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Existence of solution to scalar BSDEs with weakly L^{1+} -integrable terminal values

Ying Hu* Shanjian Tang†

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Abstract. In this paper, we study a scalar linearly growing BSDE with a weakly L^{1+} -integrable terminal value. We prove that the BSDE admits a solution if the terminal value satisfies some Ψ -integrability condition, which is weaker than the usual L^p ($p > 1$) integrability and stronger than $L \log L$ integrability. We show by a counterexample that $L \log L$ integrability is not sufficient for the existence of solution to a BSDE of a linearly growing generator.

AMS Subject Classification: 60H10

Key Words Backward stochastic differential equation, weak integrability, terminal condition, dual representation.

1 Introduction

Consider the following Backward Stochastic Differential Equation (BSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad (1.1)$$

where the function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ satisfies

$$|f(s, y, z)| \leq \alpha_s + \beta|y| + \gamma|z|,$$

with $\alpha \in L^1(0, T)$, $\beta \geq 0$ and $\gamma > 0$.

It is well known that if $\xi \in L^p$, with $p > 1$, then there exists a solution to BSDE (1.1), see e.g. [5, 4, 1]. The aim of this paper is to find a weaker integrability condition for the terminal value ξ , under which the solution still exists.

Set

$$\Psi_\lambda(x) = xe^{(\frac{2}{\lambda} \log(x+1))^{1/2}}, \quad (\lambda, x) \in (0, \infty) \times [0, \infty).$$

Our sufficient condition is: there exists $\lambda \in (0, \frac{1}{\gamma^2 T})$ such that

$$\mathbb{E}[\Psi_\lambda(|\xi|)] < +\infty.$$

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Remark 1.1 Note that the preceding Ψ_λ -integrability is stronger than L^1 , weaker than L^p for any $p > 1$, because for any $\varepsilon > 0$, we have,

$$x \leq x e^{(\frac{2}{\lambda} \log(x+1))^{1/2}} \leq x e^{\varepsilon \log(x+1) + \frac{1}{2\varepsilon\lambda}} \leq e^{\frac{1}{2\varepsilon\lambda}} x (x+1)^\varepsilon, \quad x \geq 0.$$

Moreover, for any $p \geq 1$, there exists a constant $C_p > 0$ such that

$$x e^{(\frac{2}{\lambda} \log(x+1))^{1/2}} \geq C_p x \log^p(x+1).$$

We will see that even the condition

$$\mathbb{E}[\Psi_\lambda(|\xi|)] < +\infty$$

for a certain $\lambda > \frac{1}{\gamma^2 T}$ (which implies that $|\xi| \log^p(|\xi| + 1) \in L^1$) is still too weak to ensure the existence of solution by giving a simple example in Example 2.3.

Note that if the generator f is of sublinear growth with respect to z , i.e. there exists $q \in [0, 1)$,

$$|f(t, y, z)| \leq \alpha + \beta|y| + \gamma|z|^q,$$

then there exists a solution for $\xi \in L^1$, see [1].

Our method applies the dual representation of solution to BSDE with convex generator (see, e.g. [4, 6, 3]) in order to establish some a priori estimate and then the localization procedure of real-valued BSDE [2].

Let us close this introduction by giving the notations that we will use in all the paper. For the remaining of the paper, let us fix a nonnegative real number $T > 0$. First of all, $(W_t)_{t \in [0, T]}$ is a standard Brownian motion with values in \mathbb{R}^d defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of the Brownian motion W augmented by the \mathbb{P} -null sets of \mathcal{F} . The sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted by \mathcal{P} .

Consider real valued BSDEs which are equations of type (1.1), where f (hereafter called the generator) is a random function $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ and measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1 \times d})$, and ξ (hereafter called the terminal condition or terminal value) is a real \mathcal{F}_T -measurable random variable.

Definition 1.2 By a solution to BSDE (1.1), we mean a pair $(Y_t, Z_t)_{t \in [0, T]}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$ such that \mathbb{P} -a.s., $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ belongs to $L^2(0, T)$ and $t \mapsto g(t, Y_t, Z_t)$ belongs to $L^1(0, T)$, and \mathbb{P} -a.s. (Y, Z) verifies (1.1).

By BSDE (ξ, f) , we mean the BSDE of generator f and terminal condition ξ .

