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# $\mathbb{A}^2$ -FIBRATIONS BETWEEN AFFINE SPACES ARE TRIVIAL $\mathbb{A}^2$ -BUNDLES

ADRIEN DUBOULOZ

**ABSTRACT.** We give a criterion for a flat fibration with affine plane fibers over a smooth scheme defined over a field of characteristic zero to be a Zariski locally trivial  $\mathbb{A}^2$ -bundle. An application is a positive answer to a version of the Dolgachev-Weisfeiler Conjecture for such fibrations: a flat fibration  $\mathbb{A}^m \rightarrow \mathbb{A}^n$  with all fibers isomorphic to  $\mathbb{A}^2$  is the trivial  $\mathbb{A}^2$ -bundle.

## INTRODUCTION

An  $\mathbb{A}^n$ -fibration over a scheme  $X$  is a flat affine morphism of finite presentation  $\pi : V \rightarrow X$  whose fibers, closed or not, are all isomorphic to the affine  $n$ -space  $\mathbb{A}^n$  over the corresponding residue fields. A version of the Dolgachev-Weisfeiler Conjecture [6, 3.8.3] (see also [24, Conjecture 3.14]) asks whether an  $\mathbb{A}^n$ -fibration  $\pi : V \rightarrow X$  over a normal locally noetherian integral scheme  $X$  is a form of the affine  $n$ -space  $\mathbb{A}_X^n = X \times \mathbb{A}^n$  over  $X$  for the Zariski topology, or at least for the étale topology. The conjecture in such general form remains open, and so far, only the following two special cases are known (see [4] for a survey):

1) For  $n = 1$ , every  $\mathbb{A}^1$ -fibration  $\pi : V \rightarrow X$  over a normal locally noetherian integral scheme  $X$  is a Zariski locally trivial  $\mathbb{A}^1$ -bundle by successive results of Kambayashi-Miyayoshi [16] and Kambayashi-Wright [17].

2) For  $n = 2$ , a result of Sathaye [22] asserts that an  $\mathbb{A}^2$ -fibration  $\pi : V \rightarrow X$  over the spectrum  $X$  of a rank one discrete valuation ring containing  $\mathbb{Q}$  is the trivial  $\mathbb{A}^2$ -bundle over  $X$ . The proof depends on the famous Abhyankar-Moh Theorem, and the characteristic zero hypothesis is crucial as illustrated by counter-examples in positive characteristic constructed by Asanuma [1, §5.1]. Additional results concerning the structure of  $\mathbb{A}^2$ -fibrations over spectra of one-dimensional noetherian domains containing  $\mathbb{Q}$  have been obtained later on by Asanuma-Bathwadekar [2].

In contrast, the stable structure of general  $\mathbb{A}^n$ -fibrations is quite well understood: it was established by Asanuma [1] that every such fibration  $\pi : V \rightarrow X$  over a smooth affine scheme  $X$  defined over a field of characteristic zero is stably isomorphic to the total space of vector bundle over  $X$ , in the sense that  $V \times_X \mathbb{A}_X^m \simeq E \times_X \mathbb{A}_X^m$  for some vector bundle  $p : E \rightarrow X$  of rank  $n$  over  $X$ . The question whether the  $\mathbb{A}_X^m$  factor can be “canceled” to obtain that these  $\mathbb{A}^n$ -fibrations are themselves vector bundles remains open in general.

On the other hand, by a result of Bass-Connell-Wright [3], a Zariski locally trivial  $\mathbb{A}^n$ -bundle  $\pi : V \rightarrow X$  over an affine scheme always carries the structure of a vector bundle: there exist local trivializations of  $V$  on a Zariski open cover of  $X$  for which the corresponding transition isomorphisms are linear automorphisms of  $\mathbb{A}^n$ . A well-known property of vector bundles, and more generally of affine-linear bundles  $\nu : V \rightarrow X$  - that is, locally trivial  $\mathbb{A}^n$ -bundles whose transition isomorphisms are affine automorphisms of  $\mathbb{A}^n$  - is that their relative cotangent sheaves  $\Omega_{V/X}^1$  are induced from  $X$ , i.e. isomorphic to the pull-back to  $V$  of a locally free sheaf of  $\mathcal{O}_X$ -modules. Summing up, if an  $\mathbb{A}^n$ -fibration  $\pi : V \rightarrow X$  over an affine scheme  $X$  is a Zariski locally trivial  $\mathbb{A}^n$ -bundle, then its relative cotangent sheaf is induced from  $X$ . Our main result, which can be summarized as follows, implies in particular that the converse holds for  $\mathbb{A}^2$ -fibrations over smooth affine schemes:

