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GEVREY ESTIMATES OF THE RESOLVENT AND SUB-EXPONENTIAL TIME-DECAY OF SOLUTIONS

XUE PING WANG

ABSTRACT. In this article, we study a class of non-selfadjoint Schrödinger operators H which are perturbation of some model operator H_0 satisfying a weighted coercive assumption. For the model operator H_0 , we prove that the derivatives of the resolvent satisfy some Gevrey estimates at threshold zero. As application, we establish large time expansions of semigroups e^{-tH} and e^{-itH} for $t > 0$ with subexponential time-decay estimates on the remainder, including possible presence of zero eigenvalue and real resonances.

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1. INTRODUCTION

This work is concerned with time-decay of semigroups e^{-tH} and e^{-itH} as $t \rightarrow +\infty$ where $H = -\Delta + V(x)$ is a compactly supported perturbation of some model operator $H_0 = -\Delta + V_0(x)$ with a complex-valued potential $V_0(x) = V_1(x) - iV_2(x)$ verifying a weighted coercive condition (see (2.3)). Large time behavior of solutions is closely related to low-energy spectral properties of operators. There are many works on low-energy spectral analysis and large time asymptotics for selfadjoint Schrödinger operators $-\Delta + V(x)$ with a real-valued decreasing potential $V(x)$ verifying

$$|V(x)| \leq C\langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^n, \quad (1.1)$$

for some $\rho > 0$, where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Here we only mention [3, 12] for quickly decaying potentials ($\rho > 2$), [24] for critically decaying potentials ($\rho = 2$) under an assumption of Hardy inequality for the model operator and [23] in one-dimensional case when this Hardy condition is not satisfied. For slowly decreasing potentials ($0 < \rho < 2$), there are works of [6] when the potential is negative and [17, 28, 29] when it is globally positive. When $\rho \geq 2$, threshold zero may be an eigenvalue and/or a resonance and for critically decaying potentials, threshold resonance may appear in any space dimension with arbitrary multiplicity. For slowly decreasing potentials ($0 < \rho < 2$), threshold resonance is absent and low-energy spectral analysis has not yet been done in presence of zero eigenvalue. For non-selfadjoint Schrödinger operators, we can mention works [20, 13] on the limiting absorption principle- and [7, 25] on dispersive estimates. In [7], absence of real resonances is assumed and in [25] only dissipative operators are considered. In the later case, positive resonances may exist but outgoing positive resonances (see Definition 2.4) are absent due to the dissipative condition on potential.

Known results related to the topic studied in this work concern mainly selfadjoint Schrödinger operators with globally positive and slowly decreasing potentials. Let $H_0 = -\Delta + V_0(x)$ with a real positive potential V_0 satisfying for some constants $\mu \in]0, 1[$ and $c_1, c_2 > 0$

$$c_1\langle x \rangle^{-2\mu} \leq V_0(x) \leq c_2\langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n. \quad (1.2)$$

Under some additional conditions on V_0 , it is known ([17, 28, 29])) that the spectral measure $E'_0(\lambda)$ of H_0 is smooth at $\lambda = 0$ and satisfies for any $N \geq 0$

$$\|E'_0(\lambda)\|_{L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}} = O(|\lambda|^N), \quad \lambda \rightarrow 0, \quad (1.3)$$

$$\|e^{-tH_0}\|_{L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}} = O(e^{-ct^\beta}) \quad (1.4)$$

where $\beta = \frac{1-\mu}{1+\mu}$ and c is some positive constant. In one dimensional case, if $V_0(x)$ is in addition analytic, D. Yafaev ([28]) proves that

$$\|e^{-itH_0}\|_{L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}} = O(e^{-c|t|^\beta}), \quad |t| \rightarrow +\infty. \quad (1.5)$$

The proof given in [28] is based on explicit construction of solutions to Schrödinger equation in one dimensional case which is not available in higher dimensions. Another related topic is return to equilibrium of Fokker-Planck operator with a positive, sub-linearly increasing potential. In [26], it is conjectured that the convergence rate in this situation should be subexponential in time. We learned from F. Bolley that there

exists probabilistic approaches to this problem. See the lecture notes of P. Cattiaux [4] for an overview. While polynomially decaying remainder estimate is proved in [5], the subexponential remainder estimate (2.6) is proved in a recent work of T. Li and Z. Zhang ([16]). Note that M. Klein and J. Rama ([15]) also studied Gevrey estimates in a different context to analyze large time evolution of quantum resonance states.

In this article we are mainly interested in non-selfadjoint Schrödinger operators $H = -\Delta + V(x)$, although Gevrey estimates of the model resolvent at threshold are proved for a class of second order elliptic operators. Typically the potential in the model operator $H_0 = -\Delta + V_0(x)$ is of the form: $V_0(x) = V_1(x) - iV_2(x)$ with $V_1, V_2 \geq 0$ verifying

$$c\langle x \rangle^{-2\mu} \leq V_1(x) + V_2(x) \leq C\langle x \rangle^{-2\mu}, x \in \mathbb{R}^n, \quad (1.6)$$

for some constants $c, C > 0$ and $\mu \in]0, 1[$. We assume throughout this paper that

$$W = V - V_0 \text{ is a bounded, compactly supported function.} \quad (1.7)$$

If V_0 belongs to the class of potentials \mathcal{A} (see Definition 2.3), one of the results proved in this work for the Schrödinger semigroup e^{-itH} is an asymptotic expansion of the form

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \overline{\mathbb{C}}_+} e^{-itH} \Pi_\lambda - \Pi_0(t) - \sum_{\nu \in r_+(H)} e^{-it\nu} P_\nu(t))\chi\|_{\mathcal{L}(L^2)} \leq C_\chi e^{-ct^{\frac{1-\mu}{1+\mu}}} \quad t > 0. \quad (1.8)$$

Here $\chi \in C_0^\infty(\mathbb{R}^n)$, $c > 0$ is independent of χ , $r_+(H)$ is the set of outgoing positive resonances of H (see Definition 2.4), Π_λ is the Riesz projection of H associated with λ , $\Pi_0(t)$ and $P_\nu(t)$ are operators of finite rank depending polynomially on t . See Theorem 2.4 for more precision on conditions and results. Note that if $\text{Im } V \leq 0$, then $r_+(H) = \emptyset$ and that there exists $V \in C_0^\infty(\mathbb{R}^n)$ with $\text{Im } V \geq 0$ such that $r_+(-\Delta + V) \neq \emptyset$ (see [25] for an example of incoming positive resonance with $V \in C_0^\infty$ and $\text{Im } V \leq 0$).

To prove (1.8), we use analytic deformation of H outside some sufficiently large ball in \mathbb{R}^n and prove the existence of a curve Γ in the lower half complex plane, intersecting the real axis only at point 0, such that the above this curve, the meromorphic extension of cut-off resolvent $\chi R(z)\chi$ from \mathbb{C}_+ has at most a finite number of poles and those located in $]0, +\infty[$ are precisely outgoing positive resonances. In particular, zero is not an accumulation point of quantum resonances of H located in that region. Under some conditions, we compute the resolvent expansion at threshold in presence of zero eigenvalue and prove the Gevrey estimates for the remainder. Then (1.8) is deduced by representing $\chi e^{-itH} \chi$ as sum of some residue terms and a Cauchy integral of the cut-off resolvent on Γ . The subexponential time-decay estimates is obtained from the Gevrey estimates on the remainder by expanding it at threshold up to some order N appropriately chosen according to $t > 0$.

Real resonances, called spectral singularity by J. Schwartz in [21] in more general framework, are the main obstacle to study spectral properties of non-selfadjoint Schrödinger operators near positive real half-axis. Up to now, one only knows that real resonances form bounded set with Lebesgue measure zero ([20, 21]). To study spectral

properties of non-selfadjoint Schrödinger operators near positive real half-axis, one usually either supposes the absence of real resonances or uses some kind of exponential-type decay on potentials. In this work we prove for some classes of potentials with analyticity condition, real resonances are at most a countable set with zero as the only possible accumulation point. If in addition the weighted coercive condition is satisfied, then outgoing positive resonances are at most a finite set. Each outgoing positive resonance is a pole of some meromorphic extension of the resolvent from the upper half complex plan, hence contributes to large time asymptotics of solutions as $t \rightarrow +\infty$.

A technical task in this work is Gevrey estimates on various remainders at threshold zero. To establish Gevrey estimates on the resolvent of model operator H_0 at threshold, we prove firstly an energy estimate uniformly on some parameter $s \in \mathbb{R}$. This kind of estimate for fixed s is already proved by D. Yafaev in [29]. The uniformity on $s \in \mathbb{R}$ is crucial for us, because it allows to control norms of the resolvent in weighted spaces with respect to several parameters (Theorem 3.4), from which we deduce Gevrey estimates on the model resolvent at threshold (Theorem 2.1). To estimate remainders in the asymptotic expansions of $(H - z)^{-1}$ near 0, we make use of Theorem 2.1 for the model operator and operations of operator-valued functions in Gevrey classes.

The organisation of this paper is as follows. In Section 2, we introduce conditions and definitions used and state main results obtained in this work. Sections 3 and 4 are devoted to the analysis of the model operator $H_0 = -\Delta + V_0(x)$ verifying the weighted coercive condition (2.3). In Section 3, we prove Gevrey estimates of the model resolvent at threshold (Theorem 2.1). We firstly establish a uniform energy estimate which allows to control the growth of powers of the resolvent at threshold in weighted spaces. Then Theorem 2.1 is deduced by an appropriate induction. In Section 4, we begin with evaluating the numerical range of H_0 and prove resolvent estimates for H_0 on the left of a curve located in the right half complex-plane. An estimate like (2.18) is proved for H_0 . We prove the absence of complex eigenvalues in some domaine near zero for a class of model operators H_0 . The subexponential time-decay of e^{-itH_0} is studied in subsection 4.3 when potential V_0 belongs to some analytic class \mathcal{A} introduced in Section 2. We show that there exists a contour located in the lower half complex plane passing by 0 on which the cut-off resolvent $\chi(H_0 - z)^{-1}\chi$ is uniformly bounded and that there are no quantum resonances and eigenvalues of H_0 in a sector below the positive half real axis. (2.20) for H_0 is then obtained by deforming the integral contour into the lower half complex plane. Compactly supported perturbations of the model operator H_0 are studied in Sections 5 and 6. In Section 5, we analyze properties of real resonances, study the low-energy resolvent expansion for H and prove Theorems 2.2 and 2.3. Since the method of low-energy spectral analysis used in the proof of Theorem 2.3 is well known for selfadjoint operators, we only emphasize upon Gevrey estimates on remainders. Subexponential time-decay estimates in Theorems 2.2 and 2.3 are derived from low-energy resolvent expansion by the same method as that used for H_0 . Finally in Section 6, we study the case of threshold eigenvalue for non-selfadjoint Schrödinger operators and prove Theorem 2.4. In order to obtain more precisions when zero eigenvalue is geometrically simple, We firstly establish a representation formula for the Riesz projection π_1 associated to the compact operator G_0W with eigenvalue -1 and then use

Grushin method to compute the leading term of the resolvent. The Gevrey estimates on remainders can be proved as in Section 5 and hence the details are omitted in Section 6. Results related to the model operator H_0 are announced in [27].

Notation. We denote $H^{r,s}$, $r \geq 0, s \in \mathbb{R}$ the weighted Sobolev space of order r with the weight $\langle x \rangle^s$ on \mathbb{R}^n :

$$H^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^n); \|u\|_{r,s} = \|\langle x \rangle^s (1 - \Delta)^{\frac{r}{2}} u\|_{L^2} < \infty\}.$$

For $r < 0$, $H^{r,s}$ is defined as dual space of $H^{-r,-s}$ with dual product identified with the scalar product $\langle \cdot, \cdot \rangle$ of $L^2(\mathbb{R}^n)$. Denote $H^{0,s} = L^{2,s}$. $\mathcal{L}(r, s; r', s')$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. If $(r, s) = (r', s')$, we denote $\mathcal{L}(r, s) = \mathcal{L}(r, s; r', s')$. Unless mentioned explicitly, $\|\cdot\|$ denotes norm in $L^2(\mathbb{R}^n)$ or in $\mathcal{L}(L^2)$ when no confusion is possible. \mathbb{C}_\pm denote respectively the upper and the lower open half-complex plane and $\overline{\mathbb{C}_\pm}$ their closure.

2. STATEMENT OF RESULTS

We shall prove Gevrey estimates of the resolvent at threshold for a class second order elliptic operators satisfying a weighted coercive condition. Let

$$H_0 = - \sum_{i,j=1}^n \partial_{x_i} a^{ij}(x) \partial_{x_j} + \sum_{j=1}^n b_j(x) \partial_{x_j} + V_0(x), \quad (2.1)$$

where $a^{ij}(x)$, $b_j(x)$ and $V_0(x)$ are complex-valued measurable functions. Suppose that $a^{ij}, b_j \in C_b^1(\mathbb{R}^n)$ and that there exists $c > 0$ such that

$$\operatorname{Re}(a^{ij}(x)) \geq cI_n, \quad \forall x \in \mathbb{R}^n. \quad (2.2)$$

Assume that V_0 is relatively bounded with respect to $-\Delta$ with relative bound zero, $\operatorname{Re} H_0 \geq 0$ and that there exists some constants $0 < \mu < 1$ and $c_0 > 0$ such that

$$|\langle H_0 u, u \rangle| \geq c_0 (\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2), \quad \text{for all } u \in H^2, \quad (2.3)$$

$$\sup_x |\langle x \rangle^\mu b_j(x)| < \infty, \quad j = 1, \dots, n. \quad (2.4)$$

Condition (2.3) is called weighted coercive condition.

Remark 2.1. If $H_0 = -\Delta + V_0(x)$ with $V_0(x) = V_1(x) - iV_2(x)$ with V_j real. Assume that $-\alpha\Delta + V_1(x) \geq v_1(x) \geq 0$ for some $\alpha \in]0, 1[$. If $V_2 \geq 0$ is such that for some $c > 0$

$$v_1(x) + V_2(x) \geq c\langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n. \quad (2.5)$$

then the weighted coercive condition (2.3) is satisfied. If $V_1(x)$ is globally positive and slowly decaying (i. e. $V_1(x) \geq c\langle x \rangle^{-2\mu}$ for some $\mu \in]0, 1[$ and $c > 0$), then (2.3) is satisfied by $H_0 = -\Delta + V_1(x) - iV_2(x)$ for any real function V_2 which is $-\Delta$ -bounded with relative bound zero.

Note that when we study Schrödinger operators $H_0 = -\Delta + V_0$ by technics of analytic dilation or analytic deformation, if H_0 verifies (2.3), the analytically dilated or distorted operators obtained from H_0 are of the form (2.1) and still satisfy (2.3) if the dilation or distortion parameter is small. The condition that $V_0(x)$ is $-\Delta$ -bounded with relative bound zero allows to include a class of N -body potentials.

Under the assumptions 2.2, 2.3 and 2.4, one can show that H_0 is bijective from $D(H_0) = H^2(\mathbb{R}^n)$ to $R(H_0)$ and $R(H_0)$ is dense in $L^2(\mathbb{R}^n)$. Let $G_0 : R(H_0) \rightarrow D(H_0)$ be the algebraic inverse of H_0 . Denote $L^{2,s} = L^2(\mathbb{R}_x^n; \langle x \rangle^{2s} dx)$ and $\mathcal{D} = \cap_{s \in \mathbb{R}} L^{2,s}$. Then $G_0(\mathcal{D}) \subset \mathcal{D}$ and G_0 is a densely defined, continuous from $R(H_0) \cap L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$. (See Lemma 3.3). To simplify notation, we still denote by G_0 its continuous extension by density so that G_0 is regarded as a bounded operator from $L^{2,s}$ to $L^{2,s-2\mu}$. Consequently for any $N \in \mathbb{N}$, $G_0^N : L^{2,s} \rightarrow L^{2,s-2\mu N}$ is well defined for any $s \in \mathbb{R}$. Let $R_0(z) = (H_0 - z)^{-1}$ for $z \notin \sigma(H)$. Since $\operatorname{Re} H_0 \geq 0$ on L^2 , one has

$$\text{s-} \lim_{z \in \Omega(\delta), z \rightarrow 0} R_0(z) = G_0$$

as operators from $L^{2,s}$ to $L^{2,s-2\mu}$, where $\Omega(\delta) = \{z; \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta\}$ for some $\delta > 0$.

Theorem 2.1. *Assume the conditions (2.1)-(2.4). The following estimates hold.*

(a). *For any $a > 0$, there exists $C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} G_0^N\| + \|G_0^N e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a^N N^{\gamma N}, \forall N. \quad (2.6)$$

(b). *There exists some constant $C > 0$ such that $\forall \chi \in C_0^\infty(\mathbb{R}^n)$, one has for some $C_\chi > 0$*

$$\|\chi(x) G_0^N\| + \|G_0^N \chi(x)\| \leq C_\chi C^N N^{\gamma N}, \forall N. \quad (2.7)$$

Here $\gamma = \frac{2\mu}{1-\mu}$.

Since one has at least formally

$$\frac{d^N}{dz^N} R_0(z)|_{z=0} = N! G_0^{N+1},$$

Theorem 2.1 says that derivatives of the resolvent of H_0 at threshold satisfies the Gevrey estimates of order $\sigma = 1 + \gamma$. To study large time behavior of semigroups, we introduce two classes of potentials \mathcal{V} and \mathcal{A} .

Definition 2.2. *Denote \mathcal{V} the class of complex-valued potentials V_0 such that*

$$V_0 \text{ is } -\Delta\text{-compact and (2.3) is satisfied for some } \mu \in]0, 1[. \quad (2.8)$$

and

$$\operatorname{Re} H_0 \geq -\alpha \Delta \text{ and } |\operatorname{Im} V_0(x)| \leq C \langle x \rangle^{-2\mu'} \quad (2.9)$$

for some constants $\alpha, \mu', C > 0$.

Results for the heat semigroup e^{-tH} will be proved for model potentials $V_0 \in \mathcal{V}$. To study the Schrödinger semigroup e^{-itH} we will use both technics of analytical dilation and analytical deformation, hence need the analyticity of potentials.

Definition 2.3. Let \mathcal{A} denote the class of complex-valued potentials $V_0(x) = V_1(x) - iV_2(x)$ for $x \in \mathbb{R}^n$ with $n \geq 2$ such that $V_0 \in \mathcal{V}$ satisfies the estimate (2.3) for some $\mu \in]0, 1[$ verifying

$$0 < \mu < \frac{3}{4} \text{ if } n = 2 \text{ and } 0 < \mu < 1 \text{ if } n \geq 3; \quad (2.10)$$

and that V_1 and V_2 are dilation analytic ([2]) and extend holomorphically into a complex region of the form

$$\Omega = \{x \in \mathbb{C}^n; |\operatorname{Im} x| < c|\operatorname{Re} x|\} \cup \{x \in \mathbb{C}^n; |x| > c^{-1}\}$$

for some $c > 0$ and satisfy for some $c_j > 0$ and $R \in [0, +\infty]$

$$|V_j(x)| \leq c_1 \langle \operatorname{Re} x \rangle^{-2\mu}, \quad x \in \Omega, \quad j = 1, 2, \quad (2.11)$$

$$V_2(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \quad (2.12)$$

$$x \cdot \nabla V_1(x) \leq -c_3 \frac{x^2}{\langle x \rangle^{2\mu+2}}, \quad x \in \mathbb{R}^n \text{ with } |x| \geq R, \text{ and} \quad (2.13)$$

$$V_2(x) \geq c_5 \langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n \text{ with } |x| < R. \quad (2.14)$$

Remark that when $R = 0$, (2.13) is a global virial condition on V_1 and (2.14) is void; while if $R = +\infty$, no virial condition is needed on V_1 , but (2.14) is required on the whole space which means that the dissipation is strong. Potentials of the form

$$V_0(x) = \frac{c}{\langle x \rangle^{2\mu}} - iV_2(x) \quad (2.15)$$

satisfy conditions (2.11)-(2.14) with $R = 0$, if $V_2 \geq 0$ and V_2 is holomorphic in Ω satisfying $|V_2(x)| \leq C \langle x \rangle^{-2\mu}$ for $x \in \Omega$. Conditions (2.13) and (2.14) are used to prove the non-accumulation of quantum resonances towards zero in some sector.

For $V_0 \in \mathcal{A}$, one can study quantum resonances of $H_0 = -\Delta + V_0(x)$ by both analytical dilation or analytical deformation outside some compact ([2, 10, 22]). We shall show that under the conditions (2.12), (2.13) and (2.14), there are no quantum resonances of H_0 in a sector below the positive real half-axis in complex plane.

Let $V_0 \in \mathcal{V}$ and $H_0 = -\Delta + V_0(x)$. Let $H = H_0 + W(x)$ be a compactly supported perturbation of H_0 : $W \in L_{\text{comp}}^\infty = \{u \in L^\infty(\mathbb{R}^n), \text{supp } u \text{ compact}\}$. It will be proved in Section 6 that H has only at most a finite number of discrete eigenvalues located on the left of a curve Γ of the forme

$$\Gamma = \{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\}$$

for some constants $C, \mu' > 0$ and that there exists a nice bound for the resolvent of H_0 on Γ . Note that zero may be an embedded eigenvalue, but it is never a resonance of H , i. e., if $u \in L^2(\mathbb{R}^n; \langle x \rangle^{2s} dx) \cap H_{\text{loc}}^2(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ such that $Hu = 0$, then one can show that $u \in H^2(\mathbb{R}^n)$. Complex eigenvalues of H may accumulate to zero from the right side of Γ . Let $\sigma_d(H)$ ($\sigma_p(H)$, resp.) denote the set of discrete eigenvalues of H (the set of eigenvalues of H , resp.).