For any real $p \geq 1$, \mathcal{S}^p denotes the set of real-valued, adapted and càdlàg processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right]^{1/p} < +\infty,$$

and \mathcal{M}^p denotes the set of (equivalent class of) predictable processes $(Z_t)_{t \in [0, T]}$ with values in $\mathbb{R}^{1 \times d}$ such that

$$\|Z\|_{\mathcal{M}^p} := \mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{p/2} \right]^{1/p} < +\infty.$$

The rest of the paper is organized as follows. Section 2 establishes a necessary and sufficient condition for the existence of solution to BSDE (1.1) for the typical form of generator $f(t, y, z) = \alpha_t + \beta y + \gamma|z|$. Section 3 gives the Ψ_λ integrability condition for the existence of solution to BSDE (1.1) for $f(t, y, z) = \alpha_t + \beta y + \gamma|z|$. Section 4 is devoted to the sufficiency of the Ψ_λ -integrability condition for the existence of solution to BSDE (1.1) of the general linearly growing generator.

2 Typical Case

Let us first consider the following BSDE:

$$Y_t = \xi + \int_t^T (\alpha_s + \beta Y_s + \gamma |Z_s|) ds - \int_t^T Z_s dW_s, \quad (2.2)$$

where $\alpha \in L^1(0, T)$, and $\beta \geq 0$ and $\gamma > 0$ are some real constants. We suppose further that the terminal condition ξ is nonnegative. Note that if Y is a solution belonging to class D, then as $e^{\beta t} Y_t$ is a local supermartingale, it is a supermartingale, from which we deduce that $Y \geq 0$. In this subsection, we restrict ourselves to nonnegative solution.

For $\xi \in L^p$ ($p > 1$), BSDE (2.2) has a unique solution. It has a dual representation as follows (see, e.g. [4, 3])

$$Y_t = \operatorname{ess\,sup}_{q \in \mathcal{A}} \{ \mathbb{E}_q [e^{\beta(T-t)} \xi | \mathcal{F}_t] \} + \int_t^T e^{\beta(s-t)} \alpha_s ds, \quad (2.3)$$

where \mathcal{A} is the set of progressively measurable processes q such that $|q| \leq \gamma$,

$$\frac{d\mathbb{Q}^q}{d\mathbb{P}} = M_T^q,$$

with

$$M_t^q = \exp \left\{ \int_0^t q_s dW_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right\}, \quad t \in [0, T],$$

and \mathbb{E}_q is the expectation with respect to \mathbb{Q}^q .

Theorem 2.1 *Let us suppose that $\xi \geq 0$. Then BSDE (2.2) admits a solution (Y, Z) such that $Y \geq 0$ if and only if there exists a locally bounded process \bar{Y} such that*

$$\operatorname{ess\,sup}_{q \in \mathcal{A}} \{ \mathbb{E}_q [e^{\beta(T-t)} \xi | \mathcal{F}_t] \} + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \bar{Y}_t.$$

Proof. If BSDE (2.2) admits a solution (Y, Z) such that $Y \geq 0$, then we define a sequence of stopping times

$$\sigma_n = T \wedge \inf \{ t \geq 0 : |Y_t| > n \},$$

with the convention that $\inf \emptyset = +\infty$.

As $W_s^q = W_s - \int_0^s q_r dr$ is a Brownian motion under \mathbb{Q}^q , we have

$$Y_{t \wedge \sigma_n} = Y_{\sigma_n} + \int_{t \wedge \sigma_n}^{\sigma_n} (\alpha_s + \beta Y_s + \gamma |Z_s| - Z_s q_s) ds - \int_{t \wedge \sigma_n}^{\sigma_n} Z_s dW_s^q.$$

Applying Itô's formula to $e^{\beta s} Y_s$, we deduce

$$e^{\beta(t \wedge \sigma_n)} Y_{t \wedge \sigma_n} = e^{\beta \sigma_n} Y_{\sigma_n} + \int_{t \wedge \sigma_n}^{\sigma_n} e^{\beta s} (\alpha_s + \gamma |Z_s| - Z_s q_s) ds - \int_{t \wedge \sigma_n}^{\sigma_n} e^{\beta s} Z_s dW_s^q,$$

from which we obtain

$$\mathbb{E}_q [e^{\beta(\sigma_n - t \wedge \sigma_n)} Y_{\sigma_n} + \int_{t \wedge \sigma_n}^{\sigma_n} e^{\beta(s - t \wedge \sigma_n)} \alpha_s ds | \mathcal{F}_t] \leq Y_{t \wedge \sigma_n}.$$

Fatou's lemma yields that

$$\mathbb{E}_q [e^{\beta(T-t)} \xi | \mathcal{F}_t] + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq Y_t.$$

On the other hand, if there exists a locally bounded process \bar{Y} such that

$$\operatorname{ess\,sup}_{q \in \mathcal{A}} \{\mathbb{E}_q[e^{\beta(T-t)} \xi | \mathcal{F}_t]\} + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \bar{Y}_t,$$

then we construct the solution by use of a localization method (see e.g. [2]). We describe this method here for completeness. Let (Y^n, Z^n) be the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the following BSDE

$$Y_t^n = \xi \mathbf{1}_{\{\xi \leq n\}} + \int_t^T (\alpha_s + \beta Y_s^n + \gamma |Z_s^n|) ds - \int_t^T Z_s^n dW_s.$$

