unitarily equivalent if $\operatorname{Im}\theta = \operatorname{Im}\theta'$, z_θ is independent of $\operatorname{Re}\theta$. It follows that z_θ is independent of θ for $\operatorname{Im}\theta > 0$ and $|\theta| < \delta$ (Theorem 1.9 in Chapter VII of [16]). Since $z_\theta \to -1$ as $\theta \to 0$, it follows that $z_\theta = -1$ for all $\theta \in D_+(0,\delta)$ if $\delta > 0$ is small enough. This proves that λ is an eigenvalue of H_θ if $\theta \in D_+(0,\delta)$. Therefore $r_+(H) \subset \sigma_d(H_\theta)$ which completes the proof of (5.12) with sign $+$. The equality with sign $-$ can be proved in the same way. The last affirmation is immediate since the set of positive resonances of H is equal to $r_+(H) \cup r_-(H)$. ■

In the above proof, we showed that if $\lambda \in r_+(H)$, then there exists $c > 0$ such that -1 is the only eigenvalue of K_θ inside the disk $D(-1,c)$ for all $\theta \in D_+(0,\delta)$. Therefore one can define the Riesz projection of eigenvalue -1 of K_θ by

$$\pi_\theta = \frac{1}{2\pi i}\int_{|z+1|=\frac{c}{2}} (z - K_\theta)^{-1} dz, \quad \forall \theta \in D_+(0,\delta). \tag{5.14}$$

The following result is immediate.

Corollary 5.4. *Assume the conditions of Theorem 5.3. Let $\lambda > 0$ be an outgoing resonance of $H = -\Delta + V$. Denote π_θ the Riesz projection of eigenvalue -1 of K_θ, $\theta \in D_+(0,\delta)$, $\delta > 0$. Then as operators on $L^{2,-s}$, $\frac{1}{2} < s < \frac{\rho}{2}$, π_θ is continuous for $\theta \in D_+(0,\delta)$ and holomorphic for θ in the interior of this half disk.*

5.2. **The case zero is not an eigenvalue.** In this subsection, we study the case zero is not an eigenvalue of H and prove Theorem 2.2.

Proposition 5.5. *Let $H_0 = -\Delta + V_0(x)$ with $V_0 \in \mathcal{V}$. Let $W \in L^\infty(\mathbb{R}^n)$ with compact support and $H = H_0 + W(x)$. Assume that 0 is not an eigenvalue of H. Then one has:*

(a). There exist some constants $c_1, \mu' > 0$ such that outside the set

$$\Omega_1 = \{z \in \mathbb{C}; \operatorname{Re} z \geq 0 \text{ and } |\operatorname{Im} z| \leq c_1 |\operatorname{Re} z|^{\mu'}\},$$

there are at most a finite number of discrete eigenvalues of H. There exists some $\delta > 0$ such that

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{\mu'}}} \text{ for } z \notin \Omega_1 \text{ and } |z| < \delta. \tag{5.15}$$

(b). The limit

$$R(0) = s - \lim_{z\to 0, z\notin\Omega_1} R(z) \tag{5.16}$$

exists in $\mathcal{B}(0,s;\ 0, s - 2\kappa\mu)$, $\kappa \geq \frac{1}{\mu'}$, for any $s \in \mathbb{R}$ and one has

$$\|e^{-a\langle x\rangle^{1-\mu}} R(z)^N\| \leq C_a^{N+1} N^{\gamma N} \tag{5.17}$$

for all $N \in \mathbb{N}^*$ and $z \in \Omega_-(\delta) \cup \{0\}$. Here $a > 0$ and C_a, c_a are some positive constants and $\Omega_-(\delta) = \{z \in \mathbb{C}^*; |z| < \delta, \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta\}$.

Proof. Note that $G_0 W$ is a compact operator and that 0 is not an eigenvalue of H if and only if -1 is not an eigenvalue of $G_0 W$. So if 0 is not an eigenvalue of H, operator $1 + G_0 W$ is invertible on L^2. From Proposition 4.1, one deduces that $1 + R_0(z)W$ is invertible for $|z|$ small and $z \notin \Omega_1$ and

$$\|(1 + R_0(z)W)^{-1}\| \leq C \tag{5.18}$$

uniformly for $|z|$ small and $z \notin \Omega_1$. This shows that 0 is not an accumulation point of $\sigma(H)\backslash\Omega_1$. In addition, $z \to 1+R_0(z)W$ is holomorphic in $\mathbb{C}\setminus\Omega_1$. The analytic Fredholm Theorem shows that $(1+R_0(z)W)^{-1}$ is a meromorphic function with at most a discrete set of poles in $\mathbb{C}\setminus\Omega_1$ (with c_1 replaced by a slightly bigger constant in the definition of Ω_1). Therefore the number of eigenvalues of H in $\mathbb{C}\setminus\Omega_1$ is at most finite. (5.15) follows from Proposition 4.1 and the equation

$$R(z) = (1+R_0(z)W)^{-1}R_0(z).$$

Noticing that $G_0 = \text{s-}\lim_{z\to 0, z\notin\Omega_1} R_0(z)$ exists as operators from $L^{2,s}$ to $L^{2,s-2\kappa\mu}$, one obtains (5.16) with $R(0) = (1+G_0W)^{-1}G_0$.

To prove that Gevrey estimates of the resolvent $R(z)$, we remark that if $F(z)$ and $G(z)$ are two bounded operator-valued functions on some domain Ω_0 satisfying the Gevrey estimates

$$\|F^{(N)}(z)\| \leq AC_1^N(N!)^\sigma \tag{5.19}$$

$$\|G^{(N)}(z)\| \leq BC_2^N(N!)^\sigma \tag{5.20}$$

for all $N \in \mathbb{N}$ and $z \in \Omega_0$ and for some $\sigma > 1$ and $A, B, C_1, C_2 > 0$, then $F(z)G(z)$ satisfies the Gevrey estimates

$$\|(FG)^{(N)}(z)\| \leq ABC_3^N(N!)^\sigma \tag{5.21}$$

for all $N \in \mathbb{N}$ and $z \in \Omega_0$ where

$$C_3 = D_\sigma \max\{C_1, C_2\} \text{ with } D_\sigma = \sup_{N\in\mathbb{N}} \sum_{j=0}^{N} \left(\frac{j!(N-j)!}{N!}\right)^{\sigma-1} < \infty. \tag{5.22}$$

Here $F^{(N)}(z)$ denotes the N-th derivative of $F(z)$. If $F(z)$ is invertible for $z \in \Omega_0$ with

$$\|F(z)^{-1}\| \leq M \tag{5.23}$$

uniformly in $z \in \Omega_0$, then the inverse $H(z) = F(z)^{-1}$ satisfies the Gevrey estimates

$$\|H^{(N)}(z)\| \leq MC_4^N(N!)^\sigma \tag{5.24}$$

for all $N \in \mathbb{N}$ and $z \in \Omega_0$, where $C_4 = MC_1D_\sigma$. Denote $G^\sigma(\Omega_0)$ the set of bounded operator-valued functions on Ω_0 verifying Gevrey estimes of order $\sigma > 1$. Since $e^{-a\langle x\rangle^{1-\mu}}R_0(z)$ and $\chi R_0(z)$ belong to $G^\sigma(\Omega_-(\delta))$ with

$$\sigma = 1+\gamma = \frac{1+\mu}{1-\mu}, \tag{5.25}$$

Seeing the uniform boundedness of $(1+R_0(z)W)^{-1}$ for $z \in \Omega_-(\delta)$, one deduces from (5.24) that

$$(1+R_0(z)W)^{-1} \in G^\sigma(\Omega_-(\delta)).$$

(5.17) for $z \in \Omega_-(\delta)$ is proved by using the equation

$$e^{-a\langle x\rangle^{1-\mu}}R(z) = e^{-a\langle x\rangle^{1-\mu}}R_0(z)(1+WR_0(z))^{-1}, \tag{5.26}$$

Corollary 3.5 and operations in Gevrey class G^σ. ■

Making use of Corollary 4.2, one can show that $e^{-a\langle x\rangle^{1-\mu}}R(z) \in G^{1+\kappa\gamma}(O_0(\delta))$, where $O(\delta)$ is defined by (4.3) and $O_0(\delta) = O(\delta)\cup\{0\}$.

Proposition 5.6. *Let $V_0 \in \mathcal{A}$. Assume that zero is not an eigenvalue of $H = H_0 + W(x)$. Then one has*

(a). There exists $\delta > 0$ such that H has at most a finite number of eigenvalues in the sector

$$S(-\delta, \pi+\delta) = \{z \in \mathbb{C}^*; -\delta < \arg z < \pi+\delta\}$$

and for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi R(z)\chi$ defined for $\operatorname{Im} z > 0$ extends meromorphically into $S(-\delta, \pi+\delta)$ and there exists some constant $C_\chi, c > 0$ such that

$$\|\chi R(z)\chi\| \leq C_\chi \tag{5.27}$$

for $z \in S(-\delta, \pi+\delta)$ and $|z| < c$.

(b). The limit $\chi R^{(N)}(0)\chi = \lim_{z\in S(-\delta,\pi+\delta), z\to 0} \chi R^{(N)}(z)\chi$ exists in $\mathcal{B}(L^2)$ for any $N \in \mathbb{N}$ and one has

$$\|\chi R^{(N)}(z)\chi\| \leq C_\chi C^N N^{(1+\gamma)N} \tag{5.28}$$

for any $N \in \mathbb{N}^*$ and $z \in \Omega_0(\delta, \theta)$ for some $\delta, \operatorname{Im}\theta > 0$, where $\Omega_0(\delta,\theta)$ is defined by (4.44).