More subtle is the role of real resonances. Recall that if V is of short-range, $\lambda > 0$ is called resonance of $H = -\Delta + V(x)$ if the equation $Hu = \lambda u$ admits a non-trivial

solution $u \in H_{\text{loc}}^2(\mathbb{R}^n)$ satisfying one of Sommerfeld radiation conditions:

$$u(x) = \frac{e^{\pm i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}} (a_{\pm}(\omega) + o(1)), \quad |x| \rightarrow \infty, \quad (2.16)$$

for some $a_{\pm} \in L^2(\mathbb{S}^{n-1})$, $a_{\pm} \neq 0$. λ is called an outgoing (resp., incoming) positive resonance of H if u verifies (2.16) with sign $+$ (resp. with sign $-$). It is known that if V is real, then positives resonances are absent ([1]) and if $\text{Im } V \leq 0$, outgoing resonances are absent ([18]). In this paper, we use a slight different definition for outgoing resonances. Let U_0 be a complex valued function such that $(x \cdot \nabla_x)^j U_0$, $j = 0, 1, 2$, are $-\Delta$ -compact and $\text{Im } U_0 \leq 0$. Then for any $\lambda > 0$ the boundary value of the resolvent

$$(-\Delta + U_0 - (-\lambda + i0))^{-1} = \lim_{z \rightarrow \lambda, \text{Im } z > 0} (-\Delta + U_0 - (z))^{-1} \quad (2.17)$$

exists in $\mathcal{L}(0, s; 0, -s)$ for any $s > \frac{1}{2}$ and is Hölder-continuous for $\lambda > 0$. See [19].

Definition 2.4. Let $U(x)$ be a Lebesgue measurable function such that $U(x) - U_0(x)$ is bounded and of short-range on \mathbb{R}^n . $\lambda > 0$ is called outgoing resonance of $-\Delta + U$ if -1 is an eigenvalue of the compact operator $(-\Delta + U_0 - (-\lambda + i0))^{-1}(U - U_0)$ in $L^{2,-s}$ for $s > \frac{1}{2}$ and sufficiently close to $\frac{1}{2}$. Denote $r_+(-\Delta + U)$ the set of outgoing resonances of H . For $\lambda \in r_+(H)$, define $m_+(\lambda)$ as the algebraic multiplicity of eigenvalue -1 of $(-\Delta + U_0 - (-\lambda + i0))^{-1}(U - U_0)$. Similarly if $\text{Im } U_0 \geq 0$, one can define the set of incoming positive resonances $r_-(-\Delta + U)$ and $m_-(\lambda)$ for $\lambda \in r_-(-\Delta + U)$.

If U is of short-range, then our definition coincides with the usual one. In the case that zero is not an eigenvalue of non-selfadjoint Schrödinger operator $H = H_0 + W(x)$, we prove the following

Theorem 2.2. Assume that zero is not an eigenvalue of H .

(a). Let $V_0 \in \mathcal{V}$. For any $a > 0$ there exist $c_a, C_a > 0$ such that

$$\|e^{-a(x)^{1-\mu}} (e^{-tH} - \sum_{\lambda \in \sigma_d(H), \text{Re } \lambda \leq 0} e^{-tH} \Pi_\lambda)\| \leq C_a e^{-c_a t^\beta} \quad t > 0, \quad (2.18)$$

where

$$\beta = \frac{1 - \mu}{1 + \mu}. \quad (2.19)$$

(b). Let $V_0 \in \mathcal{A}$. Then the set of outgoing resonances $r_+(H)$ of H is at most finite. There exists some constant $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ one has

$$\|\chi (e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \bar{\mathbb{C}}_+} e^{-itH} \Pi_\lambda - \sum_{\nu \in r_+(H)} e^{-it\nu} P_\nu(t))\chi\| \leq C_\chi e^{-c t^\beta} \quad t > 0, \quad (2.20)$$

Here Π_λ denotes the Riesz projection associated with the discrete eigenvalue λ of H and $P_\nu(t)$ is an operator depending polynomially on t with coefficients of rank not exceeding $m_+(\nu)$.

Consider now the case that zero is an eigenvalue of H . If H is selfadjoint, H has only a finite number of negative eigenvalues and both positive eigenvalues and positive resonances are absent. We can apply the known method in threshold spectral analysis

for selfadjoint operators to compute low-energy expansion of the resolvent. Theorem 2.1 allows to estimate remainders in Gevrey spaces and to prove the following

Theorem 2.3. *Assume that zero is an eigenvalue of H and that both H and H_0 are selfadjoint.*

(a). *If $V_0 \in \mathcal{V}$, then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}}(e^{-tH} - \sum_{\lambda \in \sigma_p(H)} e^{-t\lambda} \Pi_\lambda)\| \leq C_a e^{-c_a t^\beta} \quad t > 0, \quad (2.21)$$

(b). *Let $V_0 \in \mathcal{A}$. Then there exists some constant $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has*

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_p(H)} e^{-it\lambda} \Pi_\lambda)\chi\| \leq C_\chi e^{-c t^\beta} \quad t > 0, \quad (2.22)$$

Here Π_λ denotes the orthogonal eigenprojection of H associated with eigenvalue λ of H .

Theorem 2.3 can be applied to a class of Witten Laplacians for which zero is an eigenvalue embedded in the continuous spectrum which is equal to $[0, +\infty[$. Our result is new concerning the Schrödinger semigroup e^{-itH} in presence of zero eigenvalue. For the heat semigroup, there are results obtained by method of Markov processes ([4, 5]).

The case of zero eigenvalue in non-selfadjoint case is more difficult. There does not yet exist general method to treat this kind of problem. In our case G_0W is compact on $L^2(\mathbb{R}^n)$ and zero is an eigenvalue of H if and only if -1 is an eigenvalue of G_0W and zero eigenvalue of H , if it does exist, is of finite geometrical multiplicity. Let m denote the algebraic multiplicity of eigenvalue -1 of G_0W . Then one can show that there exists some numerical Gevrey function $\omega(z)$ such that for z near zero, $z \in \sigma_d(H)$ if and only if $\omega(z) = 0$. (See Proposition 6.1). In addition, $\omega(z)$ admits an asymptotic expansion of any order in powers of z : there exist some constants $\omega_j \in \mathbb{C}$, $j \in \mathbb{N}$, such that

$$\omega(z) = \omega_1 z + \cdots + \omega_N z^N + O(|z|^{N+1}), \quad (2.23)$$

for z near 0 and $\operatorname{Re} z < 0$ and for any $N \in \mathbb{N}^*$.

Theorem 2.4. *1. Assume that zero is an eigenvalue of H and that there exists some $\omega_k \neq 0$ such that*

$$\omega(z) = \omega_k z^k + O(|z|^{k+1}), \quad (2.24)$$

for z near 0 and $\operatorname{Re} z < 0$. Then the following results hold.

1a. *If $V_0 \in \mathcal{V}$, then for any $a > 0$, there exists some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}}(e^{-tH} - \sum_{\lambda \in \sigma_d(H), \operatorname{Re} \lambda \leq 0} e^{-t\lambda} \Pi_\lambda - \Pi_0(t))\| \leq C_a e^{-c_a t^\beta} \quad t > 0, \quad (2.25)$$

1b. Let $V_0 \in \mathcal{A}$. Then the set of outgoing resonances $r_+(H)$ of H is at most finite and there exist $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$,

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \bar{\mathbb{C}}_+} e^{-itH} \Pi_\lambda - \Pi_0(t) - \sum_{\nu \in r_+(H)} e^{-it\nu} P_\nu(t))\chi\| \leq C_\chi e^{-ct^\beta} \quad t > 0, \quad (2.26)$$

Here Π_λ and $P_\nu(t)$ have the same meaning as in Theorem 2.2 (b) and $\Pi_0(t)$ is polynomial in t of the form

$$\Pi_0(t) = \sum_{j=0}^{k-1} t^j \Pi_{0,j} \quad (2.27)$$

where $\Pi_{0,j}$, $0 \leq j \leq k-1$ is an operator of rank not exceeding m , m being the algebraic multiplicity of -1 as eigenvalue of $G_0 W$.

2. Assume that zero eigenvalue of H is geometrically simple.

2a. If (2.24) is satisfied, then $\Pi_{0,k-1}$ is of rank one, given by

$$\Pi_{0,k-1} = \langle \cdot, J\psi_0 \rangle \psi_0 \quad (2.28)$$

for some eigenfunction ψ_0 associated with zero eigenvalue of H . Here J is the complex conjugation $J : f(x) \rightarrow \overline{f(x)}$.

2b. If there exists an associated eigenfunction φ_0 such that

$$\int_{\mathbb{R}^n} (\varphi_0(x))^2 dx = 1, \quad (2.29)$$

then Condition (2.24) is satisfied with $k = 1$ and one has

$$\Pi_0(t) = \Pi_{0,0} = \langle \cdot, J\varphi_0 \rangle \varphi_0. \quad (2.30)$$

Condition (2.24) is similar to that used in [21] to study spectral properties of some non-selfadjoint operators in presence of spectral singularities. This kind of conditions can be checked at positive resonances under some analyticity assumptions (see Remark 6.1. But it is not clear if it can be satisfied at threshold even if the potential is analytic. The assumption that zero is a geometrically simple eigenvalue of H implies that there is only one Jordan block of the compact operator $G_0 W$ associated with eigenvalue -1 . This allows to construct explicitly a representation of the associated Riesz projection and to compute the leading term. The method developed in the proof of Theorem 2.4 is general and applies to some other situations. See Remarks 6.1 and 6.2. In Section 6, we give an example such that (5.82) is satisfied.

3. GEVREY ESTIMATES OF THE MODEL RESOLVENT AT THRESHOLD

The starting point of our Gevrey estimates of the resolvent of H_0 is a uniform *a priori* energy estimate for the model operator H_0 . In the sequel, we need to apply this kind of energy estimates to the Schrödinger operator $-\Delta + V_1(x) - iV_2(x)$ and to its analytically

dilated or distorted versions as well. For this purpose, we begin with a slightly more general setting where H_0 is a second order elliptic differential operator of the form

$$H_0 = - \sum_{i,j=1}^n \partial_{x_i} a^{ij}(x) \partial_{x_j} + \sum_{j=1}^n b_j(x) \partial_{x_j} + V_0(x) \quad (3.1)$$

satisfying conditions (2.2), (2.3) and (2.4).

Denote $b = (b_1, \dots, b_n)$ and

$$|a|_\infty = \max_{1 \leq i,j \leq n} \sup_{x \in \mathbb{R}^n} |a^{ij}(x)|, \quad |b|_{\mu,\infty} = \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |\langle x \rangle^\mu b_j(x)|. \quad (3.2)$$

For $s \in \mathbb{R}$, denote

$$\varphi_s(x) = (1 + \frac{|x|^2}{R_s^2})^s, \quad (3.3)$$

where $R_s = M \langle s \rangle^{\frac{1}{1-\mu}}$ with $M = M(c_0, |a|_\infty, |b|_\infty) > 1$ large enough, but independent of $s \in \mathbb{R}$. The uniformity in $s \in \mathbb{R}$ in the following lemma is important for Gevrey estimates of the model resolvent at threshold.

3.1. A uniform energy estimate.

Lemma 3.1. *Let H_0 be given by (3.1). Under the conditions (2.2), (2.3) and (2.4) with $0 < \mu < 1$, there exist some constants $C, M > 0$ depending only on $|a|_\infty, |b|_{\mu,\infty}$ and c_0 given in (2.3) such that*

$$\|\langle x \rangle^{-\mu} \varphi_s(x) u\| + \|\nabla(\varphi_s(x) u)\| \leq C \|\langle x \rangle^\mu \varphi_s(x) H_0 u\| \quad (3.4)$$

for any $s \in \mathbb{R}$ and $u \in H_{\text{loc}}^2(\mathbb{R}^n)$ with $\langle x \rangle^{s+\mu} H_0 u \in L^2$.

Proof. We calculate $\langle u, \varphi_s^2 H_0 u \rangle$ for $u \in C_0^\infty$:

$$\begin{aligned} & \langle u, \varphi_s^2 H_0 u \rangle \\ &= \langle \varphi_s u, H_0(\varphi_s u) \rangle + \langle \varphi_s u, [\sum_{i,j=1}^n \partial_{x_i} a^{ij} \partial_{x_j}, \varphi_s] u \rangle - \langle \varphi_s u, (b \cdot \nabla \varphi_s) u \rangle \\ &= I + II + III, \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} I &= \langle \varphi_s u, H_0(\varphi_s u) \rangle \\ II &= \langle \varphi_s u, \sum_{i,j=1}^n ((\partial_{x_i} \varphi_s) a^{ij} \partial_{x_j} u + \partial_{x_i} (a^{ij} (\partial_{x_j} \varphi_s) u)) \rangle \\ III &= -\langle \varphi_s u, (b \cdot \nabla \varphi_s) u \rangle. \end{aligned}$$

Since $\varphi_s \partial_{x_j} u = \partial_{x_j}(\varphi_s u) - (\partial_{x_j} \varphi_s) u$, one has

$$\begin{aligned} & |\langle \varphi_s u, (\partial_{x_i} \varphi_s) a^{ij} \partial_{x_j} u + \partial_{x_i} (a^{ij} (\partial_{x_j} \varphi_s) u) \rangle| \\ &= | \langle (\partial_{x_i} \varphi_s) u, a^{ij} (\partial_{x_j}(\varphi_s u) - (\partial_{x_j} \varphi_s) u) \rangle + \langle \varphi_s u, \partial_{x_i} (a^{ij} (\partial_{x_j} \varphi_s) u) \rangle | \\ &= | \langle (\partial_{x_i} \varphi_s) u, a^{ij} (\partial_{x_j}(\varphi_s u) - (\partial_{x_j} \varphi_s) u) \rangle - \langle \partial_{x_i}(\varphi_s u), a^{ij} (\partial_{x_j} \varphi_s) u \rangle | \\ &\leq |a|_\infty (\|(\partial_{x_i} \varphi_s) u\| (\|\partial_{x_j}(\varphi_s u)\| + \|(\partial_{x_j} \varphi_s) u\|) + \|\partial_{x_i}(\varphi_s u)\| \|(\partial_{x_j} \varphi_s) u\|) \end{aligned}$$

The term II in (3.5) can be bounded by

$$\begin{aligned} |II| &\leq |a|_\infty \left(\sum_{i=1}^n \|(\partial_{x_i} \varphi_s)u\| \right) \left(\sum_{j=1}^n (2\|\partial_{x_j}(\varphi_s u)\| + \|(\partial_{x_j} \varphi_s)u\|) \right) \\ &\leq n^2 |a|_\infty \|(\nabla \varphi_s)u\| (2\|\nabla(\varphi_s u)\| + \|(\nabla \varphi_s)u\|) \\ &\leq n^2 |a|_\infty (\epsilon \|\nabla(\varphi_s u)\|^2 + (1 + \frac{1}{\epsilon}) \|(\nabla \varphi_s)u\|^2) \end{aligned}$$

for any $\epsilon > 0$. Clearly, III verifies

$$|III| \leq |b|_{\mu, \infty} \|\langle x \rangle^{-\mu} \varphi_s u\| \|(\nabla \varphi_s)u\| \leq |b|_{\mu, \infty} (\epsilon \|\langle x \rangle^{-\mu} \varphi_s u\|^2 + \frac{1}{4\epsilon} \|(\nabla \varphi_s)u\|^2) \quad (3.6)$$

Taking $\epsilon = \epsilon(c_0, |a|_\infty, |b|_{\mu, \infty}) > 0$ appropriately small where $c_0 > 0$ is given by (2.3), it follows from (2.3) that

$$\begin{aligned} |\langle u, \varphi_s^2 H u \rangle| &\geq |I| - |II| - |III| \\ &\geq \frac{c_0}{2} (\|\nabla(\varphi_s u)\|^2 + \|\langle x \rangle^{-\mu} \varphi_s(x)u\|^2) - \langle u, W_s u \rangle \end{aligned} \quad (3.7)$$

where $W_s(x) = c_1 |\nabla \varphi_s|^2$ with $c_1 > 0$ some constant depending only on c_0 , $|a|_\infty$ and $|b|_{\mu, \infty}$. One can check that

$$\begin{aligned} |\nabla \varphi_s|^2 &= \frac{4s^2 x^2}{R_s^4 (1 + \frac{x^2}{R_s^2})^2} (1 + \frac{x^2}{R_s^2})^{2s} \\ &\leq \frac{4s^2 x^2}{(R_s^2 + x^2)^2} \varphi_s^2 \leq \frac{4s^2}{R_s^2 + x^2} \varphi_s^2 \end{aligned}$$

Since $R_s^2 + x^2 \geq 2^{-2\mu} R_s^{2(1-\mu)} \langle x \rangle^{2\mu}$ and $R_s = M \langle s \rangle^{\frac{1}{1-\mu}}$, $W(x)$ is bounded by

$$0 \leq W_s(x) \leq \frac{4c_1 \langle s \rangle^2}{R_s^2 + x^2} \varphi_s^2 \leq \frac{2^{2\mu} 4c_1}{M^{2(1-\mu)} \langle x \rangle^{2\mu}} \varphi_s^2. \quad (3.8)$$

Since $0 < \mu < 1$, one can choose $M = M(c_0, |a|_\infty, |b|_{\mu, \infty}) > 1$ large enough so that $\frac{2^{2\mu} 4c_1}{M^{2(1-\mu)}} < \frac{c_0}{4}$. Consequently, the above estimate combined with (3.7) gives

$$|\langle u, \varphi_s^2 H_0 u \rangle| \geq \frac{c_0}{4} (\|\nabla(\varphi_s u)\|^2 + \|\langle x \rangle^{-\mu} \varphi_s u\|^2). \quad (3.9)$$

Remark that

$$|\langle u, \varphi_s^2 H_0 u \rangle| \leq \|\langle x \rangle^{-\mu} \varphi_s u\| \|\langle x \rangle^\mu \varphi_s H_0 u\| \leq \frac{c_0}{8} \|\langle x \rangle^{-\mu} \varphi_s u\|^2 + \frac{2}{c_0} \|\langle x \rangle^\mu \varphi_s H_0 u\|^2.$$

It follows from (3.9) that

$$\|\langle x \rangle^\mu \varphi_s H_0 u\|^2 \geq \frac{c_0^2}{16} (\|\langle x \rangle^{-\mu} \varphi_s u\|^2 + \|\nabla(\varphi_s u)\|^2), \quad u \in C_0^\infty(\mathbb{R}^n). \quad (3.10)$$

By an argument of density, one obtains (3.4) with some constant $C > 0$ independent of $s \in \mathbb{R}$. ■

Corollary 3.2. *Under the conditions of Lemma 3.1, there exists some constant $C > 0$ such that for any $f \in L^{2,r}$ and $u \in H_{\text{loc}}^2$ such that $H_0 u = f$, one has: $u \in L^{2,r-2\mu}$, $\nabla u \in L^{2,r-\mu}$ and*

$$\|\langle x \rangle^{r-\mu} \nabla u\| + \|\langle x \rangle^{r-2\mu} u\| \leq C \|\langle x \rangle^r f\|. \quad (3.11)$$

Proof. It follows from Lemma 3.1 with $s = \frac{r-\mu}{2}$. \blacksquare

Lemma 3.1 shows that $H_0 : D(H_0) \rightarrow R(H_0) := \text{Range}(H_0) \subset L^2(\mathbb{R}^n)$ is bijective. Let G_0 denote its algebraic inverse with $D(G_0) = R(H_0)$. Then one has

$$H_0 G_0 = 1 \text{ on } R(H_0), \quad G_0 H_0 = 1 \text{ on } D(H_0) \quad (3.12)$$

Lemma 3.3. (a). *G_0 is a densely defined closed operator on $L^2(\mathbb{R}^n)$. If H_0 is self-adjoint (resp., maximally dissipative), then $-G_0$ is also selfadjoint (resp., maximally dissipative).*

(b). *There exists some C such that*

$$\|\nabla(\varphi_s G_0 \varphi_{-s} \langle x \rangle^{-\mu} w)\| + \|\langle x \rangle^{-\mu} \varphi_s G_0 \varphi_{-s} \langle x \rangle^{-\mu} w\| \leq C \|w\| \quad (3.13)$$

for all $w \in \mathcal{D}$ and $s \in \mathbb{R}$. Here $\mathcal{D} = \bigcap_{s \in \mathbb{R}} L^{2,s}$.

Proof. We firstly show that $D(G_0)$ is dense. Remark that $\text{Re } H_0 \geq 0$. Let $f \in \mathcal{D}$ and $u_\epsilon = (H_0 + \epsilon)^{-1} f$, $\epsilon > 0$. Since $\text{Re } H_0 \geq 0$ and H_0 verifies the weighted coercive condition (2.3), $H_0 + \epsilon$ satisfies also (2.3) with the same constant $c_0 > 9$ independent of $\epsilon > 0$. Following the proof of Lemma 3.1 with H_0 replaced by $H_0 + \epsilon$, one has that for any $s > 0$

$$\|\langle x \rangle^{s-\mu} \nabla u_\epsilon\| + \|\langle x \rangle^{s-2\mu} u_\epsilon\| \leq C_s \|\langle x \rangle^s f\|$$

uniformly in $\epsilon > 0$. For $s > 2\mu$, this estimate implies that the sequence $\{u_\epsilon; \epsilon \in]0, 1]\}$ is relatively compact in L^2 . Therefore there exists a subsequence $\{u_{\epsilon_k}; k \in \mathbb{N}\}$ and $u \in L^2$ such that $\epsilon_k \rightarrow 0$ and $u_{\epsilon_k} \rightarrow u$ in L^2 as $k \rightarrow +\infty$. It follows that $H_0 u = f$ in the sense of distributions. The ellipticity of H_0 implies that $u \in H^2(\mathbb{R}^n)$. Therefore $f \in R(H_0) = D(G_0)$. This shows that $\mathcal{D} \subset D(G_0)$. In particular $D(G_0)$ is dense in $L^{2,r}$ for any $r \in \mathbb{R}$. The closeness of G_0 follows from that of H_0 . The other assertions can be easily checked.