**Theorem.** *Let  $\pi : V \rightarrow X$  be an  $\mathbb{A}^2$ -fibration over a smooth locally noetherian scheme defined over a field of characteristic zero. If  $\Omega_{V/X}^1$  is induced from  $X$  then  $\pi : V \rightarrow X$  is an affine-linear bundle.*

Note that combined with the fact that vector bundles on  $\mathbb{A}_k^m$  are trivial by the Quillen-Suslin Theorem [21, 23], this characterization implies that an  $\mathbb{A}^2$ -fibration  $\pi : \mathbb{A}_k^m \rightarrow \mathbb{A}_k^n$  is isomorphic to the trivial  $\mathbb{A}^2$ -bundle  $\mathbb{A}_k^n \times \mathbb{A}_k^2$ .

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As an application in the special case  $m = 4$ , we deduce that the famous Vénéreau polynomials [25], as well as large collections of “Vénéreau-type” polynomials introduced by Daigle-Freudentburg [5] and Lewis [19], are variables of polynomial rings in four variables over a field (see § 3.2). As another application, we answer in § 3.3 a question raised by Freudentburg [8] concerning the structure of locally nilpotent derivations with a slice on polynomial rings in three variables over a base ring.

### 1. RECOLLECTION ON $\mathbb{A}^n$ -FIBRATIONS, $\mathbb{A}^n$ -BUNDLES AND AFFINE-LINEAR BUNDLES

In what follows we fix a base field  $k$  of characteristic zero. All schemes considered are defined over  $k$ .

**Definition 1.** Let  $X$  be a scheme. A flat affine morphism of finite presentation  $f : V \rightarrow X$  is called:

1) An  $\mathbb{A}^n$ -fibration if for every point  $x \in X$ , the scheme theoretic fiber  $f^{-1}(x)$  is isomorphic to the affine  $n$ -space  $\mathbb{A}_{\kappa(x)}^n$  over the residue field  $\kappa(x)$ .

2) A Zariski (resp. étale) locally trivial  $\mathbb{A}^n$ -bundle if there exists a Zariski open (resp. étale) cover  $Y \rightarrow X$  of  $X$  such that  $V \times_X Y$  is isomorphic as a scheme over  $Y$  to the trivial  $\mathbb{A}^n$ -bundle  $\mathbb{A}_Y^n = Y \times \mathbb{A}_k^n$ .

Elementary examples of Zariski locally trivial  $\mathbb{A}^n$ -bundles are vector bundles. To fix a convention, by a vector bundle of rank  $n \geq 1$  over a scheme  $X$ , we mean the relative spectrum  $p : E = \text{Spec}(\text{Sym } \mathcal{E}^\vee) \rightarrow X$  of the symmetric algebra of the dual of a locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $n$ . Recall that every vector bundle  $p : E \rightarrow X$  carries the structure of a Zariski locally constant group scheme for the law given by the addition of germs of sections. An  $E$ -torsor is an étale locally trivial principal homogeneous  $E$ -bundle, that is, a scheme  $\nu : V \rightarrow X$  equipped with an action  $\mu : E \times_X V \rightarrow V$  of  $E$  for which there exists an étale cover  $Y \rightarrow X$  such that  $V \times_X Y$  is equivariantly isomorphic to  $E \times_X Y$  acting on itself by translations.

**Definition 2.** An affine-linear bundle or rank  $n \geq 1$  over a scheme  $X$  is étale locally trivial  $\mathbb{A}^n$ -bundle  $\nu : V \rightarrow X$  which can be further equipped with the structure of an  $E$ -torsor for a suitable vector bundle  $p : E \rightarrow X$  of rank  $n$  over  $X$ .

Equivalently, an affine-linear bundle is an étale locally trivial  $\mathbb{A}^n$ -bundle  $\nu : V \rightarrow X$  for which there exists an étale cover  $f : Y \rightarrow X$  and an isomorphism  $V \times_X Y \xrightarrow{\sim} \mathbb{A}_Y^n$  such that over  $Y \times_X Y$  equipped with the two projections  $\text{pr}_1, \text{pr}_2 : Y \times_X Y \rightarrow Y$ ,  $\text{pr}_2^* \varphi \circ \text{pr}_1^* \varphi^{-1}$  is an affine automorphism of  $\mathbb{A}_{Y \times_X Y}^n$ , i.e. is given by an element  $(A, T)$  of  $\text{Aff}_n(Y \times_X Y) = \text{GL}_n(Y \times_X Y) \rtimes \mathbb{G}_a^n(Y \times_X Y)$ . The vector bundle  $E$  for which  $\nu : V \rightarrow X$  is an  $E$ -torsor is uniquely determined up to isomorphism by the fact that its class in  $H_{\text{ét}}^1(X, \text{GL}_n)$  coincides with that of the 1-cocycle  $A \in \text{GL}_n(Y \times_X Y)$  for the étale cover  $f : Y \rightarrow X$  of  $X$ . Since the affine group  $\text{Aff}_n = \text{GL}_n \rtimes \mathbb{G}_a^n$  is special [12], every affine-linear bundle is actually locally trivial in the Zariski topology. Furthermore, there is a one-to-one correspondence between isomorphism classes of  $E$ -torsors over  $X$  and elements of the cohomology group  $\check{H}^1(X, E) \simeq H_{\text{Zar}}^1(X, E) \simeq H_{\text{ét}}^1(X, E)$ , with  $0 \in \check{H}^1(X, E)$  corresponding to the trivial  $E$ -torsor  $p : E \rightarrow X$  (see e.g. [11, XI.4]). In particular, every  $E$ -torsor over an affine scheme  $X$  is isomorphic to the trivial one.