By comparison theorem, Y^n is nondecreasing with respect to n . Moreover, setting $q_s^n = \gamma \operatorname{sgn}(Z_s^n)$, we obtain

$$\begin{aligned} Y_t^n &= \mathbb{E}_{q^n} [e^{\beta(T-t)} \xi \mathbf{1}_{\{\xi \leq n\}} | \mathcal{F}_t] + \int_t^T e^{\beta(s-t)} \alpha_s ds \\ &\leq \mathbb{E}_{q^n} [e^{\beta(T-t)} \xi | \mathcal{F}_t] + \int_t^T e^{\beta(s-t)} \alpha_s ds \\ &\leq \bar{Y}_t. \end{aligned}$$

Set

$$\tau_k = T \wedge \inf\{t \geq 0 : \bar{Y}_t > k\},$$

and

$$Y_k^n(t) = Y_{t \wedge \tau_k}^n, \quad Z_k^n(t) = Z_t^n \mathbf{1}_{t \leq \tau_k}.$$

Then (Y_k^n, Z_k^n) satisfies

$$Y_k^n(t) = Y_k^n(T) + \int_t^T \mathbf{1}_{s \leq \tau_k} (\alpha_s + \beta Y_k^n(s) + \gamma |Z_k^n(s)|) ds - \int_t^T Z_k^n(s) dW_s. \quad (2.4)$$

For fixed k , Y_k^n is nondecreasing with respect to n and remains bounded by k . We can now apply the stability property of BSDE with bounded terminal data (see e.g. Lemma 3, page 611 in [2]). Setting $Y_k(t) = \sup_n Y_k^n(t)$, there exists Z_k such that $\lim_n Z_k^n = Z_k$ in \mathcal{M}^2 and

$$Y_k(t) = \sup_n Y_{\tau_k}^n + \int_t^{\tau_k} (\alpha_s + \beta Y_k(s) + \gamma |Z_k(s)|) ds - \int_t^{\tau_k} Z_k(s) dW_s. \quad (2.5)$$

Finally, noting that

$$Y_{k+1}(t \wedge \tau_k) = Y_k(t \wedge \tau_k), \quad Z_{k+1} \mathbf{1}_{t \leq \tau_k} = Z_k \mathbf{1}_{t \leq \tau_k},$$

we conclude the existence of solution (Y, Z) . □

Remark 2.2 Consider the case $d = 1$. If BSDE (2.2) admits a solution (Y, Z) such that $Y \geq 0$, by taking $q = \gamma$ and $q = -\gamma$, we deduce that both $\xi e^{\gamma W_T}$ and $\xi e^{-\gamma W_T}$ are in $L^1(\Omega)$, which implies that $\xi e^{\gamma |W_T|} \in L^1(\Omega)$, as

$$\xi e^{\gamma |W_T|} \leq \xi e^{\gamma W_T} + \xi e^{-\gamma W_T}.$$

Example 2.3 Let us set $d = 1$, $T = 1$, $\beta = 0$, $\gamma = 1$, $\mu \in (0, 1)$, and

$$\xi = e^{\frac{1}{2} W_1^2 - \mu |W_1| + \frac{1}{2} \mu^2} - 1.$$

In this case, BSDE (2.2) does not admit a solution (Y, Z) such that $Y \geq 0$, as $\xi e^{|W_1|}$ does not belong to $L^1(\Omega)$ by the following direct calculus:

$$\mathbb{E}[\xi e^{|W_1|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (e^{\frac{1}{2}|x|^2 - \mu|x| + \frac{1}{2}\mu^2} - 1) e^{|x|} e^{-\frac{1}{2}|x|^2} dx = +\infty.$$

Whereas it is straightforward to see that for any $p \geq 1$, $\xi \log^p(\xi + 1) \in L^1(\Omega)$.

For $\lambda > 1$, consider the following terminal condition

$$\xi = e^{\frac{1}{2}W_1^2 - \mu|W_1| + \frac{1}{2}\mu^2} - 1, \quad \mu \in \left(\frac{1}{\sqrt{\lambda}}, 1\right).$$

We have $\Psi_\lambda(\xi) \in L^1$ by the following straightforward calculus:

$$\mathbb{E}[\Psi_\lambda(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (e^{\frac{1}{2}|x|^2 - \mu|x| + \frac{1}{2}\mu^2} - 1) e^{\frac{1}{\sqrt{\lambda}}|x| - \mu} e^{-\frac{1}{2}|x|^2} dx < +\infty,$$

while BSDE (2.2) has no solution, in view of the fact that $\xi e^{|W_1|}$ does not belong to $L^1(\Omega)$.