The argument used above shows that for any $w \in \mathcal{D}$, one can find $u \in D(H_0)$ such that $H_0 u = \varphi_{-s} \langle x \rangle^{-\mu} w$. (3.13) follows from (3.4). \blacksquare

Lemma 3.3 shows that for any s , $\langle x \rangle^{-\mu} \varphi_s G_0 \varphi_{-s} \langle x \rangle^{-\mu}$ defined on $\mathcal{D} = \bigcap_{s \in \mathbb{R}} L^{2,s}$ can be uniquely extended to a bounded operator on $L^2(\mathbb{R}^n)$, or in other words, for any $s \in \mathbb{R}$, G_0 is bounded from $D(G_0) \cap L^{2,s}$ to $L^{2,s-2\mu}$:

$$\|\langle x \rangle^{-\mu} \varphi_s G_0 u\| \leq C \|\varphi_s \langle x \rangle^\mu u\| \quad (3.14)$$

uniformly in $u \in D(G_0) \cap L^{2,s}$ and $s \in \mathbb{R}$. This implies that $G_0 \mathcal{D} \subset \mathcal{D}$ and G_0 extends to a continuous operator from $L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$. It follows that $G_0^N(\mathcal{D}) \subset \mathcal{D}$ and by an induction, one can check that G_0^N extends to a bounded operator from $L^{2,s}$ to $L^{2,s-2N\mu}$ for any $s \in \mathbb{R}$. To simplify notation, we still denote G_0 (resp., G_0^N) its continuous extension by density as operator from $L^{2,s}$ to $L^{2,s-2\mu}$ (resp., from $L^{2,s}$ to

$L^{2,s-2N\mu}$).

3.2. Gevrey estimates for the model resolvent.

Theorem 3.4. *Let $M > 1$ be given in Lemma 3.1. Denote*

$$x_{N,r} = \frac{x}{R_{N,r}} \text{ with } R_{N,r} = R_{(2N-1+r)\mu} = M\langle(2N-1+r)\mu\rangle^{\frac{1}{1-\mu}} \quad (3.15)$$

where $N \in \mathbb{N}$ and $r \in \mathbb{R}_+$ and $M > 0$ is a constant given by Lemma 3.1. Set $\langle x_{N,r} \rangle = (1 + |x_{N,r}|^2)^{\frac{1}{2}}$. Then there exists some constant $C > 0$ such that

$$\|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0^N \langle x_{N,r} \rangle^{r\mu}\| \leq C^N \langle(2N-1+r)\mu\rangle^{\gamma N}, \quad (3.16)$$

for any integer $N \geq 1$ and any $r \geq 0$. Here

$$\gamma = \frac{2\mu}{1-\mu}. \quad (3.17)$$

Proof. Making use of Lemma 3.1, one can check that operator

$$I_N = \langle x_{N,r} \rangle^{-2N\mu-r\mu} G_0^N \langle x \rangle^{r\mu} \quad (3.18)$$

is well defined on \mathcal{D} and extends to a bounded operator on L^2 . To show the estimate (3.16), we use an induction on N . Since $\langle x \rangle \leq \frac{1}{R} \langle \frac{x}{R} \rangle$ for $R \geq 1$, it follows from (3.13) that

$$\|\langle \frac{x}{R_s} \rangle^{-s-\mu} G_0 \langle \frac{x}{R_s} \rangle^{s-\mu}\| \leq C' R_s^{2\mu} \leq C_1 \langle s \rangle^\gamma \quad (3.19)$$

uniformly in s , where $R = M \langle s \rangle^{\frac{1}{1-\mu}}$. In particular, when $s = (1+r)\mu$, one has $R_s = M \langle (1+r)\mu \rangle^{\frac{1}{1-\mu}} = R_{1,r}$ and

$$\|I_1\| \leq C_1 \langle (1+r)\mu \rangle^\gamma \quad (3.20)$$

for all $r \geq 0$, which proves (3.16) when $N = 1$. Assume now that $N \geq 2$ and that one has proved for some $C > 0$ independent of N and $r \geq 0$ that

$$\|I_{N-1}\| \leq C^{N-1} \langle(2N-3+r)\mu\rangle^{\gamma(N-1)}. \quad (3.21)$$

Write I_N as

$$I_N = \langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N-1,r} \rangle^{(2N-2+r)\mu} \cdot I_{N-1} \cdot \langle x_{N-1,r} \rangle^{-r\mu} \langle x_{N,r} \rangle^{r\mu}$$

Notice that

$$\langle x_{N,r} \rangle \leq \langle x_{N-1,r} \rangle \leq \frac{R_{N,r}}{R_{N-1,r}} \langle x_{N,r} \rangle$$

for any $N \geq 2$. Applying (3.19) with $s = (2N-1+r)\mu$, one obtains

$$\|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N,r} \rangle^{(2N-2+r)\mu}\| \leq C_1 \langle(2N-1+r)\mu\rangle^\gamma.$$

Making use of the induction hypothesis, one can estimate I_N as follows:

$$\begin{aligned}
 \|I_N\| &\leq \|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N-1} \rangle^{(2N-2+r)\mu}\| \cdot \|I_{N-1}\| \\
 &\leq \|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N,r} \rangle^{(2N-2+r)\mu}\| \cdot \left\| \left(\frac{\langle x_{N-1} \rangle}{\langle x_{N,r} \rangle} \right)^{(2N-2+r)\mu} \right\| \cdot \|I_{N-1}\| \\
 &\leq C_1 \langle (2N-1+r)\mu \rangle^\gamma \cdot \left(\frac{\langle (2N-1+r)\mu \rangle}{\langle (2N-3+r)\mu \rangle} \right)^{\gamma(N-1+\frac{r}{2})} \cdot C^{N-1} \langle (2N-3+r)\mu \rangle^{\gamma(N-1)} \\
 &\leq C_1 \left(\frac{2N-1+r}{2N-3+r} \right)^{\gamma(N-1+\frac{r}{2})} C^{N-1} \langle (2N-1+r)\mu \rangle^{\gamma N}.
 \end{aligned} \tag{3.22}$$

The sequence $\left\{ \left(\frac{2m-1+r}{2m-3+r} \right)^{\gamma(m-1+\frac{r}{2})}; m \geq 2 \right\}$ is uniformly bounded in $r \geq 0$. Hence there exists some $C_2 > 0$ such that

$$C_1 \left(\frac{2m-1+r}{2m-3+r} \right)^{\gamma(m-1+\frac{r}{2})} \leq C_2$$

for any $m \geq 2$ and $r \geq 0$. Increasing the constant C if necessary, one can suppose without loss that $C_2 \leq C$ and one obtains from (3.22) that

$$\|I_N\| \leq C^N \langle (2N-1+r)\mu \rangle^{N\gamma} \tag{3.23}$$

Theorem 3.4 is proven by induction. ■

Let $R_0(z) = (H_0 - z)^{-1}$ denote the resolvent of H_0 and

$$\Omega(\delta) = \left\{ z \in \mathbb{C}^*; \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta \right\},$$

$\delta > 0$. Since $\operatorname{Re} H_0 \geq 0$, there exists some constant $C_1 > 0$ such that

$$\|R_0(z)\| \leq \frac{C_1}{|z|}, \quad z \in \Omega(\delta).$$

From the equation $R_0(z) = G_0 + zG_0 + z^2G_0^2R(z)$, it follows that as operators from $L^{2,s}$ to $L^{2,s-2\mu}$, $s \in \mathbb{R}$, one has

$$\operatorname{s-}\lim_{z \in \Omega(\delta), z \rightarrow 0} R_0(z) = G_0 \tag{3.24}$$

for any $\delta > 0$. Similarly one can check that for any $N \in \mathbb{N}^*$, one has

$$\operatorname{s-}\lim_{z \in \Omega(\delta), z \rightarrow 0} R_0(z)^N = G_0^N. \tag{3.25}$$

as operators from $L^{2,s}$ to $L^{2,s-2N\mu}$. By an abuse of notation, we denote $R(0) = G_0$. Thus $R_0(z)$ is defined for z in $\Omega(\delta) \cup \{0\}$.

Corollary 3.5. *The following Gevrey estimates of the resolvent hold.*

(a). *For any $a > 0$, there exists some constant $C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} R_0(z)^N\| + \|R_0(z)^N e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a^N N^{\gamma N} \tag{3.26}$$

for any integer $N \geq 1$ and $z \in \Omega(\delta) \cup \{0\}$.

(b). Then there exists some constant $C > 0$ such that

$$\|\chi(x)R_0(z)^N\| + \|R_0(z)^N\chi(x)\| \leq C_\chi C^N N^{\gamma N} \quad (3.27)$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $N \geq 1$ and $z \in \Omega(\delta) \cup \{0\}$. Here $\gamma = \frac{2\mu}{1-\mu}$.

Proof. Notice that $\|zR_0(z)\|$ is uniformly bounded in $\mathcal{L}(L^2)$ for $z \in \Omega(\delta)$ ($\delta > 0$ is fixed) and that

$$R_0(z)^N = G_0^N(1 + zR_0(z))^N.$$

According to Theorem 3.4 with $r = 0$, one has for some constant $C > 0$

$$\|\langle x_{N,0} \rangle^{-2N\mu} R_0(z)^N\| \leq C^N N^{\gamma N}, \quad (3.28)$$

for any integer $N \geq 1$ and $z \in \Omega(\delta) \cup \{0\}$.

Let $a > 0$. Then

$$\|e^{-a\langle x \rangle^{1-\mu}} R_0(z)^N\| \leq \|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty} C^N N^{\gamma N}.$$

To evaluate the norm $\|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty}$, consider the function

$$f(r) = e^{-ar^{1-\mu}} \left\langle \frac{r}{R_N} \right\rangle^{2N\mu},$$

where $r = |x|$ and $R_N = R_{N,0} = M\langle (2N-1)\mu \rangle^{\frac{1}{1-\mu}}$. One calculates:

$$f'(r) = \frac{f(r)}{r^\mu(R_N^2 + r^2)} (-2a(1-\mu)(R_N^2 + r^2) + 2N\mu r^{1+\mu}), r \geq 1.$$

Let $A \geq 1$. Since $R_N \sim c'N^{\frac{1}{1-\mu}}$ for some constant $c' > 0$, one can check that $Nr^{1+\mu} \leq \frac{c}{A^{1-\mu}}r^2$ if $r \geq AR_N$ for some constant $c > 0$ independent of A , r and N . Therefore, if $A = A(\mu, a) > 1$ is chosen sufficiently large, one has

$$f'(r) < 0, \quad r > AR_N,$$

thus $f(r)$ is decreasing in $[AR_N, +\infty[$. It is now clear that

$$\|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty} \leq \sup_{0 \leq r \leq AR_N} f(r) \leq \langle A \rangle^{2N\mu}$$

This proves Part (a) of Corollary with $C_a = C\langle A \rangle^{2\mu}$.

To prove Part (b), let $\chi \in C_0^\infty(\mathbb{R}^n)$. Let $R > 0$ such that $\text{supp } \chi \subset B(0, R)$. (3.28) shows that there exists some constant $C_1 > 0$ such that

$$\|\chi(x)G_0^N(1 + zR_0(z))^N\| \leq \|\langle x_{N,0} \rangle^{2\mu N} \chi\|_{L^\infty} \times C_1^N N^{\gamma N}$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $N \geq 1$ and $z \in \Omega(\epsilon) \cup \{0\}$. Then One can check that

$$\|\langle x_{N,0} \rangle^{2\mu N} \chi\|_{L^\infty} \leq \|\chi\|_{L^\infty} \left(1 + \frac{R^2}{M^2((2N-1)\mu)^{\frac{2}{1-\mu}}}\right)^{\mu N} \leq C_2 2^{\mu N}$$

for some constant C_2 depending only on χ and R , but independent of N . This proves (3.27) with $C_\chi = C_2$ and $C = C_1 2^\mu$ which is independent of χ . \blacksquare

Theorem 2.1 is a particular case of Corollary 3.5. Corollary 3.5 shows that the model resolvent $R_0(z)$ belongs to the Gevrey class of order $\sigma = 1 + \gamma$ on $\Omega(\delta) \cup \{0\}$.

4. QUANTUM DYNAMICS GENERATED BY THE MODEL OPERATOR

4.1. Subexponential time-decays of heat semigroup. Consider now the model operator of the form $H_0 = -\Delta + V_0(x)$ with $V_0(x) = V_1(x) - iV_2(x)$, $V_1(x), V_2(x)$ being real, satisfying Condition (2.3). Denote $R_0(z) = (H_0 - z)^{-1}$. Theorem 3.4 can be used to prove subexponential time-decay for local energies of solutions to the heat and Schrödinger equations. To study the heat semigroup e^{-tH_0} , $t \geq 0$, we use Cauchy integral formula for semigroups and need some information of the resolvent on a contour in the right half complex plane passing through the origin.

Proposition 4.1. *Assume that $\operatorname{Re} H_0 \geq -a\Delta$ for some $a > 0$ and that the imaginary part of the potential $V_0(x)$ verifies the estimate*

$$|V_2(x)| \leq C\langle x \rangle^{-2\mu'}, \quad \forall x \in \mathbb{R}^n, \quad (4.1)$$

for some $0 < \mu' < \min\{\frac{n}{2}, 1\}$. Then there exists some constant $C_0 > 0$ such that the numerical range $N(H_0)$ of H_0 is contained in a region of the form $\{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| \leq C_0(\operatorname{Re} z)^{\mu'}\}$. Consequently, for any $A_0 > C_0$ there exists some constant M_0 such that

$$\|R_0(z)\| \leq \frac{M_0}{|z|^{\frac{1}{\mu'}}} \quad (4.2)$$

for $z \in \Omega := \{z \in \mathbb{C}^*; |z| \leq 1, \operatorname{Re} z < 0 \text{ or } \operatorname{Re} z \geq 0, |\operatorname{Im} z| > A_0(\operatorname{Re} z)^{\mu'}\}$.

Proof. For $z = \langle u, H_0 u \rangle \in N(H_0)$ where $u \in D(H_0)$ and $\|u\| = 1$, one has

$$\begin{aligned} \operatorname{Re} z &= \operatorname{Re} \langle u, H_0 u \rangle \geq a\|\nabla u\|^2 \\ |\operatorname{Im} z| &\leq \langle u, |V_2| u \rangle \leq C\|\langle x \rangle^{-\mu'} u\|^2. \end{aligned}$$

According to the generalized Hardy inequality ([9]), for $0 < \mu' < \frac{n}{2}$ there exists some constant $C_{\mu'}$ such that

$$\|\langle x \rangle^{-\mu'} u\|^2 \leq \| |x|^{-\mu'} u \|^2 \leq C_{\mu'} \|\nabla |^{\mu'} u\|^2. \quad (4.3)$$

Let \hat{u} denote the Fourier transform of u normalized such that $\|\hat{u}\| = \|u\|$ and $\tau = \|\nabla u\|$. Then

$$\begin{aligned} \|\nabla |^{\mu'} u\|^2 &= \|\xi |^{\mu'} \hat{u}\|^2 = \|\xi |^{\mu'} \hat{u}\|_{L^2(|\xi| \geq \tau)}^2 + \|\xi |^{\mu'} \hat{u}\|_{L^2(|\xi| < \tau)}^2 \\ &\leq \tau^{2(\mu'-1)} \|\xi | \hat{u}\|_{L^2(|\xi| \geq \tau)}^2 + \tau^{2\mu'} \|\hat{u}\|_{L^2(|\xi| < \tau)}^2 \\ &\leq 2\tau^{2\mu'} = 2\|\nabla u\|^{2\mu'}. \end{aligned}$$

This proves that $\operatorname{Re} z \geq 0$ and $|\operatorname{Im} z| \leq C_0(\operatorname{Re} z)^{\mu'}$ when $z \in N(H_0)$. The other assertions of Proposition are immediate, since $\sigma(H_0) \subset \overline{N(H_0)}$ and

$$\|R_0(z)\| \leq \frac{1}{\operatorname{dist}(z, N(H_0))}.$$

■

Making use of the equation

$$R_0(z) = \sum_{j=0}^{k-1} z^j G_0^{j+1} + z^k G_0^k R_0(z) \quad (4.4)$$

we deduce from Theorem 3.4 (with $r = 0$ and $N = k$) and Proposition 4.1 the following estimate for $k \in \mathbb{N}^*$ and $k \geq \frac{1}{\mu'}$

$$\|\langle x \rangle^{-2k\mu} R_0(z)\| \leq C \quad (4.5)$$

uniformly in $z \in \Omega$ and z near 0. Notice that under the conditions of Proposition 4.1, one can not exclude possible accumulation of complex eigenvalues towards zero. Making use of Proposition 4.1, one can prove the following uniform Gevrey estimates in a domain located in the right half complex plane.

Corollary 4.2. *Under the conditions of proposition 4.1, let κ be an integer such that $\kappa + 1 \geq \frac{1}{\mu'}$. Then for any $a > 0$ there exist $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} \frac{d^{N-1}}{dz^{N-1}} R_0(z)\| \leq c_a C_a^N N^{(1+(1+\kappa)\gamma)N}, \quad \forall N \geq 1, \quad (4.6)$$

and there exists some constant $C > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has

$$\|\chi(x) \frac{d^{N-1}}{dz^{N-1}} R_0(z)\| \leq C_\chi C^N N^{(1+(1+\kappa)\gamma)N}, \quad \forall N \geq 1, \quad (4.7)$$

uniformly in $z \in \Omega$. Here Ω is defined in Proposition 4.1.

Proof. For $z \in \Omega$, decompose $R_0(z)$ into

$$R_0(z) = A(z) + G_0^{\kappa+1} B(z)$$

with $A(z) = \sum_{j=0}^{\kappa} z^j G_0^{j+1}$ and $B(z) = z^{\kappa+1} R_0(z)$. By Proposition 4.1, $\|B(z)\|$ is uniformly bounded for $z \in \Omega$. Theorem 3.4 shows that for some constant C_1

$$\|\langle x_{\kappa+1,r} \rangle^{-(2\kappa+2+r)\mu} G_0^{\kappa+1} \langle x_{\kappa+1,r} \rangle^{r\mu}\| \leq C_1 \langle (2\kappa+1+r)\mu \rangle^{\gamma(\kappa+1)}, \quad (4.8)$$

$$\|\langle x_{\kappa+1,r} \rangle^{-(2\kappa+2+r)\mu} A(z) \langle x_{\kappa+1,r} \rangle^{r\mu}\| \leq C_1 \langle (2\kappa+1+r)\mu \rangle^{\gamma(\kappa+1)} \quad (4.9)$$

for any $r \geq 0$ and $|z| \leq 1$. Making use of the relation

$$R_0(z)^N = A(z) R_0(z)^{N-1} + G_0^{\kappa+1} R_0(z)^{N-1} B(z)$$

one can show by an induction on N that there exists some constant $C > 0$ such that

$$\|\langle x_{(\kappa+1)N,0} \rangle^{-2(\kappa+1)N\mu} R_0(z)^N\| \leq C^N N^{N\gamma(1+\kappa)} \quad (4.10)$$

for any $N \geq 1$ and $z \in \Omega$. In fact, the case $N = 1$ follows from (4.8) and (4.9). If (4.10) is proven with N replaced by $N - 1$ for some $N \geq 2$, noticing that $x_{(\kappa+1)N,0} = x_{\kappa+1,2(\kappa+1)(N-1)}$, (4.8) and (4.9) with $r = 2(\kappa+1)(N-1)$ show that

$$\begin{aligned} & \|\langle x_{(\kappa+1)N,0} \rangle^{-2(\kappa+1)N\mu} R_0(z)^N\| \\ & \leq C_1 \langle (2(\kappa+1)N-1)\mu \rangle^{\gamma(\kappa+1)} (\|\langle x_{(\kappa+1)(N-1),0} \rangle^{-2(\kappa+1)(N-1)\mu} R_0(z)^{N-1}\| \\ & \quad + \|\langle x_{(\kappa+1)(N-1),0} \rangle^{-2(\kappa+1)(N-1)\mu} R_0(z)^{N-1}\| \|B(z)\|) \\ & \leq C_2 C^{N-1} N^{N\gamma(1+\kappa)} \end{aligned}$$

for some constant C_2 independent of N . Increasing the constant C if necessary, this proves (4.10) for all $N \geq 1$ by an induction. (4.6) and (4.7) are deduced from (4.10) as in the proof of Corollary 3.5. \blacksquare

Note that in the applications, we only use the Gevrey estimates at threshold. As another consequence of Proposition 4.1, we obtain the following estimate on the expansion of the resolvent at 0:

Corollary 4.3. *Under the conditions of Proposition 4.1, assume in addition (2.3) with $\mu \in]0, 1[$. Then there exists some constant $c > 0$ such that for any $z \in \Omega$ and z near 0, one has for some N (depending on z) such that*

$$\|\langle x_{N,0} \rangle^{-2N\mu} (R_0(z) - \sum_{j=0}^N z^j G_0^{j+1})\| \leq e^{-c|z|^{-\frac{1}{\gamma}}}. \quad (4.11)$$

Here $\langle x_{N,0} \rangle$ is defined in Theorem 3.4 with $r = 0$.

Proof. Theorem 3.4 and Proposition 4.1 show that for any N , one has

$$\|\langle x_{N,0} \rangle^{-2N\mu} (R_0(z) - \sum_{j=0}^N z^j G_0^{j+1})\| \leq C^N N^{\gamma N} |z|^{N+1-\frac{1}{\mu'}}, \quad (4.12)$$

for all $z \in \Omega$ and z near 0. The remainder estimate can be minimized by choosing an appropriate N in terms of $|z|$. For fixed $M' > 1$ and $z \neq 0$, take $N = \lfloor \frac{1}{(CM'|z|)^{\frac{1}{\gamma}}} \rfloor$. Then one has for z in a small neighbourhood of zero and $z \neq 0$:

$$C^N N^{\gamma N} |z|^{N+1-\frac{1}{\mu'}} \leq e^{-c_1 N \log M} \leq e^{-c_2 |z|^{-\frac{1}{\gamma}}}$$

where c_1, c_2 are some positive constants. \blacksquare

Theorem 4.4. *Let $H_0 = -\Delta + V_0(x)$ with $V_0 \in \mathcal{V}$. Then for any $a > 0$, there exist some constant $c_a, C_a > 0$ such that*

$$\|e^{-a(x)^{1-\mu}} e^{-tH_0}\| + \|e^{-tH_0} e^{-a(x)^{1-\mu}}\| \leq C_a e^{-c_a t^\beta}, \quad t > 0, \quad (4.13)$$

with β given by (2.19).