Recall [9, 16.4.9] that given a vector bundle  $p : E = \text{Spec}(\text{Sym } \mathcal{E}^\vee) \rightarrow X$ , there exists a canonical isomorphism of  $\mathcal{O}_E$ -modules  $p^* \mathcal{E}^\vee \xrightarrow{\sim} \Omega_{E/X}^1$  defined as the composition of the canonical homomorphism  $p^* \mathcal{E}^\vee \rightarrow \mathcal{O}_E$  with the canonical derivation  $d_{E/X} : \mathcal{O}_E \rightarrow \Omega_{E/X}^1$ . A direct local calculation shows more generally that the relative cotangent sheaf  $\Omega_{V/X}^1$  of any  $E$ -torsor  $\nu : V \rightarrow X$  is isomorphic to  $\nu^* \mathcal{E}^\vee$ . This property actually characterizes affine-linear bundles among étale locally trivial  $\mathbb{A}^n$ -bundles:

**Lemma 3.** *A étale locally trivial  $\mathbb{A}^n$ -bundle  $\pi : V \rightarrow X$  over a scheme  $X$  is an affine-linear bundle if and only if there exists a locally free sheaf  $\mathcal{E}$  of rank  $n$  on  $X$  such that  $\Omega_{V/X}^1 \simeq \pi^* \mathcal{E}^\vee$ . If such a locally free sheaf  $\mathcal{E}$  exists, then  $\pi : V \rightarrow X$  is torsor under the rank  $n$  vector bundle  $p : E = \text{Spec}(\text{Sym } \mathcal{E}^\vee) \rightarrow X$  on  $X$ .*

*Proof.* Since  $\pi$  is an affine morphism of finite type,  $\mathcal{A} = \pi_* \mathcal{O}_V$  is a quasi-coherent  $\mathcal{O}_X$ -algebra of finite type. Let  $d_{V/X} : \mathcal{O}_V \rightarrow \Omega_{V/X}^1$  be the canonical  $\mathcal{O}_X$ -derivation. Since  $\Omega_{V/X}^1 \simeq \pi^* \mathcal{E}^\vee$ ,  $\pi_* \Omega_{V/X}^1$  is isomorphic to  $\pi_* \mathcal{O}_V \otimes \mathcal{E}^\vee = \mathcal{A} \otimes \mathcal{E}^\vee$  by the projection formula, and the direct image  $\pi_* d_{V/X}$  of  $d_{V/X}$  is an  $\mathcal{O}_X$ -derivation  $\partial_1 : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{E}^\vee$  of  $\mathcal{A}$  with values in  $\mathcal{A} \otimes \mathcal{E}^\vee$ . For every  $r \geq 2$ , we let  $\partial_r : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym}^r \mathcal{E}^\vee$  be the  $\mathcal{O}_X$ -linear homomorphism defined as the composition of

$$(\partial_1 \otimes \text{id}_{\text{Sym}^{r-1} \mathcal{E}^\vee}) \circ \partial_{r-1} : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym}^{r-1} \mathcal{E}^\vee \rightarrow (\mathcal{A} \otimes \mathcal{E}^\vee) \otimes \text{Sym}^{r-1} \mathcal{E}^\vee$$

with the canonical homomorphism  $\mathcal{A} \otimes (\mathcal{E}^\vee \otimes \text{Sym}^{r-1} \mathcal{E}^\vee) \rightarrow \mathcal{A} \otimes \text{Sym}^r \mathcal{E}^\vee$ . We let  $\partial_0 = \text{id}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym}^0 \mathcal{E}^\vee \simeq \mathcal{A}$ . The kernels  $\mathcal{K}er \partial_r$ ,  $r \geq 0$ , form an increasing sequence of quasi-coherent sub- $\mathcal{O}_X$ -modules of  $\mathcal{A}$ . We claim that  $\mathcal{A} = \text{colim}_{r \geq 0} \mathcal{K}er \partial_r$  and that the map