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^n)$. Let U_θ be defined by (4.35) with $R_0 > 1$ such that supp W and supp χ are contained in the ball $\{x \in \mathbb{R}^n; |x| < R_0\}$. Let $H(\theta) = U_\theta H U_\theta^{-1}$. Then $H(\theta) = H_0(\theta) + W$ and

$$\chi R(z)\chi = \chi(1 + R_0(z,\theta)W)^{-1} R_0(z,\theta)\chi, z \notin \sigma(H) \tag{5.29}$$

for θ real. $H(\theta)$ and $H_0(\theta)$ are holomorphic families of type A for $\theta \in \overline{\mathbb{C}}_+$ with $|\theta|$ small (see [16]). For $\operatorname{Im}\theta > 0$, $R_0(z,\theta)$ defined initially for $\operatorname{Im} z >> 1$ extends holomorphically into $S(-\delta, \pi+\delta)$ if $0 < \delta < c\operatorname{Im}\theta$ for some $c > 0$ small enough. Since 0 is not an eigenvalue of H, -1 is not an eigenvalue of $G_0 W$. By Proposition 4.7,

$$\|(R_0(z,\theta) - G_0)W\| \leq C(|z| + |\theta|)$$

for $z \in S(-\delta, \pi+\delta)$. Therefore -1 is not an eigenvalue of $R_0(z,\theta)W$ if $|z|+|\theta|$ is small enough and

$$\|(1 + R_0(z,\theta)W)^{-1}\| \leq C_1 \tag{5.30}$$

uniformly for $z \in S(-\delta, \pi+\delta)$, $\operatorname{Im}\theta > 0$ and $|z|+|\theta|$ small. It follows that $(1 + R_0(z,\theta)W)^{-1}$ is a meromorphic function for z in $S(-\delta, \pi+\delta)$ with only at most a finite number of poles. Since eigenvalues and positive outgoing resonances of H are among discrete eigenvalues of $H(\theta)$ with $\operatorname{Im}\theta > 0$, (5.30) implies that

$$(\sigma_d(H) \cup r_+(H)) \cap \{z; z \in S(-\delta, \pi+\delta), |z| \leq c\} = \emptyset \tag{5.31}$$

for some $c > 0$. This proves the finiteness of eigenvalues of H in $S(-\delta, \pi+\delta)$, because zero is the only possible accumulation point of eigenvalues of H in $S(-\delta, \pi+\delta)$. Estimate (5.27) follows from (4.40) and (5.29)with fixed $\operatorname{Im}\theta > 0$. (5.28) can be derived from Corollary 4.8 and (5.29). ■

Since $\operatorname{Im} V_0 \leq 0$ for $V_0 \in \mathcal{A}$, Theorem 5.1 shows that zero is the only possible accumulation point of $r_+(H)$. The following statement is an immediate consequence of (5.30) and Theorem 5.1.

Corollary 5.7. *Assume the conditions of Proposition 5.6. Then the set $r_+(H)$ is at most finite.*

Proof of Theorem 2.2 (a). Theorem 2.2 (a) can be proved in the same way as Theorem 4.4 for the model operator H_0. By Proposition 5.5, one can find a contour Γ in the right half-plane of the form

$$\Gamma = \{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\}$$

for some $C, \mu' > 0$ such that $\sigma(H) \cap \Gamma = \{0\}$ and there are only a finite number of complex eigenvalues of H located on the left of Γ. Let

$$\Lambda = \sigma(H) \cap \{z; \operatorname{Re} z < 0 \text{ or } \operatorname{Re} z \geq 0 \text{ and } |\operatorname{Im} z| > C(\operatorname{Re} z)^{\mu'}\}.$$

Then one has

$$e^{-tH} - \sum_{\lambda\in\Lambda} e^{-tH}\Pi_\lambda = \frac{i}{2\pi}\int_\Gamma e^{-tz}R(z)dz. \tag{5.32}$$

where Π_λ is the Riesz projection of H associated the eigenvalue λ. Making use of Proposition 5.5, one can prove as in Theorem 4.4 that

$$\|e^{-a\langle x\rangle^{1-\mu}}(e^{-tH} - \sum_{\lambda\in\Lambda} e^{-tH}\Pi_\lambda)\| \le C_a e^{-c_a t^\beta}, \quad t>0. \tag{5.33}$$

(2.22) is proved, because if $\lambda\in\sigma_d(H)$ with $\mathrm{Re}\,\lambda>0$, then $\|e^{-tH}\Pi_\lambda\|$ decreases exponentially in $t>1$. (2.24) can be deduced in a similar way.

(b). According to Proposition 5.6 and Corollary 5.7, there exists some $\eta>0$ such that Ω_η contains no poles with negative imaginary part of meromorphic extension of $\chi R(z)\chi$ from the upper half-plane. Under the assumptions Theorem 2.2 (b), $\chi R(z)\chi$ has only a finite number of poles in $\overline{\mathbb{C}}_+$ which are either discrete eigenvalues or positive outgoing resonances of H. Making use of (5.29) for some fixed $\theta\in\mathbb{C}_+$, one obtains the representation formula for $\chi e^{-itH}\chi$:

$$\chi e^{-itH}\chi - \sum_{\lambda \text{ poles in } \overline{\mathbb{C}}_+} \mathrm{Res}(e^{-itz}\chi R(z)\chi;\lambda) = \frac{i}{2\pi}\int_{\Gamma_\eta} e^{-itz}\chi R(z)\chi dz, \quad t>0, \tag{5.34}$$

where $\mathrm{Res}(e^{-itz}\chi R(z)\chi;\lambda)$ is the residue of $e^{-itz}\chi R(z)\chi$ at pole λ and $\Gamma_\eta = \{z=re^{-i\eta}; r\ge 0\}\cup\{z=-re^{i\eta}; r\ge 0\}$ with $\eta>0$ chosen such that there are no eigenvalues of H on Γ_η, nor between Γ_η and the real axis. It is easy to see that

$$\mathrm{Res}(e^{-itz}\chi R(z)\chi;\lambda) = \chi e^{-itH}\Pi_\lambda\chi, \text{ if } \lambda\in\sigma_d(H),$$

while for $\lambda\in r_+(H)$ we can only affirm that

$$\mathrm{Res}(e^{-itz}\chi R(z)\chi;\lambda) = \chi e^{-it\lambda}P_\lambda(t)\chi$$

for some operator of finite rank $P_\lambda(t)$ which is polynomial in t. See Remark 5.2 on the rank of coefficients of $P_\lambda(t)$. From Proposition 4.7, one deduces that

$$\|\chi R(z)\chi\| \le \frac{C}{\langle z\rangle}, \quad z\in\Gamma_\eta, |z|>>1,$$

if $\eta>0$ is small in comparing with $\mathrm{Im}\,\theta$. Therefore the part of integral in (5.34) on $\Gamma_\eta\cap\{|z|\ge\delta\}$, $\delta>0$, decreases exponentially like $O(e^{-c_\delta t})$, $t\to+\infty$, for some $c_\delta>0$. To evaluate the part of integral on $\Gamma_\eta\cap\{|z|\le\delta\}$, let $G(\theta)=(1+G_0(\theta)W)^{-1}G_0(\theta)$, $\mathrm{Im}\,\theta>0$. Then $G(\theta)$ verifies the estimates of Theorem 2.1. Notice that

$$\chi R(z)\chi = \chi\left(\sum_{j=0}^N z^j G(\theta)^{j+1} + z^{N+1}G(\theta)^{N+1}R(z,\theta)\right)\chi$$

for $\mathrm{Im}\,\theta>0$ and $z\in\Omega(\delta,\theta)$, if the analytic distortion is made outside a sufficiently large ball. The sub-exponential time-decay of the integral on $\Gamma_\eta\cap\{|z|\le\delta\}$ can be deduced from the Gevrey estimates on the powers of $G(\theta)^j$ and the resolvent bound

$$\|\langle x\rangle^{-2\mu}R(z,\theta)\|\le C$$

for $z\in\Omega(\delta,\theta)$. The details are the same as in the proof of Theorem 4.9 and are omitted here. ■

5.3. **Threshold eigenvalue in selfadjoint case.** In this subsection, we prove Theorem 2.3. Assume that zero is an eigenvalue of $H = H_0 + W$. Then -1 is an eigenvalue of G_0W and $\mathrm{Ker}(1 + G_0W)$ in L^2 coincides with the eigenspace of H with eigenvalue zero. The following decay estimate of eigenfunctions may be known somewhere. We use this result in the construction of an approximative Grushin problem for the cut-off resolvent.

Lemma 5.8. *Assume that $H_0 = -\Delta + V_0$ satisfies the condition (2.3). Then there exists some constant $\alpha_0 > 0$ such that if $u \in H^2(\mathbb{R}^n)$ verifies $Hu = 0$, then $e^{\alpha_0\langle x\rangle^{1-\mu}}u \in L^2(\mathbb{R}^n)$.*