Proof. Let Γ be the contour defined by $\Gamma = \{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\}$ oriented in anti-clockwise sense, where $C > 0$ is sufficiently large. Here $\mu' > 0$ is appropriately small such that both conditions (2.9) and (4.1) are satisfied. By Proposition 4.1, the numerical range of H_0 is located on the right hand side of Γ and one has

$$e^{-tH_0} = \frac{i}{2\pi} \int_{\Gamma} e^{-tz} R_0(z) dz. \quad (4.14)$$

Decompose Γ as $\Gamma = \Gamma_0 + \Gamma_1$ where Γ_0 is the part of Γ with $0 \leq \operatorname{Re} z \leq \delta$ while Γ_1 is the part of Γ with $\operatorname{Re} z > \delta$ where $\delta > 0$ is sufficiently small. Clearly, the integral on Γ_1 is exponentially decreasing as $t \rightarrow \infty$

$$\left\| \int_{\Gamma_1} e^{-tz} R_0(z) dz \right\| \leq C e^{-\frac{t}{c}}, t > 0,$$

for some constant $C > 0$. For $z \in \Gamma_0$, denote $f_N(z) = R_0(z) - \sum_{j=0}^N z^j G_0^{j+1}$. Then

$$f_N(z) = z^{N+1} G_0^N R_0(z).$$

Then Theorem 2.1 shows that for any $a > 0$ there exist some constants $C, C_1 > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} f_N(z)\| \leq C_1 C^N |z|^{N+1-\frac{1}{\mu'}} N^{\gamma N} \quad (4.15)$$

for $z \in \Gamma_0$. It follows that

$$\begin{aligned} & \left\| \int_{\Gamma_0} e^{-tz} e^{-a\langle x \rangle^{1-\mu}} R_0(z) dz \right\| \\ & \leq \sum_{j=0}^N \|e^{-a\langle x \rangle^{1-\mu}} G_0^{j+1}\| \left| \int_{\Gamma_0} e^{-tz} z^j dz \right| + \left\| \int_{\Gamma_0} e^{-tz} e^{-a\langle x \rangle^{1-\mu}} f_N(z) dz \right\| \\ & \leq C_2 + C_2 \sum_{j=1}^N C^j j^{\gamma j} e^{-\delta t} + C_2 C^N N^{\gamma N} \int_{\Gamma_0} |e^{-tz}| |z|^{N+1-\frac{1}{\mu'}} |dz| \end{aligned}$$

for some $C_2 > 0$ and for all $t > 0$ and $N \geq 1$. Parameterizing Γ_0 by $z = \lambda \pm ic\lambda^{\frac{1}{\mu'}}$ with $\lambda \in]0, \delta]$, one can evaluate the last integral as follows:

$$\begin{aligned} \int_{\Gamma_0} |e^{-tz}| |z|^{N+1-\frac{1}{\mu'}} |dz| & \leq C_3^N \int_0^\delta e^{-t\lambda} \lambda^{N+1-\frac{1}{\mu'}} d\lambda \\ & \leq C_3^N t^{-N-2+\frac{1}{\mu'}} \int_0^{\delta t} e^{-\tau} \tau^{N+1-\frac{1}{\mu'}} d\tau \\ & \leq C_4^N t^{-N-2+\frac{1}{\mu'}} N^N \end{aligned}$$

for some $C_3, C_4 > 0$. This proves that there exist some constants B_0 and $B_1 > 0$ such that

$$\left\| \int_{\Gamma_0} e^{-tz} \langle x_N \rangle^{-2(N+1)\mu} R_0(z) dz \right\| \leq B_0 B_1^N N^{\gamma N} (N e^{-\delta t} + N^N t^{-N-2+\frac{1}{\mu'}}) \quad (4.16)$$

for any $t > 0$ and $N \geq 1$. Choosing N in terms of t such that $N \simeq \left(\frac{t}{M_1 B_1}\right)^{\frac{1}{1+\gamma}}$ as $t \rightarrow +\infty$ for some fixed appropriate constant $M_1 > 1$, one obtains that

$$\left\| \int_{\Gamma_0} e^{-tz} \langle x_N \rangle^{-2(N+1)\mu} R_0(z) dz \right\| \leq C e^{-\delta_0 t^{\frac{1}{1+\gamma}}} \quad (4.17)$$

for some $C, \delta_0 > 0$. This proves that there exist some constants $C, c > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} e^{-tH_0}\| \leq C e^{-ct^\beta}, \quad t > 0, \quad (4.18)$$

with $\beta = \frac{1}{1+\gamma} = \frac{1+\mu}{1-\mu}$. ■

As a consequence of Theorem 4.4, one obtains that there exists some constant $c > 0$ such that

$$\|e^{-tH_0} f\| \leq C_R e^{-ct^\beta} \|f\|, \quad t > 0, \quad (4.19)$$

for all $f \in L^2(\mathbb{R}^n)$ with support contained in $\{|x| \leq R\}$, $R > 0$.

4.2. An estimate on spectral measure. For the selfadjoint Schrödinger operator H_0 with a global positive and slowly decreasing potential V_0 , it is known that under some additional conditions the spectral measure $E'_0(\lambda)$ of H_0 satisfies the estimate that for any $N > 0$

$$E'_0(\lambda) = O_N(\lambda^N) \quad (4.20)$$

in appropriate spaces as $\lambda \rightarrow 0$ (see [17]). The Gevrey estimates of the resolvent at threshold allow to give an improvement of this result. Let us begin with the following results on the boundary values of the resolvent up to real axis.

Lemma 4.5. *Let $V_0(x) = V_1(x) - iV_2(x)$ with $V_1(x), V_2(x)$ real. Assume that V_1 is of class C^2 on \mathbb{R}^n and that there exists $\mu \in]0, 1[$ and some constants $c_j > 0$, $j = 1, 2, 3$, such that*

$$c_1 \langle x \rangle^{-2\mu} \leq V_1(x) \leq c_2 \langle x \rangle^{-2\mu}, \quad (4.21)$$

$$|(x \cdot \nabla)^j V_1(x)| \leq c_2 \langle x \rangle^{-2\mu}, \quad j = 1, 2 \quad (4.22)$$

$$x \cdot \nabla V_1(x) \leq -c_3 \langle x \rangle^{-2\mu}, \quad |x| > R \text{ for some } R > 0, \quad (4.23)$$

$$|V_2(x)| \leq c_2 \langle x \rangle^{-1-\mu-\epsilon_0}, \quad \epsilon_0 > 0. \quad (4.24)$$

Then the eigenvalues of H_0 are absent in a neighbourhood of zero and the boundary values of the resolvent $R_0(\lambda \pm i0) = \lim_{z \rightarrow \lambda, \pm \text{Im } z > 0} (H_0 - z)^{-1}$ exist for $\lambda \in [0, \delta]$ for some $\delta > 0$ and are Hölder continuous as operators in $\mathcal{L}(L^{2, \frac{1+\mu}{2}}; L^{2, -\frac{1+\mu}{2}})$.

Proof. Let $H_1 = -\Delta + V_1(x)$ be the selfadjoint part of H_0 and $R_1(z) = (H_1 - z)^{-1}$. Then one knows from [17] that under the condition of this Lemma, $R_1(\lambda \pm i0)$ exists for $\lambda \in [0, \delta]$ for some $\delta > 0$ and are Hölder continuous as operators in $\mathcal{L}(L^{2, s}; L^{2, -s})$, $s > \frac{1+\mu}{2}$. Note that the smoothness assumption on the potential used in [17] is only needed for higher order resolvent estimates.

One knows that $G_{0,1} = \lim_{z \rightarrow 0, z \notin \mathbb{R}_+} R_1(z)$ exists and that $G_{0,1}V_2$ is a compact operator in $L^{2, -s}$ for $\frac{1+\mu}{2} < s < \frac{1+\mu+\epsilon_0}{2}$. Therefore the kernel of $1 + iG_{0,1}V_2$ is of finite dimension. From Lemma 3.3 applied to $G_{0,1}$, one deduces that this kernel is contained in $L^{2, r}$ for any $r > 0$. Since $(1 + iG_{0,1}V_2)u = 0$ if and only if $H_0u = 0$, Lemma 3.1 that $\text{Ker}(1 + iG_{0,1}V_2)$ in $L^{2, -s}$ is trivial. Therefore $((1 + iG_{0,1}V_2)^{-1})$ is bounded in $L^{2, -s}$. By the continuity of $R_1(z)$ for z near 0 and $z \notin \mathbb{R}_+$, one deduces that $1 + iR_1(z)V_2$ is invertible in $L^{2, -s}$ and its inverse is Hölder continuous in $\mathcal{L}(L^{2, -s})$ for z near 0 and $z \notin \mathbb{R}_+$. This implies in particular that the eigenvalues of H_0 are absent in a neighbourhood of zero and the limits $R_0(\lambda \pm i0) = \lim_{z \rightarrow \lambda, \pm \text{Im } z > 0} (H_0 - z)^{-1}$ exist for $\lambda \geq 0$ and small enough and are Hölder continuous in λ . ■

Corollary 4.6. *Under the conditions of Lemma 4.5, assume in addition that $V_2 = 0$ such that H_0 is selfadjoint. Denote by $E_0(\lambda)$ the spectral projection of H_0 associated with the interval $] -\infty, \lambda]$. Let $s > \frac{1+\mu}{2}$. Then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} E'_0(\lambda) \langle x \rangle^{-s}\| \leq C_a e^{-c_a |\lambda|^{-\frac{1}{\gamma}}}, \quad 0 < \lambda \leq \delta. \quad (4.25)$$

Proof. Since $E'_0(\lambda) = \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0))$, $\|\langle x \rangle^{-s} E'_0(\lambda) \langle x \rangle^{-s}\|$ is uniformly bounded for $\lambda > 0$ near 0 ([17]). Iterating the resolvent equation, one obtains for any $N \in \mathbb{N}$

$$E'_0(\lambda) = \lambda^N G_0^N E'_0(\lambda), \quad 0 < \lambda \leq \delta. \quad (4.26)$$

Applying (3.16 with $r = s$, one deduces as in the proof of Corollary 3.5 that for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} E'_0(\lambda) \langle x \rangle^{-s}\| \leq C_a c_a^N N^{\gamma N} \lambda^N \quad (4.27)$$

for all $N \in \mathbb{N}$ and $\lambda \in]0, \delta]$. It remains to choose N in terms of $\lambda > 0$ (it suffices to take N equal to integer part of $c\lambda^{-\frac{1}{\gamma}}$ for some appropriate $c > 0$) such that

$$c_a^N N^{\gamma N} \lambda^N \leq C' e^{-c'\lambda^{-\frac{1}{\gamma}}}, \quad 0 < \lambda \leq \delta,$$

for some constants $c', C' > 0$. (4.25) is proved. \blacksquare

4.3. Subexponential time-decay of Schrödinger semigroup. To obtain subexponential time-decay of solutions to the Schrödinger equation associated with H_0 , we shall use both techniques of analytic dilation and analytic deformation to study quantum resonances of H_0 . These different techniques give rise to the same set of quantum resonances. See [2, 8, 10, 22].

Firstly, we use the analytic dilation method to prove a global resolvent estimate in some sector below the positive real half-axis. For $V_0 \in \mathcal{A}$, denote $\tilde{H}_0(\theta) = -(1+\theta)^{-2}\Delta + V_0((1+\theta)x)$ for $\theta \in \mathbb{C}$ and θ near 0. Set $\tilde{R}_0(z, \theta) = (\tilde{H}_0(\theta) - z)^{-1}$. For θ real, $\tilde{R}_0(z, \theta)$ is holomorphic in \mathbb{C}_+ and meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$. Since $V_0 \in \mathcal{A}$, there exists some constant $\delta > 0$ such that $\{\tilde{H}_0(\theta); \theta \in \mathbb{C}, |\theta| < \delta\}$ is a holomorphic family of type A. For $\text{Im } \theta > 0$ small enough, the resolvent $\tilde{R}_0(z, \theta)$ defined for $z \in \mathbb{C}_+$ with $\text{Im } z \gg 1$ can be meromorphically extended across the positive real half-axis \mathbb{R}_+ into the sector $\{z; \arg z > -\text{Im } \theta\}$. The poles of $\tilde{R}_0(z, \theta)$ in this sector are by definition quantum resonances of H which are independent of θ ([2]).

We begin with the following elementary Hardy type inequality.

Lemma 4.7. *Let $n \geq 2$ and $0 < s < n - 1$. One has*

$$\|\langle x \rangle^{-1-\frac{s}{2}} u\|^2 \leq \frac{1}{2\sqrt{(n-1)(n-1-s)}} (\|\nabla u\|^2 + \|\langle x \rangle^{-s} u\|^2) \quad (4.28)$$

for all $u \in H^1(\mathbb{R}^n)$.

Proof. Let $x = r\omega$, $r \geq 0$ and $\omega \in \mathbb{S}^{n-1}$. For $u \in \mathcal{S}(\mathbb{R}^n)$, denote

$$F(r, \omega) = \frac{|u(r\omega)|^2 r^{n-1}}{\langle r \rangle^s}.$$

Then one has

$$F'_r(r, \omega) = \frac{((n-1)(1+r^2) - sr^2)|u(r\omega)|^2 r^{n-2}}{\langle r \rangle^{s+2}} + 2\frac{r^{n-1}}{\langle r \rangle^s} \text{Re}(u'_r(r\omega) \overline{u(r\omega)}).$$

Here $F'_r(r, \omega)$ is the derivation of $F(r, \omega)$ with respect to r . Since for $n \geq 2$, one has $\int \int_{\mathbb{R}_+ \times \mathbb{S}^{n-1}} F'(r, \omega) dr d\omega = 0$, we deduce the identity

$$\int \int_{\mathbb{R}^n} \frac{(n-1) + (n-1-s)r^2}{r \langle r \rangle^{s+2}} |u(x)|^2 dx = -2 \int \int_{\mathbb{R}^n} \langle x \rangle^{-s} \operatorname{Re}(u'_r \bar{u}) dx \quad (4.29)$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$. Inequality (4.28) follows from the trivial bounds

$$2\sqrt{(n-1)(n-1-s)} \leq \frac{(n-1) + (n-1-s)r^2}{r}$$

and

$$-2 \int \int_{\mathbb{R}^n} \langle x \rangle^{-s} \operatorname{Re}(u'_r \bar{u}) dx \leq 2 \|u'_r\| \| \langle x \rangle^{-s} u \| \leq \| \nabla u \|^2 + \| \langle x \rangle^{-s} u \|^2$$

and an argument of density. ■

Lemma 4.8. *Let $V_0 \in \mathcal{A}$. Then there exists some constant $c_0 > 0$ such that for $\theta \in \mathbb{C}$ with $|\theta|$ sufficiently small and $\operatorname{Im} \theta > 0$, one has*

$$\sigma(\tilde{H}_0(\theta)) \cap \{z \in \mathbb{C}; \operatorname{Im} z > 0 \text{ or } \arg z > -c_0 \operatorname{Im} \theta\} = \emptyset \quad (4.30)$$

and

$$\| \langle x \rangle^{-2\mu} \tilde{R}_0(z, \theta) \| \leq \frac{1}{c_0 \operatorname{Im} \theta \langle z \rangle} \quad (4.31)$$

for $z \in \mathbb{C}$ with $\arg z > -c_0 \operatorname{Im} \theta$.

Proof. We only consider the case $\theta = i\tau$ with $\tau > 0$ small enough. Since $V_0 = V_1 - iV_2 \in \mathcal{A}$, one has

$$\begin{aligned} V_0((1+i\tau)x) &= V_1(x) + \tau x \cdot \nabla V_2(x) \\ &\quad - i(V_2(x) - \tau x \cdot \nabla V_1(x) + O(\tau^2 \langle x \rangle^{-2\mu})) \end{aligned}$$

for $\tau > 0$ sufficiently small. Let $z = \langle u, \tilde{H}_0(\theta)u \rangle$, $u \in H^2$ with $\|u\| = 1$. Then,

$$\operatorname{Re} z = \frac{1-\tau^2}{(1+\tau^2)^2} \|\nabla u\|^2 + \langle u, (V_1(x) + O(\tau \langle x \rangle^{-2\mu}))u \rangle, \quad (4.32)$$

$$\begin{aligned} \operatorname{Im} z &= -\frac{2\tau}{(1+\tau^2)^2} \|\nabla u\|^2 - \langle u, (V_2(x) - \tau x \cdot \nabla V_1(x))u \rangle \\ &\quad + \langle u, O(\tau^2 \langle x \rangle^{-2\mu})u \rangle. \end{aligned} \quad (4.33)$$

This implies that there exists $c > 0$ such that

$$\operatorname{Re} z \geq c(\|\nabla u\|^2 + \| \langle x \rangle^{-\mu} u \|^2), \quad (4.34)$$

for $\tau > 0$ sufficiently small. If $R \in]0, \infty]$, one has for some $c' > 0$

$$V_2(x) - \tau x \cdot \nabla V_1(x) \geq c' \tau \langle x \rangle^{-2\mu}, \forall x \in \mathbb{R}^n,$$

which gives that

$$\operatorname{Im} z \leq -c' \tau (\|\nabla u\|^2 + \| \langle x \rangle^{-\mu} u \|^2) \quad (4.35)$$

for some $c'' > 0$. This shows that $\operatorname{Im} z \leq -C\tau \operatorname{Re} z$ ($C = c''c^{-1}$) if $R \in]0, +\infty]$.

If $R = 0$, one has $V_2(x) \geq 0$ for all $x \in \mathbb{R}^n$ and

$$V_2(x) - \tau x \cdot \nabla V_1(x) \geq c_3 \tau \frac{x^2}{\langle x \rangle^{2\mu+2}}, \quad \forall x \in \mathbb{R}^n,$$

for some $c_3 > 0$. In this case, one has

$$\operatorname{Im} z \leq -C\tau(\|\nabla u\|^2 + \langle u, \frac{x^2}{\langle x \rangle^{2\mu+2}} u \rangle) + C\tau^2 \|\langle x \rangle^{-\mu} u\|^2. \quad (4.36)$$

Lemma 4.7 with $s = \mu$ shows

$$\frac{1}{\langle x \rangle^{2\mu+2}} \leq \frac{1}{2\sqrt{(n-1)(n-1-\mu)}} \left(-\Delta + \frac{1}{\langle x \rangle^{2\mu}} \right)$$

in the sense of selfadjoint operators. For $0 < \mu < \frac{3}{4}$ when $n = 2$ and $\mu \in]0, 1[$ if $n \geq 3$, one has

$$\alpha \equiv \frac{1}{2\sqrt{(n-1)(n-1-\mu)}} < 1.$$

This proves that

$$\begin{aligned} \|\nabla u\|^2 + \langle u, \frac{x^2}{\langle x \rangle^{2\mu+2}} u \rangle &= \|\nabla u\|^2 + \langle u, \left(\frac{1}{\langle x \rangle^{2\mu}} - \frac{1}{\langle x \rangle^{2\mu+2}} \right) u \rangle \\ &\geq (1 - \alpha)(\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2). \end{aligned}$$

Consequently, one obtains

$$\operatorname{Im} z \leq -C(1 - \alpha)\tau(\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2) + C\tau^2 \|\langle x \rangle^{-\mu} u\|^2 \leq -C_1\tau \operatorname{Re} z \quad (4.37)$$

for some $C_1 > 0$ if $\tau > 0$ is small enough. This proves that the numerical range of $\tilde{H}_0(\theta)$ is included in the region $\Gamma = \{z; \operatorname{Re} z \geq 0, \operatorname{Im} z \leq -C_1\tau \operatorname{Re} z\}$ for some constant $C_1 > 0$. Since $\sigma(\tilde{H}_0(\theta)) \subset \Gamma$ and $\|\tilde{R}_0(z, \theta)\| \leq \operatorname{dist}(z, \Gamma)^{-1}$, one has

$$\|\tilde{R}_0(z, \theta)\| \leq \frac{1}{c_0 \operatorname{Im} \theta |z|} \quad (4.38)$$

for $z \in \mathbb{C}$ with $\arg z > -c_0 \operatorname{Im} \theta$ for some $c_0 > 0$. The conclusion of Lemma 4.8 follows now from Theorem 3.4 applied to $H_0 = \tilde{H}_0(\theta)$ which verifies Conditions (2.1) – (2.4) uniformly in θ with $\operatorname{Im} \theta \geq 0$ and $|\theta| < \delta$. \blacksquare

In order to obtain subexponential time-decay estimates for $\chi e^{-itH_0} \chi$, $\chi \in C_0^\infty(\mathbb{R}^n)$, we use analytical distortion outside of H_0 the support of χ . Let $R_0 > 1$ and $\rho \in C^\infty(\mathbb{R})$ with $0 \leq \rho \leq 1$ and $\rho(r) = 0$ if $r \leq 1$ and $\rho(r) = 1$ if $r \geq 2$. Define for $R_0 > 1$

$$F_\theta(x) = x \left(1 + \theta \rho \left(\frac{|x|}{R_0} \right) \right), \quad x \in \mathbb{R}^n. \quad (4.39)$$

When $\theta \in \mathbb{R}$ with $|\theta|$ sufficiently small, $x \rightarrow F_\theta(x)$ is a global diffeomorphism on \mathbb{R}^n . Set

$$U_\theta f(x) = |DF_\theta(x)|^{\frac{1}{2}} f(F_\theta(x)), \quad f \in L^2(\mathbb{R}^n), \quad (4.40)$$

where $DF_\theta(x)$ is the Jacobi matrix and $|DF_\theta(x)|$ the Jacobian of the change of variables: $x \rightarrow F_\theta(x)$. One has

$$|DF_\theta(x)| = \begin{cases} 1, & |x| < R_0; \\ (1 + \theta)^n, & |x| > 2R_0 \end{cases} \quad (4.41)$$

U_θ is unitary in $L^2(\mathbb{R}^n)$ for θ real with $|\theta|$ sufficiently small. Define the distorted operator $H_0(\theta)$ by

$$H_0(\theta) = U_\theta H_0 U_\theta^{-1}. \quad (4.42)$$

One can calculate that

$$H_0(\theta) = -\Delta_\theta + V(F_\theta(x)) \quad (4.43)$$

where $-\Delta_\theta = {}^t \nabla_\theta \cdot \nabla_\theta$ with

$$\nabla_\theta = ({}^t DF_\theta)^{-1} \cdot \nabla - \frac{1}{|DF_\theta|^2} ({}^t DF_\theta)^{-1} \cdot (\nabla |DF_\theta|) \quad (4.44)$$

In particular, $\nabla_\theta f = (1 + \theta)^{-1} \nabla f$ if f is supported outside the ball $B(0, 2R_0)$. If $V_0 \in \mathcal{A}$, $H_0(\theta)$ can be extended to a holomorphic family of type A for θ in a small complex neighbourhood of zero. $H_0(\theta)$ and $\tilde{H}_0(\theta)$ coincide outside the ball $B(0, 2R_0)$ and they have the same essential spectra. In addition their discrete eigenvalues in the region $\{z \in \mathbb{C}; \operatorname{Re} z \geq 0, \operatorname{Im} z > -c \operatorname{Im} \theta \operatorname{Re} z\}$ for some $c > 0$ small enough are the same (see [2, 10, 8]). Since $\tilde{R}_0(z, \theta)$ is holomorphic in z there, so is $R_0(z, \theta) = (H_0(\theta) - z)^{-1}$.

