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GEVREY ESTIMATES OF THE RESOLVENT AND SUB-EXPONENTIAL TIME-DECAY OF SOLUTIONS

XUE PING WANG

ABSTRACT. In this article, we study a class of nonselfadjoint Schrödinger operators which are perturbation of a model operator satisfying some weighted coercive assumption. For the model operator, we prove that the derivatives of the resolvent satisfy some Gevrey estimates at the threshold zero. As application, we establish large time expansions for the semigroups e^{-tH} and e^{-itH} for $t > 0$ with subexponential time-decay estimates on the remainder. We also study the case when zero is an embedded eigenvalue of non-selfadjoint Schrödinger operators.

1. INTRODUCTION

This work is concerned with the time-decay of semigroups e^{-tH} and e^{-itH} as $t \rightarrow +\infty$ where H is a compactly supported perturbation of some model operator $H_0 = -\Delta + V(x)$ with a complex-valued potential $V(x) = V_1(x) - iV_2(x)$, where either $V_1(x)$ or $V_2(x)$ are slowly decreasing like $\frac{1}{\langle x \rangle^{2\mu}}$ for some $0 < \mu < 1$. Here $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. There are many works on low-energy spectral analysis of selfadjoint Schrödinger operators $-\Delta + V(x)$ with a real decreasing potential $V(x)$ verifying

$$|V(x)| \leq C \langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^n, \quad (1.1)$$

for some $\rho > 0$. Here we only mention [2, 10] for quickly decaying potentials ($\rho > 2$), [15] for critically decaying potentials ($\rho = 2$) under an assumption of Hardy inequality for the model operator and [14] in one-dimensional case when this Hardy condition is not satisfied. For slowly decreasing potentials ($0 < \rho < 2$), there are works of [5] when the potential is negative and [13, 19] when it is globally positive. When $\rho \geq 2$, threshold zero may be an eigenvalue and/or a resonance and for critically decaying potentials, threshold resonance may appear in any space dimension with arbitrary multiplicity. For slowly decreasing potentials ($0 < \rho < 2$), threshold resonance is absent and low-energy spectral analysis has not yet been done in presence of zero eigenvalue. We study here a class of non-selfadjoint Schrödinger operators which are compactly supported perturbation of some model operator H_0 which satisfies a weighted coercive condition. Part of the results of this work are announced in [18].

The analysis of the class of non-selfadjoint operators considered in this work is in part motivated by large-time behavior of solutions to the Kramers-Fokker-Planck equation with a slowly increasing potential. After a change of unknowns and for appropriate

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values of physical constants, the Kramers-Fokker-Planck equation can be written into the form

$$\partial_t u(t; x, v) + Pu(t; x, v) = 0, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, \quad n \geq 1, \quad t > 0, \quad (1.2)$$

with some initial data

$$u(0; x, v) = u_0(x, v). \quad (1.3)$$

Here P is the Kramers-Fokker-Planck operator:

$$P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x - (\nabla U(x)) \cdot \nabla_v, \quad (1.4)$$

where the potential $U(x)$ is supposed to be a real-valued C^1 function. Define \mathfrak{M} by

$$\mathfrak{M}(x, v) = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{1}{2}(v^2 + U(x))}. \quad (1.5)$$

Then one has $P\mathfrak{M} = 0$. If $|\nabla U(x)| \geq C > 0$ and $U(x) > 0$ outside some compact, $U(x)$ increases at least linearly and $\mathfrak{M} \in L^2$, then 0 is in the discrete spectra of P and after suitable normalization, one has in $L^2(\mathbb{R}^{2n})$

$$e^{-tP} u_0 = \langle \mathfrak{M}, u_0 \rangle \mathfrak{M} + O(e^{-\sigma t}), \quad t \rightarrow +\infty. \quad (1.6)$$

for some $\sigma > 0$ depending on the spectral gap between 0 (see for example [7, 8, ?]). In this case when where $U(x)$ is of short-range, the essential spectrum of P is equal to $[0, \infty[$. It is proved in [17] by scattering method that for short-range potentials in dimension three one has

$$e^{-tP} u_0 = \frac{1}{(4\pi t)^{\frac{3}{2}}} \langle \mathfrak{M}, u_0 \rangle \mathfrak{M} + O(t^{-\frac{3}{2}-\epsilon}), \quad t \rightarrow +\infty, \quad (1.7)$$

in appropriately weighted L^2 -spaces. It is conjectured in [17] that when the potential $U(x)$ increases sublinearly: $U(x) \sim |x|^\tau$, $0 < \tau < 1$, then one should have

$$e^{-tP} u_0 = \langle \mathfrak{M}, u_0 \rangle \mathfrak{M} + O(e^{-\sigma t^{\frac{\tau}{2-\tau}}}), \quad t \rightarrow +\infty. \quad (1.8)$$

There exists a probabilistic approach of subexponentially convergence to invariant measures in Markov processes. See the lecture notes of P. Cattiaux [3] for an overview on this approach. While polynomially decaying remainder estimate is obtained in [4], the subexponential remainder estimate (1.18) is proved in a recent work of T. Li and Z. Zhang ([12]) using method of weak Poincaré inequality. Note that M. Klein and J. Rama ([11]) used Gevrey estimates in different context to study large time evolution of quantum resonance states. In this article we only study Schrödinger operators. The Gevrey estimate of the resolvent at threshold proved in this work is of interest in itself. It allows us to study not only semigroup of the heat equation, but also that of the Schrödinger equation under some analyticity assumption on potentials.

Before stating our results, let us recall some known results ([13, 19, 20]) for selfadjoint Schrödinger operators with globally positive and slowly decreasing potentials. Let $H_0 = -\Delta + V(x)$ be selfadjoint. Assume that there exist some constants $\mu \in]0, 1[$ and $c_1, c_2 > 0$ such that

$$c_1 \langle x \rangle^{-2\mu} \leq V(x) \leq c_2 \langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n. \quad (1.9)$$

Under some additional conditions on derivatives of V , it is known ([13, 20]) that the spectral measure $E'(\lambda)$ of H_0 is smooth at $\lambda = 0$ and satisfies for any $N \geq 0$

$$\|E'(\lambda)\|_{L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}} = O(|\lambda|^N), \quad \lambda \rightarrow 0, \quad (1.10)$$

$$\|e^{-tH_0}\|_{L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}} = O(e^{-ct^\beta}) \quad (1.11)$$

where $\beta = \frac{1-\mu}{1+\mu}$ and c is some positive constant. In one dimensional case, if $V(x)$ is in addition analytic, D. Yafaev ([19]) proves that

$$\|e^{-itH_0}\|_{L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}} = O(e^{-c|t|^\beta}), \quad |t| \rightarrow +\infty. \quad (1.12)$$

The proof of [19] is based on an explicit construction of solutions to time-dependent Schrödinger equation in one dimensional case. This kind of construction is not available in higher dimensions. When specified to the selfadjoint case ($V_2 = 0$), the Gevrey estimates of the resolvent at threshold obtained in this work allow to prove a subexponential estimate for $E'(\lambda)$ near zero and (1.12) in dimensions $n \geq 2$.

The model operator H_0 used in this work is a class second order elliptic operators satisfying a weighted coercive condition. Consider

$$H_0 = - \sum_{i,j=1}^n \partial_{x_i} a^{ij}(x) \partial_{x_j} + \sum_{j=1}^n b_j(x) \partial_{x_j} + V(x), \quad (1.13)$$

where $a^{ij}(x)$, $b_j(x)$ and $V(x)$ are complex-valued measurable functions. Suppose that $a^{ij}, b_j \in C_b^1(\mathbb{R}^n)$ and that there exists $c > 0$ such that

$$\text{Re}(a^{ij}(x)) \geq cI_n, \quad \forall x \in \mathbb{R}^n. \quad (1.14)$$

Assume that V is relatively bounded with respect to $-\Delta$ with relative bound zero, $\text{Re} H_0 \geq 0$ and that there exists some constants $0 < \mu < 1$ and $c_0 > 0$ such that

$$|\langle H_0 u, u \rangle| \geq c_0 (\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2), \quad \text{for all } u \in H^2, \quad (1.15)$$

$$\sup_x |\langle x \rangle^\mu b_j(x)| < \infty, \quad j = 1, \dots, n. \quad (1.16)$$

Condition (1.15) is called weighted coercive condition.

Remark 1.1. If $H_0 = -\Delta + V(x)$ with $V(x) = V_1(x) - iV_2(x)$ with $V_j(x) \geq 0$, the weighted coercive condition (1.15) is satisfied if

$$V_1(x) + V_2(x) \geq c \langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n. \quad (1.17)$$

If $V_1(x)$ is globally positive and slowly decaying (i. e. $V_1(x) \geq c \langle x \rangle^{-2\mu}$ for some $\mu \in]0, 1[$ and $c > 0$), then (1.15) is satisfied by $-\Delta + V_1(x) - iV_2(x)$ for any real function V_2 which is $-\Delta$ -bounded with relative bound zero.

Note also that when we study Schrödinger operators $H_0 = -\Delta + V$ by technics of analytic deformation, the deformed operator can be written in the form (1.13). Condition $V(x)$ is $-\Delta$ -bounded allows to include a class of multiparticle interactions.

Under the assumptions 1.14, 1.15 and 1.16, one can show that H_0 is bijective from $D(H_0) = H^2(\mathbb{R}^n)$ to $R(H_0)$ and $R(H_0)$ is dense in $L^2(\mathbb{R}^n)$. Let $G_0 : R(H_0) \rightarrow D(H_0)$ be the algebraic inverse of H_0 . Denote $L^{2,s} = L^2(\mathbb{R}_x^n; \langle x \rangle^{2s} dx)$ and $\mathcal{D} = \bigcap_{s \in \mathbb{R}} L^{2,s}$. Then $G_0(\mathcal{D}) \subset \mathcal{D}$ and G_0 is a densely defined, continuous from $R(H_0) \cap L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$. (See Lemma 2.3). To simplify notion, we still denote by G_0 its continuous extension by density so that G_0 is regarded as a bounded operator from $L^{2,s}$ to $L^{2,s-2\mu}$. Consequently for any $N \in \mathbb{N}$, $G_0^N : L^{2,s} \rightarrow L^{2,s-2\mu N}$ is well defined for any $s \in \mathbb{R}$. Let $R_0(z) = (H_0 - z)^{-1}$ for $z \notin \sigma(H)$. One has

$$\text{s-} \lim_{z \in \Omega(\delta), z \rightarrow 0} R_0(z) = G_0$$

as operators from $L^{2,s}$ to $L^{2,s-2\mu}$, where $\Omega(\delta) = \{z; \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta\}$ for some $\delta > 0$.

Theorem 1.1. *Assume the conditions (1.13), (1.14), (1.15) and (1.16). The following estimates hold.*

(a). *For any $a > 0$, there exists $C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} G_0^N\| + \|G_0^N e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a^N N^{\gamma N}, \forall N. \quad (1.18)$$

(b). *There exists some constant $C > 0$ such that $\forall \chi \in C_0^\infty(\mathbb{R}^n)$, one has for some $C_\chi > 0$*

$$\|\chi(x) G_0^N\| + \|G_0^N \chi(x)\| \leq C_\chi C^N N^{\gamma N}, \forall N. \quad (1.19)$$

Here $\gamma = \frac{2\mu}{1-\mu}$.

Since one has

$$\frac{d^N}{dz^N} R_0(z)|_{z=0} = N! G_0^{N+1},$$

the estimates given in Theorem 1.1 can be regarded as Gevrey estimates of the resolvent at threshold zero.

To study large time behavior of semigroups, we introduce two classes of potentials \mathcal{V} and \mathcal{A} . Let \mathcal{V} denote the class of complex-valued potentials V such that

$$V \text{ is } -\Delta\text{-compact and (1.15) is satisfied for some } \mu \in]0, 1[. \quad (1.20)$$

and

$$\text{Re } H_0 \geq -\alpha \Delta \text{ and } |\text{Im } V(x)| \leq C \langle x \rangle^{-2\mu'} \quad (1.21)$$

for some constants $\alpha, \mu', C > 0$.

To study the time-decay of the Schrödinger equation, we will use both technics of analytical dilation and analytical deformation. Let \mathcal{A} denote the class of complex-valued potentials $V(x) = V_1(x) - iV_2(x)$ for $x \in \mathbb{R}^n$ with $n \geq 2$ such that $-\Delta + V(x)$ satisfies the estimate (1.15) for some $\mu \in]0, 1[$ verifying

$$0 < \mu < \frac{3}{4} \text{ if } n = 2 \text{ and } 0 < \mu < 1 \text{ if } n \geq 3; \quad (1.22)$$

and that V_1 and V_2 are dilation analytic ([1]) and extend holomorphically into a complex region of the form

$$\Omega = \{x \in \mathbb{C}^n; |\text{Im } x| < c|\text{Re } x|\} \cup \{x \in \mathbb{C}^n; |x| > c^{-1}\}$$

for some $c > 0$ and satisfy for some $c_j > 0$ and $R \in [0, +\infty]$

$$|V_j(z)| \leq c_1 \langle \operatorname{Re} z \rangle^{-2\mu}, \quad z \in \Omega, \quad j = 1, 2, \quad (1.23)$$

$$V_2(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \quad (1.24)$$

$$x \cdot \nabla V_1(x) \leq -c_3 \frac{x^2}{\langle x \rangle^{2\mu+2}}, \quad x \in \mathbb{R}^n \text{ with } |x| \geq R, \text{ and} \quad (1.25)$$

$$V_2(x) \geq c_5 \langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n \text{ with } |x| < R. \quad (1.26)$$

Remark that when $R = 0$, (1.25) is a global virial condition on V_1 and (1.26) is void; while if $R = +\infty$, no virial condition is needed on V_1 , but (1.26) is required on the whole space which means that the dissipation is strong. Potentials of the form

$$V(x) = \frac{c}{\langle x \rangle^{2\mu}} - iV_2(x) \quad (1.27)$$

satisfy conditions (1.23)-(1.26) with $R = 0$, if $V_2 \geq 0$ and V_2 is holomorphic in Ω satisfying $|V_2(z)| \leq C \langle z \rangle^{-2\mu}$ for $z \in \Omega$.

Clearly, $\mathcal{A} \subset \mathcal{V}$. For $V \in \mathcal{A}$, one can study quantum resonances of $H_0 = -\Delta + V(x)$ by both analytical dilation or analytical deformation outside some compact ([1, 6]). We shall show that under the conditions (1.24), (1.25) and (1.26), there are no eigenvalues nor quantum resonances of H_0 in a sector below the positive real half-axis in complex plane.

Let $V \in \mathcal{V}$, $H_0 = -\Delta + V(x)$ and $H = H_0 + W(x)$ be a compactly supported perturbation of H_0 : $W \in L_{\text{comp}}^\infty = \{u \in L^\infty(\mathbb{R}^n), \text{supp } u \text{ compact}\}$. Then one can prove that H has only at most a finite number of discrete eigenvalues located at the left of a curve Γ of the forme

$$\Gamma = \{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\}$$

and there exists a nice control of the resolvent of H_0 on Γ . Note that zero may be an embedded eigenvalue, but it is never a resonance of H and that complex eigenvalues of H may accumulate to zero from the right side of Γ . Let $\sigma_d(H)$ ($\sigma_p(H)$, resp.) denote the set of discrete eigenvalues of H (the set of eigenvalues of H , resp.).

Theorem 1.2. *Assume that 0 is not an eigenvalue of H . The following statements hold.*

(a). *Let $V \in \mathcal{V}$. For any $a > 0$ there exist $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} (e^{-tH} - \sum_{\lambda \in \sigma_d(H), \operatorname{Re} \lambda \leq 0} e^{-tH} \Pi_\lambda)\| \leq C_a e^{-c_a t^{\frac{1-\mu}{1+\mu}}} \quad t > 0, \quad (1.28)$$

(b). *Let $V \in \mathcal{A}$. Then there exists some constant $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \mathbb{R}_-} e^{-itH} \Pi_\lambda) \chi\| \leq C_\chi e^{-ct^{\frac{1-\mu}{1+\mu}}} \quad t > 0, \quad (1.29)$$

Here Π_λ denotes the Riesz projection associated with the discrete eigenvalue λ of H .

When zero is an embedded eigenvalue, in the selfadjoint case, we can apply Grushin-Feshbach method to compute low-energy expansion. Theorem 1.1 allows to estimate remainders in Gevrey spaces and to prove the following

Theorem 1.3. *Assume that 0 is an embedded eigenvalue of H and that both H and H_0 are selfadjoint.*

(a). *If $V \in \mathcal{V}$, then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}}(e^{-tH} - \sum_{\lambda \in \sigma_p(H), \operatorname{Re} \lambda \leq 0} e^{-t\lambda} \Pi_\lambda)\| \leq C_a e^{-c_a t^{\frac{1-\mu}{1+\mu}}} \quad t > 0, \quad (1.30)$$

(b). *Let $V \in \mathcal{A}$. Then there exists some constant $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has*

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_p(H) \cap \mathbb{R}_-} e^{-it\lambda} \Pi_\lambda)\chi\| \leq C_\chi e^{-ct^{\frac{1-\mu}{1+\mu}}} \quad t > 0, \quad (1.31)$$

Here Π_λ denotes the eigenprojection of H associated with eigenvalue λ of H .

Theorem 1.3 can be applied to a class of Witten Laplacians for which zero is an eigenvalue embedded in the continuous spectrum which is equal to $[0, +\infty[$. Our result is new concerning the Schrödinger equation associated to this class of Witten Laplacians. See Section 6.

The case zero is an embedded eigenvalue of non-selfadjoint Schrödinger operator H is more difficult. There is not yet general method for threshold spectral analysis of non-selfadjoint operators. In this work, we only the case zero eigenvalue is geometrically simple under some condition. Remark that zero is an eigenvalue of H if and only if -1 is an eigenvalue of $G_0 W$. Since in our case $G_0 W$ is compact on $L^2(\mathbb{R}^n)$, zero eigenvalue of H is of finite geometrical multiplicity. Since this eigenvalue is embedded in the essential spectrum, we do not know how to definite its algebraic multiplicity.

Theorem 1.4. *Assume that 0 is a geometrically simple eigenvalue of H and that there exists an associated eigenfunction φ such that*

$$\int_{\mathbb{R}^n} (\varphi_0(x))^2 dx = 1. \quad (1.32)$$

Let Π_0 be defined by

$$\Pi_0 = \langle \cdot, J\varphi_0 \rangle \varphi_0 \quad (1.33)$$

where J is complex conjugation: $J : f(x) \rightarrow \overline{f(x)}$. The following results hold.

