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# Trajectory tracking for a system of two linear hyperbolic PDEs with uncertainties

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**Abstract:** We give sufficient conditions that guarantee exponential stability for an uncertain system of two coupled linear hyperbolic PDEs with the actuation proposed in (Vazquez et al. (2011)) applied at one boundary of the system. Using the backstepping approach, we map the original uncertain system to a system with a simpler structure. Using the Lyapunov approach proposed in (Bastin and Coron (2011); Diagne et al. (2012)) it is then possible to derive sufficient conditions that guarantee robustness. Using the same target system we derive a simple algorithm that ensures robust output tracking.

*Keywords:* Hyperbolic Partial Differential Equations; Uncertain linear system, Robustness, Tracking, Backstepping.

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## 1. INTRODUCTION

This article details the robustness properties of the controller proposed in (Vazquez et al. (2011)). This control law (applied at only one boundary) ensures, in the ideal case, the stabilization of a system of two heterodirectional linear first-order hyperbolic Partial Differential Equations (PDEs) in minimum time. This article provides sufficient conditions that guarantee the exponential stability of the controlled system in presence of uncertainties. Moreover, we solve the problem of output tracking for this uncertain system.

Most physical systems involving a transport phenomenon can be modeled using hyperbolic partial differential equations (PDEs): traffic flow (Amin et al. (2008)), heat exchangers (Xu and Sallet (2002)), open channel flow (Coron et al. (1999), de Halleux et al. (2003)) or multiphase flow (Di Meglio (2011), Djordjevic et al. (2010), Dudret et al. (2012)).

The backstepping approach (Hu et al. (2016); Krstic and Smyshlyaev (2008); Vazquez et al. (2011)) has enabled the design of stabilizing full-state feedback laws for these systems. These controllers are *explicit*, in the sense that they are expressed as a linear functional of the distributed state at each instant. The (distributed) gains can be computed offline. If the gap between backstepping design and existence results for stabilizing controllers is now partially bridged (Auriol and Di Meglio (2016)), many properties (including robustness properties) of the backstepping controllers remain unknown.

A natural question towards an engineering use of backstepping is whether these control designs are robust to the presence of uncertainties in the system parameters?

To investigate this question, one can rely on the numerous contributions focusing on stability conditions (Bastin and Coron (2011); Diagne et al. (2012)).

The main contribution of this paper is to derive sufficient robustness conditions guaranteeing the exponential stability of an uncertain system subject to a backstepping controller. Moreover, we derive a control law that guarantees the (robust) output tracking of the uncertain system to a reference signal  $\chi(t)$ .

Our approach is the following. Using two successive Volterra transformations, the uncertain system combined with a nominal control law is mapped to a target system with a well-known structure. The main idea is to use backstepping-like transformation not only for control design, but also for analysis. Then, using existing results, we derive sufficient conditions that guarantee exponential stability. For the tracking problem, we derive from the obtained target system an integral equation satisfied by the output. Analyzing the properties of this integral equation we derive, using a Recursive Least Squares algorithm, a simple control law that ensures robust exponential tracking.

The paper is organized as follow. In Section 2 we introduce the model equations and the notations; in particular we distinguish the nominal system (that was used to design the controller) and the real system with uncertainties. In Section 3, using the backstepping approach and successive Volterra transformations, we present a robustness result as a sufficient condition that guarantees the exponential stability of the uncertain system. Some complementary remark are given in subsection 3.5. In section 4 we focus on the tracking problem, expressing the output as the solution of an integral equation. An algorithm ensuring

robust output tracking is then presented in Section 5. Finally some simulation results are given in Section 6.

## 2. PROBLEM DESCRIPTION

We first consider the nominal-ideal problem consisting of the following 2-states linear hyperbolic system

### 2.1 Nominal problem

$$u_t(t, x) + \lambda u_x(t, x) = \sigma^{+-} v(t, x) \quad (1)$$

$$v_t(t, x) - \mu v_x(t, x) = \sigma^{-+} u(t, x) \quad (2)$$

evolving in  $\{(t, x) \mid t \geq 0, x \in [0, 1]\}$ , with the following linear boundary conditions

$$u(t, 0) = qv(t, 0), \quad v(t, 1) = ru(t, 1) + U(t) \quad (3)$$

The inside-domain coupling terms  $\sigma^{-+}$  and  $\sigma^{+-}$ , the boundary coupling terms  $q$  and  $\rho$  and the velocities  $\lambda$  and  $\mu$  are assumed to be constant. Moreover, we assume that

$$-\mu < 0 < \lambda \quad \text{and} \quad q \neq 0 \quad (4)$$

The initial conditions denoted  $u_0$  and  $v_0$  are assumed to belong to  $L^2([0, 1])$ . For this system, a feedback control law has been designed in (Coron et al. (2013)). This control law ensures the stabilization of the system in finite time. More precisely, the following theorem holds.