$$\exp(\partial_1) := \sum_{r \geq 0} \frac{\partial_r}{r!} : \mathcal{A} \rightarrow \bigoplus_{r \geq 0} \mathcal{A} \otimes \text{Sym}^r \mathcal{E}^\vee \simeq \mathcal{A} \otimes \text{Sym} \cdot \mathcal{E}^\vee$$

is a well-defined homomorphism of  $\mathcal{O}_X$ -algebra, corresponding to an action  $\mu : E \times_X V \rightarrow V$  of the vector bundle  $p : E = \text{Spec}(\text{Sym} \cdot \mathcal{E}^\vee) \rightarrow X$  on  $\pi : V \rightarrow X$ . This can be seen as follows: let  $f : Y \rightarrow X$  be an étale cover of  $X$  on which  $\pi : V \rightarrow X$  becomes trivial, say  $W = V \times_X Y \simeq \mathbb{A}_Y^n = \text{Spec}(\mathcal{O}_Y[t_1, \dots, t_n])$ . Then  $\Omega_{W/Y}^1$  is the free  $\mathcal{O}_W$ -module with basis  $dt_1, \dots, dt_n$  and since  $\Omega_{W/Y}^1 = \text{pr}_V^* \Omega_{V/X}^1 \simeq \text{pr}_Y^* f^* \mathcal{E}^\vee$ , we conclude by restricting to the zero section of  $\mathbb{A}_Y^n$  that  $f^* \mathcal{E}^\vee \simeq \mathcal{O}_Y^{\oplus n}$ . Via these isomorphisms, the homomorphism

$$f^* \partial_r : f^* \mathcal{A} = (\text{pr}_Y)_* \mathcal{O}_W \rightarrow f^*(\mathcal{A} \otimes \text{Sym}^r \mathcal{E}^\vee) \simeq (\text{pr}_Y)_* \mathcal{O}_W \otimes \text{Sym}^r f^* \mathcal{E}^\vee$$

coincides with the homomorphism of  $\mathcal{O}_Y$ -modules

$$\begin{aligned} d_r : \mathcal{O}_Y[t_1, \dots, t_n] &\longrightarrow \mathcal{O}_Y[t_1, \dots, t_n] \otimes \text{Sym}^r(\mathcal{O}_Y \langle dt_1, \dots, dt_n \rangle) \\ p(t_1, \dots, t_n) &\mapsto \sum_{I=(i_1, \dots, i_n), i_1 + \dots + i_n = r} \sum \frac{\partial^r p}{\partial t_1^{i_1} \dots \partial t_n^{i_n}} \otimes dt_1^{i_1} \dots dt_n^{i_n}. \end{aligned}$$

Since colimits commute with pull-backs, we have  $f^* \text{colim}_{r \geq 0} \mathcal{K}er \partial_r = \text{colim}_{r \geq 0} \mathcal{K}er d_r = \mathcal{O}_Y[t_1, \dots, t_n]$ . This implies that the injective homomorphism  $\text{colim}_{r \geq 0} \mathcal{K}er \partial_r \rightarrow \mathcal{A}$  is also surjective, hence that  $\exp(\partial_1)$  is indeed well-defined. Now we have to check that it satisfies the usual axioms for being the co-morphism of an action of  $E$  on  $V$ , namely, the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\exp(\partial_1)} & \mathcal{A} \otimes \text{Sym} \cdot \mathcal{E}^\vee \\ \exp(\partial_1) \downarrow & & \downarrow \exp(\partial_1) \otimes \text{id} \\ \mathcal{A} \otimes \text{Sym} \cdot \mathcal{E}^\vee & \xrightarrow{\text{id} \otimes \text{m}} & \mathcal{A} \otimes \text{Sym} \cdot \mathcal{E}^\vee \otimes \text{Sym} \cdot \mathcal{E}^\vee \end{array} \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\exp(\partial_1)} & \mathcal{A} \otimes \text{Sym} \cdot \mathcal{E}^\vee \\ \text{id} \downarrow & & \downarrow \text{id} \otimes \varepsilon \\ \mathcal{A} & \xrightarrow{\simeq} & \mathcal{A} \otimes \mathcal{O}_X \end{array}$$