Proof. Let $\varphi(x) = \alpha\langle x\rangle^{1-\mu}$, $\alpha > 0$. Let χ be a smooth cut-off on $\mathbb{R}^n$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Set

$$\varphi_R(x) = \chi(\frac{x}{R})\varphi(x), \quad R \ge 1.$$

Then

$$|\nabla\varphi_R(x)| \le \alpha(1 + \frac{C}{R^{1-\mu}})\langle x\rangle^{-\mu} \le 2\alpha\langle x\rangle^{-\mu}$$

uniformly in $R \ge R_1$ where R_1 is sufficiently large. Let $u \in H^2(\mathbb{R}^n)$. Then one has

$$\begin{aligned} |\langle e^{2\varphi_R}Hu, u\rangle| &= |\langle H(e^{\varphi_R}u), e^{\varphi_R}u\rangle - \langle[\Delta, e^{\varphi_R}]u, e^{\varphi_R}u\rangle| \\ &= |\langle H(e^{\varphi_R}u), e^{\varphi_R}u\rangle + 2\langle(|\nabla\varphi_R|^2e^{\varphi_R}u - \nabla\varphi_R\cdot\nabla(e^{\varphi_R}u), e^{\varphi_R}u\rangle| \\ &\ge |\langle H(e^{\varphi_R}u), e^{\varphi_R}u\rangle| - (8\alpha^2 + 2\alpha)\|\langle x\rangle^{-\mu}e^{\varphi_R}u\|^2 - 2\alpha\|\nabla(e^{\varphi_R}u)\|^2 \end{aligned} \tag{5.35}$$

uniformly in $R \ge R_1$. Since W is compactly supported,φ_R is bounded on supp W uniformly with respect to R. Making use of the condition (2.3), one obtains for some constants $c_0, C > 0$,

$$|\langle H(e^{\varphi_R}u), e^{\varphi_R}u\rangle| \ge c_0(\|\nabla(e^{\varphi_R}u)\|^2 + \|\langle x\rangle^{-\mu}e^{\varphi_R}u\|^2) - C\|u\|^2 \tag{5.36}$$

for all $R \ge R_1$. For $\alpha > 0$ appropriately small, one deduces that there exists some constant $C_1 > 0$ such that

$$\|\langle x\rangle^{-\mu}e^{\varphi_R}u\|^2 + \|\nabla(e^{\varphi_R}u)\|^2 \le C_1(|\langle e^{2\varphi_R}Hu, u\rangle| + \|u\|^2) \tag{5.37}$$

for any $u \in H^2(\mathbb{R}^n)$ and $R \ge R_1$. If $u \in H^2(\mathbb{R}^n)$ such that $Hu = 0$, it follows that

$$\|\langle x\rangle^{-\mu}e^{\varphi_R}u\|^2 + \|\nabla(e^{\varphi_R}u)\|^2 \le C_1\|u\|^2 \tag{5.38}$$

for all $R \ge R_1$. This proves that $\langle x\rangle^{-\mu}e^{\varphi}u \in L^2(\mathbb{R}^n)$ and $\nabla(e^{\varphi}u) \in L^2(\mathbb{R}^n)$. Lemma 5.8 is proved, provided that $0 < \alpha_0 < \alpha$. ■

Theorem 5.9. *Let $H_0 = -\Delta + V_0(x)$ and $H = H_0 + W(x)$ with $V_0 \in \mathcal{V}$ and $W \in L^\infty_{\mathrm{comp}}$. Assume that 0 is an eigenvalue of H and that both H_0 and H are selfadjoint. Let Π_0 denote the eigenprojection of H associated with eigenvalue zero. Then there exists some constant $\delta > 0$ such that*

$$R(z) = -\frac{\Pi_0}{z} + R_1(z) \tag{5.39}$$

for $z \in O(\delta)$. The remainder $R_1(z)$ satisfies the estimates

$$\|\langle x\rangle^{-s}R_1(z)\| + \|R_1(z)\langle x\rangle^{-s}\| \le C_s \tag{5.40}$$

for $s > 2\mu$ and $z \in O(\delta)$. In addition the limit $R_1(0) = \lim_{z\to 0, z\in O(\delta)} R_1(z)$ exists in $\mathcal{B}(L^2, L^{2,-s})$ and for any $a > 0$ there exist some constants $C_a, c_a > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}}R_1^{(N)}(z)\| + \|R_1^{(N)}(z)e^{-a\langle x\rangle^{1-\mu}}\| \le C_a c_a^N N^{\sigma N}, \tag{5.41}$$

for any $N \in \mathbb{N}^*$ and $z \in \Omega_0(\delta)$. Here $\sigma = 1 + \gamma$.

Proof. We use the Grushin method to study the low-energy asymptotics for the resolvent of H by using the equation

$$R(z) = (1 + R_0(z)W)^{-1}R_0(z). \tag{5.42}$$

Since the method is well-known in selfadjoint case (see [13, 27]), we shall skip over some details and emphasize on the Gevrey estimates for the remainder. Since G_0 is continuous from $L^{2,s+2\mu}$ to $L^{2,s}$ for any s, $\mathrm{Ker}_{L^{2,s}}(1 + G_0W)$ is independent of $s \in \mathbb{R}$ and coincides with the eigenspace of H associated withe the eigenvalue 0. We need only to work in $L^2(\mathbb{R}^n)$.

Let $\psi_1, \cdots, \psi_m$ be a basis of $\mathrm{Ker}(1 + G_0W)$ such that

$$\langle \psi_j, -W\psi_k\rangle = \delta_{jk}, \quad j, k = 1, \cdots, m. \tag{5.43}$$

(5.43) can be realized because the quadratic form $\phi \to \langle \phi, -W\phi\rangle = \langle \phi, H_0\phi\rangle$ is positive definite on $\mathrm{Ker}(1 + G_0W)$. Define $Q : L^2 \to L^2$ by

$$Qf = \sum_{j=1}^{m} \langle -W\psi_j, f\rangle \psi_j, \quad f \in L^2. \tag{5.44}$$

Set $Q' = 1 - Q$. Then Q commutes with $1 + G_0W$. -1 is not eigenvalue of compact operator $Q'(G_0W)Q'$, hence $Q'(1 + G_0W)Q'$ is invertible on the range of Q' with bounded inverse. From Theorem 3.4 with $N = 1$ and Proposition 4.1, one deduces that

$$(R_0(z) - G_0)W = O(|z|) \tag{5.45}$$

for $z \in O(\delta)$. It follows that if $\delta > 0$ is small enough,

$$E(z) = (Q'(1 + R_0(z)W)Q')^{-1}Q' \tag{5.46}$$

is well-defined and continuous in $z \in O(\delta)$ and

$$\|E(z)\| \leq C \tag{5.47}$$

uniformly in $z \in O(\delta)$. By Corollary 4.2 with $\kappa = 1$ (because H_0 is selfadjoint) and (5.21), $E(z)$ satisfies Gevrey estimates

$$\|E^{(N)}(z)\| \leq CC'^N N^{\sigma N} \tag{5.48}$$

for some $C' > 0$ and for all $z \in O_0(\delta)$.

Define $S : \mathbb{C}^m \to D(H)$ and $T : L^2 \to \mathbb{C}^m$ by

$$\begin{aligned} Sc &= \sum_{j=1}^{m} c_j\psi_j, \quad c = (c_1, \cdots, c_m) \in \mathbb{C}^m, \\ Tf &= (\langle -W\psi_1, f\rangle, \cdots, \langle -W\psi_m, f\rangle) \in \mathbb{C}^m, \quad f \in L^2. \end{aligned}$$

Set $W(z) = (1 + R_0(z)W)$ and

$$E_+(z) = S - E(z)W(z)S, \tag{5.49}$$

$$E_-(z) = T - TW(z)E(z), \tag{5.50}$$

$$E_{-+}(z) = -TW(z)S + TW(z)E(z)W(z)S. \tag{5.51}$$

Then one has the formula

$$(1 + R_0(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \text{ on } H^{1,-s}. \tag{5.52}$$

Since $E(z)$, $W(z)$ satisfy Gevrey estimates of the form (5.48) on $O_0(\delta)$, $E_\pm(z)$ and $E_{-+}(z)$ satisfy similar Gevrey estimates on $O_0(\delta)$. The leading term of $E_{-+}(z)$ can be explicitly calculated:

$$E_{-+}(z) = -z\Psi + z^2 r_1(z) \tag{5.53}$$

where the matrix $\Psi = (\langle \psi_j, \psi_k \rangle)_{1\leq j,k\leq m}$ is positive definite and $r_1(z)$ satisfies the Gevrey estimates of order $\sigma = 1+\gamma$ in $O_0(\delta)$. Consequently,

$$E_{-+}(z)^{-1} = -\frac{\Psi^{-1}}{z} + \widetilde{r}_1(z) \tag{5.54}$$

with $\widetilde{r}_1(z)$ uniformly bounded on $O(\delta)$ and continuous up to 0 and $\widetilde{r}_1(z)$ satisfying the Gevrey estimates of the form (5.48) in $O_0(\delta)$. It follows that $(1+R_0(z)W)^{-1}$ is of the form

$$(1+R_0(z)W)^{-1} = \frac{A_0}{z} + B(z) \tag{5.55}$$

where

$$A_0 = S\Psi^{-1}T \tag{5.56}$$

is an operator of rank m and $B(z)$ is uniformly bounded in $O(\delta)$ and satisfies the Gevrey estimates

$$\|B^{(N)}(z)\| \leq CC'^N N^{\sigma N}, \quad \forall N \in \mathbb{N}^*, \tag{5.57}$$

for z in $O(\delta)$. From the equation $R(z) = (1+R_0(z)W)^{-1}R_0(z)$ and Corollary 4.2 with $\kappa = 1$, we deduce that

$$R(z) = -\frac{\Pi_0}{z} + R_1(z) \tag{5.58}$$

where $R_1(z)$ satisfies

$$\|R_1(z)\langle x\rangle^{-2\mu}\| \leq C$$

and

$$\|R_1^{(N)}(z)e^{-a\langle x\rangle^{1-\mu}}\| \leq C_a c_a^N N^{\sigma N}$$

for z in $O_0(\delta)$ and $N \in \mathbb{N}^*$. This proves (5.40) and (5.41). ■

Theorem 5.10. *Let $H_0 = -\Delta + V_0(x)$ and $H = H_0 + W(x)$ with $V_0 \in \mathcal{A}$ and $W \in L^\infty_{\rm comp}$. Assume that 0 is an eigenvalue of H and that both H_0 and H are selfadjoint. Let Π_0 denote the eigenprojection of H associated with eigenvalue zero. Then there exist some constants $C, c, \delta > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ the cut-off resolvent $\chi R(z)\chi$ defined for $\operatorname{Im} z > 0$ extends to a holomorphic function in $\Omega(\delta,\theta)$ and one has*

$$\chi R(z)\chi = -\frac{\chi\Pi_0\chi}{z} + R_2(z) \tag{5.59}$$

for $z \in \Omega(\delta,\theta)$ where the remainder $R_2(z)$ is continuous up to $z = 0$ and satisfies the Gevrey estimates

$$\|R_2^{(N)}(z)\| \leq C_\chi C^N N^{\sigma N}, \quad N \in \mathbb{N}^*, \tag{5.60}$$

for $z \in \Omega_0(\delta,\theta)$. Here $\sigma = 1+\gamma$, $\Omega(\delta,\theta)$ and $\Omega_0(\delta,\theta)$ are the same as in Corollary 4.8.