*Theorem 1.* (Coron et al. (2013)). Consider system (1)-(2) with boundary conditions (3) and the following feedback law

$$U(t) = -ru(t, 1) + \int_0^1 [K(1, \xi)u(t, \xi) + L(1, \xi)v(t, \xi)] d\xi \quad (5)$$

where the kernels  $K$  and  $L$  are defined by the following well-posed system of equations

$$\mu K_x(x, \xi) - \lambda K_\xi(x, \xi) = \sigma^{-+} L(x, \xi) \quad (6)$$

$$\mu L_x(x, \xi) + \mu L_\xi(x, \xi) = \sigma^{+-} K(x, \xi) \quad (7)$$

and the following set of boundary conditions

$$K(x, x) = -\frac{\sigma^{-+}}{\lambda + \mu} \quad L(x, 0) = \frac{q\lambda}{\mu} K(x, 0) \quad (8)$$

Then  $v(t, 0) \equiv 0$  if  $t \geq t_F = \frac{1}{\lambda} + \frac{1}{\mu}$ .

In what follows, we are interested in the robustness of the above controller to uncertainties in the system parameters.

### 2.2 System under consideration

We now consider the following uncertain system

$$u_t(t, x) + \bar{\lambda} u_x(t, x) = \bar{\sigma}^{+-} v(t, x) \quad (9)$$

$$v_t(t, x) - \bar{\mu} v_x(t, x) = \bar{\sigma}^{-+} u(t, x) \quad (10)$$

evolving in  $\{(t, x) \mid t > 0, x \in [0, 1]\}$ , with the following linear boundary conditions

$$u(t, 0) = \bar{q}v(t, 0), \quad v(t, 1) = \bar{r}u(t, 1) + \delta_U U(t) \quad (11)$$

where

$$\bar{\lambda} = \lambda + \delta_\lambda, \quad \bar{\mu} = \mu + \delta_\mu \quad (12)$$

$$\bar{\sigma}^{+-} = \sigma^{+-} + \delta_{\sigma^{+-}}, \quad \bar{\sigma}^{-+} = \sigma^{-+} + \delta_{\sigma^{-+}} \quad (13)$$

$$\bar{q} = q + \delta_q, \quad \bar{r} = r + \delta_r \quad (14)$$

with  $-\bar{\mu} < 0 < \bar{\lambda}$  and  $\bar{q} \neq 0$ . The terms  $\delta_\lambda$  and  $\delta_\mu$  represent uncertainties on the velocities whereas the terms  $\delta_{\sigma^{+-}}$ ,

$\delta_{\sigma^{-+}}$ ,  $\delta_q$  and  $\delta_r$  represent uncertainties on the coupling parameters. The term  $\delta_U \neq 0$  represents an uncertainty on the actuation. All of these uncertainties are assumed constant.

### 2.3 Formulation of the problem

The first goal of this article is to derive sufficient conditions on the uncertainties for **exponential** stability of system (1)-(3) with the feedback law (5). This is the topic of Section 3. Then we derive a control law that ensures the **exponential** convergence of the output  $v(t, 0)$  to a reference continuous function  $\chi(t)$ . This is the topic of Section 4.

## 3. ROBUSTNESS: GENERAL SUFFICIENT CONDITIONS FOR A SYSTEM OF TWO EQUATIONS

In this section, we present the main new idea of the paper. We first apply the classical backstepping transformation to the uncertain system (9)-(11). Because of the uncertainties, there remain integral coupling terms at the boundary of the target system. We analyze these by means of a second Volterra transformation. This new approach highlights the potential of backstepping as an analysis tool, rather than only a control design tool. With these two successive transformations, system (9)-(11) is mapped to a system with a simpler structure. Then, it becomes possible, using existing results, to derive sufficient conditions for robustness.

### 3.1 First Volterra transformation: Removing inside-domain couplings

We first consider the following classical Volterra transformation from (Coron et al. (2013))

$$\alpha(t, x) = u(t, x) - \int_0^x (M^{uu}(x, \xi)u(\xi) + M^{uv}(x, \xi)v(\xi)) d\xi \quad (15)$$

$$\beta(t, x) = v(t, x) - \int_0^x (M^{vu}(x, \xi)u(\xi) + M^{vv}(x, \xi)v(\xi)) d\xi \quad (16)$$

where the kernels  $M^{uu}, M^{uv}, M^{vu}, M^{vv}$  are defined on  $\mathcal{T} = \{(x, \xi) \in [0, 1]^2 \mid \xi \leq x\}$  by the following set of hyperbolic PDEs:

$$0 = \bar{\lambda} M_x^{uu}(x, \xi) + M_\xi^{uu}(x, \xi) \bar{\lambda} + M^{uv}(x, \xi) \bar{\sigma}^{-+} \quad (17)$$

$$0 = \bar{\lambda} M_x^{uv}(x, \xi) - M_\xi^{uv}(x, \xi) \bar{\mu} - M^{uv}(x, \xi) \bar{\sigma}^{+-} \quad (18)$$

$$0 = \bar{\mu} M_x^{vu}(x, \xi) - M_\xi^{vu}(x, \xi) \bar{\lambda} - M^{vv}(x, \xi) \bar{\sigma}^{-+} \quad (19)$$

$$0 = \bar{\mu} M_x^{vv}(x, \xi) + M_\xi^{vv}(x, \xi) \bar{\mu} - M^{vu}(x, \xi) \bar{\sigma}^{+-} \quad (20)$$

with the following set of boundary conditions:

$$M^{vu}(x, x) = -\frac{\bar{\sigma}^{-+}}{\bar{\lambda} + \bar{\mu}}, \quad \bar{\mu} M^{vv}(x, 0) = \bar{\lambda} \bar{q} M^{vu}(x, 0) \quad (21)$$

$$M^{uv}(x, x) = -\frac{\bar{\sigma}^{+-}}{\bar{\lambda} + \bar{\mu}}, \quad \bar{\mu} M^{uv}(x, 0) = \bar{\lambda} \bar{q} M^{uu}(x, 0) \quad (22)$$