where  $\text{m} : \text{Sym} \cdot \mathcal{E}^\vee \rightarrow \text{Sym} \cdot \mathcal{E}^\vee \otimes \text{Sym} \cdot \mathcal{E}^\vee \simeq \text{Sym} \cdot (\mathcal{E}^\vee \oplus \mathcal{E}^\vee)$  is the homomorphism of  $\mathcal{O}_X$ -algebra induced by the diagonal homomorphism  $\mathcal{E}^\vee \rightarrow \mathcal{E}^\vee \oplus \mathcal{E}^\vee$  and where  $\varepsilon : \text{Sym} \cdot \mathcal{E}^\vee \rightarrow \mathcal{O}_X$  is the canonical homomorphism with kernel  $\mathcal{E}^\vee \cdot \text{Sym} \cdot \mathcal{E}^\vee$ . The commutativity of these diagrams can be checked locally on an étale cover of  $X$ . But by construction,  $f^* \exp(\partial_1) : f^* \mathcal{A} \rightarrow f^* \mathcal{A} \otimes \text{Sym} \cdot f^* \mathcal{E}^\vee$  coincides with  $\exp(d_1)$ , which is precisely the co-morphism of the action of  $E \times_X Y \simeq \mathbb{G}_{a,Y}^n$  on  $V \times_X Y \simeq \mathbb{A}_Y^n$  by translations. The above diagrams are thus commutative, and we conclude that  $\exp(\partial_1)$  is the co-morphism of an action  $\mu : E \times_X V \rightarrow V$  of the vector bundle  $p : E = \text{Spec}(\text{Sym} \cdot \mathcal{E}^\vee) \rightarrow X$  on  $\pi : V \rightarrow X$  for which  $V \times_X Y$  is equivariantly isomorphic to  $E \times_X Y$  acting on itself by translations. This shows that  $\pi : V \rightarrow X$  is an  $E$ -torsor, as desired.  $\square$

## 2. A CHARACTERIZATION OF AFFINE-LINEAR BUNDLES AMONG $\mathbb{A}^n$ -FIBRATIONS

In this section, we establish a more flexible characterization of affine-linear bundles among all  $\mathbb{A}^n$ -fibrations instead of only locally trivial ones. We begin with the following local version.

**Proposition 4.** *Let  $B$  be the spectrum of a noetherian normal local ring of dimension  $d \geq 2$  and let  $\pi : V \rightarrow B$  be an  $\mathbb{A}^n$ -fibration. Suppose that*

- $B$  is regular or  $\pi$  has a section,*
- $\pi$  is a Zariski locally trivial  $\mathbb{A}^n$ -bundle outside the closed point of  $B$ ,*
- $\Omega_{V/B}^1$  is trivial.*

*Then  $\pi : V \rightarrow B$  is a trivial  $\mathbb{A}^n$ -bundle.*

*Proof.* Let  $b_0$  be the closed point of  $B$  and let  $B_* = B \setminus \{b_0\}$ . Since  $\Omega_{V/B}^1 \simeq \pi^* \mathcal{O}_B^{\oplus n}$ , it follows from Lemma 3 that the restriction  $\pi : V_* \rightarrow B_*$  of  $\pi : V \rightarrow B$  over  $B_*$  is a  $\mathbb{G}_a^n$ -torsor. It suffices to show that  $\pi : V_* \rightarrow B_*$  is a trivial  $\mathbb{G}_a^n$ -torsor. Indeed, if so, then since  $V$  and  $B \times \mathbb{A}^n$  are both affine and normal, the isomorphism  $V_* \simeq B_* \times \mathbb{A}^n$  extends to an isomorphism  $V \simeq B \times \mathbb{A}^n$  of schemes over  $B$ . If  $\pi$  has a section, then  $\pi : V_* \rightarrow B_*$  has a section hence is the trivial  $\mathbb{G}_a^n$ -torsor. Now suppose that  $B$  is regular and denote by  $g = (g_1, \dots, g_n)$  the isomorphism class of  $\pi : V_* \rightarrow B_*$  in  $H^1(B_*, \mathcal{O}_B^{\oplus n}) \simeq H^1(B_*, \mathcal{O}_B)^{\oplus n}$ . Since  $B$  is affine, the vanishing of the cohomology groups  $H^i(B, \mathcal{O}_B)$ ,  $i \geq 1$ , implies that in the long exact excision sequence of cohomology

groups for the pair  $(B, b_0)$  the connecting homomorphism  $H^1(B_*, \mathcal{O}_B) \rightarrow H^2_{b_0}(B, \mathcal{O}_B)$  is an isomorphism. If  $d \geq 3$  then since  $B$  is regular,  $H^2_{b_0}(B, \mathcal{O}_B) = 0$  (see e.g. [10, Theorem 3.8]) and so,  $\pi : V_* \rightarrow B_*$  is the trivial  $\mathbb{G}_a^n$ -torsor.