Proof. It suffices to prove (5.59) for $\chi \in C_0^\infty(\mathbb{R}^n)$ with sufficiently large support. Let $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi_0(x) \leq 1$, $\chi_0(x) = 1$ for $|x| \leq 1$ and 0 for $|x| \geq 2$. Set

$$\chi_j(x) = \chi_0(\frac{x}{jR}), \quad j = 1, 2, \tag{5.61}$$

where $R > R_0$ is to be adjusted and R_0 is chosen such that supp $W \subset \{x; |x| \leq R_0\}$. Then $\chi_j W = W$ and $\chi_1\chi_2 = \chi_1$. Then one has

$$\chi_1 R(z)\chi_1 = \chi_1(1+\chi_2 R_0(z,\theta)W)^{-1}\chi_2 R_0(z,\theta)\chi_1, \tag{5.62}$$

where the analytic distortion is carried out outside the support of χ_2. (5.62) initially valid for θ real and $\operatorname{Im} z > 0$ allows to extend $z \to \chi_1 R(z)\chi_1$ meromorphically into a sector below the positive real axis when $\operatorname{Im}\theta > 0$. In the following $\theta \in \mathbb{C}$ is fixed with $\operatorname{Im}\theta > 0$. According to Corollary 4.8, $1+\chi_2 R_0(z,\theta)W$ and $\chi_2 R_0(z,\theta)\chi_1$ belong to Gevrey class $G^\sigma(\Omega_0(\delta,\theta))$.

Let $\{\psi_j, j = 1, \cdots, m\}, Q$ and Q' be defined as in the proof of Theorem 5.9. Then -1 is not an eigenvalue of the compact operator $Q'(G_0W)Q'$. Since $Q'(\chi_2 G_0 W)Q'$ converges to $Q'G_0WQ'$ in operator norm as $R \to \infty$, -1 is not an eigenvalue of $Q'\chi_2 G_0 W Q'$ for all $R \geq R_1$ where $R_1 > R_0$ is sufficiently large. Then $Q'(1 + \chi_2 G_0 W)Q'$ is invertible on Range Q'. So is $Q'(1 + \chi_2 R_0(z,\theta)W)Q'$ for $z \in \Omega(\delta,\theta)$ for some $\delta > 0$, because the analytic distortion is made outside the support of χ_2 and W. The inverse

$$E_0(z,\theta) = (Q'(1 + \chi_2 R_0(z,\theta)W)Q')^{-1}Q' \tag{5.63}$$

is uniformly bounded in $z \in \Omega(\delta,\theta)$ (see Proposition 4.7) and by (5.21) it belongs to Gevrey class $G^\sigma(\Omega(\delta,\theta))$.

Define $S_1 : \mathbb{C}^m \to L^2$ and $T_1 : L^2 \to \mathbb{C}^m$ by

$$S_1 = \chi_1 S, \quad T_1 = T\chi_1 \tag{5.64}$$

where S, T are defined in Theorem 5.9. By Lemma 5.8,

$$S_1T_1 = Q + O(e^{-cR^{1-\mu}}), \quad T_1S_1 = 1 + O(e^{-cR^{1-\mu}}) \tag{5.65}$$

for some $c > 0$. Let $W(z,\theta) = 1 + \chi_2 R_0(z,\theta)W$. Consider the Grushin problem

$$\begin{pmatrix} W(z,\theta) & S_1 \\ T_1 & 0 \end{pmatrix} : L^2 \otimes \mathbb{C}^m \to L^2 \otimes \mathbb{C}^m. \tag{5.66}$$

One has

$$\begin{pmatrix} W(z,\theta) & S_1 \\ T_1 & 0 \end{pmatrix} \begin{pmatrix} E_0(z,\theta) & S_1 \\ T_1 & -T_1W(z,\theta)S_1 \end{pmatrix} = 1 + \mathcal{R}(z,\theta) \tag{5.67}$$

where

$$\mathcal{R}(z,\theta) = \begin{pmatrix} QW(z,\theta)E_0(z,\theta) + S_1T_1 - Q & (1 - S_1T_1)W(z,\theta)S_1 \\ T_1E_0(z,\theta) & T_1S_1 - 1 \end{pmatrix}. \tag{5.68}$$

$\mathcal{R}(z,\theta)$ is sum of a nilpotent matrix and a matrix of order $O(e^{-cR^{1-\mu}})$. Consequently $1+\mathcal{R}(z)$ is invertible $z \in \Omega(\delta,\theta)$ if $R > R_1$ is sufficiently large. This proves the Grushin problem is invertible from the right. Similarly one can show it is invertible from the left, therefore it is invertible with inverse given by

$$\begin{pmatrix} E_0(z,\theta) & S_1 \\ T_1 & -T_1W(z,\theta)S_1 \end{pmatrix} (1 + \mathcal{R}(z))^{-1} := \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \tag{5.69}$$

As usual, one has the formula

$$(1 + \chi_2 R_0(z,\theta)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z). \tag{5.70}$$

$E_{-+}(z)$ is of the form

$$E_{-+}(z) = -T_1W(z,\theta)S_1(1 + O(e^{-cR^{1-\mu}})) + O(|z|^2)$$

By the choice of χ_1, χ_2, one has

$$\begin{aligned} T_1W(z,\theta)S_1 &= T_1(1 + R_0(z,\theta)W)S_1 \\ &= zT_1G_1(\theta)W)S_1 + O(|z|^2) = zT_1G_1WS_1 + O(|z|^2), \end{aligned}$$

where $G_1(\theta) = G_0(\theta)^2$ and $G_1 = G_0^2$. By the calculation made in the proof of Theorem 5.9, one sees $\Psi_1 = T_1G_1S_1$ is an invertible matrix (if R is large enough). Consequently $E_{-+}(z)$ is invertible for $z \in S(-\delta, \pi + \delta)(c)$ with inverse of the form.

$$E_{-+}(z)^{-1} = -\frac{1}{z}\Psi_1(1 + O(e^{-cR^{1-\mu}})) + B(z) \tag{5.71}$$

where $B(z)$ uniformly bounded on $\Omega(\delta,\theta)$ and continuous up to 0. This proves the existence of an asymptotic expansion for $\chi_1 R(z)\chi_1$ for $z\in\Omega(\delta,\theta)$ of the form

$$\chi_1 R(z)\chi_1 = -\frac{U}{z} + R_2(z) \tag{5.72}$$

with $R_2(z)$ uniformly bounded on $\Omega(\delta,\theta)$ and continuous up to 0 and it belongs to $G^\sigma(\Omega_0(\delta,\theta))$. To determine U, we remark that since $\mathcal{A}\subset\mathcal{V}$, Theorem 5.9 applied to $R(z)$ with $\mathrm{Re}\, z<0$ gives $U=\chi_1\Pi_0\chi_1$. ■

Proof of Theorem 2.3. Theorem 2.3 (a) and (b) are respectively deduced from Theorems 5.9 and 5.10 and the formulas for $t>0$

$$e^{-tH} - \sum_{\lambda\in\sigma_d(H),\mathrm{Re}\,\lambda\le 0} e^{-tH}\Pi_\lambda = \frac{i}{2\pi}\lim_{\epsilon\to 0_+}\int_{\Gamma(\epsilon)} e^{-tz}R(z)dz + O(e^{-ct}) \tag{5.73}$$

$$\chi\left(e^{-itH} - \sum_{\lambda\in\sigma_d(H)\cap\mathbb{R}_-} e^{-itH}\Pi_\lambda\right)\chi = \frac{i}{2\pi}\lim_{\epsilon\to 0_+}\int_{\Gamma_\eta(\epsilon)} e^{-itz}\chi R(z)\chi dz + O(e^{-ct}) \tag{5.74}$$

where $c>0$ and

$$\begin{aligned}\Gamma(\epsilon) &= \{z;|z|\ge\epsilon,\mathrm{Re}\, z\ge 0,|\mathrm{Im}\, z| = C(\mathrm{Re}\, z)^{\mu'}\}\cup\{z;|z|=\epsilon,|\arg z|\ge\omega_0\}\\ \Gamma_\eta(\epsilon) &= \{z=re^{-i\eta},r\ge\epsilon\}\cup\{z=-re^{i\theta},r\ge\epsilon\}\cup\{z;|z|=\epsilon,-\eta\le\arg z\le\pi+\eta\}\end{aligned}$$

for some appropriate constants $C,\mu'>0,\eta>0$. In particular, $\eta>0$ is chosen such that H has no eigenvalues with negative imaginary part above $\Gamma_\eta(\epsilon)$. Here ω_0 is the argument of the point z_0 with $|z_0|=\epsilon$, $\mathrm{Re}\, z_0>0$ and $\mathrm{Im}\, z_0 = C(\mathrm{Re}\, z_0)^{\mu'}$. Remark that the sub-exponential time-decay estimates are derived from Gevrey estimates of $R_1(z)$ and $R_2(z)$ at zero and their Taylor expansion of order N with N chosen appropriately in terms of $t>0$. See the proof of Theorem 4.4 for e^{-tH_0}. ■