As proved in (Coron et al. (2013)), this transformation is well-posed, i.e there exists  $M^{uu}, M^{uv}, M^{vu}$  and  $M^{vv}$  in  $L^\infty(\mathcal{T})$ . Moreover these kernels are invertible and we denote  $N^{\alpha\alpha}, N^{\alpha\beta}, N^{\beta\alpha}$  and  $N^{\beta\beta}$  the inverse kernels.

*Remark 1.* The inverse transformation of (15)-(16) can be written

$$u(t, x) = \alpha(t, x) + \int_0^x (N^{\alpha\alpha}(x, \xi)\alpha(\xi) + N^{\alpha\beta}(x, \xi)\beta(\xi))d\xi \quad (23)$$

$$v(t, x) = \beta(t, x) + \int_0^x (N^{\beta\alpha}(x, \xi)\alpha(\xi) + N^{\beta\beta}(x, \xi)\beta(\xi))d\xi \quad (24)$$

The transformation (15),(16) maps the initial system (9)-(11) to the following target system

$$\alpha_t(t, x) + \bar{\lambda}\alpha_x(t, x) = 0 \quad (25)$$

$$\beta_t(t, x) - \bar{\mu}\beta_x(t, x) = 0 \quad (26)$$

with the following linear boundary conditions

$$\alpha(t, 0) = \bar{q}\beta(t, 0) \quad (27)$$

$$\beta(t, 1) = A\alpha(t, 1) - \int_0^1 L^\alpha(\xi)\alpha(t, \xi) + L^\beta(\xi)\beta(t, \xi)d\xi \quad (28)$$

with

$$A = r + \delta_r - \delta_U r \quad (29)$$

$$L^\alpha(\xi) = N^{\beta\alpha}(1, \xi) - AN^{\alpha\alpha}(1, \xi) - \delta_U K^{vu}(1, \xi) - \delta_U \int_\xi^1 N^{\alpha\alpha}(\nu, \xi)K(1, \nu) + N^{\beta\alpha}(\nu, \xi)L(1, \nu)d\nu \quad (30)$$

$$L^\beta(\xi) = N^{\alpha\beta}(1, \xi) - AN^{\alpha\beta}(1, \xi) - \delta_U K^{vv}(1, \xi) - \delta_U \int_\xi^1 N^{\alpha\beta}(\nu, \xi)K(1, \nu) + N^{\beta\beta}(\nu, \xi)L(1, \nu)d\nu \quad (31)$$

*Proof 1.* The result directly follows from inserting (23),(24) into (9),(10)

*3.2 Second Volterra transformation: removing the integral terms*

In this section, we analyze the stability of (25)-(28). Denoting

$$\bar{U}(t) = - \int_0^1 L^\alpha(\xi)\alpha(t, \xi) + L^\beta(\xi)\beta(t, \xi)d\xi \quad (32)$$

boundary condition (28) rewrites

$$\beta(t, 1) = A\alpha(t, 1) + \bar{U}(t) \quad (33)$$

This highlights the fact that (25)-(28) has the structure of a closed loop system with (32) as the control law. We now derive the backstepping transformation that *would have resulted* in this closed-loop system. It reads

$$\Phi(t, x) = \alpha(t, x) \quad (34)$$

$$\Psi(t, x) = \beta(t, x) - \int_0^x (P^\alpha(x, \xi)\alpha(\xi) + P^\beta(x, \xi)\beta(\xi))d\xi \quad (35)$$

where the kernels  $P^\alpha$  and  $P^\beta$  are defined on  $\mathcal{T} = \{(x, \xi) \in [0, 1]^2 \mid \xi \leq x\}$  by the following set of hyperbolic PDEs

$$\bar{\mu}P_x^\alpha - P_\xi^\alpha \bar{\lambda} = 0 \quad (36)$$

$$\bar{\mu}P_x^\beta + P_\xi^\beta \bar{\mu} = 0 \quad (37)$$

with the boundary conditions

$$P^\alpha(1, \xi) = -L^\alpha(\xi), \quad P^\beta(1, \xi) = -L^\beta(\xi) \quad (38)$$

$$P^\alpha(x, 0)\bar{\lambda}\bar{q} = \bar{\mu}P^\beta(x, 0) \quad (39)$$

*Lemma 1.* Consider system (36)-(39). There exists a unique solution  $P^\alpha$  and  $P^\beta$  in  $L^\infty(\mathcal{T})$ .