(a). *If $V \in \mathcal{V}$, then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}}(e^{-tH} - \sum_{\lambda \in \sigma_d(H), \operatorname{Re} \lambda \leq 0} e^{-t\lambda} \Pi_\lambda - \Pi_0)\| \leq C_a e^{-c_a t^{\frac{1-\mu}{1+\mu}}} \quad t > 0, \quad (1.34)$$

(b). *Let $V \in \mathcal{A}$. Then there exist $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$,*

$$\|\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \mathbb{R}_-} e^{-it\lambda} \Pi_\lambda - \Pi_0)\chi\| \leq C_\chi e^{-ct^{\frac{1-\mu}{1+\mu}}} \quad t > 0, \quad (1.35)$$

Here if $\lambda \neq 0$, Π_λ denotes the Riesz projection of H associated with eigenvalue λ of H .

The organisation of this paper is as follows. Sections 2– 5 are devoted to the analysis of the model operator $H_0 = -\Delta + V(x)$ verifying condition (1.15). In Section 2, we prove Theorem 1.1. We first establish a uniform energy estimate which allows to control the growth of powers of the resolvent at threshold in weighted spaces. Then Theorem 1.1 is deduced by an appropriate induction. In Section 3, we evaluate the numerical range of H_0 and prove resolvent estimates for H_0 on a curve located in the right half complex-plane. An estimate like (1.28) is proved for H_0 . In Section 4, we prove the absence of complex eigenvalues for a class of non-selfadjoint Schrödinger operators H which are compactly supported perturbation of H_0 and we also give an improvement for the estimate on spectral measure obtained by S. Nakamura [13] in selfadjoint case. The subexponential time-decay of e^{-itH_0} is studied in section 5 when potential V belongs to the class \mathcal{A} . We show that there exists a contour located in the lower half complex plane passing by 0 on which the cut-off resolvent $\chi(H_0 - z)^{-1}\chi$ is uniformly bounded and that there are no quantum resonances and eigenvalues of H_0 in a sector below the positive half real axis. (1.29) for H_0 is then obtained by deforming the integral contour into the lower half complex plane. Compactly supported perturbations of H_0 are studied in Sections 6 and 7. In Section 6, we study the low-energy resolvent expansion for H and prove Theorems 1.2 and 1.3. Since the method of low-energy spectral analysis used in this Section is known, we emphasize upon Gevrey estimates on remainders and Theorems 1.2 and 1.3 are proved from low-energy resolvent expansion by the same methods used for H_0 . Finally in Section 7, we study threshold eigenvalue for non-selfadjoint Schrödinger operators and prove Theorem 1.4. We firstly establish a representation formula for the Riesz projection π_1 associated to the compact operator G_0W with eigenvalue -1 and then use Grushin method to compute the leading term of the resolvent. The Gevrey estimates of the remainder can be obtained as in Section 6 and hence the details are omitted in Section 7.

Notation. We denote $H^{r,s}$, $r \geq 0, s \in \mathbb{R}$ the weighted Sobolev space of order r with the weight $\langle x \rangle^s$ on \mathbb{R}^n :

$$H^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^n); \|u\|_{r,s} = \|\langle x \rangle^s (1 - \Delta)^{\frac{r}{2}} u\|_{L^2} < \infty\}.$$

For $r < 0$, $H^{r,s}$ is defined as dual space of $H^{-r,-s}$ with dual product identified with the scalar product $\langle \cdot, \cdot \rangle$ of $L^2(\mathbb{R}^n)$. Denote $H^{0,s} = L^{2,s}$. $\mathcal{L}(r, s; r', s')$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. If $(r, s) = (r', s')$, we denote $\mathcal{L}(r, s) = \mathcal{L}(r, s; r', s')$.

2. GEVREY ESTIMATES OF THE RESOLVENT AT THRESHOLD

The starting point of our Gevrey estimates of the resolvent is a uniform *a priori* energy estimate for the model operator H_0 . In the sequel, we need to apply this kind of energy estimates both to the Schrödinger operator $-\Delta + V_1(x) - iV_2(x)$ and to its analytically dilated or distorted versions as well. For this purpose, we begin with a

setting where H_0 is a second order elliptic differential operator of the form

$$H_0 = - \sum_{i,j=1}^n \partial_{x_i} a^{ij}(x) \partial_{x_j} + \sum_{j=1}^n b_j(x) \partial_{x_j} + V(x) \quad (2.1)$$

satisfying conditions (1.13), (1.14), (1.15) and (1.16).

Denote $b = (b_1, \dots, b_n)$ and

$$|a|_\infty = \max_{1 \leq i,j \leq n} \sup_{x \in \mathbb{R}^n} |a^{ij}(x)|, \quad |b|_{\mu,\infty} = \max_{1 \leq j \leq n} \sup_{x \in \mathbb{R}^n} |\langle x \rangle^\mu b_j(x)|. \quad (2.2)$$

For $s \in \mathbb{R}$, denote

$$\varphi_s(x) = (1 + \frac{|x|^2}{R_s^2})^s, \quad (2.3)$$

where $R_s = M \langle s \rangle^{\frac{1}{1-\mu}}$ with $M = M(c_0, |a|_\infty, |b|_{\mu,\infty}) > 1$ large enough, but independent of $s \in \mathbb{R}$. The uniformity in $s \in \mathbb{R}$ in the following lemma is important for Gevrey estimates of the resolvent at threshold.

Lemma 2.1. *Let H_0 be given by (2.1). Under the conditions (1.15) and (1.16) with $0 < \mu < 1$, there exist some constants $C, M > 0$ depending only on $|a|_\infty$, $|b|_{\mu,\infty}$ and c_0 given in (1.15) such that*

$$\|\langle x \rangle^{-\mu} \varphi_s(x) u\| + \|\nabla(\varphi_s(x) u)\| \leq C \|\langle x \rangle^\mu \varphi_s(x) H_0 u\| \quad (2.4)$$

for any $s \in \mathbb{R}$ and $u \in H^2(\mathbb{R}^n)$ with $\langle x \rangle^{|s|+\mu} H_0 u \in L^2$.

Proof. We calculate $\langle u, \varphi_s^2 H_0 u \rangle$ for $u \in C_0^\infty$:

$$\langle u, \varphi_s^2 H_0 u \rangle \quad (2.5)$$

$$= \langle \varphi_s u, H_0(\varphi_s u) \rangle + \langle \varphi_s u, [\sum_{i,j=1}^n \partial_{x_i} a^{ij} \partial_{x_j}, \varphi_s] u \rangle - \langle \varphi_s u, (b \cdot \nabla \varphi_s) u \rangle$$

$$= I + II + III, \quad (2.6)$$

where

$$I = \langle \varphi_s u, H_0(\varphi_s u) \rangle$$

$$II = \langle \varphi_s u, \sum_{i,j=1}^n ((\partial_{x_i} \varphi_s) a^{ij} \partial_{x_j} u + \partial_{x_i} (a^{ij} (\partial_{x_j} \varphi_s) u)) \rangle$$

$$III = -\langle \varphi_s u, (b \cdot \nabla \varphi_s) u \rangle.$$

Since $\varphi_s \partial_{x_j} u = \partial_{x_j}(\varphi_s u) - (\partial_{x_j} \varphi_s) u$, one has

$$\begin{aligned} & |\langle \varphi_s u, (\partial_{x_i} \varphi_s) a^{ij} \partial_{x_j} u + \partial_{x_i} (a^{ij} (\partial_{x_j} \varphi_s) u) \rangle| \\ &= | \langle (\partial_{x_i} \varphi_s) u, a^{ij} (\partial_{x_j}(\varphi_s u) - (\partial_{x_j} \varphi_s) u) \rangle + \langle \varphi_s u, \partial_{x_i} (a^{ij} (\partial_{x_j} \varphi_s) u) \rangle | \\ &= | \langle (\partial_{x_i} \varphi_s) u, a^{ij} (\partial_{x_j}(\varphi_s u) - (\partial_{x_j} \varphi_s) u) \rangle - \langle \partial_{x_i}(\varphi_s u), a^{ij} (\partial_{x_j} \varphi_s) u \rangle | \\ &\leq |a|_\infty (\|(\partial_{x_i} \varphi_s) u\| (\|\partial_{x_j}(\varphi_s u)\| + \|(\partial_{x_j} \varphi_s) u\|) + \|\partial_{x_i}(\varphi_s u)\| \|(\partial_{x_j} \varphi_s) u\|) \end{aligned}$$

The term II in (2.5) can be bounded by

$$\begin{aligned} |II| &\leq |a|_\infty \left(\sum_{i=1}^n \|(\partial_{x_i} \varphi_s)u\| \right) \left(\sum_{j=1}^n (2\|\partial_{x_j}(\varphi_s u)\| + \|(\partial_{x_j} \varphi_s)u\|) \right) \\ &\leq n^2 |a|_\infty \|(\nabla \varphi_s)u\| (2\|\nabla(\varphi_s u)\| + \|(\nabla \varphi_s)u\|) \\ &\leq n^2 |a|_\infty (\epsilon \|\nabla(\varphi_s u)\|^2 + (1 + \frac{1}{\epsilon}) \|(\nabla \varphi_s)u\|^2) \end{aligned}$$

for any $\epsilon > 0$. Clearly, III verifies

$$|III| \leq |b|_{\mu, \infty} \|\langle x \rangle^{-\mu} \varphi_s u\| \|(\nabla \varphi_s)u\| \leq |b|_{\mu, \infty} (\epsilon \|\langle x \rangle^{-\mu} \varphi_s u\|^2 + \frac{1}{4\epsilon} \|(\nabla \varphi_s)u\|^2) \quad (2.7)$$

Taking $\epsilon = \epsilon(c_0, |a|_\infty, |b|_{\mu, \infty}) > 0$ appropriately small where $c_0 > 0$ is given by (1.15), it follows from (1.15) that

$$\begin{aligned} |\langle u, \varphi_s^2 H u \rangle| &\geq |I| - |II| - |III| \\ &\geq \frac{c_0}{2} (\|\nabla(\varphi_s u)\|^2 + \|\langle x \rangle^{-\mu} \varphi_s(x)u\|^2) - \langle u, W_s u \rangle \end{aligned} \quad (2.8)$$

where $W_s(x) = c_1 |\nabla \varphi_s|^2$ with $c_1 > 0$ some constant depending only on c_0 , $|a|_\infty$ and $|b|_{\mu, \infty}$. One can check that

$$\begin{aligned} |\nabla \varphi_s|^2 &= \frac{4s^2 x^2}{R_s^4 (1 + \frac{x^2}{R_s^2})^2} (1 + \frac{x^2}{R_s^2})^{2s} \\ &\leq \frac{4s^2 x^2}{(R_s^2 + x^2)^2} \varphi_s^2 \leq \frac{4s^2}{R_s^2 + x^2} \varphi_s^2 \end{aligned}$$

Since $R_s^2 + x^2 \geq 2^{-2\mu} R_s^{2(1-\mu)} \langle x \rangle^{2\mu}$ and $R_s = M \langle s \rangle^{\frac{1}{1-\mu}}$, $W(x)$ is bounded by

$$0 \leq W_s(x) \leq \frac{4c_1 \langle s \rangle^2}{R_s^2 + x^2} \varphi_s^2 \leq \frac{42^\mu c_1}{M^{2(1-\mu)} \langle x \rangle^{2\mu}} \varphi_s^2. \quad (2.9)$$

Since $0 < \mu < 1$, one can choose an $M = M(c_0, |a|_\infty, |b|_{\mu, \infty}) > 1$ large enough so that $\frac{42^\mu c_1}{M^{2(1-\mu)}} < \frac{c_0}{4}$. Consequently, the above estimate combined with (2.8) gives

$$|\langle u, \varphi_s^2 H_0 u \rangle| \geq \frac{c_0}{4} (\|\nabla(\varphi_s u)\|^2 + \|\langle x \rangle^{-\mu} \varphi_s u\|^2). \quad (2.10)$$

Remark that

$$|\langle u, \varphi_s^2 H_0 u \rangle| \leq \|\langle x \rangle^{-\mu} \varphi_s u\| \|\langle x \rangle^\mu \varphi_s H_0 u\| \leq \frac{c_0}{8} \|\langle x \rangle^{-\mu} \varphi_s u\|^2 + \frac{2}{c_0} \|\langle x \rangle^\mu \varphi_s H_0 u\|^2.$$

It follows from (2.10) that

$$\|\langle x \rangle^\mu \varphi_s(x) H_0 u\|^2 \geq \frac{c_0^2}{16} (\|\langle x \rangle^{-\mu} \varphi_s(x)u\|^2 + \|\nabla(\varphi_s(x)u)\|^2), \quad u \in C_0^\infty(\mathbb{R}^n). \quad (2.11)$$

By an argument of density, one obtains (2.4) with some constant $C > 0$ independent of $s \in \mathbb{R}$. ■

Corollary 2.2. *Under the conditions of Lemma 2.1, there exists some constant $C > 0$ such that for any $f \in L^{2,r} = L^2(\mathbb{R}^n; \langle x \rangle^{2r} dx)$ and $u \in L_{\text{loc}}^2$ such that $H_0 u = f$, one has: $u \in L^{2,r-2\mu}$, $\nabla u \in L^{2,r-\mu}$ and*

$$\|\langle x \rangle^{r-\mu} \nabla u\| + \|\langle x \rangle^{r-2\mu} u\| \leq C \|\langle x \rangle^r f\|. \quad (2.12)$$

Proof. It follows from Lemma 2.1 with $s = \frac{r-\mu}{2}$. \blacksquare

Lemma 2.1 shows that $H_0 : D(H_0) \rightarrow R(H_0) := \text{Range}(H_0) \subset L^2(\mathbb{R}^n)$ is bijective. Let G_0 denote its algebraic inverse with $D(G_0) = R(H_0)$. Then one has

$$H_0 G_0 = 1 \text{ on } R(H_0), \quad G_0 H_0 = 1 \text{ on } D(H_0) \quad (2.13)$$

Lemma 2.3. (a). G_0 is a densely defined closed operator on $L^2(\mathbb{R}^n)$. If H_0 is self-adjoint (resp., maximally dissipative), then $-G_0$ is also selfadjoint (resp., maximally dissipative).

(b). There exists some C such that

$$\|\nabla(\varphi_s G_0 \varphi_{-s} \langle x \rangle^{-\mu} w)\| + \|\langle x \rangle^{-\mu} \varphi_s G_0 \varphi_{-s} \langle x \rangle^{-\mu} w\| \leq C \|w\| \quad (2.14)$$

for all $w \in \mathcal{D}$ and $s \in \mathbb{R}$. Here $\mathcal{D} = \bigcap_{s \in \mathbb{R}} L^{2,s}$.

Proof. We firstly show that $D(G_0)$ is dense. Remark that $\text{Re } H_0 \geq 0$. Let $f \in \mathcal{D}$ and $u_\epsilon = (H_0 + \epsilon)^{-1} f$, $\epsilon > 0$. Since $\text{Re } H_0 \geq 0$ and H_0 verifies the weighted coercive condition (1.15), $H_0 + \epsilon$ satisfies also (1.15) with the same constant $c_0 > 9$ independent of $\epsilon > 0$. Following the proof of Lemma 2.1 with H_0 replaced by $H_0 + \epsilon$, one has that for any $s > 0$

$$\|\langle x \rangle^{s-\mu} \nabla u_\epsilon\| + \|\langle x \rangle^{s-2\mu} u_\epsilon\| \leq C_s \|\langle x \rangle^s f\|$$

uniformly in $\epsilon > 0$. For $s > 2\mu$, this estimate implies that the sequence $\{u_\epsilon; \epsilon \in]0, 1]\}$ is relatively compact in L^2 . Therefore there exists a subsequence $\{u_{\epsilon_k}; k \in \mathbb{N}\}$ and $u \in L^2$ such that $\epsilon_k \rightarrow 0$ and $u_{\epsilon_k} \rightarrow u$ in L^2 as $k \rightarrow +\infty$. It follows that $H_0 u = f$ in the sense of distributions. The ellipticity of H_0 implies that $u \in H^2(\mathbb{R}^n)$. Therefore $f \in R(H_0) = D(G_0)$. This shows that $\mathcal{D} \subset D(G_0)$. In particular $D(G_0)$ is dense in $L^{2,r}$ for any $r \in \mathbb{R}$. The closeness of G_0 follows from that of H_0 . The other assertions can be easily checked.

The argument above shows that for any $w \in \mathcal{D}$, one can find $u \in D(H_0)$ such that $H_0 u = \varphi_{-s} \langle x \rangle^{-\mu} w$. (2.14) follows from (2.4). \blacksquare

Lemma 2.3 shows that for any s , $\langle x \rangle^{-\mu} \varphi_s G_0 \varphi_{-s} \langle x \rangle^{-\mu}$ defined on $\mathcal{D} = \bigcap_{s \in \mathbb{R}} L^{2,s}$ can be uniquely extended to a bounded operator on $L^2(\mathbb{R}^n)$, or in other words, for any $s \in \mathbb{R}$, G_0 is bounded from $D(G_0) \cap L^{2,s}$ to $L^{2,s-2\mu}$:

$$\|w x^{-\mu} \varphi_s G_0 u\| \leq C \|\varphi_s \langle x \rangle^\mu u\| \quad (2.15)$$

uniformly in $u \in D(G_0) \cap L^{2,s}$ and $s \in \mathbb{R}$. This implies that $G_0 \mathcal{D} \subset \mathcal{D}$ and G_0 extends to a continuous operator from $L^{2,s}$ to $L^{2,s-2\mu}$ for any $s \in \mathbb{R}$. It follows that $G_0^N(\mathcal{D}) \subset \mathcal{D}$ and by an induction, one can check that G_0^N extends to a bounded operator from $L^{2,s}$ to $L^{2,s-2N\mu}$ for any $s \in \mathbb{R}$. To simplify notation, we still denote G_0 (resp., G_0^N) its continuous extension by density as operator from $L^{2,s}$ to $L^{2,s-2\mu}$ (resp., from $L^{2,s}$ to

$L^{2,s-2N\mu}$.