Remark 5.1. *As an example of applications of Theorem 2.3, consider the Witten Laplacian on function defined by*

$$-\Delta_U = {}^t\nabla_U\cdot\nabla_U \tag{5.75}$$

where $\nabla_U = e^{-U}\nabla e^U$ *and* $U\in C^2(\mathbb{R}^n)$. *Then*

$$-\Delta_U = -\Delta + (\nabla U)(x)\cdot(\nabla U)(x) - \Delta U(x)$$

If $U\in C^2(\mathbb{R}^n;\mathbb{R})$ *satisfies for some* $\rho\in]0,1[$ *and* $c_1,C_1>0$,

$$U(x)\ge c_1\langle x\rangle^\rho,\quad |\nabla U(x)|\ge c_1\langle x\rangle^{\rho-1},\quad |\partial_x^\alpha U(x)|\le C_1\langle x\rangle^{\rho-|\alpha|} \tag{5.76}$$

for x *outside some compact and for* $\alpha\in\mathbb{N}^n$ *with* $|\alpha|\le 2$. *Then* $-\Delta_U$ *can be decomposed as* $-\Delta_U = H_0 + W(x)$ *where* H_0 *satisfies the conditions of Theorem 2.1 with* $\mu = 1-\rho$ *and* $W(x)$ *is of compact support. Zero is a simple eigenvalue of* $-\Delta_U$ *embedded in its continuous spectrum* $[0,+\infty[$. *As consequence of Theorem 2.3, one obtains the following result. Let* $\varphi_0(x)$ *be a normalized eigenfunction of* $-\Delta_U$ *with eigenvalue zero:*

$$\varphi_0(x) = Ce^{-U(x)},\quad \|\varphi_0\| = 1. \tag{5.77}$$

Then for any $a>0$, *there exist some constants* $C_a,c_a>0$ *such that*

$$\|e^{t\Delta_U}f - \langle\varphi_0,f\rangle\varphi_0\| \le C_a e^{-c_a t^{\frac{\rho}{2-\rho}}}\|e^{a\langle x\rangle^\rho}f\| \tag{5.78}$$

for $t>0$ *and* f *with* $e^{a\langle x\rangle^\rho}f\in L^2$. *Note that the sub-exponential convergence estimate (5.78) without explicit remainder estimate with respect to* f *is proved in* [7] *by the method of Markov processes.*

5.4. **Threshold eigenvalue in non-selfadjoint case.** Finally we study the case zero is an embedded eigenvalue of the non-selfadjoint Schrödinger operator H. Note that one can not define in a natural way Riesz projection or Jordan structure associated with eigenvalues embedded in the essential spectrum. In this Section, we study these notions for eigenvalue -1 of the compact operator $K = G_0W$, establish low-energy resolvent expansion with Gevrey estimates on the remainder and prove Theorem 2.4.

Let $V_0 \in \mathcal{V}$. Then zero is an eigenvalue of H if and only if -1 is an eigenvalue of compact operator $K = G_0W$ on $L^2(\mathbb{R}^n)$. The algebraic multiplicity m of eigenvalue -1 of K is finite. Let $\pi_1 : L^2 \to L^2$ be the associated Riesz projection of K defined by :

$$\pi_1 = \frac{1}{2\pi i}\int_{|z+1|=\epsilon} (z-K)^{-1}dz$$

for $\epsilon > 0$ small enough. Then

$$m = \text{Rank } \pi_1. \tag{5.79}$$

π_1 is continuous on $L^{2,s}$ for any $s \in \mathbb{R}$ and $\pi_1^* : L^2 \to L^2$ is the Riesz projection of K^* associated with the eigenvalue -1.

By Corollary 4.2, $R_0(z)W$ is continuous in $z \in O(\delta)$, where $O(\delta)$ is defined by (4.3). Denote $\pi_1' = 1 - \pi_1$. $\pi'(1 + G_0W)\pi'$ is injective on the range of π_1'. The Fredholm Theorem implies that $(\pi_1'(1 + G_0W)\pi_1')^{-1}$ is invertible on L^2. It follows that

$$B_1(z) = (\pi_1'(1 + R_0(z)W)\pi_1')^{-1}\pi_1' \tag{5.80}$$

is well defined on $O(\delta)$ if $\delta > 0$ is sufficiently small. In addition $B_1(z)$ is uniformly bounded there. Since $R_0(z)W$ satisfies Gevrey estimates of order σ for z near 0 with $\text{Re}\, z < 0$ and $|\text{Im}\, z| < -C\text{Re}\, z$, $C > 0$, so does $B_1(z)$. $\pi_1(1 + R_0(z)W)\pi_1$ is of finite rank. Set

$$\omega(z) = \det(\pi_1(1 + R_0(z)W)\pi_1). \tag{5.81}$$

Then $\pi_1(1+R_0(z)W)\pi_1$ is invertible if and only if $\omega(z) \neq 0$. $\omega(z)$ satisfies the Gevrey estimates of order σ at point $z = 0$ and has an asymptotic expansion of the form

$$\omega(z) = \sum_{j=1}^{N} \omega_j z^j + O(|z|^{N+1}), z \in O(\delta), \tag{5.82}$$

for any N. The following result gives a sufficient condition to find a leading term in the asymptotic expansion of $\omega(z)$ as $z \to 0$.

Lemma 5.11. *Assume that eigenvalue -1 of G_0W is semi-simple (i.e., its algebraic and geometrical multiplicities are equal). Suppose in addition that there exists a basis $\{\varphi_1, \cdots, \varphi_m\}$ of $\text{Ker}(1 + G_0W)$ such that*

$$\det(\langle \varphi_j, J\varphi_k\rangle)_{1\leq j,k\leq m} \neq 0, \tag{5.83}$$

where J is complex conjugation. Then

$$\omega(z) = \omega_m z^m + O(|z|^{m+1}), z \in O(\delta), \tag{5.84}$$

for some $\omega_m \neq 0$.

Proof. When the eigenvalue -1 is semi-simple, one has

$$\text{Rank } \pi_1 = \dim \text{Ker}(1 + G_0W) = m.$$

By Fredholm Theorem for compact operators, -1 is also a semi-simple eigenvalue of $(G_0W)^*$ and Rank $\pi_1^* = m$. It follows from relations $H_0^* = JH_0J$, $W^* = JWJ$ that

$$JW\pi_1 = \pi_1^* JW, \qquad \pi_1 JG_0 = JG_0\pi_1^* \tag{5.85}$$

$$JW(1+G_0W) = (1+(G_0W)^*)JW, \tag{5.86}$$

$$(1+G_0W)JG_0 = JG_0(1+(G_0W)^*). \tag{5.87}$$

One deduces that $JW : \mathrm{Ker}(1+G_0W) \to \mathrm{Ker}(1+(G_0W)^*)$ is bijective. Therefore π_1 can be written into the form

$$\pi_1 = \sum_{j=1}^m \langle \cdot, \psi_j\rangle \varphi_j \tag{5.88}$$

for some $\psi_j \in \mathrm{Range}\ \pi_1^*$ with $\langle \varphi_j, \psi_k\rangle = \delta_{jk}$ and $\psi_j = JW\phi_j$ for some $\phi_j \in \mathrm{Ker}(1+G_0W)$. One can calculate

$$\pi_1 G_0\varphi_k = \sum_{j=1}^m \langle G_0\varphi_k, JW\phi_j\rangle\varphi_j = -\sum_{j=1}^m \langle \varphi_k, J\phi_j\rangle\varphi_j$$

for $k = 1, \cdots, m$. Since $\{\phi_1, \cdots, \phi_m\}$ is a basis of $\mathrm{Ker}(1+G_0W)$, condition (5.83) implies that

$$a_m = \det(\langle \varphi_k, J\phi_j\rangle)_{1\le k,j\le m} \neq 0 \tag{5.89}$$

In the basis $\varphi = \{\varphi_1, \cdots, \varphi_m\}$, the matrix of $\pi_1(1+R_0(z)W)\pi_1$ on Range π_1 takes the form

$$\mathcal{M}_\varphi\left(\pi_1(1+R_0(z)W)\pi_1\right) = -z(\langle \varphi_k, J\phi_j\rangle)_{1\le k,j\le m} + O(|z|^2). \tag{5.90}$$

Thus (5.84) holds with $\omega_m = (-1)^m a_m \neq 0$. ∎

In the general case, one can prove the following

Proposition 5.12. *Let κ be given in Corollary 4.2, $O(\delta)$ be defined by (4.3) and $O_0(\delta) = O(\delta)\cup\{0\}$. Assume that*

$$\omega(z) = \omega_k z^k + O(|z|^{k+1}) \tag{5.91}$$

for some $k \in \mathbb{N}^*$ and $\omega_k \neq 0$. Then there exist operators C_{-j}, $j = 1, \cdots, k$, with rank less than or equal to m such that

$$R(z) = \frac{C_{-k}}{z^k} + \cdots + \frac{C_{-1}}{z} + R_3(z) \tag{5.92}$$

for $z \in O(\delta)$. The remainder $R_3(z)$ satisfies the estimate: $\exists C, \mu', \delta > 0$ such that

$$\|\langle x\rangle^{-2\kappa\mu} R_3(z)\| + \|R_3(z)\langle x\rangle^{-2\kappa\mu}\| \le C_s \tag{5.93}$$

for $z \in O(\delta)$. In particular,

$$\sigma_d(H) \cap O(\delta) = \emptyset. \tag{5.94}$$

In addition $R_3(z)$ is continuous up to $z = 0$ and for any $a > 0$, there exist $C_a, c_a > 0$ such that

$$\|e^{-a\langle x\rangle^{1-\mu}} R_3^{(N)}(z)\| + \|R_3^{(N)}(z) e^{-a\langle x\rangle^{1-\mu}}\| \le C_a c_a^N N^{(1+\kappa\gamma)N}, \tag{5.95}$$

for any $N \in \mathbb{N}^*$ and $z \in O_0(\delta)$.