Deriving (34) and (35) with respect to space and time leads to the following target system

$$\Phi_t(t, x) + \bar{\lambda}\Phi_x(t, x) = 0 \quad (40)$$

$$\Psi_t(t, x) - \bar{\mu}\Psi_x(t, x) = (\bar{\mu} + \bar{\lambda})P^\alpha(x, x)\Phi(t, x) \quad (41)$$

with the boundary conditions

$$\Phi(t, 0) = \bar{q}\Psi(t, 0) \quad (42)$$

$$\Psi(t, 1) = A\Phi(t, 1) \quad (43)$$

The stability of this system is much easier to analyze than (25)-(28), as illustrated in the next section.

*3.3 General sufficient robustness conditions*

Let us introduce the following notations

$$\chi = (\Phi \ \Psi)^T, \quad \Lambda = \text{diag}\{\bar{\lambda}, -\bar{\mu}\} \quad (44)$$

$$S_0 = \begin{pmatrix} 1 & -\bar{q} \\ 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 \\ -A & 1 \end{pmatrix} \quad (45)$$

and

$$M(x) = \begin{pmatrix} 0 & 0 \\ (\bar{\mu} + \bar{\lambda})P^\alpha(x, x) & 0 \end{pmatrix} \quad (46)$$

Applying the result given in ((Diagne et al., 2012, Theorem 1)) yields the following theorem

*Theorem 2.* The system (9)-(11) with control law (5) remains exponentially stable in presence of uncertainties if there exist  $\zeta > 0$ ,  $\pi_1 > 0$  and  $\pi_2 > 0$ , such that

(1) The boundary quadratic form

$$(\chi^T(t, 1) \ \chi^T(t, 0)) \begin{pmatrix} \Lambda\Pi(1) & 0 \\ 0 & -\Lambda\Pi(0) \end{pmatrix} \begin{pmatrix} \chi(t, 1) \\ \chi(t, 0) \end{pmatrix}$$

where

$$\Pi(x) = \begin{pmatrix} \pi_1 e^{-\bar{\lambda}\zeta x} & 0 \\ 0 & \pi_2 e^{\bar{\mu}\zeta x} \end{pmatrix} \quad (47)$$

is positive definite under the constraint of the linear boundary condition  $S_0 w(t, 0) + S_1 w(t, 1) = 0$ ,  $\forall t \geq 0$  along the solutions of system (1)-(3)

(2) The matrix  $-\zeta \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \bar{\mu} \end{pmatrix} \Pi(x) + M^T(x)\Pi(x) + \Pi(x)M(x)$

is negative definite  $\forall x \in [0, 1]$

*Proof 2.* (Sketch) The complete proof is given in (Diagne et al. (2012)). The LMIs given above are obtained deriving the following Lyapunov candidate function

$$V(t) = \int_0^1 \chi^T(t, x)\Pi(x)\chi(t, x)dx \quad (48)$$

For general systems of the form (9)-(11) it is rather clear that more explicit conditions can be derived only on a case-by-case basis, when the specific structure or the numerical values of the different matrices are specified. In the next section we analyze further this sufficient condition in a specific case.

*3.4 Sufficient condition when  $A = 0$*

We now assume that  $A = 0$ , which is the case when the right boundary condition is perfectly known. Then, the following theorem derived from (Bastin and Coron (2011)), holds.

*Theorem 3.* The system (9)-(11) is exponentially stable if

$$\bar{q}_0 \int_0^1 \left( \left| \frac{\bar{\lambda} + \bar{\mu}}{\bar{\mu}} P^\alpha(\xi, \xi) \right| \right) d\xi < 1 \quad (49)$$

This condition is equivalent to

$$\int_0^1 (|\bar{q}_0 L^\alpha(\xi)| + |L^\beta(\xi)|) d\xi < 1 \quad (50)$$

where  $L^\alpha$  and  $L^\beta$  are defined by (30)-(31).

*Proof 3.* This proof is derived from ((Bastin and Coron, 2011, Theorem 1, Theorem2)). A sufficient condition so that the system is (9)-(11) exponentially stable is  $\bar{q} < \frac{1}{\eta(1)}$

where  $\eta$  is the solution (defined on  $[0, 1]$ ) of

$$(C) : \quad \eta'(x) = \left| \frac{\bar{\lambda} + \bar{\mu}}{\bar{\mu}} P^\alpha(x, x) \right|, \quad \eta(0) = 0 \quad (51)$$

Integrating (36) and (37) along their characteristic lines and using the corresponding boundary conditions yields:

$$P^\alpha(x, x) = \begin{cases} L^\alpha\left(\left(1 + \frac{\bar{\lambda}}{\bar{\mu}}\right)x - \frac{\bar{\lambda}}{\bar{\mu}}\right) & \text{if } x \geq \frac{\bar{\lambda}}{\bar{\mu} + \bar{\lambda}} \\ \frac{\bar{\mu}}{\lambda \bar{q}_0} L^\beta\left(1 - \left(\frac{\bar{\mu}}{\bar{\lambda}} + 1\right)x\right) & \text{else} \end{cases} \quad (52)$$

Using this expression and integrating equation (51) yields condition (50)

*Remark 2.* Comparing with Theorem 2, condition (50) only requires an inequality to check as opposed to LMIs to be solved. In Section 6, we illustrate the simplicity of use of this equation by means of a toy problem.