Remark that the distorted operator $H_0(\theta)$ satisfies the conditions (2.3) and (2.4) with some constant $c_0 > 0$ independent of $R_0 > 1$. Consequently, Lemma 3.1 applied to $H_0(\theta)$ implies that $\langle x \rangle^{-2\mu} R_0(0, \theta)$ is defined on the range of $H_0(\theta)$ and extends to a bounded operator in $L^2(\mathbb{R}^n)$ and Theorem 3.4 holds for $G_0(\theta) = R_0(0, \theta)$ for some constant C independent of R_0 used in the analytical distortion.

Proposition 4.9. *Assume the conditions of Lemma 4.8. Denote*

$$\Omega_1(\theta) = \{z \in \mathbb{C}; \operatorname{Re} z > 0, \operatorname{Im} z \geq -c \operatorname{Im} \theta \operatorname{Re} z\}.$$

with $c > 0$ appropriately small. Then $\Omega_1(\theta)$ is contained in resolvent set of $H_0(\theta)$ and there exists some constant $C > 0$

$$\|\langle x \rangle^{-2\mu} R_0(z, \theta) \langle x \rangle^{-2\mu}\| \leq \frac{C}{\langle z \rangle}, \quad z \in \Omega_1(\theta). \quad (4.45)$$

Proof. For $z \in \Omega_1(\theta)$ and $|z|$ large, (4.45) follows from Lemma 4.8 by an argument of perturbation. For $z \in \Omega_1(\theta)$ and $|z|$ small, one compares $R_0(z, \theta)$ with $\tilde{R}_0(z, \theta)$ and $R_0(0, \theta)$.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ if $|x| \leq 2R_0$. On the support of $1 - \chi$, $H_0(\theta) = \tilde{H}_0(\theta)$. For $z \in \Omega_1(\theta)$ and $|z|$ small, one has

$$\begin{aligned} R_0(z, \theta) &= R_0(0, \theta) + z R_0(0, \theta) R_0(z, \theta) \\ &= R_0(0, \theta) + z R_0(0, \theta) (\chi(2 - \chi) + (1 - \chi)^2) R_0(z, \theta) \\ &= R_0(0, \theta) + z R_0(0, \theta) \chi(2 - \chi) R_0(z, \theta) \\ &\quad + z R_0(0, \theta) (1 - \chi) \tilde{R}_0(z, \theta) (1 - \chi) \\ &\quad + z R_0(0, \theta) (1 - \chi) \tilde{R}_0(z, \theta) [(1 + \theta)^{-2} \Delta, \chi] R_0(z, \theta) \end{aligned}$$

Recall that for $\operatorname{Im} \theta > 0$, there exists some constant $C > 0$ such that

$$\|\tilde{R}_0(z, \theta) \langle x \rangle^{-2\mu}\| \leq C, \quad \text{for } z \in \Omega_1(\theta).$$

By the ellipticity of the operator, this implies that

$$\|\widetilde{R}_0(z, \theta) \nabla \langle x \rangle^{-2\mu}\| \leq C, \quad (4.46)$$

for $z \in \Omega_1(\theta)$ and $|z| \leq 1$. Therefore there exists possibly another constant C such that

$$\|\langle x \rangle^{-2\mu} R_0(z, \theta) \langle x \rangle^{-2\mu}\| \leq C + C|z| \|\langle x \rangle^{-2\mu} R_0(z, \theta) \langle x \rangle^{-2\mu}\| \quad (4.47)$$

for $z \in \Omega_1(\theta)$ and $|z| \leq 1$. This shows that $\|\langle x \rangle^{-2\mu} R_0(z, \theta) \langle x \rangle^{-2\mu}\|$ is uniformly bounded for $z_0 \in \Omega_1(\theta)$ and $|z|$ sufficiently small. (4.45) is proved. \blacksquare

Theorem 4.10. *Let $V_0 \in \mathcal{A}$. There exists some constant $c > 0$ such that for any function $\chi \in C_0^\infty(\mathbb{R}^n)$ there exists some constant $C_\chi > 0$ such that*

$$\|\chi(x) e^{-itH_0} \chi(x)\| \leq C_\chi e^{-c|t|^\beta}, \quad t > 0. \quad (4.48)$$

Proof. Let $R_1 > 0$ such that $\text{supp} \chi \subset B(0, R_1)$. Let $U(\theta)$ be defined as before with $R_0 > R_1$. Then one has

$$\chi(x) e^{-itH_0} \chi(x) = \chi(x) e^{-itH_0(\theta)} \chi(x)$$

for $\theta \in \mathbb{R}$ with $|\theta|$ small. For $\theta \in \mathbb{C}$ with θ near zero and $\text{Im} \theta > 0$, $H_0(\theta)$ is strictly sectorial and the resolvent $R(z, \theta)$ is holomorphic in $z \in \mathbb{C}$ with $-c \text{Im} \theta < \arg z < \pi + c$ for some $c > 0$. Making use of Proposition 4.9, one can check that

$$\chi(x) e^{-itH_0} \chi(x) = \frac{i}{2\pi} \int_{\Gamma'} e^{-itz} \chi(x) R(z, \theta) \chi(x) dz \quad (4.49)$$

where

$$\Gamma' = \{z = r e^{-i\delta}; r \geq 0\} \cup \{z = -r e^{i\delta}, r \geq 0\}$$

for $\delta = \delta(\text{Im} \theta) > 0$ small enough. Γ' is oriented in anti-clockwise sense.

The remaining part of the proof of (4.48) is the same as in Theorem 4.4 and will not be repeated here. We just indicate that if one denotes $G_0(\theta) = R(0, \theta)$, then one has

$$R_0(z, \theta) = \sum_{j=0}^N z^j G_0(\theta)^j + z^{N+1} G_0(\theta)^N R_0(z, \theta)$$

for $z \in \Gamma'$ and z near 0, and Theorem 3.4 with $r = 2$ and Proposition 4.9 show that

$$\begin{aligned} & \|\chi(x) G_0(\theta)^N R(z, \theta) \chi(x)\| \\ & \leq C_\chi \|\langle x_{N,0} \rangle^{-2(N+1)\mu} G_0(\theta)^N R_0(z, \theta) \langle x_{N,0} \rangle^{-2\mu}\| \leq C_{\chi, \text{Im} \theta} C^N N^\gamma \end{aligned} \quad (4.50)$$

with some constant C independent of χ and θ . By choosing appropriately N in terms of t as in the proof of Theorem 4.4, one obtain some $c > 0$ independent of χ such that (4.48) holds. \blacksquare

Theorem 4.10 generalizes in particular a result of D. Yafaev [28] in one-dimensional selfadjoint case to higher dimensions.

5. COMPACTLY SUPPORTED PERTURBATIONS OF THE MODEL OPERATOR

Consider non-selfadjoint Schrödinger operator H of the form

$$H = H_0 + W(x). \quad (5.1)$$

where $H_0 = -\Delta + V_0(x)$ with V_0 in \mathcal{V} or \mathcal{A} and $W \in L^\infty_{\text{comp}}(\mathbb{R}^n)$. Then the essential spectrum of H is equal to $[0, +\infty[$ and the possible accumulation points of complex eigenvalues of H are contained in \mathbb{R}_+ . We begin with the analysis of positive resonances for a class of non-selfadjoint Schrödinger operators $H = -\Delta + V(x)$ with $V(x)$ holomorphic outside some compact set. Since we are considering behavior of solutions as $t \rightarrow +\infty$, the main attention is paid to here outgoing positive resonances. Incoming positive resonances are invisible in the limit $t \rightarrow +\infty$.

5.1. Positive resonances of non-selfadjoint Schrödinger operators. Consider a class of non-selfadjoint Schrödinger operators $H = -\Delta + V(x)$ which are compactly supported perturbations of $H_0 = -\Delta + V_0$ with $\text{Im } V_0(x) \leq 0$ and $V_0(x)$ extends to a holomorphic function a a region of the form $\{x \in \mathbb{C}^n; -x| > R, |\text{Im } x| < \delta|\text{Re } x|\}$ and satisfies there

$$|V_0(x)| \leq C\langle \text{Re } x \rangle^{-\rho} \quad (5.2)$$

for some constants $R, \delta, C, \rho > 0$. Suppose in addition $(x \cdot \nabla_x)^j V_0$, $j = 0, 1, 2$, are $-\Delta$ -compact. Then the set $r_+(H)$ of outgoing positive resonances of H is well defined by Definition 2.4. For the model operator H_0 , one has $r_+(H_0) = \emptyset$. Since H_0 is dissipative, the boundary value of the resolvent

$$R_0(\lambda + i0 = \lim_{z \rightarrow \lambda, \text{Im } z > 0} (H_0 - z)^{-1} \quad (5.3)$$

exists in $\mathcal{L}(-1, s; 1, -s)$, $s > \frac{1}{2}$, for $\lambda > 0$ and is continuous in λ ([19]). Let U_θ be the analytic distortion defined by (4.40) with R_0 sufficiently large. Then H_θ

$$H_\theta = U_\theta H U_\theta^{-1}$$

defined for θ real can be extended to a holomorphic family of type A for θ in a complex neighbourhood of zero. For $\text{Im } \theta > 0$, spectrum of H_θ is $\{z, \arg z > -c\text{Im } \theta\}$ is discrete for some constant $c > 0$ and is independent of the function ρ used in the distortion ([10]).

Theorem 5.1. *There exists some constant $\theta_0 > 0$ such that*

$$\sigma_d(H(\theta)) \cap \mathbb{R}_+ = r_+(H) \quad (5.4)$$

for $\text{Im } \theta > 0$ and $|\theta| < \theta_0$. In particular, outgoing positive resonances of H is at most a countable set with zero as the only possible accumulation point.

Proof. Let $\lambda_0 > 0$. If $\lambda_0 \notin r_+(H)$, then -1 is not an eigenvalue of the compact operator $R_0(\lambda_0 + i0)W$ on $L^{2,-s}$, $\frac{1}{2} < s < \rho - \frac{1}{2}$. Therefore $(1 + R_0(\lambda_0 + i0)W)$ is invertible on $L^{2,-s}$. Since $\lambda \rightarrow R_0(\lambda + i0)$ is continuous as operator from $L^{2,s}$ to $L^{2,-s}$, we deduce that -1 is not an eigenvalue of $R_0(\lambda + i0)W$ for $\lambda \in \mathbb{R}$ and λ sufficiently near λ_0 . It follows that the boundary value of the resolvent $R(\lambda + i0) = \lim_{z \in \mathbb{C}_+, z \rightarrow \lambda} R(z)$ exists and

$$\|\langle x \rangle^{-s} R(\lambda + i0) \langle x \rangle^{-s}\| \leq C \quad (5.5)$$

for λ near λ_0 and $s > \frac{1}{2}$. This proves that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi R(z)\chi$ is bounded for $z \in \mathbb{C}_+$ and z near λ_0 . Therefore the meromorphic extension of $\chi R(z)\chi$ from the upper half complex plane is in fact holomorphic in neighbourhood of λ_0 . It follows that $\sigma_d(H_\theta) \cap \mathbb{R}_+ \subset r_+(H)$.

Conversely if $\lambda_0 \in r_+(H)$, then there exists a non zero solution $u \in L^{2,-s}$ for any $s > \frac{1}{2}$ such that

$$u = -R_0(\lambda_0 + i0)Wu.$$

Let $R_0(z, \theta) = U_\theta R_0(z)U_\theta^{-1}$ and $u_\theta = U_\theta u$, for $\theta \in \mathbb{R}$. Then one has

$$u_\theta = -R_0(\lambda_0 + i0, \theta)Wu \quad (5.6)$$

if the analytic distorsion is made outside the support of W . Since outgoing resonances of the dissipative operator H_0 are absent, $R_0(z, \theta)$ defined for $\text{Im } z > 0$ and θ real can be holomorphically extended for $\theta \in \mathbb{C}$ with $\text{Im } \theta > 0$ and $|\theta| < \theta_0$ for some $\theta_0 > 0$ depending on domain of the analyticity of V_0 . After this extension in θ , $R_0(z, \theta)$ is holomorphic for z near λ_0 and $\text{Im } z > -c\text{Im } \theta \text{Re } z$ for some $c > 0$. By (5.6), u_θ can be extended in θ for $\text{Im } \theta > 0$. By (5.6), $u_\theta \in L^2$ so long as $\text{Im } \theta > 0$. In addition $u_\theta \neq 0$ because

$$\|\langle x \rangle^{-s'}(u_\theta - u)\| \leq \|\langle x \rangle^{-s}(R_0(\lambda_0 + i0, \theta) - R_0(\lambda_0 + i0, 0))Wu\| \leq C|\theta|^\eta \quad (5.7)$$

for some $\eta > 0$ if $s' > \frac{1}{2}$. This proves λ_0 is an eigenvalue of H_θ with u_θ as an eigenfunction when $\text{Im } \theta > 0$ and $|\theta|$ is small enough. Therefore $r_+(H) \subset \sigma_d(H_\theta)$. \blacksquare

Remark 5.1. *In Theorem 5.1, the condition $\text{Im } V_0 \leq 0$ is used because potential V_0 may have a long-range tail. Similarly if $\text{Im } V_0 \geq 0$ one can prove*

$$\sigma_d(H_\theta) \cap \mathbb{R}_+ = r_-(H) \quad (5.8)$$

for $\text{Im } \theta < 0$ and $|\theta| < \theta_0$. In particular if $V_0(x)$ is real when $x \in \mathbb{R}^n$ and satisfies (5.2), then real resonances of H are at most a countable set with zero as the only possible accumulation point.

The sign restriction on $\text{Im } V_0$ is not necessary if V_0 is of short-range. To be simple we only give a result in dilation analytic case. Let $H = -\Delta + V(x)$ with V a dilation analytic short-range potential: $V_\theta(x) = V(e^\theta x)$ defined for θ real extends to a holomorphic function for θ in a complex neighbourhood of zero:

$$|V_\theta(x)| \leq \langle x \rangle^{-\rho}, \quad (5.9)$$

for $x \in \mathbb{R}^n$ and $|\theta| < \theta_0$ for some $C, \theta_0 > 0$ and $\rho > 1$. Then $H_\theta = -e^{-2\theta}\Delta + V_\theta$ is a holomorphic family of type A for $\theta \in \mathbb{C}$, $|\theta| < \theta_0$

Theorem 5.2. *Let V be dilation analytic and of short-range. Then for $|\theta| < \theta_0$ one has*

$$\sigma_d(H_\theta) \cap \mathbb{R}_+ = r_\pm(H), \pm \text{Im } \theta > 0. \quad (5.10)$$

In particular, positive resonances of H are at most a countable set with zero as the only possible accumulation point.

Proof. The inclusion $\sigma_d(H_\theta) \subset r_+(H)$, $\text{Im } \theta > 0$ can be proved in the same way as the first part of the proof of Theorem 5.1. It remains to show that $r_+(H) \subset \sigma_d(H_\theta)$, $\text{Im } \theta > 0$. Let $\lambda \in r_+(H)$. Then -1 is an eigenvalue of the compact operator $K = (-\Delta - \lambda - i0)^{-1}V$ on $L^{2,-s}$, $\frac{1}{2} < s < \frac{\rho}{2}$. Let

$$K_\theta = (-e^{-2\theta}\Delta - \lambda - i0)^{-1}V_\theta = e^{2\theta}(-\Delta - e^{2\theta}(\lambda + i0))^{-1}V_\theta,$$

for $\text{Im } \theta \geq 0$, $|\theta| < \theta_0$. K_θ is a family of compact operators on $L^{2,-s}$ continuous with respect to θ in the half disk $D_+(0, \theta_0) = \{\theta \in \mathbb{C}; \text{Im } \theta \geq 0, |\theta| < \theta_0\}$ and holomorphic for θ in its interior. In addition

$$\|\langle x \rangle^{-s}(K_\theta - K)\langle x \rangle^s\| \leq C|\theta|^\eta \quad (5.11)$$

for $\frac{1}{2} < s < \frac{\rho}{2}$ and for some $\eta > 0$, because $(-\Delta - e^{2\theta}(\lambda + i0))^{-1}$ is Hölder continuous in $\theta \in D_+(0, \theta_0)$ as operator-valued function from $L^{2,s}$ to $L^{2,-s}$. It follows that in any small neighbourhood of -1 , K_θ has at least one eigenvalue z_θ for $\theta \in D_+(0, \delta)$ if $\delta > 0$ is sufficiently small. Since K_θ and $K_{\theta'}$ are unitarily equivalent if $\text{Im } \theta = \text{Im } \theta'$, z_θ is independent of $\text{Re } \theta$. It follows that z_θ is independent of θ for $\text{Im } \theta > 0$ and $|\theta| < \delta$ (Theorem 1.9 in Ch. VII of [14]). Since $z_\theta \rightarrow -1$ as $\theta \rightarrow 0$, one concludes that $z_\theta = -1$ for $\theta \in D_+(0, \delta)$ if $\delta > 0$ is small enough. This proves that λ is an eigenvalue of H_θ if $\text{Im } \theta > 0$ and $|\theta| < \delta$. Therefore $r_+(H) \subset \sigma_d(H_\theta)$, $\text{Im } \theta > 0$ which completes the proof of (5.10) with sign $+$. The equality with sign $-$ can be proved in the same way. The last affirmation is immediate since the set of positive resonances of H is equal to $r_+(H) \cup r_-(H)$. \blacksquare

In the above proof, we showed that if $\lambda \in r_+(H)$, then there exists $c > 0$ such that -1 is the only eigenvalue of K_θ inside the disk $D(-1, c)$ for all $\theta \in D_+(\lambda, \delta)$. Then one can define the Riesz projection of eigenvalue -1 of K_θ by

$$\pi_\theta = \frac{1}{2\pi i} \int_{|z+1|=\frac{c}{2}} (z - K_\theta)^{-1} dz, \quad \theta \in D_+(\lambda, \delta). \quad (5.12)$$

The following result is immediate.

Corollary 5.3. *Assume the conditions of Theorem 5.2. Let $\lambda > 0$ be an outgoing resonance of $H = -\Delta + V$. Denote π_θ the Riesz projection of eigenvalue -1 of K_θ , $\theta \in D_+(\lambda, \delta)$, $\delta > 0$. Then as operators on $L^{2,-s}$, $\frac{1}{2} < s < \frac{\rho}{2}$, π_θ is continuous for $\theta \in D_+(\lambda, \delta)$ and holomorphic for θ in the interior of this half disk.*

5.2. Proof of Theorem 2.2.

Proposition 5.4. *Let $H_0 = -\Delta + V_0(x)$ with $V_0 \in \mathcal{V}$. Let $W \in L^\infty(\mathbb{R}^n)$ with compact support and $H = H_0 + W(x)$. Assume that 0 is not an eigenvalue of H . Then one has:*

(a). *There exist some constants $c_1, \mu' > 0$ such that outside the set*

$$\Omega_1 = \{z \in \mathbb{C}; \text{Re } z \geq 0 \text{ and } |\text{Im } z| \leq c_1 |\text{Re } z|^{\mu'}\},$$

there are at most a finite number of discrete eigenvalues of H . There exists some $\delta > 0$ such that

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{\mu'}}} \text{ for } z \notin \Omega_1 \text{ and } |z| < \delta. \quad (5.13)$$

(b). *The limit*

$$R(0) = \lim_{z \rightarrow 0, z \notin \Omega_1} R(z) \quad (5.14)$$

exists in $\mathcal{L}(L^{2,s}; L^{2,s-2\mu})$ for any $s \in \mathbb{R}$ and one has

$$\|e^{-a\langle x \rangle^{1-\mu}} R(z)^N\| \leq C_a^{N+1} N^{\gamma N}, \quad (5.15)$$

$$\|\chi R(z)^N\| \leq C_\chi C^N N^{\gamma N} \quad (5.16)$$

for all $N \in \mathbb{N}^*$ and $z \in \Omega_0 = \{z \in \mathbb{C}; \operatorname{Re} z < 0, |\operatorname{Im} z| < -M \operatorname{Re} z\} \cup \{0\}$, $M > 0$. Here $a > 0$ and $\chi \in C_0^\infty(\mathbb{R}^n)$, C_a, c_a, C_χ are some positive constants and $C > 0$ is independent of χ .