Proof. Since $\omega_k \neq 0$, $\pi_1(1+R_0(z)W)\pi_1$ is invertible on the range of π_1 for $z \in O(\delta)$ with $\delta > 0$ small enough. By Corollary 4.2), it belongs to $G^{1+\kappa\gamma}(O(\delta)$. Set

$$B_0(z) = (\pi_1(1+R_0(z)W)\pi_1)^{-1}\pi_1.$$

Then $\omega(z)B_0(z)$ has the same continuity properties as $\pi_1(1+R_0(z)W)\pi_1$ and

$$B_0(z) = z^{-k}B_{-k}^{(0)} + \cdots z^{-1}B_{-1}^{(0)} + R^{(0)}(z) \tag{5.96}$$

where $B_{-j}^{(0)}$, $j = 1, \cdots, k$, are operators of rank $\leq m$ and $R^{(0)}(z)$ is uniformly bounded for $z \in O(\delta)$ and belongs to $G^{1+\kappa\gamma}(O(\delta))$. Let $\pi_1' = 1 - \pi_1$. Let $B_1(z)$ be defined by (5.80). One can check that

$$\begin{aligned}(1 + R_0(z)W)(B_0(z) + B_1(z)) &= 1 + O(|z|),\\ (B_0(z) + B_1(z))(1 + R_0(z)W) &= 1 + O(|z|)\end{aligned}$$

in $\mathcal{B}(L^2)$. Therefore $(1 + R_0(z)W)$ is invertible for $z \in O(\delta)$ if $\delta > 0$ is small enough and

$$(1 + R_0(z)W)^{-1} = (B_0(z) + B_1(z))(1 + O(|z|)) = C_0(z) + R^{(1)}(z), \quad z \in O(\delta). \tag{5.97}$$

where

$$C_0(z) = z^{-k}C_{-k}^{(0)} + \cdots z^{-1}C_{-1}^{(0)} \tag{5.98}$$

where $C_{-j}^{(0)}$, $j = 1, \cdots, k$, are operators of rank $\leq m$ and $R^{(1)}(z)$ belongs to $G^{1+\kappa\gamma}(O(\delta)$ as bounded operator-valued function. (5.92) and (5.95) can now be derived from the equation $R(z) = (1 + R_0(z)W)^{-1}R_0(z)$. ■

Combining proofs of Lemma 5.11 and Proposition 5.12, one concludes that if eigenvalue -1 of G_0W is semi-simple and if (5.83) is satisfied, then condition (5.91) is true with $k = m$ and (5.92) holds. A direct calculation for $B_0(z)$ using (5.90) gives that $C_{-m} = \cdots = C_{-2} = 0$ and C_{-1} is an operator of rank m.

Remark 5.2. *Under the conditions of Theorem 5.1, let $\lambda > 0$ be an outgoing positive resonance of H. Let π_1 denote the Riesz projection of eigenvalue -1 of $R_0(\lambda + i0)W$ as operator on $L^{2,-s}$, $\frac{1}{2} < s < \rho - \frac{1}{2}$. Denote*

$$\omega(z) = \det\left(\pi_1(1 + \chi R_0(z)W)\pi_1\right)$$

for $z \in \mathbb{C}_+$, where $\chi \in C_0^\infty$ with $\chi W = W$. Then one can show that $\omega(\lambda) = 0$ and $\omega(z) \neq 0$ for z with $\operatorname{Im} z >> 1$. In addition, $\omega(z)$ extends to a holomorphic function into $S(-\delta, \pi + \delta)$ for some $\delta > 0$. Since λ is in the interior of $S(-\delta, \pi + \delta)$, from the uniqueness of holomorphic functions, one concludes that the order of zero of $\omega(z)$ at $z = \lambda$ is finite. Therefore there exist some $k \in \mathbb{N}^*$ and some $\omega_k \neq 0$ such that

$$\omega(z) = \omega_k(z - \lambda)^k + O(|z - \lambda|^{k+1}) \tag{5.99}$$

for z in a complex neighborhood of λ. This means a condition analogous to (5.91) is satisfied for positive resonances under some analyticity condition on potentials. The proof of Proposition 5.12, along with (5.29) for some fixed $\operatorname{Im}\theta > 0$, allows to conclude that the meromorphic extension from $\mathbb{C}_+$ of $\chi R(z)\chi$ admits an expansion around λ of the form:

$$\chi R(z)\chi = \chi\left(\frac{C_{-k}}{(z - \lambda)^k} + \cdots + \frac{C_{-1}}{z - \lambda}\right)\chi + \tilde{R}_3(z) \tag{5.100}$$

for z near λ, where C_{-j} is of rank less than or equal to $m_+(\lambda)$ and $\tilde{R}_3(z)$ is holomorphic in a neighborhood of λ. Here $m_+(\lambda) = \operatorname{Rank}\pi_1$ is the algebraic multiplicity of eigenvalue -1 of $R_0(\lambda + i0)W$. This shows that for $\nu \in r_+(H)$, the coefficients of $P_\nu(t)$ in Theorem 2.2 (b) are of rank not exceeding $m_+(\nu)$.

In order to give more precisions on the resolvent expansion of Proposition 5.92, we study in more details the Riesz projection π_1 associated with eigenvalue -1 of operator $K = G_0W$. Assume from now on that eigenvalue -1 of K is geometrically simple:

$$\dim\operatorname{Ker}(1 + K) = 1, \quad \operatorname{Rank}\pi_1 = m. \tag{5.101}$$

This condition is equivalent to requiring threshold eigenvalue of H to be geometrically simple. Operator $1+K$ being nilpotent on Range π_1, there exists some function $\phi_m \in$ range π_1 such that

$$\phi_j = (1+K)^{m-j}\phi_m \neq 0, \quad j = 1, \cdots, m. \tag{5.102}$$

One has

$$(1+K)\phi_1 = 0, \quad (1+K)\phi_j = \phi_{j-1}, \quad 2 \leq j \leq m. \tag{5.103}$$

$\phi_1, \cdots, \phi_m$ are linearly independent. Denote J the operation of complex conjugation $J : f \to \overline{f}$. Remark that $H_0^* = JH_0J$, $H^* = JHJ$. One has

$$JWK = K^*\overline{W}J. \tag{5.104}$$

It follows that

$$JW\pi_1 = \pi_1^* JW. \tag{5.105}$$

Denote

$$\phi_j^* = \overline{W\phi_j}. \tag{5.106}$$

Then

$$(1+K^*)\phi_1^* = 0, \quad (1+K^*)\phi_j^* = \phi_{j-1}^*, \quad 2 \leq j \leq m. \tag{5.107}$$

Since $\phi_1^* \neq 0$, it follows that $\phi_j^* \neq 0$ for all $1 \leq j \leq m$. From this, we deduce that $\{\phi_j^*, j = 1, \cdots, m\}$ is linearly independent and JW is injective from Range π_1 into Range π_1^*. Since

$$\text{Rank } \pi_1 = \text{Rank } \pi_1^* = m, \tag{5.108}$$

JW is a bijection from Range π_1 onto Range π_1^*.

Lemma 5.13. *Assume that threshold eigenvalue of H is geometrically simple. Then the bilinear form $B(\cdot,\cdot)$ defined on* Range π_1 *by*

$$B(\varphi, \psi) = \langle \varphi, JW\psi \rangle = \int_{\mathbb{R}^n} W(x)\varphi(x)\psi(x)\, dx \tag{5.109}$$

is non-degenerate.

Proof. Let $\phi \in$ Range π_1 such that

$$\int_{\mathbb{R}^n} W(x)\phi(x)\varphi(x)\, dx = 0$$

for all $\varphi \in$ Range π_1. Since JW is a bijection from Range π_1 onto Range π_1^*, this means that $\phi \in (\text{Range } \pi_1^*)^\perp = \text{Ker } \pi_1$, which implies that $\phi = \pi_1\phi = 0$. So $B(\cdot,\cdot)$ is non-degenerate. ■

As a consequence of Lemma 5.13, one obtains the following

Corollary 5.14. *If eigenvalue -1 of K is simple (i. e., $m = 1$), then the eigenfunction φ of H associated with threshold eigenvalue satisfies*

$$\int_{\mathbb{R}^n} W(x)(\varphi(x))^2\, dx \neq 0. \tag{5.110}$$

Proof. If $m = 1$, Range $\pi_1 = \text{Ker}(1+K)$ is one dimensional and is spanned by φ. The non-degeneracy of $B(\cdot,\cdot)$ on Range π_1 implies $\int_{\mathbb{R}^n} W(x)(\varphi(x))^2\, dx \neq 0$. ■

Lemma 5.15. *There exist $\chi_j \in \text{Ker}(1+K)^{m-j+1}$, $j = 1, \cdots, m$, such that*

$$\langle \phi_i, \chi_j^* \rangle = B(\phi_i, \chi_j) = \delta_{ij}, \quad 1 \leq i, j \leq m, \tag{5.111}$$

where $\chi_j^* = JW\chi_j$, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Proof. We use an induction to prove that for any $1 \leq l \leq m$, there exist $\varphi_j \in \text{Ker } (1+K)^j$, $1 \leq j \leq l$ such that

$$B(\varphi_i, \phi_{m-j+1}) = \delta_{ij}, \quad 1 \leq j \leq i \leq l. \tag{5.112}$$

Since $\phi_1 \in \text{Ker } (1+K)$ and $\phi_j^* \in \text{Range } (1+K^*)$ for $1 \leq j \leq m-1$, one has $\langle \phi_1, \phi_j^* \rangle = 0$ for $j = 1, \cdots, m-1$. By lemma 5.13, one has necessarily $c_1 = \langle \phi_1, \phi_m^* \rangle \neq 0$. Set

$$\varphi_1 = \frac{1}{c_1} \phi_1. \tag{5.113}$$

Then $\varphi_1 \in \text{Ker } (1+K)$ and $B(\varphi_1, \phi_m) = 1$. (5.112) is true for $l = 1$. Assume now that (5.112) is true for some $l = k-1$, $2 \leq k \leq m$. Set

$$\phi_k' = \phi_k - \sum_{j=1}^{k-1} B(\phi_k, \phi_{m-j+1}) \varphi_j \tag{5.114}$$

Then $\phi_k' \neq 0$, $\phi_k' \in \text{Ker } (1+K)^k$ and

$$B(\phi_k', \phi_{m-j+1}) = 0, \quad j = 1, \cdots, k-1.$$

Since $\phi_k' \in \text{Ker } (1+K)^k$, one has also

$$\langle \phi_k', \phi_j^* \rangle = B(\phi_k', \phi_j) = 0 \tag{5.115}$$

for $j = 1, \cdots, m-k$, because $\phi_j^* = (1+K^*)^{m-j} \phi_m^*$ belongs to the range of $(1+K^*)^k$ if $1 \leq j \leq m-k$. By Lemma 5.13, the constant $c_k = B(\phi_k', \phi_{m-k+1})$ must be nonzero. Set

$$\varphi_k = \frac{1}{c_k} \phi_k'. \tag{5.116}$$

Then (5.112) is proved for $l = k$. By an induction, one can construct φ_j, $1 \leq j \leq m$, such that (5.112) holds with $l = m$. By (5.115), one has also $B(\varphi_i, \phi_{m-j+1}) = 0$ if $i > j$. Lemma 5.15 is proved by taking $\chi_k = \varphi_{m-k+1}$, $1 \leq k \leq m$. ∎

One has the following representation of the Riesz projection π_1.