It thus remains to consider the case where  $B$  is the spectrum of a noetherian 2-dimensional regular local ring. Suppose that there exists a index  $i$  such that  $g_i \neq 0$ , say  $i = 1$ . Then there exist a nontrivial  $\mathbb{G}_a$ -torsor  $\pi_1 : W_1 \rightarrow B_*$  with isomorphy class  $g_1$  and a  $\mathbb{G}_a^{n-1}$ -torsor  $\pi_2 : W_2 \rightarrow B_*$  with isomorphy class  $(g_2, \dots, g_n)$  such that  $V_* \simeq W_1 \times_{B_*} W_2$ . Since  $\pi_1 : W_1 \rightarrow B_*$  is nontrivial, the same argument as in the proof of Proposition 1.2 in [7] shows that  $W_1$  is an affine scheme. Thus  $V_*$  is an affine scheme too as the projection  $\text{pr}_1 : V_* \rightarrow W_1$  is a  $\mathbb{G}_a^{n-1}$ -torsor, hence an affine morphism. But  $V \setminus V_* = \pi^{-1}(b_0)$  has codimension 2 in  $V$ , in contradiction to the fact that the complement of an affine open subscheme in a locally noetherian scheme has pure codimension 1. So  $\pi : V_* \rightarrow B_*$  is a trivial  $\mathbb{G}_a^n$ -torsor as desired.  $\square$

As a consequence of the previous proposition, we obtain the following characterization:

**Corollary 5.** *Let  $\pi : V \rightarrow X$  be an  $\mathbb{A}^n$ -fibration over a smooth locally noetherian scheme  $X$ . Suppose that:*

- 1)  *$\pi$  is a Zariski locally trivial  $\mathbb{A}^n$ -bundle outside a closed subset  $Z \subset X$  of codimension  $\geq 2$ ,*
- 2)  *$\Omega^1_{V/B} = \pi^* \mathcal{E}^\vee$  for some locally free sheaf  $\mathcal{E}$  of rank  $n$  on  $X$ .*

*Then  $\pi : V \rightarrow X$  is a torsor under the vector bundle  $p : E = \text{Spec}(\text{Sym} \mathcal{E}^\vee) \rightarrow X$  on  $X$ .*

*Proof.* Let  $U \subset X$  be the largest open subset over which  $\pi : V \rightarrow X$  restricts to a Zariski locally trivial  $\mathbb{A}^n$ -bundle. In view of Lemma 3, it suffices to show that  $U = X$ . By hypothesis,  $U$  contains all codimension 1 points of  $X$ . Now let  $x$  be a point of codimension 2 in  $X$  and let  $j : B = \text{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$  be the corresponding open immersion. Since  $X$  is smooth,  $\mathcal{O}_{X,x}$  is a noetherian regular local ring of dimension 2 and the pull-back  $\text{pr}_B : B \times_X V \rightarrow B$  is an  $\mathbb{A}^n$ -fibration restricting to a Zariski locally trivial  $\mathbb{A}^n$ -bundle over the complement of the closed point of  $B$ . Furthermore, since  $j^* \mathcal{E}^\vee \simeq \mathcal{O}_B^{\oplus n}$ ,  $\Omega^1_{B \times_X V/B} \simeq \text{pr}_B^* j^* \mathcal{E}^\vee$  is trivial. Thus  $\text{pr}_B : B \times_X V \rightarrow B$  is a trivial  $\mathbb{A}^n$ -bundle by virtue of Proposition 4, implying that  $U$  contains all codimension 2 points of  $X$ . By descending induction on the codimension of the points of  $X$ , we conclude by the same argument that  $U$  contains all points of  $X$ .  $\square$

**Corollary 6.** *Assume that every  $\mathbb{A}^n$ -fibration over the spectrum of a rank one discrete valuation ring containing  $k$  is a trivial  $\mathbb{A}^n$ -bundle. Then an  $\mathbb{A}^n$ -fibration  $\pi : V \rightarrow X$  over a smooth locally noetherian scheme  $X$  is an affine-linear bundle if and only if  $\Omega^1_{V/X} = \pi^* \mathcal{E}^\vee$  for some locally free sheaf  $\mathcal{E}$  of rank  $n$  on  $X$ .*

*Proof.* Since  $X$  is smooth, for every point  $x$  of codimension 1 in  $X$ , the local ring  $\mathcal{O}_{X,x}$  is a rank one discrete valuation ring. The hypothesis implies that the restriction of  $\pi : V \rightarrow X$  over the image of the open embedding  $\text{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$  is a trivial  $\mathbb{A}^n$ -bundle. The largest open subset  $U$  of  $X$  over which  $\pi : V \rightarrow X$  restricts to a Zariski locally trivial  $\mathbb{A}^n$ -bundle thus contains all points of codimension 1 of  $X$ , and the assertion then follows from Corollary 5.  $\square$