Theorem 2.4. *Let $M > 1$ be given in Lemma 2.1. Denote*

$$x_{N,r} = \frac{x}{R_{N,r}} \text{ with } R_{N,r} \equiv R_{(2N-1+r)\mu} = M\langle(2N-1+r)\mu\rangle^{\frac{1}{1-\mu}} \quad (2.16)$$

where $N \in \mathbb{N}$ and $r \in \mathbb{R}_+$ and $M > 0$ is a constant given by Lemma 2.1. Set $\langle x_{N,r} \rangle = (1 + |x_{N,r}|^2)^{\frac{1}{2}}$. Then there exists some constant $C > 0$ such that

$$\|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0^N \langle x_{N,r} \rangle^{r\mu}\| \leq C^N \langle(2N-1+r)\mu\rangle^{\gamma N}, \quad (2.17)$$

for any integer $N \geq 1$ and any $r \geq 0$. Here

$$\gamma = \frac{2\mu}{1-\mu}. \quad (2.18)$$

Proof. Making use of Lemma 2.1, one can check that operator

$$I_N = \langle x_{N,r} \rangle^{-2N\mu-r\mu} G_0^N \langle x \rangle^{r\mu} \quad (2.19)$$

is well defined on $\mathcal{D} = \cap_{s \in \mathbb{R}} L^{2,s}$ and extends to a bounded operator in L^2 . To show the estimate (2.17), we use an induction on N . Since $\langle x \rangle \leq \frac{1}{R} \langle \frac{x}{R} \rangle$ for $R \geq 1$, it follows from (2.14) that

$$\|\langle \frac{x}{R_s} \rangle^{-s-\mu} G_0 \langle \frac{x}{R_s} \rangle^{s-\mu}\| \leq C' R_s^{2\mu} \leq C_1 \langle s \rangle^\gamma \quad (2.20)$$

uniformly in s , where $R = M \langle s \rangle^{\frac{1}{1-\mu}}$. In particular, when $s = (1+r)\mu$, one has $R_s = M \langle(1+r)\mu\rangle^{\frac{1}{1-\mu}} = R_{1,r}$ and

$$\|I_1\| \leq C_1 \langle(1+r)\mu\rangle^\gamma \quad (2.21)$$

for all $r \geq 0$, which proves (2.17) when $N = 1$. Assume now that $N \geq 2$ and that one has proved for some $C > 0$ independent of N and $r \geq 0$ that

$$\|I_{N-1}\| \leq C^{N-1} \langle(2N-3+r)\mu\rangle^{\gamma(N-1)}. \quad (2.22)$$

Write I_N as

$$I_N = \langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N-1,r} \rangle^{(2N-2+r)\mu} \cdot I_{N-1} \cdot \langle x_{N-1,r} \rangle^{-r\mu} \langle x_{N,r} \rangle^{r\mu}$$

Notice that

$$\langle x_{N,r} \rangle \leq \langle x_{N-1,r} \rangle \leq \frac{R_{N,r}}{R_{N-1,r}} \langle x_{N,r} \rangle$$

for any $N \geq 2$. Applying (2.20) with $s = (2N-1+r)\mu$, one obtains

$$\|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N,r} \rangle^{(2N-2+r)\mu}\| \leq C_1 \langle(2N-1+r)\mu\rangle^\gamma.$$

Making use of the induction hypothesis, one can estimate I_N as follows:

$$\begin{aligned}
\|I_N\| &\leq \|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N-1} \rangle^{(2N-2+r)\mu}\| \cdot \|I_{N-1}\| \\
&\leq \|\langle x_{N,r} \rangle^{-(2N+r)\mu} G_0 \langle x_{N,r} \rangle^{(2N-2+r)\mu}\| \cdot \left\| \left(\frac{\langle x_{N-1} \rangle}{\langle x_{N,r} \rangle} \right)^{(2N-2+r)\mu} \right\| \cdot \|I_{N-1}\| \\
&\leq C_1 \langle (2N-1+r)\mu \rangle^\gamma \cdot \left(\frac{\langle (2N-1+r)\mu \rangle}{\langle (2N-3+r)\mu \rangle} \right)^{\gamma(N-1+\frac{\epsilon}{2})} \cdot C^{N-1} \langle (2N-3+r)\mu \rangle^{\gamma(N-1)} \\
&\leq C_1 \left(\frac{2N-1+r}{2N-3+r} \right)^{\gamma(N-1+\frac{\epsilon}{2})} C^{N-1} \langle (2N-1+r)\mu \rangle^{\gamma N}. \tag{2.23}
\end{aligned}$$

Since the sequence $\left\{ \left(\frac{2m-1+r}{2m-3+r} \right)^{\gamma(m-1+\frac{\epsilon}{2})}; m \geq 2 \right\}$ is uniformly bounded in $r \geq 0$, there exists some $C_2 > 0$ such that

$$C_1 \left(\frac{2m-1+r}{2m-3+r} \right)^{\gamma(m-1+\frac{\epsilon}{2})} \leq C_2$$

for any $m \geq 2$ and $r \geq 0$. Increasing the constant C if necessary, one can suppose without loss that $C_2 \leq C$ and one obtains from (2.23) that

$$\|I_N\| \leq C^N \langle (2N-1+r)\mu \rangle^{N\gamma} \tag{2.24}$$

Theorem 2.4 is proven by induction. ■

Let $R_0(z)$ denote the resolvent of H_0 and

$$\Omega(\delta) = \left\{ z \in \mathbb{C}^*; \frac{\pi}{2} + \delta < \arg z < \frac{3\pi}{2} - \delta \right\},$$

$\delta > 0$. Since $\operatorname{Re} H_0 \geq 0$, there exists some $C_1 > 0$ such that

$$\|R_0(z)\| \leq \frac{C_1}{|z|}, \quad z \in \Omega.$$

From the equation $R_0(z) = G_0 + zG_0 + z^2G_0^2R_0(z)$, it follows that as operators from $L^{2,s}$ to $L^{2,s-2\mu}$, $s \in \mathbb{R}$, one has

$$\operatorname{s-}\lim_{z \in \Omega(\delta), z \rightarrow 0} R_0(z) = G_0 \tag{2.25}$$

for any $\delta > 0$. Similarly one can check that for any $N \in \mathbb{N}^*$, one has

$$\operatorname{s-}\lim_{z \in \Omega(\delta), z \rightarrow 0} R_0(z)^N = G_0^N. \tag{2.26}$$

as operators from $L^{2,s}$ to $L^{2,s-2N\mu}$. By an abuse of notation, we denote $R(0) = G_0$. Thus the resolvent $R_0(z)$ is defined for z in $\Omega(\delta) \cup \{0\}$. As a consequence of Theorem 2.4, one can deduce the following

Corollary 2.5. *The following Gevrey estimates of the resolvent hold.*

(a). *For any $a > 0$, there exists some constant $C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} R_0(z)^N\| + \|R_0(z)^N e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a^N N^{\gamma N} \tag{2.27}$$

for any integer $N \geq 1$ and $z \in \Omega(\delta) \cup \{0\}$.

(b). Then there exists some constant $C > 0$ such that

$$\|\chi(x)R_0(z)^N\| + \|R_0(z)^N\chi(x)\| \leq C_\chi C^N N^{\gamma N} \quad (2.28)$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $N \geq 1$ and $z \in \Omega(\delta) \cup \{0\}$. Here $\gamma = \frac{2\mu}{1-\mu}$.

Proof. Notice that $\|zR_0(z)\|$ is uniformly bounded in $\mathcal{L}(L^2)$ for $z \in \Omega(\epsilon)$ and that

$$R_0(z)^N = G_0^N(1 + zR_0(z))^N.$$

According to Theorem 2.4 with $r = 0$, one has for some constant $C > 0$

$$\|\langle x_{N,0} \rangle^{-2N\mu} R_0(z)^N\| \leq C^N N^{\gamma N}, \quad (2.29)$$

for any integer $N \geq 1$ and $z \in \Omega(\epsilon) \cup \{0\}$.

Let $a > 0$. Then

$$\|e^{-a\langle x \rangle^{1-\mu}} R_0(z)^N\| \leq \|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty} C^N N^{\gamma N}.$$

To evaluate the norm $\|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty}$, consider the function

$$f(r) = e^{-ar^{1-\mu}} \left\langle \frac{r}{R_N} \right\rangle^{2N\mu},$$

where $r = |x|$ and $R_N = R_{N,0} = M\langle (2N-1)\mu \rangle^{\frac{1}{1-\mu}}$. One calculates:

$$f'(r) = \frac{f(r)}{r^\mu(R_N^2 + r^2)} (-2a(1-\mu)(R_N^2 + r^2) + 2N\mu r^{1+\mu}), r \geq 1.$$

Let $A \geq 1$. Since $R_N \sim c'N^{\frac{1}{1-\mu}}$ for some constant $c' > 0$, one can check that $Nr^{1+\mu} \leq \frac{c}{A^{1-\mu}}r^2$ if $r \geq AR_N$ for some constant $c > 0$ independent of A , r and N . Therefore, if $A = A(\mu, a) > 1$ is chosen sufficiently large, one has

$$f'(r) < 0, \quad r > AR_N,$$

thus $f(r)$ is decreasing in $[AR_N, +\infty[$. It is now clear that

$$\|e^{-a\langle x \rangle^{1-\mu}} \langle x_{N,0} \rangle^{2N\mu}\|_{L^\infty} \leq \sup_{0 \leq r \leq AR_N} f(r) \leq \langle A \rangle^{2N\mu}$$

This proves Part (a) of Corollary with $C_a = C\langle A \rangle^{2\mu}$.

To prove Part (b), let $\chi \in C_0^\infty(\mathbb{R}^n)$. Let $R > 0$ such that $\text{supp } \chi \subset B(0, R)$. (5.1) shows that there exists some constant $C_1 > 0$ such that

$$\|\chi(x)G_0^N(1 + zR_0(z))^N\| \leq \|\langle x_{N,0} \rangle^{2\mu N} \chi\|_{L^\infty} \times C_1^N N^{\gamma N}$$

for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $N \geq 1$ and $z \in \Omega(\epsilon) \cup \{0\}$. Then One can check that

$$\|\langle x_{N,0} \rangle^{2\mu N} \chi\|_{L^\infty} \leq \|\chi\|_{L^\infty} \left(1 + \frac{R^2}{M^2((2N-1)\mu)^{\frac{2}{1-\mu}}}\right)^{\mu N} \leq C_2 2^{\mu N}$$

for some constant C_2 depending only on χ and R , but independent of N . This proves (2.28) with $C_\chi = C_2$ and $C = C_1 2^\mu$ which is independent of χ . \blacksquare

Theorem 1.1 is a particular case of Corollary 2.5. Corollary 2.5 shows that the resolvent $R_0(z)$ belongs to some Gevrey class of order $1 + \gamma$ on $\Omega(\delta) \cup \{0\}$

3. SUBEXPONENTIAL TIME-DECAY OF e^{-tH_0}

From now on, we consider the model Schrödinger operator $H_0 = -\Delta + V(x)$ with $V(x) = V_1(x) - iV_2(x)$, $V_1(x), V_2(x)$ being real. Denote $R_0(z) = (H_0 - z)^{-1}$. Theorem 2.4 can be used to prove subexponential time-decay for local energies of solutions to the heat and Schrödinger equations. To do this, we use Cauchy integral formula for semigroups and need some information of the resolvent on a contour in the right half complex plane passing through the origin.

Proposition 3.1. *Assume that $\operatorname{Re} H_0 \geq -a\Delta$ for some $a > 0$ and that the imaginary part of the potential $V(x)$ verifies the estimate*

$$|V_2(x)| \leq C\langle x \rangle^{-2\mu'}, \quad \forall x \in \mathbb{R}^n, \quad (3.1)$$

for some $0 < \mu' < \min\{\frac{n}{2}, 1\}$. Then there exists some constant $C_0 > 0$ such that the numerical range $N(H_0)$ of H_0 is contained in a region of the form $\{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| \leq C_0(\operatorname{Re} z)^{\mu'}\}$. Consequently, for any $A_0 > C_0$ there exists some constant M_0 such that

$$\|R_0(z)\| \leq \frac{M_0}{|z|^{\frac{1}{\mu'}}} \quad (3.2)$$

for $z \in \Omega := \{z \in \mathbb{C}^*; |z| \leq 1, \operatorname{Re} z < 0 \text{ or } \operatorname{Re} z \geq 0, |\operatorname{Im} z| > A_0(\operatorname{Re} z)^{\mu'}\}$.

Proof. For $z = \langle u, H_0 u \rangle \in N(H_0)$ where $u \in D(H_0)$ and $\|u\| = 1$, one has

$$\begin{aligned} \operatorname{Re} z &= \operatorname{Re} \langle u, H_0 u \rangle \geq a\|\nabla u\|^2 \\ |\operatorname{Im} z| &\leq \langle u, |V_2| u \rangle \leq C\|\langle x \rangle^{-\mu'} u\|^2. \end{aligned}$$

According to the generalized Hardy inequality ([9]), one has for $0 < \mu' < \frac{n}{2}$

$$\|\langle x \rangle^{-\mu'} u\|^2 \leq \| |x|^{-\mu'} u \|^2 \leq \frac{\Gamma(\frac{n-2\mu'}{4})^2}{2^{2\mu'} \Gamma(\frac{n+2\mu'}{4})^2} \|\nabla |^{\mu'} u\|^2. \quad (3.3)$$

Let \hat{u} denote the Fourier transform of u normalized such that $\|\hat{u}\| = \|u\|$ and $\tau = \|\nabla u\|$. Then

$$\begin{aligned} \|\nabla |^{\mu'} u\|^2 &= \|\xi |^{\mu'} \hat{u}\|^2 = \|\xi |^{\mu'} \hat{u}\|_{L^2(|\xi| \geq \tau)}^2 + \|\xi |^{\mu'} \hat{u}\|_{L^2(|\xi| < \tau)}^2 \\ &\leq \tau^{2(\mu'-1)} \|\xi \hat{u}\|_{L^2(|\xi| \geq \tau)}^2 + \tau^{2\mu'} \|\hat{u}\|_{L^2(|\xi| < \tau)}^2 \\ &\leq 2\tau^{2\mu'} = 2\|\nabla u\|^{2\mu'}. \end{aligned}$$

In the first inequality above, the condition $0 < \mu' \leq 1$ is used. This proves that $|\operatorname{Im} z| \leq C_0(\operatorname{Re} z)^{\mu'}$ when $z \in N(H_0)$. The other assertions of Proposition are immediate, since $\sigma(H_0) \subset \overline{N(H_0)}$ and

$$\|R_0(z)\| \leq \frac{1}{\operatorname{dist}(z, N(H_0))}.$$

■

Notice that under the conditions of Proposition 3.1, one can not exclude possible accumulation of complex eigenvalues towards zero. Making use of Proposition 3.1, one can prove the following uniform Gevrey estimates in a domain located in the right half

complex plane.

Corollary 3.2. *Under the conditions of proposition 3.1, let κ be an integer such that $\kappa + 1 \geq \frac{1}{\mu'}$. Then for any $a > 0$ there exist $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} \frac{d^{N-1}}{dz^{N-1}} R_0(z)\| \leq c_a C_a^N N^{(1+(1+\kappa)\gamma)N}, \quad \forall N \geq 1, \quad (3.4)$$

and there exists some constant $C > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, one has

$$\|\chi(x) \frac{d^{N-1}}{dz^{N-1}} R_0(z)\| \leq C_\chi C^N N^{(1+(1+\kappa)\gamma)N}, \quad \forall N \geq 1, \quad (3.5)$$

uniformly in $z \in \Omega$. Here Ω is defined as in Proposition 3.1.

Proof. For $z \in \Omega$, decompose $R_0(z)$ into

$$R_0(z) = A(z) + G_0^{\kappa+1} B(z)$$

with $A(z) = \sum_{j=0}^{\kappa} z^j G_0^{j+1}$ and $B(z) = z^{\kappa+1} R_0(z)$. By Proposition 3.1, $\|B(z)\|$ is uniformly bounded for $z \in \Omega$. Theorem 2.4 shows that for some constant C_1

$$\|\langle x_{\kappa+1,r} \rangle^{-(2\kappa+2+r)\mu} G_0^{\kappa+1} \langle x_{\kappa+1,r} \rangle^{r\mu}\| \leq C_1 \langle (2\kappa+1+r)\mu \rangle^{\gamma(\kappa+1)}, \quad (3.6)$$

$$\|\langle x_{\kappa+1,r} \rangle^{-(2\kappa+2+r)\mu} A(z) \langle x_{\kappa+1,r} \rangle^{r\mu}\| \leq C_1 \langle (2\kappa+1+r)\mu \rangle^{\gamma(\kappa+1)} \quad (3.7)$$

for any $r \geq 0$ and $|z| \leq 1$. Making use of the relation

$$R_0(z)^N = A(z)R_0(z)^{N-1} + G_0^{\kappa+1} R_0(z)^{N-1} B(z)$$

one can show by an induction on N that there exists some constant $C > 0$ such that

$$\|\langle x_{(\kappa+1)N,0} \rangle^{-2(\kappa+1)N\mu} R_0(z)^N\| \leq C^N N^{N\gamma(1+\kappa)} \quad (3.8)$$

for any $N \geq 1$ and $z \in \Omega$. In fact, the case $N = 1$ follows from (3.6) and (3.7). If (3.8) is proven with N replaced by $N - 1$ for some $N \geq 2$, noticing that $x_{(\kappa+1)N,0} = x_{\kappa+1,2(\kappa+1)(N-1)}$, (3.6) and (3.7) with $r = 2(\kappa+1)(N-1)$ show that

$$\begin{aligned} & \|\langle x_{(\kappa+1)N,0} \rangle^{-2(\kappa+1)N\mu} R_0(z)^N\| \\ & \leq C_1 \langle (2(\kappa+1)N-1)\mu \rangle^{\gamma(\kappa+1)} (\|\langle x_{(\kappa+1)(N-1),0} \rangle^{-2(\kappa+1)(N-1)\mu} R_0(z)^{N-1}\| \\ & \quad + \|\langle x_{(\kappa+1)(N-1),0} \rangle^{-2(\kappa+1)(N-1)\mu} R_0(z)^{N-1}\| \|B(z)\|) \\ & \leq C_2 C^{N-1} N^{N\gamma(1+\kappa)} \end{aligned}$$

for some constant C_2 independent of N . Increasing the constant C if necessary, this proves (3.8) for all $N \geq 1$ by an induction. (3.4) and (3.5) are deduced from (3.8) as in the proof of Corollary 2.5. \blacksquare

As another consequence of Proposition 3.1, we obtain the following estimate on the expansion of the resolvent at 0:

Corollary 3.3. *Under the conditions of Proposition 3.1, assume in addition (1.15) with $\mu \in]0, 1[$. Then there exists some constant $c > 0$ such that for any $z \in \Omega$ and z near 0,*

one has for some N (depending on z) such that

$$\|\langle x_{N,0} \rangle^{-2N\mu} (R_0(z) - \sum_{j=0}^N z^j G_0^{j+1})\| \leq e^{-c|z|^{-\frac{1}{\gamma}}}. \quad (3.9)$$

Here $\langle x_{N,0} \rangle$ is defined in Theorem 2.4 with $r = 0$.