Corollary 5.16. *One has*

$$\pi_1 u = \sum_{j=1}^{m} \langle u, \chi_j^* \rangle \phi_j, \quad u \in L^2(\mathbb{R}^n). \tag{5.117}$$

Proof. Denote π the operator $\pi : u \to \sum_{j=1}^m \langle u, \chi_j^* \rangle \phi_j$. Then it is clear that $\pi^2 = \pi$ and Range π = Range π_1. It is trivial that Ker $\pi_1 \subset$ Ker π. If $u \in$ Ker π, then $\langle u, \chi_j^* \rangle = 0$ for $j = 1, \cdots, m$. Therefore $u \in (\text{Range } \pi_1^*)^\perp = \text{Ker } \pi_1$ which implies that Ker $\pi \subset$ Ker π_1. This shows that Ker $\pi_1 =$ Ker π. This proves $\pi = \pi_1$. ∎

From the proof of Lemma 5.15, one sees that if -1 is a simple eigenvalue of K ($m = 1$), then the associated Riesz projection is given by

$$\pi_1 = \langle \cdot, \varphi^* \rangle \varphi \tag{5.118}$$

where φ is an eigenfunction of K with eigenvalue -1 normalized by

$$\int_{\mathbb{R}^n} W(x) (\varphi(x))^2 \, dx = 1. \tag{5.119}$$

The existence of such an eigenfunction is guaranteed by Corollary 5.14.

To study the singularity of the resolvent $R(z)$ at threshold zero, we use the resolvent equation

$$R(z) = (1 + R_0(z) W)^{-1} R_0(z)$$

for $z \notin \sigma(H)$ and study the following Grushin problem in $L^2 \times \mathbb{C}^m$:

$$\begin{pmatrix} 1+R_0(z)W & S \\ T & 0 \end{pmatrix} : L^2 \times \mathbb{C}^m \to L^2 \times \mathbb{C}^m \tag{5.120}$$

where

$$S: \quad \mathbb{C}^m \to L^2, c = (c_1, \cdots, c_m) \to Sc = \sum_{j=1}^m c_j \phi_j, \tag{5.121}$$

$$T: \quad L^2 \to \mathbb{C}^m, f \to Tf = (\langle f, \chi_1^* \rangle, \cdots, \langle f, \chi_m^* \rangle). \tag{5.122}$$

$$\tag{5.123}$$

Then $ST = \pi_1$ and $TS = I_n$. Since K commutes with its Riesz projection π_1 and since $1+K$ is injective on Range π_1' where $\pi_1' = 1 - \pi_1$, $1+K$ is invertible on the range of π_1'. By an argument of perturbation, $\pi_1'(1+R_0(z)W)\pi_1'$ is invertible on range of π_1' for $z \in O(\delta)$ if $\delta > 0$ is appropriately small and its inverse $E(z)$ is uniformly bounded on $O(\delta)$ where

$$E(z) = (\pi_1'(1+R_0(z)W)\pi_1')^{-1}\pi_1' \tag{5.124}$$

By the arguments used in Section 6.1, $E(z)$ belongs to the Gevrey class $G^\sigma(O_0(\delta))$ with $\sigma = 1+\gamma$. One can check that for $z \in O(\delta)$,

$$\begin{pmatrix} 1+R_0(z)W & S \\ T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \tag{5.125}$$

where

$$E_+(z) = (1 - E(z)R_0(z)W)S \tag{5.126}$$

$$E_-(z) = T(1 - R_0(z)WE(z)) \tag{5.127}$$

$$E_{-+}(z) = -T(1+R_0(z)W)S + TR_0(z)WE(z)R_0(z)WS. \tag{5.128}$$

It follows that $z \notin \sigma(P)$ if and only if $\det E_{-+}(z) \neq 0$ and one has

$$(1+R_0(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z). \tag{5.129}$$

By operations on Gevrey functions ((5.19) -(5.24)), $E_{-+}(z)$ is $m \times m$-matrix valued Gevrey function for $z \in O_0(\delta)$. Therefore it has an asymptotic expansion near 0 up to any order

$$E_{-+}(z) = B_0 + B_1 z + \cdots + B_N z^n + O(|z|^{N+1}) \tag{5.130}$$

where B_j is some $m \times m$ matrix. More precisely, since $TR_0(z)WE(z)R_0(z)WS = O(|z|^2)$, $E_{-+}(z)$ verifies

$$E_{-+}(z) = \left(-\langle (1+R_0(z)W)\phi_k, \chi_j^* \rangle\right)_{1 \le j,k \le m} + O(|z|^2) \tag{5.131}$$

$$= -\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - z \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm} \end{pmatrix} + O(|z|^2)$$

where

$$b_{jk} = \langle G_1 W \phi_k, \chi_j^* \rangle. \tag{5.132}$$

Note that ϕ_1 and χ_m belong to Ker $(1+G_0W)$ and $\chi_m^* = JW\chi_m$, they are rapidly decreasing, by Lemma 5.8. One can calculate

$$\begin{aligned} b_{m1} &= \lim_{\lambda\to 0_-} \langle \frac{1}{\lambda}(1+R_0(\lambda)W)\phi_1, JW\chi_m\rangle \\ &= -\lim_{\lambda\to 0_-} \langle R_0(\lambda)\phi_1, JW\chi_m\rangle \\ &= -\lim_{\lambda\to 0_-} \langle \phi_1, JR_0(\lambda)W\chi_m\rangle \\ &= \langle \phi_1, J\chi_m\rangle. \end{aligned} \tag{5.133}$$

Similarly one can calculate for $2 \le j \le m$

$$b_{mj} = -\langle W\phi_j, JG_0\chi_m\rangle = \langle \phi_j - \phi_{j-1}, J\chi_m\rangle. \tag{5.134}$$

Summing up, we have proved the following

Proposition 5.17. $\det E_{-+}(z)$ *is a Gevrey function of order* σ *for* $z \in O_0(\delta)$ *and has an asymptotic expansion in powers of* z

$$\det E_{-+}(z) = \sigma_1 z + \cdots \sigma_N z^N + O(|z|^{N+1}) \tag{5.135}$$

for any N*, where*

$$\sigma_1 = -b_{m1}. \tag{5.136}$$

Theorem 5.18. *Let* $H_0 = -\Delta + V_0(x)$ *and* $H = H_0 + W(x)$ *with* $V_0 \in \mathcal{V}$ *and* $W \in L^\infty_{\rm comp}$*. Assume that threshold eigenvalue of* H *is geometrically simple.*

(a). Suppose that

$$\det E_{-+}(z) = \sigma_k z^k + O(|z|^{k+1}) \tag{5.137}$$

for some $\sigma_k \neq 0$*,* $k \ge 1$*. Then there exist operators* C_j*,* $j = -k, \cdots, -1$ *with ranks less than or equal to* m *such that*

$$R(z) = \frac{C_{-k}}{z^k} + \cdots + \frac{C_{-1}}{z} + R_3(z) \tag{5.138}$$

for $z \in O(\delta)$*, where* C_{-j}*,* $1 \le j \le k-1$*, are of rank less than or equal to* m *and* C_{-k} *is a rank one operator given by*

$$C_{-k} = \langle \cdot, J\varphi_1\rangle\varphi_1, \tag{5.139}$$

with φ_1 *an eigenfunction of* H *associated with threshold eigenvalue. The remainder* $R_3(z)$ *satisfies the estimates:* $\exists C, \mu', \delta > 0$ *such that*

$$\|\langle x\rangle^{-s} R_3(z)\| + \|R_3(z)\langle x\rangle^{-s}\| \le C_s \tag{5.140}$$

for $s \ge 2\kappa\mu$ *and* $z \in O(\delta)$*; and for any* $a > 0$*,* $\exists C_a, c_a > 0$ *such that*

$$\|e^{-a\langle x\rangle^{1-\mu}} R_3^{(N)}(z)\| + \|R_3^{(N)}(z) e^{-a\langle x\rangle^{1-\mu}}\| \le C_a c_a^N N^{(1+\kappa\gamma)N}, \tag{5.141}$$

for any $N \in \mathbb{N}^*$ *and* $z \in O(\delta)$*.* κ *is given in Corollary 4.2 and* $O(\delta)$ *is defined by (4.3).*

(b). Suppose in addition that there exists an eigenfunction φ_0 *of* H *associated with eigenvalue zero such that*

$$\int_{\mathbb{R}^n} (\varphi_0(x))^2 dx = 1. \tag{5.142}$$

Then Condition (5.137) is satisfied with $k = 1$ *and one has*

$$C_{-1} = -\langle \cdot, J\varphi_0\rangle\varphi_0. \tag{5.143}$$

Proof. (a). The existence of the resolvent expansion is proved in Proposition 5.12 and the Gevrey estimates of the remainder can be obtained in the same way as in Theorem 5.9, making use of Corollary 4.2. We only calculate C_{-k}. Under the condition (5.137), one has

$$E_{-+}(z)^{-1} = \frac{{}^t\mathrm{Com}\, E_{-+}(z)}{\det E_{-+}(z)} = z^{-k}C + O(|z|^{-k+1}) \tag{5.144}$$

for $z \in O(\delta)$, where

$$C = \begin{pmatrix} 0 & \cdots & 0 & (-1)^{m+1}\sigma_k^{-1} \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

From (5.129), one obtains

$$(1 + R_0(z)W)^{-1} = -z^{-k}SCT + O(|z|^{-k+1}). \tag{5.145}$$

Using the definition of S and T, one sees

$$SCTf = \frac{(-1)^{m+1}}{\sigma_k}\langle f, \chi_m^*\rangle\phi_1 \tag{5.146}$$

Noticing that

$$\langle G_0 f, \chi_m^*\rangle = \langle f, G_0^* JW\chi_m\rangle = \langle f, JG_0W\chi_m\rangle = -\langle f, J\chi_m\rangle,$$

we deduce from (5.42) that

$$R(z) = \frac{C_{-k}}{z^k} + O(|z|^{-k+1}) \tag{5.147}$$

for $z \in O(\delta)$, where C_{-k} is of rank one, given by

$$C_{-k}f = \frac{(-1)^{m+1}}{\sigma_k}\langle f, J\chi_m\rangle\phi_1 \tag{5.148}$$

Since χ_m and ϕ_1 belong to the one dimensional space Ker $(1 + K)$, C_{-k} can written as

$$C_{-k}f = \langle f, J\varphi_1\rangle\varphi_1 \tag{5.149}$$

where φ_1 is an eigenfunction of H with eigenvalue zero. This proves part (a).