### 3.5 Complementary remarks

In this section, we give some fundamental complementary remarks that are not detailed here due to the space constraints.

*Bounds for the uncertainties* As is, Theorem 2 gives a sufficient condition for stability for a fixed value of each uncertain parameter. The following theorem gives a result for uncertainty *within a certain interval* for **one** uncertain parameter.

*Theorem 4.* Consider system (9)-(11) where we have one uncertainty  $\delta \in [-\delta_{max}, \delta_{max}]$  corresponding to only a single parameter. If (9)-(11) remains exponentially stable for  $\delta = -\delta_{max}$  and  $\delta = \delta_{max}$  then it remains exponentially stable for any  $\delta \in [-\delta_{max}, \delta_{max}]$

*Proof 4.* We only give here a sketch of the proof due to lack of space: first, using Lumer-Phillips' Theorem (see (Lumer and Phillips (1961))) for instance, one can prove that the operators corresponding to extreme values of the uncertainty and their adjoints are dissipative and closed. Then, one can show by computation that dissipativity and closure are preserved for uncertainties within the extreme values. Finally, Lumer-Phillips' theorem yields the result.

Let us now consider system (9)-(11) with uncertainties on each parameters. We have no more than 7 uncertain coefficients. We can rewrite this set of coefficients  $(x_1, \dots, x_7)$  (for instance  $x_1 = \sigma^{+-}$ ,  $x_2 = \lambda$ ). The order in which these coefficients are taken does not matter. Let us now denote for each coefficient  $x_i$  the known bound for the corresponding uncertainty as  $\delta_i$ . Finally, we denote  $\mathcal{I} = \{-1, 1\}$ . We then have the following result:

*Corollary 1.* If for all set  $r = (r_1, r_2, \dots, r_7) \in \mathcal{I}^7$ , system (9)-(11) where the coefficients  $(x_1 \dots x_7)$  are replaced by  $(x_1 + r_1 \delta_1, \dots, x_7 + r_7 \delta_7)$  satisfies Theorem 2, then for all  $s_i \in [-1, 1]$  system (9)-(11) where the coefficients  $(x_1 \dots x_7)$  are replaced by  $(x_1 + s_1 \delta_1, \dots, x_k + s_7 \delta_7)$  is exponentially stable.

*Remark 3.* This theorem means that knowing that the system remains exponentially stable for set of uncertainties located on the  $2^7$  vertices of an 7-orthotope (i.e a hyper-rectangle) implies that the system remains exponentially stable for uncertainties inside the 7-orthotope.

*Robustness of the output feedback controller* We have made as an implicit assumption that the distributed states  $u$  and  $v$  are available as measurements. Actually, in (Vazquez et al. (2011)), the control law (5) is combined with an observer to synthesize an output feedback controller that requires only measurements from the uncollocated boundary. The robustness of the controller does not imply that of the output feedback controller. We also need to prove that the observer remains robust to the model uncertainties, but this topic is out of the scope of this paper.

## 4. TRACKING PROBLEM: ANALYSIS OF THE OUTPUT $V(T, 0)$

In this section we focus on the following trajectory tracking problem. Given  $\chi(t)$  a reference continuous function, we want to find a control law  $U(t)$  so that the output  $v(t, 0)$  converges exponentially to  $\chi(t)$ . In the case of the nominal problem it is proved in (Hu et al. (2016)) that for (1)-(2) with boundary conditions (3) and the following feedback law

$$U(t) = -ru(t, 1) + \chi\left(t + \frac{1}{\mu}\right) + \int_0^1 [K(1, \xi)u(t, \xi) + L(1, \xi)v(t, \xi)] d\xi \quad (53)$$

Then  $v(t, 0) \equiv \chi(t)$  if  $t \geq t_F = \frac{1}{\lambda} + \frac{1}{\mu}$ . In this section we analyze the properties of the output  $v(t, 0)$  of the uncertain system with the control law (53) for a particular reference trajectory  $\chi(t)$ . More precisely, we prove that if  $\chi$  is a polynomial function of time, then  $x(t) = v(t, 0)$  is a polynomial function of the same order.

### 4.1 An integral equation for $v(t, 0)$

First, notice that following the same approach as above with control law (53) the uncertain system (9)-(11) with control law (53) is equivalent to (40)-(43) where (43) is replaced by

$$\Psi(t, 1) = A\Phi(t, 1) + \delta_U \chi\left(t + \frac{1}{\mu}\right) \quad (54)$$

Using the method of characteristics (i.e integrating (40)-(41) along their characteristic lines and using the boundary conditions (42)-(54)), yields

$$\Phi(t, x) = \Phi\left(t - \frac{x}{\lambda}, 0\right) = \bar{q}\Psi\left(t - \frac{x}{\lambda}, 0\right) \quad (55)$$

$$\Psi\left(t + \frac{1}{\mu}, 0\right) = \Psi(t, 1) + \int_0^{\frac{1}{\mu}} (\bar{\lambda} + \bar{\mu}) P^\alpha(1 - \bar{\mu}s, 1 - \bar{\mu}s) \Phi(t + s, 1 - \bar{\mu}s) ds \quad (56)$$