Proof. Theorem 2.4 and Proposition 3.1 show that for any N , one has

$$\|\langle x_{N,0} \rangle^{-2N\mu} (R_0(z) - \sum_{j=0}^N z^j G_0^{j+1})\| \leq C^N N^{\gamma N} |z|^{N+1-\frac{1}{\mu}}, \quad (3.10)$$

for all $z \in \Omega$ and z near 0. The remainder estimate can be minimized by choosing an appropriate N in terms of $|z|$. For fixed $M' > 1$ and $z \neq 0$, take $N = \lfloor \frac{1}{(CM'|z|^{\frac{1}{\gamma}})} \rfloor$. Then one has for z in a small neighbourhood of zero and $z \neq 0$:

$$C^N N^{\gamma N} |z|^{N+1-\frac{1}{\mu}} \leq e^{-c_1 N \log M} \leq e^{-c_2 |z|^{-\frac{1}{\gamma}}}$$

where c_1, c_2 are some positive constants. ■

Theorem 3.4. Let $H_0 = -\Delta + V(x)$ with $V \in \mathcal{V}$. Set

$$\beta = \frac{1-\mu}{1+\mu}. \quad (3.11)$$

Then for any $a > 0$, there exist some constant $c_a, C_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} e^{-tH_0}\| + \|e^{-tH_0} e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a e^{-c_a t^\beta}, \quad t > 0. \quad (3.12)$$

Proof. Let Γ be the contour defined by $\Gamma = \{z; \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^\mu\}$ oriented in anti-clockwise sense, where $C > 0$ is sufficiently large. By Proposition 3.1, the numerical range of H_0 is located on the right hand side of Γ and one has

$$e^{-tH_0} = \frac{i}{2\pi} \int_{\Gamma} e^{-tz} R_0(z) dz. \quad (3.13)$$

Decompose Γ as $\Gamma = \Gamma_0 + \Gamma_1$ where Γ_0 is the part of Γ with $0 \leq \operatorname{Re} z \leq \delta$ while Γ_1 is the part of Γ with $\operatorname{Re} z > \delta$ where $\delta > 0$ is sufficiently small. Clearly, the integral on Γ_1 is exponentially decreasing as $t \rightarrow \infty$

$$\left\| \int_{\Gamma_1} e^{-tz} R_0(z) dz \right\| \leq C e^{-\frac{t}{C}}, \quad t > 0,$$

for some constant $C > 0$. For $z \in \Gamma_0$, denote $f_N(z) = R_0(z) - \sum_{j=0}^N z^j G_0^{j+1}$. Then

$$f_N(z) = z^{N+1} G_0^N R_0(z).$$

Let $x_N = x_{N,0}$ be defined as in Theorem 2.4 with $r = 0$. Then Theorem 2.4 shows that there exist some constants $C, C_1 > 0$ such that According to Theorem 1.1 and Proposition 3.1, for any $a > 0$ there exist some constant $C, C_1 > 0$ such that

$$\|\langle x_N \rangle^{-2N\mu} f_N(z)\| \leq C_1 C^N |z|^{N+1-\frac{1}{\mu}} N^{\gamma N} \quad (3.14)$$

for $z \in \Gamma_0$. It follows that

$$\begin{aligned}
 & \left\| \int_{\Gamma_0} e^{-tz} \langle x_N \rangle^{-2N\mu} R_0(z) dz \right\| \\
 & \leq \sum_{j=0}^N \left\| \langle x_N \rangle^{-2(N+1)\mu} G_0^{j+1} \right\| \left\| \int_{\Gamma_0} e^{-tz} z^j dz \right\| + \left\| \int_{\Gamma_0} e^{-tz} \langle x_N \rangle^{-2(N+1)\mu} f_N(z) dz \right\| \\
 & \leq \sum_{j=0}^N C^j j^{\gamma j} e^{-\delta t} + C_1 C^N N^{\gamma N} \int_{\Gamma_0} |e^{-tz}| |z|^{N+1-\frac{1}{\mu'}} |dz|
 \end{aligned}$$

for all $t > 0$ and $N \geq 1$. Parameterizing Γ_0 by $z = \lambda \pm ic\lambda^{\frac{1}{\mu'}}$ with $\lambda \in]0, \delta]$, one can evaluate the last integral as follows:

$$\begin{aligned}
 \int_{\Gamma_0} |e^{-tz}| |z|^{N+1-\frac{1}{\mu'}} |dz| & \leq C_2^N \int_0^\delta e^{-t\lambda} \lambda^{N+1-\frac{1}{\mu'}} d\lambda \\
 & \leq C_2^N t^{-N-2+\frac{1}{\mu'}} \int_0^{\delta t} e^{-\tau} \tau^{N+1-\frac{1}{\mu'}} d\tau \\
 & \leq C_3^N t^{-N-2+\frac{1}{\mu'}} N^N.
 \end{aligned}$$

This proves that there exist some different constants C_1 and $C > 0$ such that

$$\left\| \int_{\Gamma_0} e^{-tz} \langle x_N \rangle^{-2(N+1)\mu} R_0(z) dz \right\| \leq C_1 C^N N^{\gamma N} (N e^{-\delta t} + N^N t^{-N-2+\frac{1}{\mu'}}) \quad (3.15)$$

for any $t > 0$ and $N \geq 1$. Choosing N in terms of $t \gg 1$ such that $N \simeq \left(\frac{t}{M_1 C}\right)^{\frac{1}{1+\gamma}}$ for some fixed appropriate $M_1 > 1$, one obtains that

$$\left\| \int_{\Gamma_0} e^{-tz} \langle x_N \rangle^{-2(N+1)\mu} R_0(z) dz \right\| \leq C e^{-\delta_0 t^{\frac{1}{1+\gamma}}} \quad (3.16)$$

for some $C, \delta_0 > 0$. This proves that there exists some constants $C, c > 0$ independent of N and t such that

$$\left\| \langle x_N \rangle^{-2(N+1)\mu} e^{-tH_0} \right\| \leq C e^{-ct^\beta}, \quad t > 0, \quad (3.17)$$

with $N = N(t)$ chosen as above and $\beta = \frac{1}{1+\gamma} = \frac{1-\mu}{1+\mu}$. (3.12) is deduced from (3.17) by noticing that for any $a > 0$, there exists some constant $C_a > 0$ such that

$$\left\| e^{-a\langle x \rangle^{1-\mu}} \langle x_N \rangle^{2N\mu} \right\|_{L^\infty} \leq C_a^N.$$

See the proof of Corollary 2.5. ■

As a consequence of Theorem 3.4, one obtains that there exists some constant $c > 0$ such that

$$\left\| e^{-tH_0} f \right\| \leq C_R e^{-ct^\beta} \|f\|, \quad t > 0, \quad (3.18)$$

for all $f \in L^2(\mathbb{R}^n)$ with support contained in $\{|x| \leq R\}$, $R > 0$.

4. AN ESTIMATE ON THE SPECTRAL MEASURE

For the selfadjoint Schrödinger operator with a global positive and slowly decreasing potential, it is known that under some additional conditions the spectral measure $E'(\lambda)$ satisfies the estimate that for any $N > 0$

$$E'(\lambda) = O(\lambda^N) \quad (4.1)$$

in appropriate spaces as $\lambda \rightarrow 0$ (see [13]). The Gevery estimates of the resolvent at threshold allow to give an improvement of this result. Let us begin with the following results on the boundary values of the resolvent up to real axis.

Lemma 4.1. *Let $V(x) = V_1(x) - iV_2(x)$ with $V_1(x), V_2(x)$ real. Assume that V_1 is of class C^2 on \mathbb{R}^n and that there exists $\mu \in]0, 1[$ and some constants $c_j > 0$, $j = 1, 2, 3$, such that*

$$c_1 \langle x \rangle^{-2\mu} \leq V_1(x) \leq c_2 \langle x \rangle^{-2\mu}, \quad (4.2)$$

$$|(x \cdot \nabla)^j V_1(x)| \leq c_2 \langle x \rangle^{-2\mu}, \quad j = 1, 2 \quad (4.3)$$

$$x \cdot \nabla V_1(x) \leq -c_3 \langle x \rangle^{-2\mu}, \quad |x| > R \text{ for some } R > 0, \quad (4.4)$$

$$|V_2(x)| \leq c_2 \langle x \rangle^{-1-\mu-\epsilon_0}, \quad \epsilon_0 > 0. \quad (4.5)$$

Then the eigenvalues of H_0 are absent in a neighbourhood of zero and the boundary values of the resolvent $R_0(\lambda \pm i0) = \lim_{z \rightarrow \lambda, \pm \text{Im } z > 0} (H_0 - z)^{-1}$ exist for $\lambda \in [0, \delta]$ for some $\delta > 0$ and are Hölder continuous as operators in $\mathcal{L}(L^{2, \frac{1+\mu}{2}}; L^{2, -\frac{1+\mu}{2}})$.

Proof. Let $H_1 = -\Delta + V_1(x)$ be the selfadjoint part of H_0 and $R_1(z) = (H_1 - z)^{-1}$. Then one knows from [13] that under the condition of this Lemma, $R_1(\lambda \pm i0)$ exists for $\lambda \in [0, \delta]$ for some $\delta > 0$ and are Hölder continuous as operators in $\mathcal{L}(L^{2, s}; L^{2, -s})$, $s > \frac{1+\mu}{2}$. Note that the smoothness assumption on the potential used in [13] is only needed for higher order resolvent estimates.

One knows that $G_{0,1} = \lim_{z \rightarrow 0, z \notin \mathbb{R}_+} R_1(z)$ exists and that $G_{0,1}V_2$ is a compact operator in $L^{2, -s}$ for $\frac{1+\mu}{2} < s < \frac{1+\mu+\epsilon_0}{2}$. Therefore the kernel of $1 + iG_{0,1}V_2$ is of finite dimension. From Lemma 2.3 applied to $G_{0,1}$, one deduces that this kernel is contained in $L^{2, r}$ for any $r > 0$. Since $(1 + iG_{0,1}V_2)u = 0$ if and only if $H_0u = 0$, Lemma 2.1 that $\ker(1 + iG_{0,1}V_2)$ in $L^{2, -s}$ is trivial. Therefore $((1 + iG_{0,1}V_2)^{-1})$ is bounded in $L^{2, -s}$. By the continuity of $R_1(z)$ for z near 0 and $z \notin \mathbb{R}_+$, one deduces that $1 + iR_1(z)V_2$ is invertible in $L^{2, -s}$ and its inverse is Hölder continuous in $\mathcal{L}(L^{2, -s})$ for z near 0 and $z \notin \mathbb{R}_+$. This implies in particular that the eigenvalues of H_0 are absent in a neighbourhood of zero and the limits $R_0(\lambda \pm i0) = \lim_{z \rightarrow \lambda, \pm \text{Im } z > 0} (H_0 - z)^{-1}$ exist for $\lambda \geq 0$ and small enough and are Hölder continuous in λ . ■

Theorem 4.2. *Under the conditions of Lemma 4.1, assume in addition that $V_2 = 0$ such that H_0 is selfadjoint. Denote by $E(\lambda)$ the spectral projection of H_0 associated with the interval $] -\infty, \lambda]$. Let $s > \frac{1+\mu}{2}$. Then for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that*

$$\|e^{-a\langle x \rangle^{1-\mu}} E'(\lambda) \langle x \rangle^{-s}\| \leq C_a e^{-c_a |\lambda|^{-\frac{1}{\gamma}}}, \quad 0 < \lambda \leq \delta. \quad (4.6)$$

Proof. Since $E'(\lambda) = \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0))$, $\|\langle x \rangle^{-s} E'(\lambda) \langle x \rangle^{-s}\|$ is uniformly bounded for $\lambda > 0$ near 0 ([13]). Iterating the resolvent equation, one obtains for any $N \in \mathbb{N}$

$$E'(\lambda) = \lambda^N G_0^N E'(\lambda), \quad 0 < \lambda \leq \delta. \quad (4.7)$$

Applying (2.17 with $r = s$, one deduces as in the proof of Corollary 2.5 that for any $a > 0$, there exist some constants $c_a, C_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} E'(\lambda) \langle x \rangle^{-s}\| \leq C_a c_a^N N^{\gamma N} \lambda^N \quad (4.8)$$

for all $N \in \mathbb{N}$ and $\lambda \in]0, \delta]$. It remains to choose N in terms of $\lambda > 0$ (it suffices to take N equal to integer part of $c\lambda^{-\frac{1}{\gamma}}$ for some appropriate $c > 0$) such that

$$c_a^N N^{\gamma N} \lambda^N \leq C' e^{-c'\lambda^{-\frac{1}{\gamma}}}, \quad 0 < \lambda \leq \delta,$$

for some constants $c', C' > 0$. (4.6) is proved. \blacksquare

5. SUBEXPONENTIAL TIME-DECAY OF e^{-itH_0}

To obtain subexponential time-decay of solutions to the Schrödinger equation associated with H_0 , we shall use the technique of resonances and deform the integral contour into the lower half-complex plane.

Let \mathcal{A} denote the class of complex-valued potentials $V = V_1 - iV_2$ such that $V \in \mathcal{A}$, V_1 and V_2 extend holomorphically into a region of the form $\Omega = \{x \in \mathbb{C}^n; |\operatorname{Im} x| < c|\operatorname{Re} x|\} \cup \{x \in \mathbb{C}^n; |x| < c\}$ for some $c > 0$ and satisfy for some $c_j > 0$ and $R \in [0, +\infty]$

$$|V_j(x)| \leq c_1 \langle \operatorname{Re} x \rangle^{-2\mu}, \quad x \in \Omega, \quad j = 1, 2, \quad (5.1)$$

$$V_2(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \quad (5.2)$$

$$x \cdot \nabla V_1(x) \leq -c_3 \frac{x^2}{\langle x \rangle^{2\mu+2}}, \quad x \in \mathbb{R}^n \text{ with } |x| \geq R, \text{ and} \quad (5.3)$$

$$V_2(x) \geq c_5 \langle x \rangle^{-2\mu}, \quad x \in \mathbb{R}^n \text{ with } |x| < R. \quad (5.4)$$

When $R = 0$, (5.3) is a global virial condition on V_1 and (5.4) is void; while if $R = +\infty$, no virial condition is needed on V_1 , but (5.4) is required on the whole space which means that the dissipation is strong. For $V \in \mathcal{A}$, one can define the resonances of H_0 by both analytical dilation or analytical deformation methods ([1, 6]). We shall show that conditions (5.2), (5.3) and (5.4) allow to prove the absence of eigenvalues and resonances in a sector below the positive real half-axis.

Firstly, we use the analytic dilation method. For $V \in \mathcal{A}$, denote $\tilde{H}_0(\theta) = -(1 + \theta)^{-2}\Delta + V((1 + \theta)x)$ for $\theta \in \mathbb{C}$ and θ near 0. Set $\tilde{R}_0(z, \theta) = (\tilde{H}_0(\theta) - z)^{-1}$. For θ real, $\tilde{R}_0(z, \theta)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$. $\{\tilde{H}_0(\theta); \theta \in \mathbb{C}, |\theta| < \delta\}$ is a holomorphic family of type A in the sense of T. Kato. For $\operatorname{Im} \theta > 0$ small enough, the resolvent $\tilde{R}_0(z, \theta)$ defined for $z \in \mathbb{C}_+$ with $\operatorname{Im} z \gg 1$ can be meromorphically extended across the positive real half-axis \mathbb{R}_+ into the sector $\{z; \arg z > -\operatorname{Im} \theta\}$ (c.f. [1]). The poles of $\tilde{R}_0(z, \theta)$ in this sector are by definition the resonances of H which are independent of θ ([1]).

We begin with the following elementary Hardy type inequality.

Lemma 5.1. *Let $n \geq 2$ and $0 < s < n - 1$. One has*

$$\|\langle x \rangle^{-1-\frac{s}{2}}u\|^2 \leq \frac{1}{2\sqrt{(n-1)(n-1-s)}}(\|\nabla u\|^2 + \|\langle x \rangle^{-s}u\|^2) \quad (5.5)$$

for all $u \in H^1(\mathbb{R}^n)$.

Proof. Let $x = r\omega$, $r \geq 0$ and $\omega \in \mathbb{S}^{n-1}$. For $u \in \mathcal{S}(\mathbb{R}^n)$, denote

$$F(r, \omega) = \frac{|u(r\omega)|^2 r^{n-1}}{\langle r \rangle^s}.$$

Then one has

$$F'_r(r, \omega) = \frac{((n-1)(1+r^2) - sr^2)|u(r\omega)|^2 r^{n-2}}{\langle r \rangle^{s+2}} + 2\frac{r^{n-1}}{\langle r \rangle^s} \operatorname{Re}(u'_r(r\omega)\overline{u(r\omega)}).$$

Here $F'_r(r, \omega)$ is the derivation of $F(r, \omega)$ with respect to r . Since for $n \geq 2$, one has $\int \int_{\mathbb{R}_+ \times \mathbb{S}^{n-1}} F'(r, \omega) dr d\omega = 0$, we deduce the identity

$$\int \int_{\mathbb{R}^n} \frac{(n-1) + (n-1-s)r^2}{r\langle r \rangle^{s+2}} |u(x)|^2 dx = -2 \int \int_{\mathbb{R}^n} \langle x \rangle^{-s} \operatorname{Re}(u'_r \bar{u}) dx \quad (5.6)$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$. Inequality (5.5) follows from the trivial bounds

$$2\sqrt{(n-1)(n-1-s)} \leq \frac{(n-1) + (n-1-s)r^2}{r}$$

and

$$-2 \int \int_{\mathbb{R}^n} \langle x \rangle^{-s} \operatorname{Re}(u'_r \bar{u}) dx \leq 2\|u'_r\| \|\langle x \rangle^{-s}u\| \leq \|\nabla u\|^2 + \|\langle x \rangle^{-s}u\|^2$$

and an argument of density. ■

Lemma 5.2. *Let $n \geq 2$ and $V \in \mathcal{A}$ with $0 < \mu < \frac{3}{4}$ when $n = 2$ and $\mu \in]0, 1[$ if $n \geq 3$. Then there exists some constant $c_0 > 0$ such that for $\theta \in \mathbb{C}$ with $|\theta|$ sufficiently small and $\operatorname{Im} \theta > 0$, one has*

$$\sigma(\tilde{H}_0(\theta)) \cap \{z \in \mathbb{C}; \operatorname{Im} z > 0 \text{ or } \arg z > -c_0 \operatorname{Im} \theta\} = \emptyset \quad (5.7)$$

and

$$\|\langle x \rangle^{-2\mu} \tilde{R}_0(z, \theta)\| \leq \frac{1}{c_0 \operatorname{Im} \theta \langle z \rangle} \quad (5.8)$$

for $z \in \mathbb{C}$ with $\arg z > -c_0 \operatorname{Im} \theta$.