If (5.142) is satisfied, then $m = 1$ and one has $\chi_m = d_1\varphi_0$ and $\phi_1 = d_2\varphi_0$ for some constants $d_j \neq 0$. Therefore

$$\sigma_1 = -\langle\phi_1, J\chi_m\rangle = -d_1d_2 \neq 0.$$

Condition (5.137) is satisfied with $k = 1$. Set

$$\psi_0 = \sqrt{\frac{d_1}{b_{m1}}}\phi_1. \tag{5.150}$$

Then $C_{-1} = -\langle\cdot, J\psi_0\rangle\psi_0$. ψ_0 is an eigenfunction of H with eigenvalue zero and

$$\int_{\mathbb{R}^n} (\psi_0(x))^2\, dx = \frac{\langle\phi_1, J\chi_m\rangle}{b_{m1}} = 1.$$

Since threshold eigenvalue of H is geometrically simple, one has $\psi_0 = \pm\varphi_0$. This proves

$$C_{-1} = -\langle\cdot, J\varphi_0\rangle\varphi_0. \tag{5.151}$$

■

Remark 5.19. *If zero were a discrete eigenvalue of H, then its algebraic multiplicity might be defined. In this case Condition (5.142) is equivalent with the simplicity of threshold eigenvalue of H. In Theorem 5.18, we do not make assumption on the geometric multiplicity of eigenvalue -1 of G_0W. In addition, (5.142) does not follow from the simplicity of t eigenvalue -1 of K (see Corollary 5.14). From Formula (5.131) for $E_{-+}(z)$, one sees that if threshold eigenvalue of H is geometrically simple and if the eigenfunction associated with threshold eigenvalue verifies*

$$\int_{\mathbb{R}^n} \varphi_0(x)^2 \, dx = 0,$$

then $b_{m1} = 0$ and Condition (5.137) can only be satisfied for some $k \geq 2$. In this case, the leading singularity of resolvent $R(z)$ near 0 may take the form $\frac{C_{-k}}{z^k}$ for some $k \geq 2$.

To study e^{-itH} in the case H is non-selfadjoint and zero is an eigenvalue of H, we need the following analog of Theorem 5.10 for selfadjoint operators.

Theorem 5.20. *$V_0 \in \mathcal{A}$. Assume that zero is a geometrically simple eigenvalue of H. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\Omega(\delta, \theta)$, $\operatorname{Im}\theta > 0$, be defined as in Corollary 4.8. Under the condition (5.137), the meromorphic extension of $\chi R(z)\chi$ from $\mathbb{C}_+$ verifies*

$$\chi R(z)\chi = \chi \left(\frac{C_{-k}}{z^k} + \cdots + \frac{C_{-1}}{z} \right) \chi + R_4(z) \tag{5.152}$$

for $z \in \Omega(\delta, \theta)$, where C_{-j} is the same as in Theorem 5.18 and the remainder $R_4(z)$ is continuous up to $z = 0$ and satisfies the Gevrey estimates

$$\|R_4^{(N)}(z)\| \leq C_\chi C^N N^{\sigma N} \tag{5.153}$$

for $z \in \Omega_0(\delta, \theta)$. In addition if (5.142) is true, then Condition (5.137) is satisfied with $k = 1$ and (5.152) holds with $C_{-1} = -\langle \cdot, J\varphi_0 \rangle \varphi_0$.

Theorem 5.20 is derived by combining methods used in Theorem 5.10 and Theorem 5.18. Note that when $V_0 \in \mathcal{A}$, one can apply Corollary 4.8 instead of Corollary 4.2 to estimate the remainder. The details are omitted.

Remark 5.3. *(a). Assume that eigenvalue -1 of G_0W is semi-simple with multiplicity m. Under the condition (5.83), the condition (2.28) is satisfied with $k = m$ according to Lemma 5.11. Using formula (5.88), one can construction a Grushin problem as in the proof of Theorem 5.18 and (5.144) becomes in this case*

$$E_{-+}(z)^{-1} = \frac{C}{z} + O(1) \tag{5.154}$$

where C is a matrix of rank m. Following the proofs of Theorems 5.18 and 5.20, one concludes that the resolvent expansions given in Theorems 5.18 and 5.20 still hold with $k = m$ and one has

$$C_{-m} = \cdots = C_{-2} = 0, \quad C_{-1} = \sum_{j=1}^{m} \langle \cdot, J\psi_j \rangle \varphi_j \tag{5.155}$$

where $\{\psi_j, j = 1, \cdots, m\}$ and $\{\varphi_j, j = 1, \cdots, m\}$ are two basis of the eigenspace of H associated with threshold eigenvalue.

(b). Similarly to Proposition 5.12, one can show that in general case of threshold eigenvalue of H, Theorem 5.20 still holds if condition (2.28) is satisfied and $V_0 \in \mathcal{A}$. But we can then only affirm that C_{-j} is of rank $\leq m$, for $j = 1, \cdots, k$, as in Proposition 5.12.

Proof of Theorem 2.4. If threshold eigenvalue of H is geometrically simple, Theorem 5.20 implies that outgoing positive resonances are absent in neighborhood of zero. Therefore $r_+(H)$ is at most a finite set. The result of Theorem 2.4 for e^{-tH} in the case threshold eigenvalue of H is geometrically simple can be derived from Theorem 5.18 and formula (5.73) when $V_0 \in \mathcal{V}$ and that for e^{-itH} is obtained from Theorem 5.20 and formula (5.74) when $V_0 \in \mathcal{A}$.

Taking notice of Proposition 5.12 and Remark 5.3 (b), one can prove in the same way the results of Theorem 2.4 in general case.

Remark 5.3 (a) allows to affirm that if eigenvalue -1 of G_0W is semi-simple, under the condition (5.83), one has

$$\Pi_{0,j} = 0, j = 1, \cdots, m-1 \text{ and } \Pi_{0,0} = \sum_{j=1}^{m} \langle \cdot, J\psi_j \rangle \varphi_j \tag{5.156}$$

where $\{\varphi_j; j = 1, \cdots, m\}$ and $\{\psi_j; j = 1, \cdots, m\}$ are basis of the eigenspace of H associated with eigenvalue zero. ■

Remark 5.4. *The methods of this subsection can be applied to other threshold spectral problems. For example for non-selfadjoint Schrödinger operator $H = -\Delta + V(x)$ with a quickly decreasing complex potential $V(x)$ on $\mathbb{R}^3$:*

$$|V(x)| \leq C\langle x \rangle^{-\rho}, \rho > 2, \tag{5.157}$$

our method allows to calculate the low-energy asymptotics of the resolvent $(H-z)^{-1}$ if zero is a resonance but not an eigenvalue. In fact using the same reduction scheme and similar calculations, one can show in this case $E_{-+}(z)$ takes the form

$$\begin{aligned} & E_{-+}(z) \\ &= -\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - z^{\frac{1}{2}} \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm} \end{pmatrix} + O(|z|^{\frac{1}{2}+\epsilon}). \end{aligned}$$

The characterization of resonant state ensures that $b_{m1} \neq 0$. Therefore one can explicitly calculate the leading term of the asymptotic expansion of $(H-z)^{-1}$ for z near 0 in the case zero is a resonance but not an eigenvalue.

In [1], Maha Aafarani applies the methods developed here to non-selfadjoint Schrödinger operators with rapidly decreasing potential and establishes large-time asymptotics of solutions in cases zero may be a resonance and/or an eigenvalue with arbitrary geometric multiplicity.

Let us end this article with an example of non-selfadjoint operator whose threshold eigenvalue is geometrically simple and condition (5.142) is satisfied.

Example 5.5. *Consider the non-selfadjoint Witten Laplacian*

$$-\Delta_U = -\Delta + (\nabla U)(x) \cdot (\nabla U)(x) - \Delta U(x)$$

where $U \in C^2(\mathbb{R}^n; \mathbb{C})$. Set $U(x) = U_1(x) + iU_2(x)$ with U_1, U_2 real valued functions. Assume that U_1 satisfies the condition (5.76) with U replaced by U_1 and that U_2 is of compact support with $\|\partial_x^\alpha U_2\|_{L^\infty}$ sufficiently small for $|\alpha| \leq 2$. Considering $-\Delta_U$ as a perturbation of $-\Delta_{U_1}$, one can show that $-\Delta_U$ has only one eigenvalue in a neighborhood of zero which is in addition geometrically simple. Therefore the eigenfunctions associated threshold eigenvalue of $-\Delta_U$

are of the form $ce^{-U(x)}$ *for some* $c \neq 0$ *and one sees that the condition (5.142) is satisfied if* $\|U_2\|_{L^\infty}$ *is sufficiently small.*

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