### 3. APPLICATIONS

#### 3.1. Dolgachev-Weisfeiler problem for $\mathbb{A}^2$ -fibrations.

**Theorem 7.** *Let  $V$  be a scheme on which every rank 2 vector bundle is trivial. Then every  $\mathbb{A}^2$ -fibration  $\pi : V \rightarrow X$  over a smooth locally noetherian scheme  $X$  is a  $\mathbb{G}_a^2$ -torsor.*

*Proof.* By [22] every  $\mathbb{A}^2$ -fibration over the spectrum of a discrete valuation ring containing  $k$  is a trivial  $\mathbb{A}^2$ -bundle. On the other hand, since  $\pi : V \rightarrow X$  is a smooth morphism [9, 17.5.1],  $\Omega^1_{V/X}$  is a locally free sheaf of rank 2 on  $V$ , hence is free by hypothesis. The assertion then follows from Proposition 4 and Corollary 5.  $\square$

As a consequence of Quillen-Suslin Theorem [21, 23], we obtain:

**Corollary 8.** *An  $\mathbb{A}^2$ -fibration  $\pi : \mathbb{A}^m \rightarrow \mathbb{A}^n$  is a trivial  $\mathbb{A}^2$ -bundle.*

*Proof.* By the previous theorem  $\pi : \mathbb{A}^m \rightarrow \mathbb{A}^n$  is a  $\mathbb{G}_a^2$ -torsor, hence is trivial since  $\mathbb{A}^n$  is affine.  $\square$

*Remark 9.* Combining the non-existence of nontrivial forms of the affine plane over fields of characteristic zero [15] with Lefschetz principle arguments (see e.g. [18, Lemma 1]), we get the following characterization, perhaps of more geometric nature:

*Let  $k$  be an algebraically closed field of infinite transcendence degree over  $\mathbb{Q}$ . Then a morphism  $\pi : \mathbb{A}_k^m \rightarrow \mathbb{A}_k^n$  whose closed fibers are all isomorphic to  $\mathbb{A}_k^2$  is a trivial  $\mathbb{A}^2$ -bundle.*

**3.2. Variables in a four dimensional polynomial ring.** Generalizing a construction due to Vénéreau [25], Daigle-Freudentburg [5] and Lewis [19] introduced families of “coordinate-like” polynomials in four variables obtained as follows: given a polynomial ring  $k[x, y, z, u]$  in four variables over a field  $k$  of characteristic zero and a polynomial  $p(x, y, z, u) = yu + \lambda(x, z)$  where  $\lambda = z^2 + r(x)z + s(x)$  for some polynomials  $r, s \in k[x]$ , we set  $v = xz + yp$  and  $w = x^2u - 2\frac{\partial \lambda}{\partial z}p - yp^2$ . A *Vénéreau-type polynomial* is a polynomial of the form  $h = y + xQ(x, v, w) \in k[x][v, w] \subset k[x][y, z, u]$ .

**Example 10.** For  $p = yu + z^2$  and  $Q = x^{n-1}v$ , one gets the famous *Vénéreau polynomials*

$$v_n = y + x^n(xz + y(yu + z^2)), \quad n \geq 1.$$

The choice  $p = yu + z^2 + z$  and  $Q = x^{n-1}v$  yields the family of polynomials  $b_n = y + x^n(xz + y(yu + z^2 + z))$ ,  $n \geq 1$ , which appeared earlier in the work of Bhatwadekar and Dutta [4, Example 4.13].

It is easily checked that  $k[x]_x[y, z, u] = k[x]_x[h, v, w]$  and that  $k[x]/(x)[y, z, u] = k[x]/(x)[h, z, u]$ . It was successively proven in several papers [25, 13, 14, 5, 19] by clever explicit computations that some of these Vénéreau-type polynomials  $h$  are *x-variables* of  $k[x, y, z, u]$ , i.e. that there exists polynomials  $h_2, h_3 \in k[x][y, z, u]$  such that  $k[x][h, h_2, h_3] = k[x][y, z, u]$ . The fact that  $v_n$ ,  $n \geq 3$  is an *x-variable* was established first by Vénéreau [25], and generalized for arbitrary  $p = yu + \lambda(x, z)$  as above by Daigle-Freudentburg [5] to all Vénéreau-type polynomials of the form  $y + x^n v$ ,  $n \geq 3$ . The fact that  $v_2$  is an *x-variable* was established by Lewis [19]. In the same article, he proved more generally that for  $p = yu + z^2$ , all Vénéreau-type polynomials of the form  $h = y + x^2Q(x, v, w) + x^3vQ_2(x, v^2, w)$  are *x-variables*.