Proof. We only consider the case $\theta = i\tau$ with $\tau > 0$ small enough. Since $V = V_1 - iV_2 \in \mathcal{A}$, one has

$$\begin{aligned} V((1+i\tau)x) &= V_1(x) + \tau x \cdot \nabla V_2(x) \\ &\quad - i(V_2(x) - \tau x \cdot \nabla V_1(x) + O(\tau^2 \langle x \rangle^{-2\mu})) \end{aligned}$$

for $\tau > 0$ sufficiently small. Let $z = \langle u, \tilde{H}_0(\theta)u \rangle$, $u \in H^2$ with $\|u\| = 1$. Then,

$$\operatorname{Re} z = \frac{1 - \tau^2}{(1 + \tau^2)^2} \|\nabla u\|^2 + \langle u, (V_1(x) + O(\tau \langle x \rangle^{-2\mu}))u \rangle, \quad (5.9)$$

$$\begin{aligned} \operatorname{Im} z &= -\frac{2\tau}{(1 + \tau^2)^2} \|\nabla u\|^2 - \langle u, (V_2(x) - \tau x \cdot \nabla V_1(x))u \rangle \\ &\quad + \langle u, O(\tau^2 \langle x \rangle^{-2\mu})u \rangle. \end{aligned} \quad (5.10)$$

This implies that there exists $c > 0$ such that

$$\operatorname{Re} z \geq c(\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2), \quad (5.11)$$

for $\tau > 0$ sufficiently small. If $R \in]0, \infty]$, one has for some $c' > 0$

$$V_2(x) - \tau x \cdot \nabla V_1(x) \geq c' \tau \langle x \rangle^{-2\mu}, \quad \forall x \in \mathbb{R}^n,$$

which gives that

$$\operatorname{Im} z \leq -c'' \tau (\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2) \quad (5.12)$$

for some $c'' > 0$. This shows that $\operatorname{Im} z \leq -C\tau \operatorname{Re} z$ ($C = c''c^{-1}$) if $R \in]0, +\infty[$.

If $R = 0$, one has $V_2(x) \geq 0$ for all x and

$$V_2(x) - \tau x \cdot \nabla V_1(x) \geq c_3 \tau \frac{x^2}{\langle x \rangle^{2\mu+2}}, \quad \forall x \in \mathbb{R}^n,$$

for some $c_3 > 0$. In this case, one has

$$\operatorname{Im} z \leq -C\tau (\|\nabla u\|^2 + \langle u, \frac{x^2}{\langle x \rangle^{2\mu+2}} u \rangle) + C\tau^2 \|\langle x \rangle^{-\mu} u\|^2. \quad (5.13)$$

Lemma 5.1 with $s = \mu$ shows

$$\frac{1}{\langle x \rangle^{2\mu+2}} \leq \frac{1}{2\sqrt{(n-1)(n-1-\mu)}} \left(-\Delta + \frac{1}{\langle x \rangle^{2\mu}} \right)$$

in the sense of selfadjoint operators. For $0 < \mu < \frac{3}{4}$ when $n = 2$ and $\mu \in]0, 1[$ if $n \geq 3$, one has

$$\alpha \equiv \frac{1}{2\sqrt{(n-1)(n-1-\mu)}} < 1.$$

This proves that

$$\begin{aligned} \|\nabla u\|^2 + \langle u, \frac{x^2}{\langle x \rangle^{2\mu+2}} u \rangle &= \|\nabla u\|^2 + \langle u, \left(\frac{1}{\langle x \rangle^{2\mu}} - \frac{1}{\langle x \rangle^{2\mu+2}} \right) u \rangle \\ &\geq (1 - \alpha) (\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2). \end{aligned}$$

Consequently, one obtains

$$\operatorname{Im} z \leq -C(1 - \alpha)\tau (\|\nabla u\|^2 + \|\langle x \rangle^{-\mu} u\|^2) + C\tau^2 \|\langle x \rangle^{-\mu} u\|^2 \leq -C_1\tau \operatorname{Re} z \quad (5.14)$$

for some $C_1 > 0$ if $\tau > 0$ is small enough. This proves that the numerical range of $\tilde{H}_0(\theta)$ is included in the region $\Gamma = \{z; \operatorname{Re} z \geq 0, \operatorname{Im} z \leq -C_1\tau \operatorname{Re} z\}$. Since $\sigma(\tilde{H}_0(\theta)) \subset \Gamma$ and $\|\tilde{R}_0(z, \theta)\| \leq \operatorname{dist}(z, \Gamma)^{-1}$, one has

$$\|\tilde{R}_0(z, \theta)\| \leq \frac{1}{c_0 \operatorname{Im} \theta |z|} \quad (5.15)$$

for $z \in \mathbb{C}$ with $\arg z > -c_0 \operatorname{Im} \theta$ for some $c_0 > 0$. The conclusion of Lemma 5.2 follows now from Theorem 2.4 (with $H = \tilde{H}_0(\theta)$). \blacksquare

In order to obtain subexponential time-decay estimates for e^{-itH_0} , we use the method of analytical distortion. Let $R_0 > 1$ and $\rho \in C^\infty(\mathbb{R})$ with $0 \leq \rho \leq 1$ and $\rho(r) = 0$ if $r \leq 1$ and $\rho(r) = 1$ if $r \geq 2$. Define for $R_0 > 1$

$$F_\theta(x) = x(1 + \theta\rho(\frac{|x|}{R_0})), \quad x \in \mathbb{R}^n. \quad (5.16)$$

When $\theta \in \mathbb{R}$ with $|\theta|$ sufficiently small, $x \rightarrow F_\theta(x)$ is a global diffeomorphism on \mathbb{R}^n . Set

$$U_\theta f(x) = |DF_\theta(x)|^{\frac{1}{2}} f(F_\theta(x)), \quad f \in L^2(\mathbb{R}^n), \quad (5.17)$$

where $DF_\theta(x)$ is the Jacobi matrix and $|DF_\theta(x)|$ the Jacobian of $x \rightarrow F_\theta(x)$. One has

$$|DF_\theta(x)| = \begin{cases} 1, & |x| < R_0; \\ (1 + \theta)^n, & |x| > 2R_0 \end{cases} \quad (5.18)$$

U_θ is unitary in $L^2(\mathbb{R}^n)$ for θ real with $|\theta|$ sufficiently small. Define the distorted operator $H(\theta)$ by

$$H_0(\theta) = U_\theta H_0 U_\theta^{-1}. \quad (5.19)$$

One can calculate that

$$H_0(\theta) = -\Delta_\theta + V(F_\theta(x)) \quad (5.20)$$

where $-\Delta_\theta = {}^t \nabla_\theta \cdot \nabla_\theta$ with

$$\nabla_\theta = ({}^t DF_\theta)^{-1} \cdot \nabla - \frac{1}{|DF_\theta|^2} ({}^t DF_\theta)^{-1} \cdot (\nabla |DF_\theta|) \quad (5.21)$$

In particular, $\nabla_\theta f = (1 + \theta)^{-1} \nabla f$ if f is supported outside the ball $B(0, 2R_0)$. If $V \in \mathcal{A}$, $H_0(\theta)$ can be extended to a holomorphic family of type A in sense of T. Kato for θ in a small complex neighbourhood of zero. $H_0(\theta)$ and $\tilde{H}_0(\theta)$ coincide outside the ball $B(0, 2R_0)$ and they have the same essential spectra. In addition their complex eigenvalues in the region $\{z \in \mathbb{C}; \operatorname{Re} z \geq 0, \operatorname{Im} z > -c \operatorname{Im} \theta \operatorname{Re} z\}$ for some $c > 0$ small enough are the same ([6]). Since $\tilde{R}_0(z, \theta)$ is holomorphic in z there, so is $R_0(z, \theta) = (H_0(\theta) - z)^{-1}$. Remark that the distorted operator $H_0(\theta)$ satisfies the conditions (1.15) and (1.16) with some constant $c_0 > 0$ independent of $R_0 > 1$. Lemma 2.1 applied to $H_0(\theta)$ implies that $\langle x \rangle^{-2\mu} R_0(0, \theta)$ is defined on the range of $H_0(\theta)$ and extends to a bounded operator in $L^2(\mathbb{R}^n)$ and Theorem 2.4 holds for $G_0(\theta) = R_0(0, \theta)$ for some constant C independent of R_0 used in the analytical distortion.

Proposition 5.3. *Assume the conditions of Lemma 5.2. Denote*

$$\Omega_1(\theta) = \{z \in \mathbb{C}; \operatorname{Re} z > 0, \operatorname{Im} z \geq -c \operatorname{Im} \theta \operatorname{Re} z\}.$$

with $c > 0$ appropriately small. Then $\Omega_1(\theta)$ is contained in resolvent set of $H_0(\theta)$ and there exists some constant $C > 0$

$$\|\langle x \rangle^{-2\mu} R_0(z, \theta) \langle x \rangle^{-2\mu}\| \leq \frac{C}{\langle z \rangle}, \quad z \in \Omega_1(\theta). \quad (5.22)$$

Proof. For $z \in \Omega_1(\theta)$ and $|z|$ large, (5.22) follows from Lemma 5.2 by an argument of perturbation. For $z \in \Omega_1(\theta)$ and $|z|$ small, one compares $R_0(z, \theta)$ with $\tilde{R}_0(z, \theta)$ and $R_0(0, \theta)$.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ if $|x| \leq 2R_0$. On the support of $1 - \chi$, $H_0(\theta) = \tilde{H}_0(\theta)$. For $z \in \Omega_1(\theta)$ and $|z|$ small, one has

$$\begin{aligned} R_0(z, \theta) &= R_0(0, \theta) + zR_0(0, \theta)R_0(z, \theta) \\ &= R_0(0, \theta) + zR_0(0, \theta)(\chi(2 - \chi) + (1 - \chi)^2)R_0(z, \theta) \\ &= R_0(0, \theta) + zR_0(0, \theta)\chi(2 - \chi)R_0(z, \theta) \\ &\quad + zR_0(0, \theta)(1 - \chi)\tilde{R}_0(z, \theta)(1 - \chi) \\ &\quad + zR_0(0, \theta)(1 - \chi)\tilde{R}_0(z, \theta)[(1 + \theta)^{-2}\Delta, \chi]R_0(z, \theta) \end{aligned}$$

Recall that for $\text{Im } \theta > 0$, there exists some constant $C > 0$ such that

$$\|\tilde{R}_0(z, \theta)\langle x \rangle^{-2\mu}\| \leq C, \text{ for } z \in \Omega_1(\theta).$$

By the ellipticity of the operator, this implies that

$$\|\tilde{R}_0(z, \theta)\nabla\langle x \rangle^{-2\mu}\| \leq C, \quad (5.23)$$

for $z \in \Omega_1(\theta)$ and $|z| \leq 1$. Therefore there exists possibly another constant C such that

$$\|\langle x \rangle^{-2\mu}R_0(z, \theta)\langle x \rangle^{-2\mu}\| \leq C + C|z|\|\langle x \rangle^{-2\mu}R_0(z, \theta)\langle x \rangle^{-2\mu}\| \quad (5.24)$$

for $z \in \Omega_1(\theta)$ and $|z| \leq 1$. This shows that $\|\langle x \rangle^{-2\mu}R_0(z, \theta)\langle x \rangle^{-2\mu}\|$ is uniformly bounded for $z_0 \in \Omega_1(\theta)$ and $|z|$ sufficiently small. (5.22) is proved. \blacksquare

Theorem 5.4. *Assume $n \geq 2$. Let $V \in \mathcal{A}$ with $0 < \mu < \frac{3}{4}$ if $n = 2$ and $0 < \mu < 1$ if $n \geq 3$. There exists some constant $c > 0$ such that for any function $\chi \in C_0^\infty(\mathbb{R}^n)$ there exists some constant $C_\chi > 0$ such that*

$$\|\chi(x)e^{-itH_0}\chi(x)\| \leq C_\chi e^{-ct|t|^\beta}, \quad t > 0, \quad (5.25)$$

where $\beta = \frac{1-\mu}{1+\mu}$.

Proof. Let $R_1 > 0$ such that $\text{supp } \chi \subset B(0, R_1)$. Let $U(\theta)$ be defined as before with $R_0 > R_1$. Then one has

$$\chi(x)e^{-itH_0}\chi(x) = \chi(x)e^{-itH_0(\theta)}\chi(x)$$

for $\theta \in \mathbb{R}$ with $|\theta|$ small. For $\theta \in \mathbb{C}$ with θ near zero and $\text{Im } \theta > 0$, $H_0(\theta)$ is strictly sectorial and the resolvent $R(z, \theta)$ is holomorphic in $z \in \mathbb{C}$ with $-c\text{Im } \theta < \arg z < \pi + c$ for some $c > 0$. Making use of Proposition 5.3, one can check that

$$\chi(x)e^{-itH_0}\chi(x) = \frac{i}{2\pi} \int_{\Gamma'} e^{-itz} \chi(x)R(z, \theta)\chi(x)dz \quad (5.26)$$

where

$$\Gamma' = \{z = re^{-i\delta}; r \geq 0\} \cup \{z = -re^{i\delta}, r \geq 0\}$$

for $\delta = \delta(\text{Im } \theta) > 0$ small enough. Γ' is oriented in anti-clockwise sense.

The remaining part of the proof of (5.25) is the same as in Theorem 3.4 and will not be repeated here. We just indicate that if one denotes $G_0(\theta) = R(0, \theta)$, then one has

$$R_0(z, \theta) = \sum_{j=0}^N z^j G_0(\theta)^j + z^{N+1} G_0(\theta)^N R_0(z, \theta)$$

for $z \in \Gamma'$ and z near 0, and Theorem 2.4 with $r = 2$ and Proposition 5.3 show that

$$\begin{aligned} & \|\chi(x) G_0(\theta)^N R(z, \theta) \chi(x)\| \\ & \leq C_\chi \|\langle x_{N,0} \rangle^{-2(N+1)\mu} G_0(\theta)^N R_0(z, \theta) \langle x_{N,0} \rangle^{-2\mu}\| \leq C_{\chi, \text{Im } \theta} C^N N^\gamma \end{aligned} \quad (5.27)$$

with some constant C independent of χ and θ . By choosing appropriately N in terms of t as in the proof of Theorem 3.4, one obtain some $c > 0$ independent of χ such that (5.25) holds. \blacksquare

6. COMPACTLY SUPPORTED PERTURBATIONS OF H_0

Consider operator H of the form

$$H = H_0 + W(x). \quad (6.1)$$

where $H_0 = -\Delta + V(x)$ is a non-selfadjoint Schrödinger operator with $V \in \mathcal{V}$ and $W \in L^\infty_{\text{comp}}(\mathbb{R}^n)$. Then the essential spectrum of H is equal to $[0, +\infty[$ and the accumulation points of the complex eigenvalues of H are contained in \mathbb{R}_+ . Denote $R(z) = (H - z)^{-1}$, $z \notin \sigma(H)$.

6.1. Proof of Theorem 1.2.

Proposition 6.1. *Let $H_0 = -\Delta + V(x)$ with $V \in \mathcal{V}$. Let $W \in L^\infty(\mathbb{R}^n)$ with compact support and $H = H_0 + W(x)$. Assume that 0 is not an eigenvalue of H . Then one has:*

(a). *There exist some constants $c_1, \mu' > 0$ such that outside the set*

$$\Omega_1 = \{z \in \mathbb{C}; \text{Re } z \geq 0 \text{ and } |\text{Im } z| \leq c_1 |\text{Re } z|^{\mu'}\},$$

there are at most a finite number of discrete eigenvalues of H . There exist some $\delta > 0$ such that

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{\mu'}}} \text{ for } z \notin \Omega_1 \text{ and } |z| < \delta. \quad (6.2)$$

(b). *The limit*

$$R(0) = \lim_{z \rightarrow 0, z \notin \Omega_1} R(z) \quad (6.3)$$

exists in $\mathcal{L}(L^{2,s}; L^{2,s-2\mu})$ for any $s \in \mathbb{R}$ and one has the Gevrey estimates

$$\|e^{-a\langle x \rangle^{1-\mu}} R(z)^N\| \leq C_a^{N+1} N^{\gamma N}, \quad (6.4)$$

$$\|\chi R(z)^N\| \leq C_\chi C^N N^{\gamma N} \quad (6.5)$$

for all $N \in \mathbb{N}^$ and $z \in \Omega_0 = \{z \in \mathbb{C}; \text{Re } z < 0, |\text{Im } z| < -M \text{Re } z\} \cup \{0\}$, $M > 0$. Here $a > 0$ and $\chi \in C_0^\infty(\mathbb{R}^n)$, C_a, c_a, C_χ are some positive constants and $C > 0$ is independent of χ .*