Proof. Note that $G_0 W$ is a compact operator and that 0 is not an eigenvalue of H if and only if -1 is not an eigenvalue of $G_0 W$. So if 0 is not an eigenvalue of H , operator $1 + G_0 W$ is invertible on L^2 . From Proposition 4.1, one deduces that $1 + R_0(z)W$ is invertible for $|z|$ small and $z \notin \Omega_1$. This shows that 0 is not an accumulation point of $\sigma(H) \setminus \Omega_1$. In addition, $z \rightarrow 1 + R_0(z)W$ is holomorphic in $\mathbb{C} \setminus \Omega_1$. The analytic Fredholm Theorem shows that $(1 + R_0(z)W)^{-1}$ is a meromorphic function with at most a discrete set of poles in $\mathbb{C} \setminus \Omega_1$. These poles are eigenvalues of H . Since 0 is not an accumulation point of eigenvalues of H in $\mathbb{C} \setminus \Omega_1$, the number of eigenvalues of H in $\mathbb{C} \setminus \Omega_1$ is finite. (5.13) follows from Proposition 4.1 and the equation

$$R(z) = (1 + R_0(z)W)^{-1} R_0(z).$$

To prove that Gevrey estimates of the resolvent $R(z)$ at threshold, we remark that if $F(z)$ and $G(z)$ are two bounded operator-valued functions on Ω_0 satisfies the Gevrey estimates

$$\|F^{(N)}(z)\| \leq AC_1^N (N!)^\sigma \quad (5.17)$$

$$\|G^{(N)}(z)\| \leq BC_2^N (N!)^\sigma \quad (5.18)$$

for all $N \in \mathbb{N}$ and $z \in \Omega_0$ and for some $\sigma > 1$ and $A, B, C_1, C_2 > 0$, then $F(z)G(z)$ satisfies the Gevrey estimates

$$\|(FG)^{(N)}(z)\| \leq ABC_3^N (N!)^\sigma \quad (5.19)$$

for all $N \in \mathbb{N}$ and $z \in \Omega_0$ where

$$C_3 = D_\sigma \max\{C_1, C_2\} \text{ with } D_\sigma = \sup_{N \in \mathbb{N}} \sum_{j=0}^N \left(\frac{j!(N-j)!}{N!} \right)^{\sigma-1} < \infty; \quad (5.20)$$

and if $F(z)$ is invertible for $z \in \Omega_0$ with uniformly bounded inverse:

$$\|F(z)^{-1}\| \leq M \quad (5.21)$$

uniformly in $z \in \Omega_0$, then the inverse $H(z) = F(z)^{-1}$ satisfies the Gevrey estimates

$$\|H^{(N)}(z)\| \leq MC_4^N (N!)^\sigma \quad (5.22)$$

for all $N \in \mathbb{N}$ and $z \in \Omega_0$, where $C_4 = MC_1 D_\sigma$. Denote $G^\sigma(\Omega_0)$ the set of bounded operator-valued functions on Ω_0 verifying Gevrey estimates of order $\sigma > 1$. Since $e^{-a\langle x \rangle^{1-\mu}} R_0(z)$ and $\chi R_0(z)$ belong to G^σ with

$$\sigma = 1 + \gamma = \frac{1 + \mu}{1 - \mu}, \quad (5.23)$$

Estimates (5.15) and (5.16) follow respectively from equations

$$e^{-a(x)^{1-\mu}}R(z) = (1 + e^{-a(x)^{1-\mu}}R_0(z)W e^{a(x)^{1-\mu}})^{-1}e^{-a(x)^{1-\mu}}R_0(z) \quad (5.24)$$

$$\chi R(z) = (1 + \chi R_0(z)W)^{-1}\chi R_0(z) \quad (5.25)$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is taken such that $\chi(x) = 1$ on $\text{supp}W$. \blacksquare

Proposition 5.5. *Let $V_0 \in \mathcal{A}$. Assume that 0 is not an eigenvalue of $H = H_0 + W(x)$. Then one has*

(a). *There exists $\delta > 0$ such that H has at most a finite number of eigenvalues in*

$$\Omega_\delta = \{z \in \mathbb{C} \setminus \{0\}; -\delta \leq \arg z \leq \pi + \delta\}$$

and for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi R(z)\chi$ defined for $\text{Im} z > 0$ extends meromorphically into Ω_δ and there exists some constant $C_\chi, c > 0$ such that

$$\|\chi R(z)\chi\| \leq C_\chi \quad (5.26)$$

for $z \in \Omega_\delta$ and $|z| < c$.

(b). *The limit $R(0) = \lim_{z \in \Omega_\delta, z \rightarrow 0} R(z)$ in $\mathcal{L}(L^{2,s+2\mu}; L^{2,s-2\mu})$ for any $s \in \mathbb{R}$ and one has*

$$\|\chi R(z)^N \chi\| \leq C_\chi C^N N^{\gamma N} \quad (5.27)$$

for any $N \in \mathbb{N}^$ and $z \in \Omega_\delta \cup \{0\}$ with $|z| < c$.*

Proof. Since $V_2 \geq 0$, one has for $\text{Im} z \gg 1$

$$\chi R(z)\chi = (1 + \chi R_0(z)W)^{-1}\chi R_0(z)\chi \quad (5.28)$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi(x) = 1$ on $\text{supp}W$. Let U_θ be defined by (4.40) with $R_0 \gg 1$ such that $\text{supp} \chi \subset \{x; |x| < R_0\}$. Let $H_0(\theta) = U_\theta H_0 U_\theta^{-1}$ and $R_0(z, \theta) = (H_0(\theta) - z)^{-1}$. Then one has for $\theta \in \mathbb{C}$, $\text{Im} \theta > 0$ and $|\theta|$ small,

$$\chi R_0(z)W = \chi R_0(z, \theta)W, \quad \chi R_0(z)\chi = \chi R_0(z, \theta)\chi.$$

For a fixed $\theta \in \mathbb{C}$ with $\text{Im} \theta > 0$, Proposition 4.9 shows that $\chi R_0(z)W$ and $\chi R_0(z)\chi$ are holomorphic in Ω_δ for some $\delta > 0$. The analytic Fredholm Theorem implies that $\chi R(z)\chi$ extends to a meromorphic in Ω_δ given by

$$\chi R(z)\chi = (1 + \chi R_0(z, \theta)W)^{-1}\chi R_0(z, \theta)\chi. \quad (5.29)$$

Zero is the only possible accumulation point of these poles. To show that zero is in fact not an accumulation point, we firstly prove that for each χ , -1 is not an eigenvalue of the compact operator $\chi G_0 W$. In fact if -1 were an eigenvalue of $\chi G_0 W$, then -1 would also be an eigenvalue of $W G_0 \chi = J(\chi G_0 W)^* J$ where J is the complex conjugaison: $Jf(x) = \overline{f(x)}$. Let $\psi \in L^2$ with $\psi \neq 0$ such that

$$\psi = -W G_0 \chi \psi.$$

Then $\chi \psi = \psi$, $G_0 \psi \in L^2$ and $H G_0 \psi = (1 + W G_0 \chi)\psi = 0$ which gives $\psi = \chi \psi = 0$, since 0 is not an eigenvalue of H and G_0 is injective. This is in contradiction with $\psi \neq 0$. therefore -1 is not an eigenvalue of $\chi G_0 W$ and $1 + \chi G_0 W$ is invertible with bounded inverse. Secondly by an argument of compactness, one deduces that if $\chi_R, R > 1$, is a

family of smooth cut-offs such that $\chi_R(x) = \chi(\frac{x}{R})$ for some function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ for $|x| \leq 1$, then there exists some constant $C > 0$ such that

$$\|(1 + \chi_R G_0 W)^{-1}\| \leq C \quad (5.30)$$

uniformly in $R > 1$, because $\|(1 + \chi_R G_0 W)^{-1} - (1 + G_0 W)^{-1}\| \rightarrow 0$ as $R \rightarrow +\infty$. According to Proposition 4.9, one has

$$1 + \chi_R R_0(z, \theta) W = 1 + \chi_R G_0(\theta) W + O(|z|)$$

in $\mathcal{L}(L^{2, -2\mu}; L^{2, -2\mu})$ for $z \in \Omega_\delta$, where $O(|z|)$ is uniform in $R > 1$. Consequently there exists some constant $c > 0$ independent of R such that the inverse $(1 + \chi_R R_0(z, \theta) W)^{-1}$ exists and is holomorphic for $z \in \Omega_\delta$ with $|z| < c$ and there exists some constant $C > 0$ such that

$$\|(1 + \chi_R R_0(z, \theta) W)^{-1}\|_{\mathcal{L}(L^{2, -2\mu}, L^{2, -2\mu})} \leq C \quad (5.31)$$

uniformly in $R > 1$ and $z \in \Omega_\delta$ with $|z| < c$. This proves that there is no pole of the meromorphic extension of $\chi_R R(z) \chi$ in $\Omega_\delta \cap \{|z| \leq c\}$ for some $c > 0$ independent of R . Since discret eigenvalues and positive outgoing resonances of H are poles of some cut-off resolvent $\chi_R R(z) \chi_R$ if $R > 1$ is large enough, (5.31) implies

$$(\sigma_d(H) \cup r_+(H)) \cap \{z; z \in \Omega_\delta, |z| \leq c\} = \emptyset \quad (5.32)$$

This proves the finiteness of eigenvalues of H in Ω_δ , because zero is the only possible accumulation point of eigenvalues of H in Ω_δ . Estimate (5.26) follows from (4.45) and (5.29).

Part (b) can be derived from (5.29), Proposition 4.9 and Theorem 2.1 applied to $G_0(\theta)$. \blacksquare

Since for $V_0 \in \mathcal{A}$, $\text{Im } V_0 \leq 0$, Theorem 5.1 can be applied which implies that zero is the only possible accumulation point of $r_+(H)$. As consequence of Theorem 5.1 and Proposition 5.5, one obtains the following

Corollary 5.6. *Assume the conditions of Proposition 5.5. Then the set $r_+(H)$ is finite.*

Proof of Theorem 2.2 (a). Theorem 2.2 (a) can be proved in the same way as Theorem 4.4 for the model operator H_0 . By Proposition 5.4, one can find a contour Γ in the right half complex plane of the form

$$\Gamma = \{z; \text{Re } z \geq 0, |\text{Im } z| = C(\text{Re } z)^{\mu'}\}$$

for some $C, \mu' > 0$ such that $\sigma(H) \cap \Gamma = \{0\}$ and there are only a finite number of complex eigenvalues of H located on the left of Γ . Let

$$\Lambda = \sigma(H) \cap \{z; \text{Re } z < 0 \text{ or } \text{Re } z \geq 0 \text{ and } |\text{Im } z| > C(\text{Re } z)^{\mu'}\}.$$

Then one has

$$e^{-tH} - \sum_{\lambda \in \Lambda} e^{-tH} \Pi_\lambda = \frac{i}{2\pi} \int_\Gamma e^{-tz} R(z) dz. \quad (5.33)$$

where Π_λ is the Riesz projection of H associated the eigenvalue λ . Making use of Proposition 5.4, one can prove as in Theorem 4.4 that

$$\|e^{-a\langle x \rangle^{1-\mu}} (e^{-tH} - \sum_{\lambda \in \Lambda} e^{-tH} \Pi_\lambda)\| \leq C_a e^{-c_a t^\beta}, \quad t > 0. \quad (5.34)$$

(2.18) follows since if $\lambda \in \sigma_d(H)$ with $\operatorname{Re} \lambda > 0$, $\|e^{-tH}\Pi_\lambda\|$ decreases exponentially. (2.20) is deduced in a similar way.

(b). According to Proposition 5.5 and Corollary 5.6, there exists some $\eta > 0$ such that Ω_η contains no poles with negative imaginary part of meromorphic extension of $\chi R(z)\chi$ from the upper half complex plane. Under the assumptions Theorem 2.2 (b), $\chi R(z)\chi$ has only a finite number of poles in $\overline{\mathbb{C}}_+$ which are either discrete eigenvalues or positive outgoing resonances of H . Making use of (5.29) for some fixed $\theta \in \mathbb{C}_+$, one obtains the the representation formula:

$$\chi e^{-itH}\chi - \sum_{\lambda \text{ poles in } \overline{\mathbb{C}}_+} \operatorname{Res}(e^{-itz}\chi R(z)\chi; \lambda) = \frac{i}{2\pi} \int_{\Gamma_\eta} e^{-itz}\chi R(z)\chi dz, \quad t > 0, \quad (5.35)$$

where $\operatorname{Res}(e^{-itz}\chi R(z)\chi; \lambda)$ is the residue of $e^{-itz}\chi R(z)\chi$ at pole λ and $\Gamma_\eta = \{z = re^{-i\eta}; r \geq 0\} \cup \{z = -re^{i\eta}; r \geq 0\}$ where $\eta > 0$ is chosen such that there are no eigenvalues of H on Γ_η , nor between Γ_η and the real axis. It is easy to see that

$$\operatorname{Res}(e^{-itz}\chi R(z)\chi; \lambda) = \chi e^{-itH}\Pi_\lambda\chi, \quad \text{if } \lambda \in \sigma_d(H),$$

while for $\lambda \in r_+(H)$ we can only affirm that

$$\operatorname{Res}(e^{-itz}\chi R(z)\chi; \lambda) = \chi e^{-it\lambda}P_\lambda(t)\chi$$

for some operator of finite rank $P_\lambda(t)$ which is polynomial in t . See Remark 6.1 on the rank of coefficients of $P_\lambda(t)$. The subexponential time-decay of the contour integral can be deduced from the Gevrey estimates for the cut-off resolvent (5.27). The details are the same as in the proof of Theorem 4.10 and are omitted here. \blacksquare

5.3. Proof of Theorem 2.3. Assume now that zero is an eigenvalue of $H = H_0 + W$. Then -1 is an eigenvalue of G_0W and $\operatorname{Ker}(1 + G_0W)$ in L^2 coincides with the eigenspace of H with eigenvalue zero. We begin with a decay estimate of the associated eigenfunctions.

Lemma 5.7. *Assume that $H_0 = -\Delta + V_0$ satisfies the condition (2.3). Then there exists some constant $\alpha_0 > 0$ such that if $u \in H^2(\mathbb{R}^n)$ such that $Hu = 0$, then $e^{\alpha_0\langle x \rangle^{1-\mu}}u \in L^2(\mathbb{R}^n)$.*

Proof. Let $\varphi(x) = \alpha\langle x \rangle^{1-\mu}$, $\alpha > 0$. Let χ be a smooth cut-off on \mathbb{R}^n such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Set

$$\varphi_R(x) = \chi\left(\frac{x}{R}\right)\varphi(x), \quad R \geq 1.$$

Then

$$|\nabla\varphi_R(x)| \leq \alpha\left(1 + \frac{C}{R^{1-\mu}}\right)\langle x \rangle^{-\mu} \leq 2\alpha\langle x \rangle^{-\mu}$$

uniformly in $R \geq R_1$ where R_1 is sufficiently large. Let $u \in H^2(\mathbb{R}^n)$. Then one has

$$\begin{aligned} |\langle e^{2\varphi_R}Hu, u \rangle| &= |\langle H(e^{\varphi_R}u), e^{\varphi_R}u \rangle - \langle [\Delta, e^{\varphi_R}]u, e^{\varphi_R}u \rangle| \\ &= |\langle H(e^{\varphi_R}u), e^{\varphi_R}u \rangle + 2\langle (|\nabla\varphi_R|^2 e^{\varphi_R}u - \nabla\varphi_R \cdot \nabla(e^{\varphi_R}u)), e^{\varphi_R}u \rangle| \\ &\geq |\langle H(e^{\varphi_R}u), e^{\varphi_R}u \rangle| - (8\alpha^2 + 2\alpha)\|\langle x \rangle^{-\mu}e^{\varphi_R}u\|^2 - 2\alpha\|\nabla(e^{\varphi_R}u)\|^2 \end{aligned} \quad (5.36)$$

uniformly in $R \geq R_1$. Since W is compactly supported, φ_R is bounded on $\text{supp } W$ uniformly with respect to R . Making use of the condition (2.3), one obtains for some constants $c_0, C > 0$,

$$|\langle H(e^{\varphi_R}u), e^{\varphi_R}u \rangle| \geq c_0(\|\nabla(e^{\varphi_R}u)\|^2 + \|\langle x \rangle^{-\mu} e^{\varphi_R}u\|^2) - C\|u\|^2 \quad (5.37)$$

for all $R \geq R_1$. For $\alpha > 0$ appropriately small, one deduces that there exists some constant $C_1 > 0$ such that

$$\|\langle x \rangle^{-\mu} e^{\varphi_R}u\|^2 + \|\nabla(e^{\varphi_R}u)\|^2 \leq C_1(|\langle e^{2\varphi_R}Hu, u \rangle| + \|u\|^2) \quad (5.38)$$

for any $u \in H^2$ and $R \geq R_1$. If $u \in H^2$ such that $Hu = 0$, it follows that

$$\|\langle x \rangle^{-\mu} e^{\varphi_R}u\|^2 + \|\nabla(e^{\varphi_R}u)\|^2 \leq C_1\|u\|^2 \quad (5.39)$$

for all $R \geq R_1$. This proves that $\langle x \rangle^{-\mu} e^{\varphi}u \in L^2(\mathbb{R}^n)$ and $\nabla(e^{\varphi}u) \in L^2$. Lemma 5.7 is proved, provided that $0 < \alpha_0 < \alpha$. \blacksquare

Theorem 5.8. *Let $H_0 = -\Delta + V_0(x)$ and $H = H_0 + W(x)$ with $V_0 \in \mathcal{V}$ and $W \in L_{\text{comp}}^\infty$. Assume that 0 is an eigenvalue of H and that both H_0 and H are selfadjoint. Let Π_0 denote the eigenprojection of H associated with eigenvalue zero. Then there exist some constants $C, \mu', \delta > 0$ such that*

$$R(z) = -\frac{\Pi_0}{z} + R_1(z) \quad (5.40)$$

for $z \in \Omega_1(\delta)$ where

$$\Omega_1(\delta) = \{z \in \mathbb{C}; |z| < \delta, \text{ either } \text{Re } z < 0 \text{ or } \text{Re } z \geq 0 \text{ and } |\text{Im } z| > C|\text{Re } z|^{\mu'}\}. \quad (5.41)$$

The remainder $R_1(z)$ satisfies the estimates

$$\|\langle x \rangle^{-s} R_1(z)\| + \|R_1(z)\langle x \rangle^{-s}\| \leq C_s \quad (5.42)$$

for $s > 2\mu + \frac{1}{\mu'}$ and $z \in \Omega_1(\delta)$; and for any $a, M > 0$ there exist some constants $C_a, c_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} R_1^{(N)}(z)\| + \|R_1^{(N)}(z)e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a c_a^N N^{\sigma N}, \quad (5.43)$$

for any $N \in \mathbb{N}$ and $z \in \Omega_-$ where $\Omega_- = \{z; \text{Re } z < 0 \text{ and } |\text{Im } z| \leq -M\text{Re } z\} \cup \{0\}$, $M > 0$. Here and in the following, $R_1^{(N)}(z)$ denotes the N -th derivative of $R_1(z)$ and $\sigma = 1 + \gamma = \frac{1+\mu}{1-\mu}$.

Proof. We use the Grushin method to study the low-energy asymptotics for the resolvent of H by using the equation

$$R(z) = (1 + R_0(z)W)^{-1}R_0(z). \quad (5.44)$$

Since the method is well-known in selfadjoint case, we shall skip over some details and emphasize on the Gevrey estimates of the remainder. Note that $\text{Ker}_{L^{2,s}}(1 + G_0W)$ is independent of $s \in \mathbb{R}$ and coincides with the eigenspace of H associated with the eigenvalue 0. We need only to work in $L^2(\mathbb{R}^n)$.

Let ψ_1, \dots, ψ_m be a basis of $\text{Ker}(1 + G_0W)$ such that

$$\langle \psi_j, -W\psi_k \rangle = \delta_{jk}, \quad j, k = 1, \dots, m. \quad (5.45)$$

(5.45) can be realized because the quadratic form $\phi \rightarrow \langle \phi, -W\phi \rangle = \langle \phi, H_0\phi \rangle$ is positive definite on $\text{Ker}(1 + G_0W)$. Define $Q : L^2 \rightarrow L^2$ by

$$Qf = \sum_{j=1}^m \langle -W\psi_j, f \rangle \psi_j, \quad f \in L^2. \quad (5.46)$$

Set $Q' = 1 - Q$. Then Q commutes with $1 + G_0W$. -1 is not eigenvalue of compact operator $Q'(G_0W)Q'$, hence $Q'(1 + G_0W)Q'$ is invertible on the range of Q' with bounded inverse. From Theorem 3.4 with $N = 1$ and Proposition 4.1, one deduces that

$$(R_0(z) - G_0)W = O(|z|) \quad (5.47)$$

for $z \in \Omega_1(\delta)$. It follows that if $\delta > 0$ is small enough,

$$E(z) = (Q'(1 + R_0(z)W)Q')^{-1}Q' \quad (5.48)$$

is well-defined and continuous in $z \in \Omega_1(\delta)$ and is uniformly bounded:

$$\|E(z)\| \leq C \quad (5.49)$$

uniformly in $z \in \Omega_1(\delta)$. By Corollary 3.5 and (5.19), $E(z)$ satisfies Gevrey estimates

$$\|E^{(N)}(z)\| \leq CC'^N N^{\sigma N} \quad (5.50)$$

for some $C' > 0$ and for all $z \in \Omega_-$.