Combining these two equations along with the boundary condition (39) and denoting  $y(t) = \Psi(t, 0)$  and  $R(s) = -\bar{q}(\bar{\lambda} + \bar{\mu})P^\alpha(1 - \bar{\mu}s, 1 - \bar{\mu}s)$  yields

$$y\left(t + \frac{1}{\bar{\mu}}\right) = A\bar{q}y\left(t - \frac{1}{\bar{\lambda}}\right) + \delta_U \chi\left(t + \frac{1}{\bar{\mu}}\right) - \int_0^{\frac{1}{\bar{\mu}}} R(s)y\left(t + s - \frac{1}{\bar{\lambda}} + \frac{\bar{\mu}}{\bar{\lambda}}s\right)ds \quad (57)$$

In the following we will denote  $C_1 = 1 - A\bar{q} + \int_0^{\frac{1}{\bar{\mu}}} R(s)ds$  and we assume that  $C_1 \neq 0$ . We make the following natural assumption.

*Assumption 1.* The uncertain system is assumed to remain exponentially stable, i.e. we assume that the conditions of Theorem 2 are satisfied. It yields that the solutions of (57) with  $\chi(t) \equiv 0$  converge exponentially to zero.

#### 4.2 Constant reference trajectory

We start by the simplest case assuming that  $\chi(t) = a_0$  for all  $t > 0$ , with  $a_0 \in \mathfrak{R}$ . Denoting  $z(t) = y(t) - \frac{\delta_U}{C_1}a_0$  yields

$$z\left(t + \frac{1}{\bar{\mu}}\right) = A\bar{q}z\left(t - \frac{1}{\bar{\lambda}}\right) - \int_0^{\frac{1}{\bar{\mu}}} R(s)z\left(t + s - \frac{1}{\bar{\lambda}} + \frac{\bar{\mu}}{\bar{\lambda}}s\right)ds$$

Using Assumption 1, we have that  $z$  converges exponentially to 0 and that consequently  $y$  converges exponentially to  $\frac{\delta_U a_0}{C_1}$ .

#### 4.3 $\chi(t)$ is polynomial function

We now assume that  $\chi(t) = \sum_{k=0}^n a_k t^k$  ( $a_k \in \mathfrak{R}$ ) is a polynomial function. We prove that it implies that  $y(t)$  converges exponentially to a polynomial function of the same order. To do so, let us first notice that Equation (57) is linear. We recall the following lemma.

*Lemma 2.* If a  $C^1$  function  $y(t)$  is such that  $y'(t)$  converges exponentially to 0 then  $y$  converges exponentially to a constant. More precisely, if a  $C^k$  function  $y(t)$  is such that  $y^{(k)}(t)$  converges exponentially to a constant then  $y$  converges to a polynomial function of degree  $k$ .

*Proof 5.* We have that

$$\forall t > 0, \quad y(t) - y(0) = \int_0^t y'(\tau)d\tau \quad (58)$$

Since  $y'$  converges exponentially to 0, there exist  $\gamma > 0$  such that  $y'(t)e^{\gamma t}$  converges to 0. Consequently the integral  $\int_0^t y'(\tau)d\tau$  converges exponentially to a constant. So does  $y(t)$ .

If  $y^{(k)}$  converges exponentially to a constant  $d_0$ , then  $(y^{(k-1)}(t) - d_0 t)$  converges exponentially to a constant  $d_1$ . So,  $(y^{(k-1)}(t) - d_0 t - d_1)$  converges exponentially to zero. By recursion, one can easily prove that  $y$  converges exponentially to a polynomial function of degree  $k$ .

Using the linearity, we only need to prove that  $\forall k > 0$ , if  $\chi(t) = t^k$  then  $y$  converges to a polynomial function of degree  $k$ . Let us formally derive  $k$  times equation (57) with respect to time (we look for solutions that have the same regularity as  $\chi$ ). It yields

$$y^{(k)}\left(t + \frac{1}{\bar{\mu}}\right) = A\bar{q}y^{(k)}\left(t - \frac{1}{\bar{\lambda}}\right) + \delta_U k! - \int_0^{\frac{1}{\bar{\mu}}} R(s)y^{(k)}\left(t + s - \frac{1}{\bar{\lambda}} + \frac{\bar{\mu}}{\bar{\lambda}}s\right)ds \quad (59)$$

Consequently (using the previous subsection),  $y^{(k)}$  converges exponentially to  $\frac{k!\delta_U}{C_1}$ . Using Lemma 2 yields that  $y$  converges to a polynomial function of degree  $k$ . Consequently, we have the following theorem.

*Theorem 5.* If  $\chi(t) = \sum_{k=0}^n a_k t^k$ , then  $y(t)$  converges exponentially (i.e is equivalent) to the function  $\sum_{k=0}^n b_k t^k$ .

Moreover, injecting this expression in (57) yields to an explicit expression of each  $b_k$  in function of the parameters of the problems  $(\mu, \bar{\mu}, A, \bar{\lambda}, R, \dots)$ .