The cases of the Vénéreau polynomial  $v_1$  and the Bhatwadekar-Dutta polynomial  $b_1$  remained open so far, but since for every Vénéreau-type polynomial  $h$ , the morphism  $f = (x, h) : \mathbb{A}_k^4 \rightarrow \mathbb{A}_k^2$  is an  $\mathbb{A}^2$ -fibration [5, Proposition 1.1.], Corollary 8 implies the following:

**Proposition 11.** *Every Vénéreau-type polynomial  $h \in k[x, y, z, u]$  is an *x-variable* of  $k[x, y, z, u]$ .*

**3.3. Locally nilpotent derivations with a slice.** Recall that given a ring  $R$ , an  $R$ -derivation  $D$  of the polynomial ring  $R[x, y, z]$  in three variables over  $R$  is called locally nilpotent if  $R[x, y, z] = \bigcup_{n \geq 0} \text{Ker} D^n$ . A slice for  $D$  is an element  $s \in R[x, y, z]$  such that  $Ds = 1$ . The following proposition answers a question of Freudentburg [8] concerning kernels of locally nilpotent  $R$ -derivations of  $R[x, y, z]$  with a slice.

**Proposition 12.** *Let  $R$  be a regular ring essentially of finite type over a field  $k$  of characteristic zero. Then the kernel of a locally nilpotent  $R$ -derivation of  $R[x, y, z]$  with a slice is isomorphic to the symmetric algebra  $\text{Sym}_R M$  of a 1-stably free projective  $R$ -module of rank 2.*

*Proof.* The kernel  $B \subset R[x, y, z]$  of such a derivation  $D$  is an  $R$ -algebra of finite type such that  $B[s] = R[x, y, z]$  for a slice  $s \in R[x, y, z]$  of  $D$ . By [8, Theorem 1.1], the inclusion  $R \subset B$  defines an  $\mathbb{A}^2$ -fibration  $f : \text{Spec}(B) \rightarrow \text{Spec}(R)$ . Since  $R[x, y, z] = B[s]$ ,  $B$  is a smooth  $R$ -algebra and so,  $\Omega_{B/R}^1$  is a projective  $B$ -module of rank 2. The map which associates to a finitely generated projective  $B$ -module  $N$  the projective  $R[x, y, z]$ -module  $N \otimes_B R[x, y, z]$  is injective. On the other hand, since  $R$  is regular and essentially of finite type, it follows from Lindell [20] that the map which associates to a finitely generated projective  $R$ -module  $M$  the projective  $R[x, y, z]$ -module  $M \otimes_B R[x, y, z]$  is bijective. Consequently, there exists a projective  $R$ -module  $M$  of rank 2 such that  $\Omega_{B/R}^1 \simeq B \otimes_R M$ . By Corollary 6,  $B$  is isomorphic to  $\text{Sym}_R M$ . The fact that  $M$  is stably free follows from the canonical split exact sequence

$$0 \rightarrow \Omega_{B/R}^1 \otimes_B R[x, y, z] \rightarrow \Omega_{R[x, y, z]/R}^1 \simeq R[x, y, z]^{\oplus 3} \rightarrow \Omega_{R[x, y, z]/B}^1 \simeq R[x, y, z] \rightarrow 0$$

which implies that  $\Omega_{B/R}^1 \otimes_B R[x, y, z] \simeq M \otimes_R R[x, y, z]$  is 1-stably free, whence that  $M$  is 1-stably free.  $\square$

*Remark 13.* Note that conversely, given a 1-stably free projective module  $M$  of rank 2 on a ring  $R$ , the symmetric algebra  $\text{Sym}_R(M \oplus R)$  is  $R$ -isomorphic to  $R[x, y, z]$  and can be equipped via the isomorphism  $\text{Sym}_R(M \oplus R) \simeq \text{Sym}_R(M) \otimes_R R[s]$  with the locally nilpotent  $R$ -derivation  $D = \frac{\partial}{\partial s}$  whose kernel is isomorphic to  $\text{Sym}_R(M)$  and which has  $s$  as a slice.

As a consequence of Quillen-Suslin Theorem [21, 23], we obtain the following solution to Question 1 in [8]:

**Corollary 14.** *Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring in  $n \geq 2$  variables. Then the kernel of a locally nilpotent  $R$ -derivation of  $R[x, y, z]$  with a slice is isomorphic to a polynomial ring in 2 variables over  $R$ . In other words, such a derivation is conjugate to  $\partial/\partial x$  by an  $R$ -automorphism of  $R[x, y, z]$ .*

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