Define $S : \mathbb{C}^m \rightarrow D(H)$ and $T : L^2 \rightarrow \mathbb{C}^m$ by

$$Sc = \sum_{j=1}^m c_j \psi_j, \quad c = (c_1, \dots, c_m) \in \mathbb{C}^m,$$

$$Tf = (\langle -W\psi_1, f \rangle, \dots, \langle -W\psi_m, f \rangle) \in \mathbb{C}^m, \quad f \in L^2.$$

Set $W(z) = (1 + R_0(z)W)$ and

$$E_+(z) = S - E(z)W(z)S, \quad (5.51)$$

$$E_-(z) = T - TW(z)E(z), \quad (5.52)$$

$$E_{-+}(z) = -TW(z)S + TW(z)E(z)W(z)S. \quad (5.53)$$

Then one has the formula

$$(1 + R_0(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \text{ on } H^{1,-s}. \quad (5.54)$$

Since $E(z)$, $W(z)$ satisfy Gevrey estimates of the form (5.50) on Ω_- , $E_{\pm}(z)$ and $E_{-+}(z)$ satisfy similar Gevrey estimates on Ω_- . The leading term of $E_{-+}(z)$ can be explicitly calculated:

$$E_{-+}(z) = -z\Psi + z^2r_1(z) \quad (5.55)$$

where the matrix $\Psi = (\langle \psi_j, \psi_k \rangle)_{1 \leq j, k \leq m}$ is positive definite and $r_1(z)$ satisfies the Gevrey estimates in Ω_- . Consequently,

$$E_{-+}(z)^{-1} = -\frac{\Psi^{-1}}{z} + \tilde{r}_1(z) \quad (5.56)$$

with $\tilde{r}_1(z)$ uniformly bounded on $\Omega_1(\delta)$ and $\tilde{r}_1(z)$ satisfying the Gevrey estimates of the form (5.50) in Ω_- . Consequently $(1 + R_0(z)W)^{-1}$ is of the form

$$(1 + R_0(z)W)^{-1} = \frac{A_0}{z} + B(z) \quad (5.57)$$

where

$$A_0 = S\Psi^{-1}T \quad (5.58)$$

is an operator of rank m and $B(z)$ is uniformly bounded $\Omega_1(\delta)$ and satisfies the Gevrey estimates

$$\|B^{(N)}(z)\| \leq CC'^N N^{\sigma N}, \quad \forall N \in \mathbb{N}, \quad (5.59)$$

for z in Ω_- . From the equation $R(z) = (1 + R_0(z)W)^{-1}R_0(z)$ and Corollary 3.5, we deduce that

$$R(z) = -\frac{\Pi_0}{z} + R_1(z) \quad (5.60)$$

where $R_1(z)$ satisfies

$$\|R_1(z)\langle x \rangle^{-2k\mu}\| \leq C$$

for $z \in \Omega_-$ if $k \in \mathbb{N}$ and $k \geq \frac{1}{\mu}$ and

$$\|R_1^{(N)}(z)e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a c_a^N N^{\sigma N}$$

for z in Ω_- and $N \in \mathbb{N}$. This proves (5.42) and (5.43). \blacksquare

Theorem 5.9. *Let $H_0 = -\Delta + V_0(x)$ and $H = H_0 + W(x)$ with $V_0 \in \mathcal{A}$ and $W \in L_{\text{comp}}^\infty$. Assume that 0 is an eigenvalue of H and that both H_0 and H are selfadjoint. Let Π_0 denote the eigenprojection of H associated with eigenvalue zero. Let Ω_δ be defined as in Proposition 5.5 and $\Omega_\delta(c) = \Omega_\delta \cap \{|z| < c\}$. Then there exist some constants $C, c, \mu', \delta > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ the cut-off resolvent $\chi R(z)\chi$ defined for $\text{Im } z > 0$ extends to a holomorphic function in $\Omega_\delta(c)$ and one has*

$$\chi R(z)\chi = -\frac{\chi \Pi_0 \chi}{z} + R_2(z) \quad (5.61)$$

for $z \in \Omega_\delta(c)$ where the remainder $R_2(z)$ is continuous up to $z = 0$ and satisfies the Gevrey estimates

$$\|\chi R_2^{(N)}(z)\chi\| \leq C_\chi C^N N^{\sigma N} \quad (5.62)$$

for $z \in \Omega_\delta(c) \cup \{0\}$ and for all $N \in \mathbb{N}^*$.

Proof. It suffices to prove (5.61) for $\chi \in C_{\text{at}}^\infty(\mathbb{R}^n)$ with sufficiently large support. Let $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi_1(x) \leq 1$, $\chi_1(x) = 1$ for $|x| \leq 1$ and 0 for $|x| \geq 2$. Set

$$\chi_j(x) = \chi_0\left(\frac{x}{jR}\right), \quad j = 1, 2, \quad (5.63)$$

where $R > R_0$ is to be adjusted and R_0 is such that $\text{supp } W \subset \{x; |x| \leq R_0\}$. Then $\chi_j W = W$ and $\chi_1 \chi_2 = \chi_1$. Then one has

$$\chi_1 R(z)\chi_1 = (1 + \chi_2 R_0(z, \theta)W)^{-1} \chi_2 R_0(z, \theta)\chi_1, \quad (5.64)$$

where the analytical distortion is carried out outside the support of χ_2 . (5.64) initially valid for θ real and $\text{Im } z > 0$ allows to extend $z \rightarrow \chi R(z)\chi$ into a sector below the positive real axis when $\text{Im } \theta > 0$. In the following $\theta \in \mathbb{C}$ is fixed with $\text{Im } \theta > 0$. $1 + \chi_2 R_0(z, \theta)W$ and $\chi_2 R_0(z, \theta)\chi_1$ belong to Gevrey class $G^\beta(\Omega_\delta)$ where Ω_δ is defined in Proposition 5.5.

Let $\{\psi_j, j = 1, \dots, m\}, Q, Q'$ be defined as in the proof of Theorem 5.8. Then -1 is not an eigenvalue of compact operator $Q'(G_0 W)Q'$. Since $Q'(\chi_2 G_0 W)Q'$ converges

to $Q'G_0WQ'$ in operator norm as $R \rightarrow \infty$, -1 is not an eigenvalue of $Q'\chi_1G_0WQ'$ if $R \geq R_1$ for some $R_1 \geq R_0$, R_1 sufficiently large. Then $Q'(1 + \chi_2G_0W)Q'$ is invertible on $\text{Range } Q'$, so is $Q'(1 + \chi_2R_0(z, \theta)W)Q'$ for $z \in \Omega_\delta(c) = \Omega_\delta \cap \{z; |z| < c\}$ for some $c > 0$. The inverse

$$E_0(z, \theta) = (Q'(1 + \chi_2R_0(z, \theta)W)Q')^{-1}Q' \quad (5.65)$$

is uniformly bounded in z (see Proposition 4.9) and by (5.19) it belongs to Gevrey class $G^\beta(\Omega_\delta(c))$.

Define $S_1 : \mathbb{C}^m \rightarrow L^2$ and $T_1 : L^2 \rightarrow \mathbb{C}^m$ by

$$S_1 = \chi_1 S, \quad T_1 = T\chi_1 \quad (5.66)$$

where S, T are defined in Theorem 5.8. By Lemma 5.7,

$$S_1 T_1 = Q' + O(e^{-cR^{1-\mu}}), \quad T_1 S_1 = 1 + O(e^{-cR^{1-\mu}}) \quad (5.67)$$

for some $c > 0$. Let $W(z, \theta) = 1 + \chi_2 R_0(z, \theta)W$. Consider the Grushin problem

$$\begin{pmatrix} W(z, \theta) & S_1 \\ T_1 & 0 \end{pmatrix} : L^2 \otimes \mathbb{C}^m \rightarrow L^2 \otimes \mathbb{C}^m. \quad (5.68)$$

One has

$$\begin{pmatrix} W(z, \theta) & S_1 \\ T_1 & 0 \end{pmatrix} \begin{pmatrix} E_0(z, \theta) & S_1 \\ T_1 & -T_1 W(z, \theta) S_1 \end{pmatrix} = 1 + \mathcal{R}(z, \theta) \quad (5.69)$$

where

$$\mathcal{R}(z, \theta) = \begin{pmatrix} QW(z, \theta)E_0(z, \theta) + S_1 T_1 - Q & (1 - T_1)W(z, \theta)S_1 \\ T_1 E_0(z, \theta) & T_1 S_1 - 1 \end{pmatrix}. \quad (5.70)$$

$\mathcal{R}(z, \theta)$ is sum of a nilpotent matrix and a matrix of order $O(e^{-cR^{1-\mu}})$. Hence $1 + \mathcal{R}(z)$ is invertible $z \in \Omega_\delta(c)$ if $R > R_1$ is sufficiently large. This proves the Grushin problem is invertible from the right. Similarly one can show it is invertible from the left, therefore it is invertible with inverse given by

$$\begin{pmatrix} E_0(z, \theta) & S_1 \\ T_1 & -T_1 W(z, \theta) S_1 \end{pmatrix} (1 + \mathcal{R}(z))^{-1} := \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \quad (5.71)$$

As usual, one has the formula

$$(1 + \chi_2 R_0(z, \theta)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z). \quad (5.72)$$

$E_{-+}(z)$ is of the form

$$E_{-+}(z) = -T_1 W(z, \theta) S_1 (1 + O(e^{-cR^{1-\mu}})) + O(|z|^2)$$

By the choice of χ_1, χ_2 , one has

$$T_1 W(z, \theta) S_1 = T_1 (1 + R_0(z, \theta)W) S_1 \quad (5.73)$$

$$= z T_1 G_1(\theta) W S_1 + O(|z|^2) = z T_1 G_1 W S_1 + O(|z|^2). \quad (5.74)$$

By the calculation made in the proof of Theorem 5.8, one sees $\Psi_1 = T_1 G_1 S_1$ is an invertible matrix (if R is large enough). Consequently $E_{-+}(z)$ is invertible for $z \in \Omega_\delta(c)$ with inverse of the form.

$$E_{-+}(z)^{-1} = -\frac{1}{z} \Psi_1 (1 + O(e^{-cR^{1-\mu}})) + B(z) \quad (5.75)$$

where $B(z)$ belongs to $G^\sigma(\Omega_\delta(c))$. This proves the existence of an asymptotic expansion for $\chi_1 R(z)\chi_1$ for $z \in \Omega_\delta(c)$ of the form

$$\chi_1 R(z)\chi_1 = -\frac{U}{z} + R_2(z) \quad (5.76)$$

with $R_2(z)$ satisfying Gevrey estimates of order σ on $\Omega_\delta(c)$. To determine U , we remark that since $\mathcal{A} \subset \mathcal{V}$, Theorem 5.8 applied to $R(z)$ with $z \in \Omega_\delta(c) \cap \{\operatorname{Re} z \leq 0\}$ gives $U = \chi_1 \Pi_0 \chi$. \blacksquare

Proof of Theorem 2.3. Theorem 2.3 (a) and (b) are respectively deduced from Theorems 5.8 and 5.9 and the formulas for $t > 0$

$$e^{-tH} - \sum_{\lambda \in \sigma_d(H), \operatorname{Re} \lambda \leq 0} e^{-tH} \Pi_\lambda = \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\Gamma(\epsilon)} e^{-tz} R(z) dz + O(e^{-ct}) \quad (5.77)$$

$$\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \mathbb{R}_-} e^{-itH} \Pi_\lambda) \chi = \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\Gamma_\eta(\epsilon)} e^{-itz} \chi R(z) \chi dz + O(e^{-ct}) \quad (5.78)$$

where $c > 0$ and

$$\begin{aligned} \Gamma(\epsilon) &= \{z; |z| \geq \epsilon, \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\} \cup \{z; |z| = \epsilon, |\arg z| \geq \omega_0\} \\ \Gamma_\eta(\epsilon) &= \{z = re^{-i\eta}, r \geq \epsilon\} \cup \{z = -re^{i\theta}, r \geq \epsilon\} \cup \{z; |z| = \epsilon, -\eta \leq \arg z \leq \pi + \eta\} \end{aligned}$$

for some appropriate constants $C, \mu' > 0, \eta > 0$. In particular, $\eta > 0$ is chosen such that H has no eigenvalues with negative imaginary part above $\Gamma_\eta(\epsilon)$. Here ω_0 is the argument of the point z_0 with $|z_0| = \epsilon$, $\operatorname{Re} z_0 > 0$ and $\operatorname{Im} z_0 = C(\operatorname{Re} z_0)^{\mu'}$. Remark that the subexponential time-decay estimates are derived from Gevrey estimates of $R_1(z)$ and $R_2(z)$ at zero and their Taylor expansion of order N with N chosen appropriately in terms of $t > 0$. See the proof of Theorem 4.4 for e^{-tH_0} . \blacksquare

Remark 5.2. As an example of applications of Theorem 2.3, consider the Witten-Laplacian

$$-\Delta_U = {}^t \nabla_U \cdot \nabla_U \quad (5.79)$$

where $\nabla_U = e^{-U} \nabla e^U$ and $U \in C^2(\mathbb{R}^n)$. Then

$$-\Delta_U = -\Delta + (\nabla U)(x) \cdot (\nabla U)(x) - \Delta U(x)$$

If $U \in C^2(\mathbb{R}^n; \mathbb{R})$ satisfies for some $\rho \in]0, 1[$ and $c_1, C_1 > 0$,

$$U(x) \geq c_1 \langle x \rangle^\rho, \quad |\nabla U(x)| \geq c_1 \langle x \rangle^{\rho-1}, \quad |\partial_x^\alpha U(x)| \leq C_1 \langle x \rangle^{\rho-|\alpha|} \quad (5.80)$$

for x outside some compact and for $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$. Then $-\Delta_U$ can be decomposed as $-\Delta_U = H_0 + W(x)$ where H_0 satisfies the conditions of Theorem 2.1 with $\mu = 1 - \rho$ and $W(x)$ is of compact support. Zero is a simple eigenvalue of $-\Delta_U$ embedded in its continuous spectrum $[0, +\infty[$. As consequence of Theorem 2.3, one obtains the following result. Let $\varphi_0(x)$ be a normalized eigenfunction of $-\Delta_U$ with eigenvalue zero:

$$\varphi_0(x) = C e^{-U(x)}, \quad \|\varphi_0\| = 1. \quad (5.81)$$

Then for any $a > 0$, there exist some constants $C_a, c_a > 0$

$$\|e^{t\Delta_U} f - \langle \varphi_0, f \rangle \varphi_0\| \leq C_a e^{-c_a t^{\frac{\rho}{2-\rho}}} \|e^{a(x)} f\| \quad (5.82)$$

for $t > 0$ and f such that $e^{a(x)^p} f \in L^2$. Note that the subexponential convergence estimate (5.82) without the explicit remainder estimate on f is proved in [5] by method of Markov processes.

6. THRESHOLD SPECTRAL ANALYSIS IN NON-SELFADJOINT CASE

6.1. The general case. Finally we study the case zero is an embedded eigenvalue of the non-selfadjoint Schrödinger operator H . Let $V_0 \in \mathcal{V}$. Then zero is an eigenvalue of H if and only if -1 is an eigenvalue of compact operator $K = G_0 W$ on $L^2(\mathbb{R}^n)$. The algebraic multiplicity m of eigenvalue -1 of K is finite, although we do not know how to define the algebraic multiplicity of zero eigenvalue of H . Let $\pi_1 : L^2 \rightarrow L^2$ be the associated Riesz projection of K defined by :

$$\pi_1 = \frac{1}{2\pi i} \int_{|z+1|=\epsilon} (z - K)^{-1} dz$$

for $\epsilon > 0$ small enough. Then

$$m = \text{Rank } \pi_1. \quad (6.1)$$

π_1 is continuous on $L^{2,s}$ for any $s \in \mathbb{R}$ and $\pi_1^* : L^2 \rightarrow L^2$ is the Riesz projection of K^* associated with the eigenvalue -1 .

By Corollary 4.2, $R_0(z)W$ is continuous in $z \in \Omega(\delta)$, where

$$\Omega(\delta) = \{|z| < \delta; \text{Re } z < 0 \text{ or } \text{Re } z \geq 0 \text{ and } |\text{Im } z| > M(\text{Re } z)^{\mu'}\}$$

for some M, δ and $\mu' > 0$. Denote $\pi'_1 = 1 - \pi_1$. $\pi'_1(1 + G_0 W)\pi'_1$ is injective on the range of π'_1 . The Fredholm Theorem implies that $(\pi'_1(1 + G_0 W)\pi'_1)^{-1}$ is invertible on L^2 . It follows that

$$B_1(z) = (\pi'_1(1 + R_0(z)W)\pi'_1)^{-1}\pi'_1 \quad (6.2)$$

is well defined on $\Omega(\delta)$ if $\delta > 0$ is sufficiently small. In addition $B_1(z)$ is uniformly bounded there. Since $R_0(z)W$ satisfies Gevrey estimates of order σ for z near 0 with $\text{Re } z < 0$ and $|\text{Im } z| < -C\text{Re } z$, $C > 0$, so does $B_1(z)$. $\pi_1(1 + R_0(z)W)\pi_1$ is of finite rank. Set

$$\omega(z) = \det(\pi_1(1 + R_0(z)W)\pi_1). \quad (6.3)$$

Then $\pi_1(1 + R_0(z)W)\pi_1$ is invertible if and only if $\omega(z) \neq 0$. $\omega(z)$ satisfies the Gevrey estimates of order σ at point $z = 0$ and has an asymptotic expansion of the form

$$\omega(z) = \sum_{j=1}^N \omega_j z^j + O(|z|^{N+1}), z \in \Omega(\delta), \quad (6.4)$$

for any N .

Proposition 6.1. *Assume that*

$$\omega(z) = \omega_k z^k + O(|z|^{k+1}) \quad (6.5)$$

for some $k \in \mathbb{N}^*$ and $\sigma_k \neq 0$. Let -1 be an eigenvalue of G_0W with algebraic multiplicity m . Then there exist operators C_{-j} , $j = 1, \dots, k$ with rank less than or equal to m such that

$$R(z) = \frac{C_{-k}}{z^k} + \dots + \frac{C_{-1}}{z} + R_3(z) \quad (6.6)$$

for $z \in \Omega(\delta)$. The remainder $R_3(z)$ satisfies the estimates: $\exists C, \mu', \delta > 0$ such that

$$\|\langle x \rangle^{-s} R_3(z)\| + \|R_3(z) \langle x \rangle^{-s}\| \leq C_s \quad (6.7)$$

for $s > 2\mu + \frac{1}{\mu'}$ and $z \in \Omega(\delta)$.

Proof. Since $\omega_k \neq 0$, $\pi_j(1 + R_0(z)W)\pi_1$ is invertible on the range of π_1 for $z \in \Omega(\delta)$ with $\delta > 0$ small enough. Set $B_0(z) = (\pi_j(1 + R_0(z)W)\pi_1)^{-1}\pi_1$. Then $\omega(z)B_0(z)$ has same continuity properties as $\pi_j(1 + R_0(z)W)\pi_1$ and

$$B_0(z) = z^{-k}B_{-k}^{(0)} + \dots + z^{-1}B_{-1}^{(0)} + R^{(0)}(z) \quad (6.8)$$

where $B_{-j}^{(0)}$, $j = 1, \dots, k$, are operators of rank $\leq m$ and $R^{(0)}(z)$ is uniformly bounded for $z \in \Omega(\delta)$. One can check that

$$(1 + R_0(z)W)(B_0(z) + B_1(z)) = 1 + O(|z|), \quad (B_0(z) + B_1(z))(1 + R_0(z)W) = 1 + O(|z|) \quad (6.9)$$

in $\mathcal{L}(L^2)$. Therefore $(1 + R_0(z)W)$ is invertible for $z \in \Omega(\delta)$ if $\delta > 0$ is small enough and

$$((1 + R_0(z)W)^{-1} = B_0(z)(1 + O(|z|))^{-1} + O(1), \quad z \in \Omega(\delta). \quad (6.10)$$

(6.6) can be now derived from the equation $R(z) = (1 + R_0(z)W)^{-1}R_0(z)$. \blacksquare

Remark 6.1. Under the conditions of Theorem 5.1, let $\lambda \in r_+(H)$ be an outgoing positive resonance of H . Let π_1 denote the Riesz projection of eigenvalue -1 of $R_0(\lambda + i0)W$ as operator on $L^{2,-s}$, $\frac{1}{2} < s < \rho - \frac{1}{2}$. Denote

$$\omega(z) = \det(\pi_1(1 + \chi R_0(z)W)\pi_1)$$

for $z \in \mathbb{C}_+$, where $\chi \in C_0^\infty$ with $\chi W = W$. Then one can show that $\omega(\lambda) = 0$ and $\omega(z) \neq 0$ for z with $\text{Im } z \gg 1$. In addition, $\omega(z)$ extends to a holomorphic function into a complex neighbourhood of λ . One concludes that there exist some $k \in \mathbb{N}^*$ and some $\omega_k \neq 0$ such that

$$\omega(z) = \omega_k(z - \lambda)^k + O(|z - \lambda|^{k+1}) \quad (6.11)$$

for z in a complex neighbourhood of λ . This means a condition analogous to (6.5) is satisfied for positive resonances under some analyticity condition on potentials. The proof of Proposition 6.1 and formula (5.29) allow to conclude that the meromorphic extension from \mathbb{C}_+ of the cut-off resolvent $\chi R(z)\chi$ admits an expansion around λ of the form:

$$\chi R(z)\chi = \chi \left(\frac{C_{-k}}{(z - \lambda)^k} + \dots + \frac{C_{-1}}{z - \lambda} \right) + \tilde{R}_3(z)\chi \quad (6.12)$$

for z near λ , where C_{-j} is of rank less than or equal to $m_+(\lambda)$ and $\tilde{R}_3(z)$ is holomorphic in a neighbourhood of λ . Here $m_+(\lambda) = \text{Rank } \pi_1$ is the algebraic multiplicity of eigenvalue -1 of $R_0(\lambda + i0)W$.