Proof. Note that G_0W is a compact operator and that 0 is not an eigenvalue of H if and only if -1 is not an eigenvalue of G_0W . So if 0 is not an eigenvalue of H , operator $1 + G_0W$ is invertible on L^2 . From Proposition 3.1, one deduces that $1 + R_0(z)W$ is invertible for $|z|$ small and $z \notin \Omega_1$. This shows that 0 is not an accumulation point of $\sigma(H) \setminus \Omega_1$. In addition, $z \rightarrow 1 + R_0(z)W$ is holomorphic in $\mathbb{C} \setminus \Omega_1$. The analytic Fredholm Theorem shows that $(1 + R_0(z)W)^{-1}$ is a meromorphic function with at most a discrete set of poles in $\mathbb{C} \setminus \Omega_1$. These poles are eigenvalues of H . Since we have seen that 0 is not an accumulation point of eigenvalues of H in $\mathbb{C} \setminus \Omega_1$, one concludes that the number of eigenvalues of H in $\mathbb{C} \setminus \Omega_1$ is finite. (6.2) follows from Proposition 3.1 and the equation

$$R(z) = (1 + R_0(z)W)^{-1}R_0(z).$$

To prove that Gevrey estimates of the resolvent $R(z)$, we remark that if $F(z)$ and $G(z)$ are two bounded operator valued functions on Ω_0 satisfies the Gevrey estimates with some $\beta > 1$

$$\|F^{(n)}(z)\| \leq AC_1^n(n!)^\beta \quad (6.6)$$

$$\|G^{(n)}(z)\| \leq BC_2^n(n!)^\beta \quad (6.7)$$

for all $n \in \mathbb{N}$ and $z \in \Omega_0$ and for some $\gamma > 1$ and $A, B, C_1, C_2 > 0$, then $F(z)G(z)$ satisfies the Gevrey estimates

$$\|(FG)^{(n)}(z)\| \leq ABC_3^n(n!)^\beta \quad (6.8)$$

for all $n \in \mathbb{N}$ and $z \in \Omega_0$ where

$$C_3 = D_\beta \max\{C_1, C_2\} \text{ with } D_\beta = \sup_{n \in \mathbb{N}} \sum_{j=0}^n \left(\frac{j!(n-j)!}{n!} \right)^{\beta-1} < \infty; \quad (6.9)$$

and if $F(z)$ is invertible for $z \in \Omega_0$ with uniformly bounded inverse:

$$\|F(z)^{-1}\| \leq M \quad (6.10)$$

for all $z \in \Omega_0$, then the inverse $H(z) = F(z)^{-1}$ satisfies the Gevrey estimates

$$\|H^{(n)}(z)\| \leq MC_4^n(n!)^\beta \quad (6.11)$$

for all $n \in \mathbb{N}$ and $z \in \Omega_0$, where $C_4 = MC_1D_\beta$. Denote $G^\beta(\Omega_0)$ the set of bounded operator-valued functions on Ω_0 verifying Gevrey estimates of order $\beta > 1$. Since $e^{-a\langle x \rangle^{1-\mu}}R_0(z)$ and $\chi R_0(z)$ belong to G^β with $\beta = 1 + \gamma = \frac{1+\mu}{1-\mu}$, Estimates (6.4) and (6.5) follow respectively from equations

$$e^{-a\langle x \rangle^{1-\mu}}R(z) = (1 + e^{-a\langle x \rangle^{1-\mu}}R_0(z)W e^{a\langle x \rangle^{1-\mu}})^{-1}e^{-a\langle x \rangle^{1-\mu}}R_0(z) \quad (6.12)$$

$$\chi R(z) = (1 + \chi R_0(z)W)^{-1}\chi R_0(z) \quad (6.13)$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is taken such that $\chi(x) = 1$ on $\text{supp}W$. ■

Proof of Theorem 1.2 (a). Theorem 1.2 (a) can be proved in the same way as Theorem 3.4 for the model operator H_0 . By Proposition 6.1, one can find a contour Γ in the right half complex plane of the form

$$\Gamma = \{z; \text{Re } z \geq 0, |\text{Im } z| = C(\text{Re } z)^{\mu'}\}$$

for some $C, \mu' > 0$ such that $\sigma(H) \cap \Gamma = \{0\}$ and there are only a finite number of complex eigenvalues of H located at the left of Γ . Let

$$\Lambda = \sigma(H) \cap \{z; \operatorname{Re} z < 0 \text{ or } \operatorname{Re} z \geq 0 \text{ and } |\operatorname{Im} z| > C(\operatorname{Re} z)^{\mu'}\}.$$

Then one has

$$e^{-tH} - \sum_{\lambda \in \Lambda} e^{-tH} \Pi_\lambda = \frac{i}{2\pi} \int_{\Gamma} e^{-tz} R(z) dz. \quad (6.14)$$

where Π_λ is the Riesz projection of H associated the eigenvalue λ . Making use of Proposition 6.1, one can prove as in Theorem 3.4 that

$$\|e^{-a(x)^{1-\mu}} (e^{-tH} - \sum_{\lambda \in \Lambda} e^{-tH} \Pi_\lambda)\| \leq C_a e^{-c_a t^\beta}, \quad t > 0. \quad (6.15)$$

(1.28) follows since if $\lambda \in \sigma_d(H)$ with $\operatorname{Re} \lambda > 0$, $\|e^{-tH} \Pi_\lambda\|$ decreases exponentially. (1.29) is deduced in a similar way. \blacksquare

Proposition 6.2. *Let $H_0 = -\Delta + V(x)$ with $V \in \mathcal{A}$ and $H = H_0 + W(x)$ for some $W \in L_{\text{comp}}^\infty$. Assume that 0 is not an eigenvalue of H . Then one has*

(a). *There exists $\delta > 0$ such that H has at most a finite number of eigenvalues in*

$$\Omega_\delta = \{z \in \mathbb{C} \setminus \{0\}; -\delta \leq \arg z \leq \pi + \delta\}$$

and for any $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi R(z)\chi$ defined for $\operatorname{Im} z > 0$ extends meromorphically into Ω_δ and there exists some constant $C_\chi, c > 0$ such that

$$\|\chi R(z)\chi\| \leq C_\chi \quad (6.16)$$

for $z \in \Omega_\delta$ and $|z| < c$.

(b). *The limit $R(0) = \lim_{z \in \Omega_\delta, z \rightarrow 0} R(z)$ in $\mathcal{L}(L^{2,s+2\mu}; L^{2,s-2\mu})$ for any $s \in \mathbb{R}$ and one has*

$$\|\chi R(z)^N \chi\| \leq C_\chi C^N N^{\gamma N} \quad (6.17)$$

for any $N \in \mathbb{N}^$ and $z \in \Omega_\delta \cup \{0\}$ with $|z| < c$.*

Proof. Since $V_2 \geq 0$, one has for $\operatorname{Im} z > 0$

$$\chi R(z)\chi = (1 + \chi R_0(z)W)^{-1} \chi R_0(z)\chi \quad (6.18)$$

if $\chi \in C_0^\infty(\mathbb{R}^n)$ is taken such that $\chi(x) = 1$ on $\operatorname{supp} W$. Let U_θ be defined by (5.17) with $R_0 \gg 1$ such that $\operatorname{supp} \chi \subset \{x; |x| < R_0\}$. Let $H_0(\theta) = U_\theta H_0 U_\theta^{-1}$ and $R_0(z, \theta) = (H_0(\theta) - z)^{-1}$. Then one has for $\theta \in \mathbb{C}$, $\operatorname{Im} \theta > 0$ and $|\theta|$ small,

$$\chi R_0(z)W = \chi R_0(z, \theta)W, \quad \chi R_0(z)\chi = \chi R_0(z, \theta)\chi.$$

For a fixed $\theta \in \mathbb{C}$ with $\operatorname{Im} \theta > 0$, Proposition 5.3 shows that $\chi R_0(z)W$ and $\chi R_0(z)\chi$ are holomorphic in Ω_δ for some $\delta > 0$. The analytic Fredholm Theorem implies that $\chi R(z)\chi$ extends to a meromorphic in Ω_δ given by

$$\chi R(z)\chi = (1 + \chi R_0(z, \theta)W)^{-1} \chi R_0(z, \theta)\chi. \quad (6.19)$$

0 is the only possible accumulation point of these poles. To show that 0 is in fact not an accumulation point, we firstly prove that for each χ , -1 is not an eigenvalue of the compact operator $\chi G_0 W$. In fact if -1 is an eigenvalue of $\chi G_0 W$, then -1 is also an

eigenvalue of $WG_0\chi = J(\chi G_0 W)^* J$ where J is the complex conjugation: $Jf(x) = \overline{f(x)}$. Let $\psi \in L^2$ with $\psi \neq 0$ such that

$$\psi = -WG_0\chi\psi.$$

Then $\chi\psi = \psi$ and $HG_0\psi = (1 + WG_0\chi)\psi = 0$ which gives $\psi = 0$, since 0 is not an eigenvalue of H and G_0 is injective. This contradiction shows that $1 + \chi G_0 W$ is invertible with bounded inverse. Secondly by an argument of compactness, one deduces that if χ_R , $R > 0$, is a family of cut-offs such that $\chi_R(x) = \chi(\frac{x}{R})$ for some function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ for $|x| \leq 1$, then there exists some constant $C > 0$ such that

$$\|(1 + \chi_R G_0 W)^{-1}\| \leq C \quad (6.20)$$

uniformly in $R > 1$. According to Proposition 5.3, one has

$$1 + \chi_R R_0(z, \theta)W = 1 + \chi_R G_0(\theta)W + O(|z|)$$

in $\mathcal{L}(L^{2, -2\mu}; L^{2, -2\mu})$ for $z \in \Omega_\delta$, where $O(|z|)$ is uniform in $R > 1$. Consequently there exists some constant $c > 0$ independent of R such that the inverse $(1 + \chi_R R_0(z, \theta)W)^{-1}$ exists and is holomorphic for $z \in \Omega_\delta$ with $|z| < c$ and there exists some constant $C > 0$ such that

$$\|(1 + \chi_R R_0(z, \theta)W)^{-1}\|_{\mathcal{L}(L^{2, -2\mu}; L^{2, -2\mu})} \leq C \quad (6.21)$$

uniformly $R > 1$ and $z \in \Omega_\delta$ with $|z| < c$. Finally since discrete eigenvalues of H are poles of the resolvent $(H(\theta) - z)^{-1}$ and each pole of the resolvent is a pole of $\chi_R R(z)\chi_R$ if $R > 1$ is large enough, (6.21) implies that there are no poles of $(H(\theta) - z)^{-1}$ (hence no eigenvalues of H) in $z \in \Omega_\delta$ with $|z| < c$ for some $c > 0$ independent of $R > 0$. This proves the finiteness of eigenvalues of H in Ω_δ , because zero is the only possible accumulation point of eigenvalues of H in Ω_δ . Estimate (6.16) follows from (5.22).

Part (b) can be derived from (6.19), Proposition 5.3 and Theorem 1.1 applied to $G_0(\theta)$. \blacksquare

Proof of Theorem 1.2 (b). According to Proposition 6.2, there exists some $0 < \eta\delta$ such that Ω_η contains no eigenvalues of H with negative imaginary part. Under the assumptions Theorem 1.3, positive eigenvalues of H is absent. If $\lambda \in \sigma_d(H)$ with $\text{Im } \lambda > 0$, then $e^{-itH}\Pi_\lambda = O(e^{-t\text{Im } \lambda})$ decreases exponentially as $t \rightarrow +\infty$. Theorem 1.2 (b) follows from the representation formula:

$$\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \mathbb{R}_-} e^{-itH}\Pi_\lambda)\chi = \frac{i}{2\pi} \int_{\Gamma_\eta} e^{-itz} \chi R(z)\chi dz + O(e^{-ct}), \quad t > 0, \quad (6.22)$$

where $c > 0$, $\Gamma_\eta = \{z = re^{-i\eta}; r \geq 0\} \cup \{z = -re^{i\eta}; r \geq 0\}$ where $0 < \eta \leq \delta$ is chosen such that there are no eigenvalues with negative imaginary part between the real axis and Γ_η . The details are the same as in Theorem 5.4 and are omitted here. \blacksquare

6.2. Proof of Theorem 1.3. Assume now that 0 is an eigenvalue of $H = H_0 + W$. Then -1 is an eigenvalue of $G_0 W$ and $\ker(1 + G_0 W)$ in L^2 coincides with the eigenspace of H with eigenvalue zero.

Lemma 6.3. *Assume that H satisfied the condition (1.15). Then there exists some constant $\alpha_0 > 0$ such that if $u \in H^2(\mathbb{R}^n)$ such that $Hu = 0$, then $e^{\alpha_0 \langle x \rangle^{1-\mu}} u \in L^2(\mathbb{R}^n)$.*

Proof. Let $\varphi(x) = \alpha \langle x \rangle^{1-\mu}$, $\alpha > 0$. Then

$$|\nabla \varphi(x)| = \alpha(1-\mu) \frac{|x|}{\langle x \rangle^{1+\mu}} \leq \alpha \langle x \rangle^{-\mu}.$$

For $u \in H^2(\mathbb{R}^n)$ with compact support, one has

$$\begin{aligned} |\langle e^{2\varphi} Hu, u \rangle| &= |\langle H(e^\varphi u), e^\varphi u \rangle - \langle [\Delta, e^\varphi] u, e^\varphi u \rangle| \\ &= |\langle H(e^\varphi u), e^\varphi u \rangle + 2\langle (|\nabla \varphi|^2 e^\varphi u - \nabla \varphi \cdot \nabla(e^\varphi u)), e^\varphi u \rangle| \\ &\geq |\langle (H_0 + W)(e^\varphi u), e^\varphi u \rangle| - (2\alpha^2 + \alpha) \|\langle x \rangle^{-\mu} e^\varphi u\|^2 - \alpha \|\nabla(e^\varphi u)\|^2 \end{aligned} \quad (6.23)$$

Since W is compactly supported, making use of the condition (1.15), one obtains for some $c_0, C > 0$,

$$|\langle (H_0 + W)(e^\varphi u), e^\varphi u \rangle| \geq c_0 (\|\nabla(e^\varphi u)\|^2 + \|\langle x \rangle^{-\mu} e^\varphi u\|^2) - C \|u\|^2 \quad (6.24)$$

For $\alpha > 0$ such that

$$2\alpha^2 + \alpha < c_0, \quad (6.25)$$

there exists some constant $C_1 > 0$ such that

$$\|\langle x \rangle^{-\mu} e^\varphi u\|^2 + \|\nabla(e^\varphi u)\|^2 \leq C_1 (|\langle e^{2\varphi} Hu, u \rangle| + \|u\|^2) \quad (6.26)$$

for any $u \in H^2$ compactly supported. If $u \in H^2$ such that $Hu = 0$, one can apply the above estimate to $u_R = \chi_R u$ where $\chi_R(x) = \chi(\frac{x}{R})$, $R > 1$ and χ is a smooth cut-off such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Then one can check that

$$|\langle e^{2\varphi} H u_R, u_R \rangle| \leq C' R^{-(1-\mu)} (\|\langle x \rangle^{-\mu} e^\varphi u\|^2 + \|\nabla(e^\varphi u)\|^2).$$

It follows that

$$\|\langle x \rangle^{-\mu} e^\varphi u_R\|^2 + \|\nabla(e^\varphi u_R)\|^2 \leq C_2 \|u\|^2 \quad (6.27)$$

uniformly in $R \gg 1$ which implies that $\langle x \rangle^{-\mu} e^\varphi u \in L^2(\mathbb{R}^n)$ and $\nabla(e^\varphi u) \in L^2$. Lemma 6.3 is proved with $0 < \alpha_0 < \alpha$. \blacksquare

Theorem 6.4. *Let $H_0 = -\Delta + V(x)$ and $H = H_0 + W(x)$ with $V \in \mathcal{V}$ and $W \in L_{\text{comp}}^\infty$. Assume that 0 is an eigenvalue of H and that both H_0 and H are selfadjoint. Let Π_0 denote the eigenprojection of H associated with eigenvalue zero. Then there exist some constants $C, \mu', \delta > 0$ such that*

$$R(z) = -\frac{\Pi_0}{z} + R_1(z) \quad (6.28)$$

for $z \in \Omega_1(\delta)$ where

$$\Omega_1(\delta) = \{z \in \mathbb{C}; |z| < \delta, \text{ either } \operatorname{Re} z < 0 \text{ or } \operatorname{Re} z \geq 0 \text{ and } |\operatorname{Im} z| > C |\operatorname{Re} z|^{\mu'}\}. \quad (6.29)$$

The remainder $R_1(z)$ satisfies the estimates

$$\|\langle x \rangle^{-s} R_1(z)\| + \|R_1(z) \langle x \rangle^{-s}\| \leq C_s \quad (6.30)$$

for $s > 2\mu + \frac{1}{\mu'}$ and $z \in \Omega_1(\delta)$; and for any $a, M > 0$ there exist some constants $C_a, c_a > 0$ such that

$$\|e^{-a \langle x \rangle^{1-\mu}} R_1^{(N)}(z)\| + \|R_1^{(N)}(z) e^{-a \langle x \rangle^{1-\mu}}\| \leq C_a c_a^N N^{\beta N}, \quad (6.31)$$

for any $N \in \mathbb{N}$ and $z \in \Omega_-$ where $\Omega_- = \{z; \operatorname{Re} z < 0 \text{ and } |\operatorname{Im} z| \leq -M \operatorname{Re} z\} \cup \{0\}$, $M > 0$. Here and in the following, $R_1^{(n)}(z)$ denotes the derivative of order n of $R_1(z)$ and $\beta = 1 + \gamma = \frac{1+\mu}{1-\mu}$.

Proof. We use the Grushin method to study the low-energy asymptotics for the resolvent of H by using the equation

$$R(z) = (1 + R_0(z)W)^{-1}R_0(z). \quad (6.32)$$

Since the method is well-known in selfadjoint case, we shall skip over some details and emphasize on the Gevrey estimates of the remainder. Note that $\ker_{L^{2,s}}(1 + G_0W)$ is independent of $s \in \mathbb{R}$ and coincides with the eigenspace of H associated with the eigenvalue 0. We need only to work in $L^2(\mathbb{R}^n)$.

Let ψ_1, \dots, ψ_m be a basis of $\ker(1 + G_0W)$ such that

$$\langle \psi_j, -W\psi_k \rangle = \delta_{jk}, \quad j, k = 1, \dots, m. \quad (6.33)$$

(6.33) can be realized because the quadratic form $\phi \rightarrow \langle \phi, -W\phi \rangle = \langle \phi, H_0\phi \rangle$ is positive definite on $\ker(1 + G_0W)$. Define $Q : L^2 \rightarrow L^2$ by

$$Qf = \sum_{j=1}^m \langle -W\psi_j, f \rangle \psi_j, \quad f \in L^2. \quad (6.34)$$

Set $Q' = 1 - Q$. Then Q commutes with $1 + G_0W$. -1 is not eigenvalue of compact operator $Q'(G_0W)Q'$, hence $Q'(1 + G_0W)Q'$ is invertible on the range of Q' with bounded inverse. From Theorem 2.4 with $N = 1$ and Proposition 3.1, one deduces that

$$(R_0(z) - G_0)W = O(|z|) \quad (6.35)$$

for $z \in \Omega_1(\delta)$. It follows that if $\delta > 0$ is small enough,

$$E(z) = (Q'(1 + R_0(z)W)Q')^{-1}Q' \quad (6.36)$$

is well-defined and continuous in $z \in \Omega_1(\delta)$ and is uniformly bounded:

$$\|E(z)\| \leq C \quad (6.37)$$

uniformly in $z \in \Omega_1(\delta)$. By Corollary 2.5 and (6.8), $E(z)$ satisfies Gevrey estimates

$$\|E^{(N)}(z)\| \leq CC'^N N^{\gamma N} \quad (6.38)$$

for some $C' > 0$ and for all $z \in \Omega_-$.