#### 4.4 Recursive formulation of polynomial outputs

In this subsection we consider the two polynomial functions  $\chi_1(t) = t^n$  and  $\chi_2(t) = t^{n-1}$  ( $n > 0$ ). As seen above the corresponding outputs  $y_1(t)$  and  $y_2(t)$  converge to polynomial functions that can respectively be expressed as

$$P_1(t) = \sum_{k=0}^n a_k t^k, \quad P_2(t) = \sum_{k=0}^{n-1} b_k t^k \quad (60)$$

We then have the following theorem

*Theorem 6.* The coefficients of  $P_2$  can be expressed as functions of the coefficients of  $P_1$ :

$$\forall 0 \leq k \leq n-1, \quad b_k = \frac{k+1}{n} a_{k+1} \quad (61)$$

*Proof 6.* The (recursive) proof is omitted due to space constraints.

Theorem 6 is extremely important since, combining it with linearity, it means that if one knows the polynomial function to which  $x$  converges when  $\chi(t) = t^n$  then, it is possible to know the output for any polynomial function of degree  $n$  or lower as input.

#### 4.5 General expression of the output for $\chi(t) = \sum_{k=0}^n a_k t^k$

We assume here that  $\chi(t) = \sum_{k=0}^n a_k t^k$  is a polynomial function of degree  $n$ . We denote  $A = (a_0 \dots a_n)^T$ . Using Theorem 5, we have that the output  $x(t)$  converges to a polynomial function of degree  $n$ :  $\sum_{k=0}^n b_k t^k$ . Denoting  $B = (b_0 \dots b_n)^T$  one has, due to Theorem 5

$$B = FA \quad (62)$$

where  $F$  is an upper triangular matrix depending only on the uncertainties. Moreover, Theorem 6 implies that  $F$  is entirely defined by its last column and that  $\forall 1 \leq i \leq j \leq n+1$ ,

$$F_{i,j} = \frac{i}{j} F_{i+1,j+1} \quad (63)$$

*Remark 4.*  $F$  is always invertible since, as proved above, it is upper triangular and all its diagonal coefficients are non-null

If one is able to determine  $F$ , then, inverting it, it becomes possible to track any polynomial function. This is the purpose of the next section.

## 5. ROBUST OUTPUT TRACKING

The purpose of this subsection is for a given polynomial signal  $\chi_0(t) = \sum_{k=0}^n a_k t^k$  to derive the control law  $U(t)$  such that  $x(t)$  converges exponentially to  $\chi_0(t)$ .

### 5.1 Identification of the matrix $F$

The first step is to identify the coefficients of the constant upper-triangular matrix  $F$  defined in section 4.4. To do so let us send  $\chi(t) = \chi_0(t)$  as an input. As seen above the output  $v(t, 0) = x(t)$  converges exponentially to the polynomial function  $\sum_{j=0}^n b_j t^j$ . Using a Recursive Least Squares algorithm (RLS) on the measured output it is possible to identify the coefficients  $b_j$ .

We then denote  $B = (b_0 \dots b_n)^T$  and  $A = (a_0 \dots a_n)^T$ . Using the results of the previous section yields

$$B = FA \quad (64)$$

$F$  is entirely defined by its last column:

$$F_n : (F_{1,n+1} \dots F_{n+1,n+1})^T \quad (65)$$

Consequently, knowing  $A$  and  $B$ , it becomes possible to explicitly determine  $F_n$  and then  $F$ .

### 5.2 Tracking of $\chi_0(t)$

We now want  $x(t)$  to converge to  $\chi_0$ . Let us define the coefficients  $Q_j$  as

$$\forall 0 \leq j \leq k \quad Q_j = ((F^{-1})(a_0 \ a_1 \ \dots \ a_k)^T)_j \quad (66)$$

These coefficients are well defined since  $F$  is invertible. We define  $\mathcal{Q} = (Q_1, \dots, Q_n)^T$ . We now consider the input function  $\chi_1(t)$  defined by

$$\chi_1(t) = \sum_{j=0}^n Q_j \chi_j(t) \quad (67)$$

Using the property of linearity and the results of the previous sections, we have that  $x(t)$  converges to the polynomial function  $R(t)$  whose coefficients are defined by

$$\forall 0 \leq j \leq n \quad R_j = (F\mathcal{Q})_j \quad (68)$$

It yields  $\forall 0 \leq j \leq n, R_j = a_j$  and that consequently  $R = \chi_0$ . We then have the following theorem:

*Theorem 7.* Considering an arbitrary polynomial function  $P(t) = \sum_{j=0}^n a_j t^j$ , and the input

$$\chi(t) = \sum_{j=0}^n ((F^{-1})(a_0 \ a_1 \ \dots \ a_n)^T)_j t^j \quad (69)$$

then the output  $v(t, 0)$  of system (9)-(11) with the control law (5) converges exponentially to  $\chi_0$ .

### 5.3 Practical implementation

The proposed algorithm is divided in two parts

- First we send as an input  $\chi_0(t)$  and we use the RLS algorithm on the output.
- Once we have enough points so that the estimation is accurate, we can compute  $F$  and  $\chi_1$  so that the output converges to  $\chi_0$ .