6.2. Representation of the Riesz projection. In order to give some more precisions on the resolvent expansion given in Proposition 6.6, we study in more details the Riesz projection π_1 associated with eigenvalue -1 . Assume from now on that this eigenvalue is geometrically simple. Set $K = G_0W$. Then

$$\dim \text{Ker}(1 + K) = 1, \quad \text{Rank } \pi_1 = m. \quad (6.13)$$

Operator $1 + K$ being nilpotent on $\text{Range } \pi_1$, there exists some function $\phi_m \in \text{range } \pi_1$ such that

$$\phi_j = (1 + K)^{m-j} \phi_m \neq 0, \quad j = 1, \dots, m. \quad (6.14)$$

One has

$$(1 + K)\phi_1 = 0, \quad (1 + K)\phi_j = \phi_{j-1}, \quad 2 \leq j \leq m. \quad (6.15)$$

ϕ_1, \dots, ϕ_m are linearly independent. Denote J the operation of complex conjugaison $J : f \rightarrow \bar{f}$. Remark that $H_0^* = JH_0J$, $H^* = JHJ$. One has

$$JWK = K^*\bar{W}J. \quad (6.16)$$

It follows that

$$JW\pi_1 = \pi_1^*JW. \quad (6.17)$$

Denote

$$\phi_j^* = \overline{W\phi_j}. \quad (6.18)$$

Then

$$(1 + K^*)\phi_1^* = 0, \quad (1 + K^*)\phi_j^* = \phi_{j-1}^*, \quad 2 \leq j \leq m. \quad (6.19)$$

Since $\phi_1^* \neq 0$, it follows that $\phi_j^* \neq 0$ for all $1 \leq j \leq m$. From this, we deduce that $\{\phi_j^*, j = 1, \dots, m\}$ is linearly independent and that $\text{Rank } \pi_1 \leq \text{Rank } \pi_1^*$. Similarly, using the relation

$$JG_0\pi_1^* = \pi_1JG_0 \quad (6.20)$$

one can prove that $\text{Rank } \pi_1 \geq \text{Rank } \pi_1^*$, which gives

Lemma 6.2. *One has*

$$\text{Rank } \pi_1 = \text{Rank } \pi_1^* = m \quad (6.21)$$

and JW is a bijection from $\text{Range } \pi_1$ onto $\text{Range } \pi_1^*$.

Lemma 6.3. *The bilinear form $B(\cdot, \cdot)$ defined on $\text{Range } \pi_1$ by*

$$B(\varphi, \psi) = \langle \varphi, JW\psi \rangle = \int_{\mathbb{R}^n} W(x)\varphi(x)\psi(x) dx \quad (6.22)$$

is non-degenerate.

Proof. Let $\phi \in \text{Range } \pi_1$ such that

$$\int_{\mathbb{R}^n} W(x)\phi(x)\varphi(x) dx = 0$$

for all $\varphi \in \text{Range } \pi_1$. This means that $\phi \in (\text{Range } \pi_1^*)^\perp = \text{Ker}\pi_1$, which implies that $\phi = \pi_1\phi = 0$. So $B(\cdot, \cdot)$ is non-degenerate. \blacksquare

As a consequence of Lemma 6.2, if $m = 1$, then eigenfunction φ of H associated with zero eigenvalue satisfies

$$\int_{\mathbb{R}^n} W(x)(\varphi(x))^2 dx \neq 0. \quad (6.23)$$

Lemma 6.4. *There exist $\chi_j \in \text{Ker}(1 + K)^{m-j+1}$, $j = 1, \dots, m$, such that*

$$\langle \phi_i, \chi_j^* \rangle = B(\phi_i, \chi_j) = \delta_{ij}, \quad 1 \leq i, j \leq m, \quad (6.24)$$

where $\chi_j^* = JW\chi_j$, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Proof. We use an induction to prove that for any $1 \leq l \leq m$, there exist $\varphi_j \in \text{Ker}(1 + K)^j$, $1 \leq j \leq l$ such that

$$B(\varphi_i, \phi_{m-j+1}) = \delta_{ij}, \quad 1 \leq j \leq i \leq l. \quad (6.25)$$

Since $\phi_1 \in \text{Ker}(1 + K)$ and $\phi_j^* \in \text{Range}(1 + K^*)$ for $1 \leq j \leq m-1$, one has $\langle \phi_1, \phi_j^* \rangle = 0$ for $j = 1, \dots, m-1$. By lemma 6.3, one has necessarily $c_1 = \langle \phi_1, \phi_m^* \rangle \neq 0$. Set

$$\varphi_1 = \frac{1}{c_1} \phi_1. \quad (6.26)$$

Then $\varphi_1 \in \text{Ker}(1 + K)$ and $B(\varphi_1, \phi_m) = 1$. (6.25) is true for $l = 1$. Assume now that (6.25) is true for some $l = k-1$, $2 \leq k \leq m$. Set

$$\phi'_k = \phi_k - \sum_{j=1}^{k-1} B(\phi_k, \phi_{m-j+1}) \varphi_j \quad (6.27)$$

Then $\phi'_k \neq 0$, $\phi'_k \in \text{Ker}(1 + K)^k$ and

$$B(\phi'_k, \phi_{m-j+1}) = 0, \quad j = 1, \dots, k-1.$$

Since $\phi'_k \in \text{Ker}(1 + K)^k$, one has also

$$\langle \phi'_k, \phi_j^* \rangle = B(\phi'_k, \phi_j) = 0 \quad (6.28)$$

for $j = 1, \dots, m-k$, because $\phi_j^* = (1 + K^*)^{m-j} \phi_m^*$ belongs to the range of $(1 + K^*)^k$ if $1 \leq j \leq m-k$. By Lemma 6.3, the constant $c_k = B(\phi'_k, \phi_{m-k+1})$ must be nonzero. Set

$$\varphi_k = \frac{1}{c_k} \phi'_k. \quad (6.29)$$

Then (6.25) is proved for $l = k$. By an induction, one can construct φ_j , $1 \leq j \leq m$, such that (6.25) holds with $l = m$. By (6.28), one has also $B(\varphi_i, \phi_{m-j+1}) = 0$ if $i > j$. Lemma 6.4 is proved by taking $\chi_k = \varphi_{m-k+1}$, $1 \leq k \leq m$. \blacksquare

One has the following representation of the Riesz projection π_1 .

Corollary 6.5. *One has*

$$\pi_1 u = \sum_{j=1}^m \langle u, \chi_j^* \rangle \phi_j, \quad u \in H^{1,-s}, s > 1. \quad (6.30)$$

Proof. Denote π the operator $\pi : u \rightarrow \sum_{j=1}^m \langle u, \chi_j^* \rangle \phi_j$. Then it is clear that $\pi^2 = \pi$ and $\text{Range } \pi = \text{Range } \pi_1$. It is trivial that $\text{Ker } \pi_1 \subset \text{Ker } \pi$. If $u \in \text{Ker } \pi$, then $\langle u, \chi_j^* \rangle = 0$ for $j = 1, \dots, m$. Therefore $u \in (\text{Range } \pi_1^*)^\perp = \text{Ker } \pi_1$ which implies that $\text{Ker } \pi \subset \text{Ker } \pi_1$. This shows that $\text{Ker } \pi_1 = \text{Ker } \pi$. This proves $\pi = \pi_1$. \blacksquare

From the proof of Lemma 6.4, one sees that if -1 is a simple eigenvalue of K ($m = 1$), then the associated Riesz projection is given by

$$\pi_1 = \langle \cdot, \varphi^* \rangle \varphi \quad (6.31)$$

where φ is an eigenfunction of K with eigenvalue -1 normalized by

$$\int_{\mathbb{R}^n} W(x)(\varphi(x))^2 dx = 1.$$

6.3. Resolvent expansion at threshold. To study the singularity of the resolvent $R(z)$ at threshold zero, we use the resolvent equation

$$R(z) = (1 + R_0(z)W)^{-1}R_0(z)$$

for $z \notin \sigma(H)$ and study the following Grushin problem in $L^2 \times \mathbb{C}^m$:

$$\begin{pmatrix} 1 + R_0(z)W & S \\ T & 0 \end{pmatrix} : L^2 \times \mathbb{C}^m \rightarrow L^2 \times \mathbb{C}^m \quad (6.32)$$

where

$$S : \mathbb{C}^m \rightarrow L^2, c = (c_1, \dots, c_m) \rightarrow Sc = \sum_{j=1}^m c_j \phi_j, \quad (6.33)$$

$$T : L^2 \rightarrow \mathbb{C}^m, f \rightarrow Tf = (\langle f, \chi_1^* \rangle, \dots, \langle f, \chi_m^* \rangle). \quad (6.34)$$

$$(6.35)$$

Then $ST = \pi_1$ and $TS = I_n$. Since K commutes with its Riesz projection π_1 and since $1 + K$ is injective on $\text{Range } \pi_1'$ where $\pi_1' = 1 - \pi_1$, $1 + K$ is invertible on the range of π_1' . By an argument of perturbation, $\pi_1'(1 + R_0(z)W)\pi_1'$ is invertible on range of π_1' for $z \in \Omega_1(\delta)$ if $\delta > 0$ is appropriately small and its inverse $E(z)$ is uniformly bounded on $\Omega_1(\delta)$ where

$$E(z) = (\pi_1'(1 + R_0(z)W)\pi_1')^{-1}\pi_1' \quad (6.36)$$

By the arguments used in Section 6.1, $E(z)$ belongs to the Gevrey class $G^\sigma(\Omega_1(\delta))$ with $\sigma = 1 + \gamma$. One can check that for $z \in \Omega_1(\delta)$,

$$\begin{pmatrix} 1 + R_0(z)W & S \\ T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \quad (6.37)$$

where

$$E_+(z) = (1 - E(z)R_0(z)W)S \quad (6.38)$$

$$E_-(z) = T(1 - R_0(z)WE(z)) \quad (6.39)$$

$$E_{-+}(z) = -T(1 + R_0(z)W)S + TR_0(z)WE(z)R_0(z)WS. \quad (6.40)$$

It follows that $z \notin \sigma(P)$ if and only if $\det E_{-+}(z) \neq 0$ and one has

$$(1 + R_0(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z). \quad (6.41)$$

By operations on Gevrey functions ((5.17) -(5.22)), $E_{-+}(z)$ is $m \times m$ -matrix valued Gevrey function for $z \in \Omega_1(\delta)$. Therefore it has an asymptotic expansion near 0 up to any order

$$E_{-+}(z) = B_0 + B_1z + \dots + B_Nz^N + O(|z|^{N+1}) \quad (6.42)$$

where B_j is some $m \times m$ matrix. More precisely, since $TR_0(z)WE(z)R_0(z)WS = O(|z|^2)$, $E_{-+}(z)$ verifies

$$\begin{aligned} E_{-+}(z) &= (-\langle (1 + R_0(z)W)\phi_k, \chi_j^* \rangle)_{1 \leq j, k \leq m} + O(|z|^2) \\ &= - \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - z \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm} \end{pmatrix} + O(|z|^2) \end{aligned} \quad (6.43)$$

where

$$b_{jk} = \langle G_1 W \phi_k, \chi_j^* \rangle. \quad (6.44)$$

Note that ϕ_1 and χ_m belong to $\text{Ker}(1 + G_0 W)$ and $\chi_m^* = JW\chi_m$, they are rapidly decreasing, by Lemma 5.7. One can calculate

$$\begin{aligned} b_{m1} &= \lim_{\lambda \rightarrow 0^-} \langle \frac{1}{\lambda} (1 + R_0(\lambda)W)\phi_1, JW\chi_m \rangle \\ &= - \lim_{\lambda \rightarrow 0^-} \langle R_0(\lambda)\phi_1, JW\chi_m \rangle \\ &= - \lim_{\lambda \rightarrow 0^-} \langle \phi_1, JR_0(\lambda)W\chi_m \rangle \\ &= \langle \phi_1, J\chi_m \rangle. \end{aligned}$$

Similarly one can calculate for $2 \leq j \leq m$

$$b_{mj} = -\langle W\phi_j, JG_0\chi_m \rangle = \langle \phi_j - \phi_{j-1}, J\chi_m \rangle.$$

Summing up, we have proved the following

Proposition 6.6. *$\det E_{-+}(z)$ is a Gevrey function of order σ for $z \in \Omega_1(\delta)$ and has an asymptotic expansion in powers of z*

$$\det E_{-+}(z) = \sigma_1 z + \cdots + \sigma_N z^N + O(|z|^{N+1}) \quad (6.45)$$

for any N , where

$$\sigma_1 = -b_{m1}. \quad (6.46)$$

Theorem 6.7. *Let $H_0 = -\Delta + V_0(x)$ and $H = H_0 + W(x)$ with $V_0 \in \mathcal{V}$ and $W \in L_{\text{comp}}^\infty$. Assume that zero eigenvalue of H is geometrically simple.*

(a). *Suppose that*

$$\det E_{-+}(z) = \sigma_k z^k + O(|z|^{k+1}) \quad (6.47)$$

for some $\sigma_k \neq 0$, $k \geq 1$. Then there exist operators C_j , $j = -k, \dots, -1$ with ranks less than or equal to m such that

$$R(z) = \frac{C_{-k}}{z^k} + \cdots + \frac{C_{-1}}{z} + R_3(z) \quad (6.48)$$

for $z \in \Omega_1(\delta)$, where C_{-j} , $1 \leq j \leq k-1$, are of rank less than or equal to m and C_{-k} is a rank one operator given by

$$C_{-k} = \langle \cdot, J\varphi_1 \rangle \varphi_1, \quad (6.49)$$

with φ_1 an eigenfunction of H associated with zero eigenvalue. The remainder $R_3(z)$ satisfies the estimates: $\exists C, \mu', \delta > 0$ such that

$$\|\langle x \rangle^{-s} R_3(z)\| + \|R_3(z) \langle x \rangle^{-s}\| \leq C_s \quad (6.50)$$

for $s > 2\mu + \frac{1}{\mu'}$ and $z \in \Omega_1(\delta)$; and for any $a > 0$, $\exists C_a, c_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} R_3^{(N)}(z)\| + \|R_3^{(N)}(z) e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a c_a^N N^{\sigma N}, \quad (6.51)$$

for any $N \in \mathbb{N}$ and $z \in \Omega_-$ where $\Omega_1(\delta)$ and Ω_- are the same as in Theorem 5.8.

(b). Suppose in addition that there exists an eigenfunction φ_0 of H associated with eigenvalue zero such that

$$\int_{\mathbb{R}^n} (\varphi_0(x))^2 dx = 1. \quad (6.52)$$

Then Condition (6.47) is satisfied with $k = 1$ and one has

$$C_{-1} = -\langle \cdot, J\varphi_0 \rangle \varphi_0. \quad (6.53)$$

Proof. (a). The existence of the resolvent expansion is proved in Proposition 6.1 and the Gevrey estimates of the remainder can be obtained in the same way as in Theorem 5.8. We only calculate C_{-k} . Under the condition (6.47), one has

$$E_{-+}(z)^{-1} = \frac{{}^t \text{Com } E_{-+}(z)}{\det E_{-+}(z)} = z^{-k} C + O(|z|^{-k+1}) \quad (6.54)$$

for $z \in \Omega_1(\delta)$, where

$$C = \begin{pmatrix} 0 & \cdots & 0 & \sigma_k^{-1} \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

From (6.41), one obtains

$$(1 + R_0(z)W)^{-1} = -z^{-k} SCT + O(|z|^{-k+1}). \quad (6.55)$$

Using the definition of S and T , one sees

$$SCTf = \frac{1}{\sigma_k} \langle f, \chi_m^* \rangle \phi_1 \quad (6.56)$$

Noticing that

$$\langle G_0 f, \chi_m^* \rangle = \langle f, G_0^* J W \chi_m \rangle = \langle f, J G_0 W \chi_m \rangle = -\langle f, J \chi_m \rangle,$$

we deduce from (5.44) that

$$R(z) = \frac{C_{-k}}{z^k} + O(|z|^{-k+1}) \quad (6.57)$$

for $z \in \Omega_1(\delta)$, where C_{-k} is of rank one, given by

$$C_{-k} f = \frac{1}{\sigma_k} \langle f, J \chi_m \rangle \phi_1 \quad (6.58)$$

Since χ_m and ϕ_1 belong to the one dimensional space $\text{Ker}(1 + K)$, C_{-k} can be written as

$$C_{-k} f = \langle f, J \varphi_1 \rangle \varphi_1 \quad (6.59)$$

where φ_1 is an eigenfunction of H with eigenvalue zero. This proves part (a).

If (6.52) is satisfied, then one has $\chi_m = d_1\varphi_0$ and $\phi_1 = d_2\varphi_0$ for some constants $d_j \neq 0$. Therefore

$$\sigma_1 = -\langle \phi_1, J\chi_m \rangle = -d_1d_2 \neq 0.$$

Condition (6.47) is satisfied with $k = 1$. Set

$$\psi_0 = \sqrt{\frac{d_1}{b_{m1}}} \phi_1. \quad (6.60)$$

Then $C_{-1} = -\langle \cdot, J\psi_0 \rangle \psi_0$. ψ_0 is an eigenfunction of H with eigenvalue zero and

$$\int_{\mathbb{R}^n} (\psi_0(x))^2 dx = \frac{\langle \phi_1, J\chi_m \rangle}{b_{m1}} = 1.$$

Since zero eigenvalue of H is geometrically simple, one has $\psi_0 = \pm\varphi_0$. This proves

$$C_{-1} = -\langle \cdot, J\varphi_0 \rangle \varphi_0. \quad (6.61)$$

■

Remark 6.2. *The methods used here can be applied to other threshold spectral problems. In particular for non-selfadjoint Schrödinger operator $H = -\Delta + V(x)$ with a quickly decreasing complex potential $V(x)$ on \mathbb{R}^3 :*

$$|V(x)| \leq C\langle x \rangle^{-\rho}, \rho > 2, \quad (6.62)$$

our method allows to calculate the low-energy asymptotics of the resolvent $(H - z)^{-1}$ if zero is a resonance but not an eigenvalue. In fact using the same reduction scheme and similar calculations, one can show in this case $E_{-+}(z)$ takes the form

$$E_{-+}(z) = - \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - z^{\frac{1}{2}} \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm} \end{pmatrix} + O(|z|^{\frac{1}{2}+\epsilon}).$$

The characterization of resonant state ensures that $b_{m1} \neq 0$. See [12] in the selfadjoint case. Therefore one can explicitly calculate the leading term of the asymptotic expansion of $(H - z)^{-1}$ for z near 0 in the case zero is a resonance but not an eigenvalue.

Theorem 6.8. $V_0 \in \mathcal{A}$. Assume that zero is a geometrically simple eigenvalue of H . Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\Omega_\delta(c)$ be defined as in Theorem 5.9 Under the condition (6.47), the meromorphic extension of $\chi R(z)\chi$ from \mathbb{C}_+ verifies

$$\chi R(z)\chi = \chi \left(\frac{C_{-k}}{z^k} + \cdots + \frac{C_{-1}}{z} + R_4(z) \right) \chi \quad (6.63)$$

for $z \in \Omega_\delta(c)$, where C_{-j} is the same as in Theorem 6.7 and the remainder $R_4(z)$ is continuous up to $z = 0$ and satisfies the Gevrey estimates

$$\|\chi R_4^{(N)}(z)\chi\| \leq C_\chi C^N N^{\sigma N} \quad (6.64)$$

for $z \in \Omega_\delta(c) \cup \{0\}$. In addition if (6.52) is true, Condition (6.47) is satisfied with $k = 1$ and (6.63) holds with $C_{-1} = -\langle \cdot, J\varphi_0 \rangle \varphi_0$.

Theorem 6.8 is derived by combining methods used in Theorem 5.9 and Theorem 6.7. The details are omitted.

Remark 6.3. *If zero eigenvalue of H is not geometrically simple, combining methods used in Proposition 6.1 and Theorems 5.8 and 5.9, one can show that the resolvent expansions given in Theorems 6.7 and 6.8 still hold with the corresponding Gevrey estimates on remainders. But in this case we can only affirm that C_{-j} is of rank $\leq m$ for $j = 1, \dots, k$, as in Proposition 6.1.*

Proof of Theorem 2.4. Theorem 6.8 implies that outgoing positive resonances are absent in neighbourhood of zero. Therefore under the conditions of this theorem, $r_+(H)$ is at most a finite set. If zero eigenvalue is geometrically simple, the results of Theorem 2.4 for e^{-tH} can be derived from Theorem 6.7 and formula (5.77) when $V_0 \in \mathcal{V}$ and those for e^{-itH} are obtained from Theorem 6.8 and formula (5.78) when $V_0 \in \mathcal{A}$. Taking notice of Remark 6.3, one can prove in the same way the results of Theorem 2.4 when zero eigenvalue is not geometrically simple. ■

Example 6.4. *Consider the non-selfadjoint Witten Laplacian*

$$-\Delta_U = -\Delta + (\nabla U)(x) \cdot (\nabla U)(x) - \Delta U(x)$$

where $U \in C^2(\mathbb{R}^n; \mathbb{C})$. Set $U(x) = U_1(x) + iU_2(x)$ with U_1, U_2 real valued functions. Assume that U_1 satisfies the condition (5.80) with U replaced by U_1 and that U_2 is of compact support with $\|\partial_x^\alpha U_2\|_{L^\infty}$ sufficiently small for $|\alpha| \leq 2$. Considering $-\Delta_U$ as a perturbation of $-\Delta_{U_1}$, one can show that $-\Delta_U$ has only one eigenvalue in a neighbourhood of zero which is in addition geometrically simple. Therefore the eigenfunctions associated zero eigenvalue of $-\Delta_U$ are of the form $ce^{-U(x)}$ for some $c \neq 0$, one concludes that the condition (6.52) is satisfied if $\|U_2\|_{L^\infty}$ is sufficiently small.

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