Define $S : \mathbb{C}^m \rightarrow D(H)$ and $T : L^2 \rightarrow \mathbb{C}^m$ by

$$Sc = \sum_{j=1}^m c_j \psi_j, \quad c = (c_1, \dots, c_m) \in \mathbb{C}^m,$$

$$Tf = (\langle -W\psi_1, f \rangle, \dots, \langle -W\psi_m, f \rangle) \in \mathbb{C}^m, \quad f \in L^2.$$

Set $W(z) = (1 + R_0(z)W)$ and

$$E_+(z) = S - E(z)W(z)S, \quad (6.39)$$

$$E_-(z) = T - TW(z)E(z), \quad (6.40)$$

$$E_{-+}(z) = -TW(z)S + TW(z)E(z)W(z)S. \quad (6.41)$$

Then one has the formula

$$(1 + R_0(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \text{ on } H^{1,-s}. \quad (6.42)$$

Since $E(z)$, $W(z)$ satisfy Gevrey estimates of the form (6.38) on Ω_- , $E_{\pm}(z)$ and $E_{-+}(z)$ satisfy similar Gevrey estimates on Ω_- . The leading term of $E_{-+}(z)$ can be explicitly calculated:

$$E_{-+}(z) = -z\Psi + z^2r_1(z) \quad (6.43)$$

where the matrix $\Psi = (\langle \psi_j, \psi_k \rangle)_{1 \leq j, k \leq m}$ is positive definite and $r_1(z)$ satisfies the Gevrey estimates in Ω_- . Consequently,

$$E_{-+}(z)^{-1} = -\frac{\Psi^{-1}}{z} + \tilde{r}_1(z) \quad (6.44)$$

with $\tilde{r}_1(z)$ uniformly bounded on $\Omega_1(\delta)$ and $\tilde{r}_1(z)$ satisfying the Gevrey estimates of the form (6.38) in Ω_- . Consequently $(1 + R_0(z)W)^{-1}$ is of the form

$$(1 + R_0(z)W)^{-1} = \frac{A_0}{z} + B(z) \quad (6.45)$$

where

$$A_0 = S\Psi^{-1}T \quad (6.46)$$

is an operator of rank m and $B(z)$ is uniformly bounded $\Omega_1(\delta)$ and satisfies the Gevrey estimates

$$\|B^{(N)}(z)\| \leq CC'^N N^{\beta N}, \quad \forall N \in \mathbb{N}, \quad (6.47)$$

for z in Ω_- . From the equation $R(z) = (1 + R_0(z)W)^{-1}R_0(z)$ and Corollary 2.5, we deduce that

$$R(z) = -\frac{\Pi_0}{z} + R_1(z) \quad (6.48)$$

where $R_1(z)$ satisfies

$$\|R_1(z)\langle x \rangle^{-2k\mu}\| \leq C$$

for $z \in \Omega_1(\delta)$ if $k \in \mathbb{N}$ and $k \geq \frac{1}{\mu'}$ and

$$\|R_1^{(N)}(z)e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a c_a^N N^{\beta N}$$

for z in Ω_- . This proves (6.30) and (6.31). ■

Theorem 6.5. *Let $H_0 = -\Delta + V(x)$ and $H = H_0 + W(x)$ with $V \in \mathcal{A}$ and $W \in L_{\text{comp}}^{\infty}$. Assume that 0 is an eigenvalue of H and that both H_0 and H are selfadjoint. Let Π_0 denote the eigenprojection of H associated with eigenvalue zero. Let Ω_{δ} be defined as in Proposition 6.2 and $\Omega_{\delta}(c) = \Omega_{\delta} \cap \{|z| < c\}$. Then there exist some constants $C, c, \mu', \delta > 0$ such that for any $\chi \in C_0^{\infty}(\mathbb{R}^n)$ the cut-off resolvent $\chi R(z)\chi$ defined for $\text{Im } z > 0$ extends to a holomorphic function in $\Omega_{\delta}(c)$ and one has*

$$\chi R(z)\chi = -\frac{\chi \Pi_0 \chi}{z} + R_2(z) \quad (6.49)$$

for $z \in \Omega_{\delta}(c)$ where the remainder $R_2(z)$ is continuous up to $z = 0$ and satisfies the Gevrey estimates

$$\|\chi R_2^{(N)}(z)\chi\| \leq C_{\chi} C^N N^{\beta N} \quad (6.50)$$

for $z \in \Omega_{\delta}(c) \cup \{0\}$.

Proof. It suffices to prove (6.49) for $\chi \in C_{\text{at}}^\infty(\mathbb{R}^n)$ with sufficiently large support. Let $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi_1(x) \leq 1$, $\chi_1(x) = 1$ for $|x| \leq 1$ and 0 for $|x| \geq 2$. Set

$$\chi_j(x) = \chi_0\left(\frac{x}{jR}\right), \quad j = 1, 2, \quad (6.51)$$

where $R > R_0$ is to be adjusted and R_0 is such that $\text{supp } W \subset \{x; |x| \leq R_0\}$. Then $\chi_j W = W$ and $\chi_1 \chi_2 = \chi_1$. Then one has

$$\chi_1 R(z) \chi_1 = (1 + \chi_2 R_0(z, \theta) W)^{-1} \chi_2 R_0(z, \theta) \chi_1, \quad (6.52)$$

where the analytical distortion is carried out outside the support of χ_2 . (6.52) initially valid for θ real and $\text{Im } z > 0$ allows to extend $z \rightarrow \chi R(z) \chi$ into a sector below the positive real axis when $\text{Im } \theta > 0$. In the following $\theta \in \mathbb{C}$ is fixed with $\text{Im } \theta > 0$. $1 + \chi_2 R_0(z, \theta) W$ and $\chi_2 R_0(z, \theta) \chi_1$ belong to Gevrey class $G^\beta(\Omega_\delta)$ where Ω_δ is defined in Proposition 6.2.

Let $\{\psi_j, j = 1, \dots, m\}, Q, Q'$ be defined as in the proof of Theorem 6.4. Then -1 is not an eigenvalue of compact operator $Q'(G_0 W)Q'$. Since $Q'(\chi_2 G_0 W)Q'$ converges to $Q'G_0 W Q'$ in operator norm as $R \rightarrow \infty$, -1 is not an eigenvalue of $Q'\chi_1 G_0 W Q'$ if $R \geq R_1$ for some $R_1 \geq R_0$, R_1 sufficiently large. Then $Q'(1 + \chi_2 G_0 W)Q'$ is invertible on $\text{Range } Q'$, so is $Q'(1 + \chi_2 R_0(z, \theta) W)Q'$ for $z \in \Omega_\delta(c) = \Omega_\delta \cap \{z; |z| < c\}$ for some $c > 0$. The inverse

$$E_0(z, \theta) = (Q'(1 + \chi_2 R_0(z, \theta) W)Q')^{-1} Q' \quad (6.53)$$

is uniformly bounded in z (see Proposition 5.3) and by (6.8) it belongs to Gevrey class $G^\beta(\Omega_\delta(c))$.

Define $S_1 : \mathbb{C}^m \rightarrow L^2$ and $T_1 : L^2 \rightarrow \mathbb{C}^m$ by

$$S_1 = \chi_1 S, \quad T_1 = T \chi_1 \quad (6.54)$$

where S, T are defined in Theorem 6.4. By Lemma ??,

$$S_1 T_1 = Q' + O(e^{-cR^{1-\mu}}), \quad T_1 S_1 = 1 + O(e^{-cR^{1-\mu}}) \quad (6.55)$$

for some $c > 0$. Let $W(z, \theta) = 1 + \chi_2 R_0(z, \theta) W$. Consider the Grushin problem

$$\begin{pmatrix} W(z, \theta) & S_1 \\ T_1 & 0 \end{pmatrix} : L^2 \otimes \mathbb{C}^m \rightarrow L^2 \otimes \mathbb{C}^m. \quad (6.56)$$

One has

$$\begin{pmatrix} W(z, \theta) & S_1 \\ T_1 & 0 \end{pmatrix} \begin{pmatrix} E_0(z, \theta) & S_1 \\ T_1 & -T_1 W(z, \theta) S_1 \end{pmatrix} = 1 + \mathcal{R}(z, \theta) \quad (6.57)$$

where

$$\mathcal{R}(z, \theta) = \begin{pmatrix} QW(z, \theta)E_0(z, \theta) + S_1 T_1 - Q & (1 - T_1)W(z, \theta)S_1 \\ T_1 E_0(z, \theta) & T_1 S_1 - 1 \end{pmatrix}. \quad (6.58)$$

$\mathcal{R}(z, \theta)$ is sum of a nilpotent matrix and a matrix of order $O(e^{-cR^{1-\mu}})$. Hence $1 + \mathcal{R}(z)$ is invertible $z \in \Omega_\delta(c)$ if $R > R_1$ is sufficiently large. This proves the Grushin problem is

invertible from the right. Similarly one can show it is invertible from the left, therefore it is invertible with inverse given by

$$\begin{pmatrix} E_0(z, \theta) & S_1 \\ T_1 & -T_1 W(z, \theta) S_1 \end{pmatrix} (1 + \mathcal{R}(z))^{-1} := \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \quad (6.59)$$

As usual, one has the formula

$$(1 + \chi_2 R_0(z, \theta) W)^{-1} = E(z) - E_+(z) E_{-+}(z)^{-1} E_-(z). \quad (6.60)$$

$E_{-+}(z)$ is of the form

$$E_{-+}(z) = -T_1 W(z, \theta) S_1 (1 + O(e^{-cR^{1-\mu}})) + O(|z|^2)$$

By the choice of χ_1, χ_2 , one has

$$T_1 W(z, \theta) S_1 = T_1 (1 + R_0(z, \theta) W) S_1 \quad (6.61)$$

$$= z T_1 G_1(\theta) W S_1 + O(|z|^2) = z T_1 G_1 W S_1 + O(|z|^2). \quad (6.62)$$

By the calculation made in the proof of Theorem 6.4, one sees $\Psi_1 = T_1 G_1 S_1$ is an invertible matrix (if R is large enough). Consequently $E_{-+}[z]$ is invertible for $z \in \Omega_\delta(c)$ with inverse of the form.

$$E_{-+}(z)^{-1} = -\frac{1}{z} \Psi_1 (1 + O(e^{-cR^{1-\mu}})) + B(z) \quad (6.63)$$

where $B(z)$ belongs to $G^\beta(\Omega_\delta(c))$. This proves the existence of an expansion for $\chi_1 R(z) \chi_1$ for $z \in \Omega_\delta(c)$ of the form

$$\chi_1 R(z) \chi_1 = -\frac{U}{z} + R_2(z) \quad (6.64)$$

with $R_2(z)$ satisfying Gevrey estimates of order β on $\Omega_\delta(c)$. Since $\mathcal{A} \subset \mathcal{V}$, Theorem 6.4 applied to $R(z)$ with $z \in \Omega_\delta(c) \cap \{\operatorname{Re} z \leq 0\}$ gives $U = \chi_1 \Pi_0 \chi$. ■

Proof of Theorem 1.3. Theorem 1.3 (a) and (b) are respectively deduced from Theorems 6.4 and 6.5 and the formulas for $t > 0$

$$e^{-tH} - \sum_{\lambda \in \sigma_d(H), \operatorname{Re} \lambda \leq 0} e^{-tH} \Pi_\lambda = \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\Gamma(\epsilon)} e^{-tz} R(z) dz + O(e^{-ct}) \quad (6.65)$$

$$\chi(e^{-itH} - \sum_{\lambda \in \sigma_d(H) \cap \mathbb{R}_-} e^{-itH} \Pi_\lambda) \chi = \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\Gamma_\eta(\epsilon)} e^{-itz} \chi R(z) \chi dz + O(e^{-ct}) \quad (6.66)$$

where $c > 0$ and

$$\Gamma(\epsilon) = \{z; |z| \geq \epsilon, \operatorname{Re} z \geq 0, |\operatorname{Im} z| = C(\operatorname{Re} z)^{\mu'}\} \cup \{z; |z| = \epsilon, |\arg z| \geq \omega_0\}$$

$$\Gamma_\eta(\epsilon) = \{z = r e^{-i\eta}, r \geq \epsilon\} \cup \{z = -r e^{i\theta}, r \geq \epsilon\} \cup \{z; |z| = \epsilon, -\eta \leq \arg z \leq \pi + \eta\}$$

for some appropriate constants $C, \mu' > 0, \eta > 0$. In particular, $\eta > 0$ is chosen such that H has no eigenvalues with negative imaginary part above $\Gamma_\eta(\epsilon)$. Here ω_0 is the argument of the point z_0 with $|z_0| = \epsilon, \operatorname{Re} z_0 > 0$ and $\operatorname{Im} z_0 = C(\operatorname{Re} z_0)^{\mu'}$. Remark that the subexponential time-decay estimates are derived from Gevrey estimates of $R_1(z)$ and $R_2(z)$ at zero and their Taylor expansion of order N with N chosen appropriately in terms of $t > 0$. See the proof of Theorem 3.4 for e^{-tH_0} . ■

Remark 6.1. *As an example of applications of Theorem 1.3, consider the Witten-Laplacian*

$$-\Delta_U = {}^t \nabla_U \cdot \nabla_U \quad (6.67)$$

where $\nabla_U = e^{-U} \nabla e^U$ and $U \in C^2(\mathbb{R}^n)$. Then

$$-\Delta_U = -\Delta + (\nabla U)(x) \cdot (\nabla U)(x) - \Delta U(x)$$

If $U \in C^2(\mathbb{R}^n; \mathbb{R})$ satisfies for some $\sigma \in]0, 1[$ and $c_1, C_1 > 0$,

$$U(x) \geq c_1 \langle x \rangle^\sigma, \quad |\nabla U(x)| \geq c_1 \langle x \rangle^{\sigma-1}, \quad |\partial_x^\alpha U(x)| \leq C_1 \langle x \rangle^{\sigma-|\alpha|} \quad (6.68)$$

for x outside some compact and for $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$. Then $-\Delta_U$ can be decomposed as $-\Delta_U = H_0 + W(x)$ where H_0 satisfies the conditions of Theorem 1.1 with $\mu = 1 - \sigma$ and $W(x)$ is of compact support. Zero is a simple eigenvalue of $-\Delta_U$ embedded in its continuous spectrum $[0, +\infty[$. As consequence of Theorem 1.3, one obtains the following result. Let $\varphi_0(x)$ be a normalized eigenfunction of $-\Delta_U$ with eigenvalue zero:

$$\varphi_0(x) = C e^{-U(x)}, \quad \|\varphi_0\| = 1. \quad (6.69)$$

Then for any $a > 0$, there exist some constants $C_a, c_a > 0$

$$\|e^{t\Delta_U} f - \langle \varphi_0, f \rangle \varphi_0\| \leq C_a e^{-c_a t^{\frac{\sigma}{2-\sigma}}} \|e^{a(x)} f\| \quad (6.70)$$

for $t > 0$ and f such that $e^{a(x)} f \in L^2$. Note that the subexponential convergence estimate (6.70) without the explicit remainder estimate on f is proved in [4] by method of Markov processes. If $U(x) = c_1 \langle x \rangle^\sigma + U_1(x)$ where $c_1 > 0, \sigma \in]0, 1[$ and $U_1 \in C^2(\mathbb{R}^n)$ with compact support and $n \geq 3$ and $\sigma \in]0, 1[$, then all conditions of Theorem 1.3 (b) are satisfied for $H = -\Delta_U$. It follows that there exists some constant $c > 0$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ and $R > 0$, one has

$$\|\chi(e^{it\Delta_U} f - \langle \varphi_0, f \rangle \varphi_0)\| \leq C_{\chi, R} e^{-ct^{\frac{\sigma}{2-\sigma}}} \|f\|, \quad (6.71)$$

for $t > 0$ and $f \in L^2(\mathbb{R}^n)$ with $\text{supp } f \subset \{x; |x| < R\}$.

7. THRESHOLD EIGENVALUE OF NON-SELFADJOINT OPERATORS

Finally we study the case zero is an embedded eigenvalue of the non-selfadjoint Schrödinger operator H which is compactly supported perturbation of a model operator $H_0 = -\Delta + V(x)$. Let $V \in \mathcal{V}$. Then 0 is an eigenvalue of H if and only if -1 is an eigenvalue of compact operator $K = G_0 W$. The algebraic multiplicity m of eigenvalue -1 of K is finite. Let $\pi_1 : L^2 \rightarrow L^2$ be the associated Riesz projection of K defined by :

$$\pi_1 = \frac{1}{2\pi i} \int_{|z+1|=\epsilon} (z - K)^{-1} dz$$

for $\epsilon > 0$ small enough. Then

$$m = \text{rank } \pi_1. \quad (7.1)$$

π_1 is continuous on $L^{2,s}$ for any $s \in \mathbb{R}$ and $\pi_1^* : L^2 \rightarrow L^2$ is the Riesz projection of K^* associated with the eigenvalue -1 .

Theorem 7.1. *Let $H_0 = -\Delta + V(x)$ and $H = H_0 + W(x)$ with $V \in \mathcal{V}$ and $W \in L_{\text{comp}}^\infty$. Assume that 0 is a geometrically simple eigenvalue of H and that the associated eigenfunction φ_0 verifies*

$$\int (\varphi_0(x))^2 dx = 1. \quad (7.2)$$

(a). *One has*

$$R(z) = -\frac{\Pi_0}{z} + R_3(z) \quad (7.3)$$

for $z \in \Omega_1(\delta)$ where

$$\Pi_0 = \langle \cdot, J\varphi_0 \rangle \varphi_0 \quad (7.4)$$

and J is complex conjugaison: $J : f \rightarrow \bar{f}$.