This approach is summarized on Figure 1.

*Remark 5.* The time required on phase 1 strongly depends on the sampling rate of the measurement.

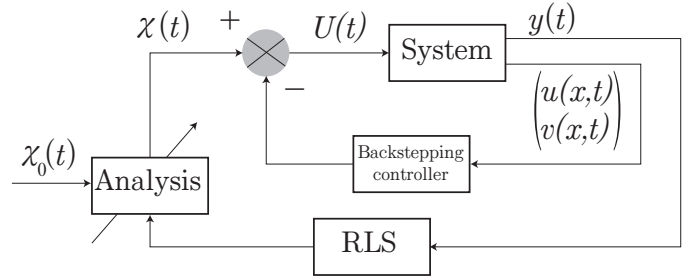


Fig. 1. Description of the algorithm

### 5.4 Extension to any continuous function

Let us consider an arbitrary continuous function  $f(t)$  defined on the compact  $[0, T]$ . Using Stone-Weierstrass' theorem it is then possible to approximate it by a polynomial function  $P_f$ . It is then sufficient to track this polynomial function  $P_f$ . Another solution would be to consider sliding window so that  $F$  can be approximated by a polynomial function of small degree on each window. The size of the first window must be large enough to be sure that the RLS algorithm converges, in order to identify the correct  $F$ .

## 6. SIMULATION RESULTS

In this section we illustrate our tracking result with simulations on a toy problem. The numerical values of the parameters are as follow

$$\lambda = 1, \quad \mu = 1 \quad \sigma^{+-} = 1, \quad \sigma^{-+} = 1 \quad (70)$$

$$q = -1, \quad r = \delta_r = 0, \quad \delta_q = -0.25, \quad \delta_U = 1.1 \quad (71)$$

$$\delta_\lambda = 0.25, \quad \delta_\mu = 0.25, \quad \delta_{\sigma^{+-}} = -\delta_{\sigma^{-+}} = 0.25 \quad (72)$$

For this set of uncertainties we get

$$\int_0^1 (|\bar{q}_0 L^\alpha(\xi)| + |L^\beta(\xi)|) d\xi = 0.121 < 1 \quad (73)$$

Consequently, the sufficient condition provided by Theorem 2 is satisfied and the uncertain system is exponentially stable. We choose as reference  $\chi_0(t) = 0.0225t^2 - 10^{-3}t^3$  so that  $\chi_0(0) = \chi_0'(0) = \chi_0'(20) = 0$  and  $\chi_0(20) = 3$ . Figure 2 pictures the reference signal, and the output  $v(t, 0)$  in three different cases. The first one is using the backstepping controller without adaption (i.e the one defined by (5)). The second one is the adaptive backstepping controller that is derived in the previous sections. Finally we consider a simple PI controller that is tuned to ensure the tracking in the nominal case. In the case of the adaptive controller, the data from the output are obtained using the RLS algorithm during the 10 first seconds of the simulation. In the same time the input is  $\chi_0(t)$ . Since the system is slightly different from the nominal one, the output progressively diverges from the reference. At time  $t = 7$ , the matrix  $F$  is computed and the input is corrected to take into account uncertainties. Due to the dynamic of the system the effect on the output is visible 1 second later. As expected the output quickly converges to the reference signal.

## 7. CONCLUSION

Using the backstepping approach on an uncertain system of first-order hyperbolic linear PDEs controlled at one boundary, we have presented a change of coordinates that enables to derive sufficient conditions guaranteeing the

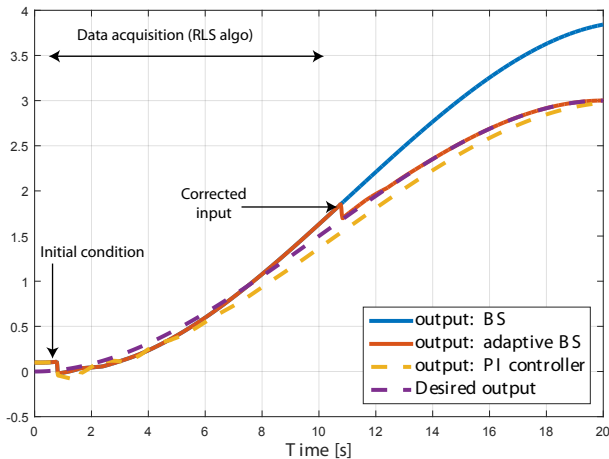


Fig. 2. Time evolution of the output for different controllers

exponential stability of the system. Then, expressing the output  $v(t, 0)$  as the solution of a simple integral equation and analyzing the properties of this integral equation, we have derived an algorithm based on a RLS method to track any continuous function.

The presented result raises several important questions about the properties of the backstepping controller. In particular, would it be possible to determine some of the uncertainties comparing the real output (obtained with the RLS algorithm) with the theoretical one?

The generalization of this result to derive sufficient conditions guaranteeing robustness of the output feedback controller for a general class of systems of hyperbolic PDEs will be the purpose of future contribution.

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