(b). *There exist some constants $C, \mu', \delta > 0$ such that*

$$\|\langle x \rangle^{-s} R_3(z)\| + \|R_3(z) \langle x \rangle^{-s}\| \leq C_s \quad (7.5)$$

for $s > 2\mu + \frac{1}{\mu'}$ and $z \in \Omega_1(\delta)$; and for any $a > 0$ there exist some constants $C_a, c_a > 0$ such that

$$\|e^{-a\langle x \rangle^{1-\mu}} R_3^{(N)}(z)\| + \|R_3^{(N)}(z) e^{-a\langle x \rangle^{1-\mu}}\| \leq C_a c_a^N N^{\beta N}, \quad (7.6)$$

for any $N \in \mathbb{N}$ and $z \in \Omega_-$ where $\Omega_1(\delta)$ and Ω_- are the same as in Theorem 6.4.

Under the conditions of Theorem 7.1, -1 is a geometrically simple eigenvalue of $K = G_0 W$. One has

$$\dim \ker(1 + K) = 1 \text{ and } \text{rank } \pi_1 = m. \quad (7.7)$$

Since $1 + K$ is nilpotent on $\text{Range } \pi_1$, there exists some function $\phi_m \in \text{range } \pi_1$ such that

$$\phi_j = (1 + K)^{m-j} \phi_m \neq 0, \quad j = 1, \dots, m. \quad (7.8)$$

Then one has

$$(1 + K)\phi_1 = 0, \quad (1 + K)\phi_j = \phi_{j-1}, \quad 2 \leq j \leq m. \quad (7.9)$$

ϕ_1, \dots, ϕ_m are linearly independent. Denote J the operation of complex conjugaison $J : f \rightarrow \bar{f}$. Remark that $H_0^* = JH_0J$, $H^* = JHJ$. One has

$$JWK = K^* \bar{W} J. \quad (7.10)$$

It follows that

$$JW\pi_1 = \pi_1^* JW. \quad (7.11)$$

Denote

$$\phi_j^* = \overline{W\phi_j}. \quad (7.12)$$

Then

$$(1 + K^*)\phi_1^* = 0, \quad (1 + K^*)\phi_j^* = \phi_{j-1}^*, \quad 2 \leq j \leq m. \quad (7.13)$$

Since $\phi_1^* \neq 0$, it follows that $\phi_j^* \neq 0$ for all $1 \leq j \leq m$. From this, we deduce that $\{\phi_j^*, j = 1, \dots, m\}$ is linearly independent and that $\text{rank } \pi_1 \leq \text{rank } \pi_1^*$. Similarly, using the relation

$$JG_0\pi_1^* = \pi_1 JG_0 \quad (7.14)$$

one can prove that $\text{rank } \pi_1 \geq \text{rank } \pi_1^*$, which gives

Lemma 7.2. *One has*

$$\text{rank } \pi_1 = \text{rank } \pi_1^* = m \quad (7.15)$$

and JW is a bijection from $\text{Range } \pi_1$ onto $\text{Range } \pi_1^*$.

Lemma 7.3. *The bilinear form $B(\cdot, \cdot)$ defined on $\text{Range } \pi_1$ by*

$$B(\varphi, \psi) = \langle \varphi, JW\psi \rangle = \int_{\mathbb{R}^n} W(x)\varphi(x)\psi(x) dx \quad (7.16)$$

is non-degenerate.

Proof. Let $\phi \in \text{Range } \pi_1$ such that

$$\int W(x)\phi(x)\varphi(x) dx = 0$$

for all $\varphi \in \text{Range } \pi_1$. This means that $\phi \in (\text{Range } \pi_1^*)^\perp = \ker \pi_1$, which implies that $\phi = \pi_1\phi = 0$. So $B(\cdot, \cdot)$ is non-degenerate. \blacksquare

As a consequence of Lemma 7.2, if $m = 1$, then eigenfunction φ of H associated with zero eigenvalue satisfies

$$\int W(x)(\varphi(x))^2 dx \neq 0. \quad (7.17)$$

Lemma 7.4. *There exist $\chi_j \in \ker(1 + K)^{m-j+1}$, $j = 1, \dots, m$, such that*

$$\langle \phi_i, \chi_j^* \rangle = B(\phi_i, \chi_j) = \delta_{ij}, \quad 1 \leq i, j \leq m, \quad (7.18)$$

where $\chi_j^* = JW\chi_j$, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Proof. We use an induction to prove that for any $1 \leq l \leq m$, there exist $\varphi_j \in \ker(1 + K)^j$, $1 \leq j \leq l$ such that

$$B(\varphi_i, \phi_{m-j+1}) = \delta_{ij}, \quad 1 \leq j \leq i \leq l. \quad (7.19)$$

Since $\phi_1 \in \ker(1 + K)$ and $\phi_j^* \in \text{Range } (1 + K^*)$ for $1 \leq j \leq m - 1$, one has $\langle \phi_1, \phi_j^* \rangle = 0$ for $j = 1, \dots, m - 1$. By lemma 7.3, one has necessarily $c_1 = \langle \phi_1, \phi_m^* \rangle \neq 0$. Set

$$\varphi_1 = \frac{1}{c_1}\phi_1. \quad (7.20)$$

Then $\varphi_1 \in \ker(1 + K)$ and $B(\varphi_1, \phi_m) = 1$. (7.19) is true for $l = 1$. Assume now that (7.19) is true for some $l = k - 1$, $2 \leq k \leq m$. Set

$$\phi'_k = \phi_k - \sum_{j=1}^{k-1} B(\phi_k, \phi_{m-j+1})\varphi_j \quad (7.21)$$

Then $\phi'_k \neq 0$, $\phi'_k \in \ker(1 + K)^k$ and

$$B(\phi'_k, \phi_{m-j+1}) = 0, \quad j = 1, \dots, k - 1.$$

Since $\phi'_k \in \ker(1 + K)^k$, one has also

$$\langle \phi'_k, \phi_j^* \rangle = B(\phi'_k, \phi_j) = 0 \quad (7.22)$$

for $j = 1, \dots, m - k$, because $\phi_j^* = (1 + K^*)^{m-j} \phi_m^*$ belongs to the range of $(1 + K^*)^k$ if $1 \leq j \leq m - k$. By Lemma 7.3, the constant $c_k = B(\phi_k', \phi_{m-k+1})$ must be nonzero. Set

$$\varphi_k = \frac{1}{c_k} \phi_k'. \quad (7.23)$$

Then (7.19) is proved for $l = k$. By an induction, one can construct φ_j , $1 \leq j \leq m$, such that (7.19) holds with $l = m$. By (7.22), one has also $B(\varphi_i, \phi_{m-j+1}) = 0$ if $i > j$. Lemma 7.4 is proved by taking $\chi_k = \varphi_{m-k+1}$, $1 \leq k \leq m$. \blacksquare

One has the following representation of the Riesz projection π_1 .

Corollary 7.5. *One has*

$$\pi_1 u = \sum_{j=1}^m \langle u, \chi_j^* \rangle \phi_j, \quad u \in H^{1,-s}, s > 1. \quad (7.24)$$

Proof. Denote π the operator $\pi : u \rightarrow \sum_{j=1}^m \langle u, \chi_j^* \rangle \phi_j$. Then it is clear that $\pi^2 = \pi$ and $\text{Range } \pi = \text{Range } \pi_1$. It is trivial that $\ker \pi_1 \subset \ker \pi$. If $u \in \ker \pi$, then $\langle u, \chi_j^* \rangle = 0$ for $j = 1, \dots, m$. Therefore $u \in (\text{Range } \pi_1^*)^\perp = \ker \pi_1$ which implies that $\ker \pi \subset \ker \pi_1$. This shows that $\ker \pi_1 = \ker \pi$. This proves $\pi = \pi_1$. \blacksquare

From the proof of Lemma 7.4, one sees that if -1 is a simple eigenvalue of K ($m = 1$), then the associated Riesz projection is given by

$$\pi_1 = \langle \cdot, \varphi^* \rangle \varphi \quad (7.25)$$

where φ is an eigenfunction of K with eigenvalue -1 normalized by

$$\int W(x) (\varphi(x))^2 dx = 1.$$

Proof of Theorem 7.1. Note that $R(z) = (1 + R_0(z)V)^{-1}R_0(z)$ for $z \notin \sigma(H)$. Study the Grushin problem in $L^2 \times \mathbb{C}^m$

$$\begin{pmatrix} 1 + R_0(z)W & S \\ T & 0 \end{pmatrix} : L^2 \times \mathbb{C}^m \rightarrow L^2 \times \mathbb{C}^m \quad (7.26)$$

where

$$S : \mathbb{C}^m \rightarrow L^2, c = (c_1, \dots, c_m) \rightarrow Sc = \sum_{j=1}^m c_j \phi_j, \quad (7.27)$$

$$T : L^2 \rightarrow \mathbb{C}^m, f \rightarrow Tf = (\langle f, \chi_1^* \rangle, \dots, \langle f, \chi_m^* \rangle). \quad (7.28)$$

$$(7.29)$$

Then $ST = \pi_1$ and $TS = I_n$. Since K commutes with its Riesz projection π_1 and $1 + K$ is injection on $\text{Range } \pi_1$ where $\pi_1' = 1 - \pi_1$, $1 + K$ is invertible on the range of π_1' . By an argument of perturbation, $\pi_1'(1 + R_0(z)W)\pi_1'$ is invertible on range of π_1' for $z \in \Omega_1(\delta)$ if $\delta > 0$ is appropriately small and its inverse $E(z)$ is uniformly bounded on $\Omega_1(\delta)$ where

$$E(z) = (\pi_1'(1 + R_0(z)W)\pi_1')^{-1} \pi_1' \quad (7.30)$$

By the arguments used in Section 6.1, $E(z)$ belongs to the Gevrey class $G^\beta(\Omega_1(\delta))$ with $\beta = 1 + \gamma$. One can check that for $z \in \Omega_1(\delta)$,

$$\begin{pmatrix} 1 + R_0(z)W & S \\ T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \quad (7.31)$$

where

$$E_+(z) = (1 - E(z)R_0(z)W)S \quad (7.32)$$

$$E_-(z) = T(1 - R_0(z)WE(z)) \quad (7.33)$$

$$E_{-+}(z) = -T(1 + R_0(z)W)S + TR_0(z)WE(z)R_0(z)WS. \quad (7.34)$$

It follows that $z \notin \sigma(P)$ if and only if $\det E_{-+}(z) \neq 0$ and one has

$$(1 + R_0(z)W)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z). \quad (7.35)$$

Since $TR_0(z)WE(z)R_0(z)WS = O(|z|^2)$, $E_{-+}(z)$ is an $m \times m$ matrix verifying

$$\begin{aligned} E_{-+}(z) &= (-\langle (1 + R_0(z)W)\phi_k, \chi_j^* \rangle)_{1 \leq j, k \leq m} + O(|z|^2) \\ &= - \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - z \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm} \end{pmatrix} + O(|z|^2) \end{aligned} \quad (7.36)$$

where

$$b_{jk} = \langle G_1 W \phi_k, \chi_j^* \rangle. \quad (7.37)$$

Note that ϕ_1 and χ_m belong to $\ker(1 + G_0 W)$ and $\chi_m^* = JW\chi_m$, they are rapidly decreasing, by Lemma 6.3. One can calculate

$$\begin{aligned} b_{m1} &= \lim_{\lambda \rightarrow 0^-} \left\langle \frac{1}{\lambda} (1 + R_0(\lambda)W)\phi_1, JW\chi_m \right\rangle \\ &= - \lim_{\lambda \rightarrow 0^-} \langle R_0(\lambda)\phi_1, JW\chi_m \rangle \\ &= - \lim_{\lambda \rightarrow 0^-} \langle \phi_1, JR_0(\lambda)W\chi_m \rangle \\ &= \langle \phi_1, J\chi_m \rangle. \end{aligned}$$

Similarly one can calculate for $2 \leq j \leq m$

$$b_{mj} = -\langle W\phi_j, JG_0\chi_m \rangle = \langle \phi_j - \phi_{j-1}, J\chi_m \rangle.$$

By the condition (7.2), one has $b_{m1} \neq 0$.

$$\det E_{-+}(z) = (-1)^m z b_{m1} + O(|z|^2) \neq 0 \quad (7.38)$$

for $z \in \Omega_1(\delta)$ which shows the invertibility of $E_{-+}(z)$. A direct computation gives that

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ z b_{m1} & \cdots & \cdots & z b_{mm} \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{b_{m2}}{b_{m1}} & -\frac{b_{m3}}{b_{m1}} & \cdots & \cdots & \frac{1}{z b_{m1}} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

It follows that

$$E_{-+}(z)^{-1} = -\frac{1}{z}C + O(1) \quad (7.39)$$

where C is the matrix of rank one given by

$$C = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{b_{m1}} \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}. \quad (7.40)$$

From (7.35), one obtains

$$(1 + R_0(z)W)^{-1} = \frac{1}{z}SCT + O(1). \quad (7.41)$$

Using the definition of S and T , one obtains

$$SCTf = \frac{1}{b_{m1}}\langle f, \chi_m^* \rangle \phi_1 \quad (7.42)$$

Noticing that

$$\langle G_0f, \chi_m^* \rangle = \langle f, G_0^*JW\chi_m \rangle = \langle f, JG_0W\chi_m \rangle = -\langle f, J\chi_j m \rangle,$$

we deduce from (6.32) that

$$R(z) = -\frac{\Pi_0}{z} + O(1) \quad (7.43)$$

for $z \in \Omega_1(\delta)$. Here Π_0 is given by

$$\Pi_0 = \frac{1}{b_{m1}}\langle f, J\chi_m \rangle \phi_1 \quad (7.44)$$

Both χ_m and ϕ_1 belong to $\ker(1 + K)$ which is of dimension one. There exists some constant $d_1 \neq 0$ such that $\chi_m = d_1\phi_1$. Set

$$\varphi_1 = \sqrt{\frac{d_1}{b_{m1}}}\phi_1. \quad (7.45)$$

Then $\Pi_0 = \langle \cdot, J\varphi_1 \rangle \varphi_1$. φ_1 is an eigenfunction of H verifying $H\varphi_1 = 0$ and

$$\int (\varphi_1(x))^2 dx = 1.$$

Since zero eigenvalue of H is geometrically simple, one has $\varphi_1 = \pm\varphi_0$. This proves

$$\Pi_0 = \langle \cdot, J\varphi_0 \rangle \varphi_0 \quad (7.46)$$

The estimates on remainder $R_3(z)$ can be proved in the same way as in Theorem 6.4. \blacksquare

Theorem 7.6. *Let $H_0 = -\Delta + V(x)$ and $H = H_0 + W(x)$ with $V \in \mathcal{A}$ and $W \in L_{r, mcomp}^\infty$. Assume that 0 is a geometrically simple eigenvalue of H and that the associated eigenfunction φ_0 verifies*

$$\int (\varphi_0(x))^2 dx = 1. \quad (7.47)$$

Let $\Omega_\delta(c)$ be defined as in Theorem 6.5 and $\Pi_0 = \langle \cdot, J\varphi_0 \rangle \varphi_0$. Then one has for any $\chi \in C_0^\infty(\mathbb{R}^n)$

$$\chi R(z)\chi = -\frac{\chi \Pi_0 \chi}{z} + R_4(z) \quad (7.48)$$

for $z \in \Omega_\delta(c)$ where the remainder $R_4(z)$ is continuous up to $z = 0$ and satisfies the Gevrey estimates

$$\|\chi R_4^{(N)}(z)\chi\| \leq C_\chi C^N N^{\gamma N} \quad (7.49)$$

for $z \in \Omega_\delta(c) \cup \{0\}$.

Theorem 7.6 is derived by combining methods used in Theorem 6.5 and Theorem 7.1. The details are omitted. \blacksquare

Theorem 1.4 (a) (Theorem 1.4 (b), respectively) is derived from Theorem 7.1 and formula (6.65) (Theorem 7.6 and formula (6.66), respectively). \blacksquare

Consider the non-selfadjoint Witten Laplacian

$$-\Delta_U = -\Delta + (\nabla U)(x) \cdot (\nabla U)(x) - \Delta U(x)$$

where $U \in C^2(\mathbb{R}^n; \mathbb{C})$. Set $U(x) = U_1(x) + iU_2(x)$ with U_1, U_2 real valued functions. Assume that U_1 satisfies the condition (6.68) with U replaced by U_1 and that U_2 is of compact support with $\|\partial_x^\alpha U_2\|_{L^\infty}$ sufficiently small for $|\alpha| \leq 2$. Considering $-\Delta_U$ as a perturbation of $-\Delta_{U_1}$, one can show that $-\Delta_U$ has only one eigenvalue in a neighbourhood of zero which is in addition simple. Since $-\Delta_U e^{-U} = 0$, one concludes that 0 is a geometrically simple eigenvalue of $-\Delta_U$ and the condition (7.2) is satisfied for $\varphi_0(x) = ce^{-U(x)}$ with some constant $c \neq 0$. Therefore Theorem 1.4 (a) can be applied to the non-selfadjoint Witten Laplacian $-\Delta_U$. Theorem 1.4 (b) can be also applied to $-\Delta_U$ under some additional conditions on U .

Remark 7.1. *The methods used in this Section here can be applied to other threshold spectral problems. In particular for non-selfadjoint Schrödinger operator $H = -\Delta + V(x)$ with a quickly decreasing complex potential $V(x)$ on \mathbb{R}^n :*

$$|V(x)| \leq C \langle x \rangle^{-\rho}, \rho > 2. \quad (7.50)$$

with $n = 3, 4$, our method allows to calculate the low-energy asymptotics of the resolvent $(H - z)^{-1}$ if zero is a resonance but not an eigenvalue. In fact one can show that for $n = 3, 4$, zero resonance, if it exists, is geometrically simple. The Grushin method presented here can reduce that resolvent expansion for $(H - z)^{-1}$ to that of $E_{-+}(z)^{-1}$. The calculation given in this Section shows that $E_{-+}(z)$ admits an expansion which for $n = 3$ takes the form

$$\begin{aligned} E_{-+}(z) &= \left(-\langle (1 + R_0(z)V)\phi_k, \chi_j^* \rangle \right)_{1 \leq j, k \leq m} + O(|z|^2) \\ &= - \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - z^{\frac{1}{2}} \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm} \end{pmatrix} + O(|z|^{\frac{1}{2}+\epsilon}). \end{aligned} \quad (7.51)$$

Here $R_0(z) = (-\Delta - z)^{-1}$. The characterization of resonant states ensures that $b_{m1} \neq 0$ without any assumption of the type (7.2), hence the matrix $E_{-+}(z)$ is invertible for z small and $z \neq 0$. Consequently low-energy resolvent expansion of $(H - z)^{-1}$ be calculated as in the proof of Theorem 1.4 without any assumption of the type (7.2).

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