3 Sufficient Condition

Let us now look for a sufficient condition for the existence of a locally bounded process \bar{Y} such that

$$\operatorname{ess\,sup}_{q \in \mathcal{A}} \{\mathbb{E}_q[e^{\beta(T-t)} \xi | \mathcal{F}_t]\} + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \bar{Y}_t.$$

For $\lambda > 0$, define the functions Φ_λ and Ψ_λ :

$$\begin{aligned} \Phi_\lambda(x) &= e^{\frac{1}{2}\lambda \log^2(x)}, \quad x > 0, \\ \Psi_\lambda(y) &= ye^{\left(\frac{2}{\lambda} \log(y+1)\right)^{1/2}}, \quad y \geq 0. \end{aligned}$$

Then we have

Proposition 3.1 For any $x \in \mathbb{R}$ and $y \geq 0$, we have

$$e^x y \leq \Phi_\lambda(e^x) + e^{\frac{2}{\lambda}} \Psi_\lambda(y).$$

Proof. Set

$$z = \left(\frac{2}{\lambda} \log(y+1)\right)^{1/2} \geq 0,$$

then

$$y = e^{\frac{\lambda}{2}z^2} - 1.$$

It is sufficient to prove that for any $x \in \mathbb{R}$ and $z \geq 0$,

$$e^{\frac{1}{2}\lambda x^2 - x} + (e^{\frac{\lambda}{2}z^2} - 1)(e^{z + \frac{2}{\lambda} - x} - 1) \geq 0.$$

It is evident to see that the above inequality holds when $z + \frac{2}{\lambda} - x \geq 0$.

Consider the case $z + \frac{2}{\lambda} - x < 0$. Then $x > z + \frac{2}{\lambda} > 0$. Hence

$$\begin{aligned} & e^{\frac{1}{2}\lambda x^2 - x} + (e^{\frac{\lambda}{2}z^2} - 1)(e^{z + \frac{2}{\lambda} - x} - 1) \\ &= e^{\frac{1}{2}\lambda(x - \frac{1}{\lambda})^2 - \frac{1}{2\lambda}} + e^{\frac{\lambda}{2}z^2 + z + \frac{2}{\lambda} - x} + 1 - e^{z + \frac{2}{\lambda} - x} - e^{\frac{\lambda}{2}z^2} \\ &\geq e^{\frac{1}{2}\lambda(z + \frac{1}{\lambda})^2 - \frac{1}{2\lambda}} - e^{\frac{\lambda}{2}z^2} \\ &\geq 0. \end{aligned}$$

□

Proposition 3.2 Let $0 < \lambda < \frac{1}{\gamma^2 T}$. For any $q \in \mathcal{A}$,

$$\mathbb{E}[\Phi_\lambda(e^{\int_t^T q_s dW_s}) | \mathcal{F}_t] \leq \frac{1}{\sqrt{1 - \lambda \gamma^2 (T - t)}}.$$

Proof. Firstly, by use of Girsanov's lemma, for $\theta \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E}[e^{\theta \int_t^T q_s dW_s} | \mathcal{F}_t] \\ &= \mathbb{E}[e^{\theta \int_t^T q_s dW_s - \frac{\theta^2}{2} \int_t^T |q_s|^2 ds} e^{\frac{\theta^2}{2} \int_t^T |q_s|^2 ds} | \mathcal{F}_t] \\ &\leq e^{\frac{\theta^2 \gamma^2}{2} (T-t)}. \end{aligned}$$

Then we apply

$$e^{\lambda \frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\sqrt{\lambda} y x - \frac{y^2}{2}} dy$$

to deduce that

$$\begin{aligned} & \mathbb{E}[\Phi_\lambda(e^{\int_t^T q_s dW_s}) | \mathcal{F}_t] \\ &= \mathbb{E}[e^{\frac{\lambda}{2} (\int_t^T q_s dW_s)^2} | \mathcal{F}_t] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbb{E}[e^{\sqrt{\lambda} y \int_t^T q_s dW_s - \frac{y^2}{2}} | \mathcal{F}_t] dy \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{(\sqrt{\lambda} y \gamma)^2}{2} (T-t) - \frac{y^2}{2}} dy \\ &= \frac{1}{\sqrt{1 - \lambda \gamma^2 (T - t)}}. \end{aligned}$$

□

Applying the above two propositions, we deduce the following sufficient condition.

Theorem 3.3 Let us suppose that there exists $\lambda \in (0, \frac{1}{\gamma^2 T})$ such that

$$\mathbb{E}[\Psi_\lambda(\xi)] < +\infty.$$

Then

$$\operatorname{ess\,sup}_{q \in \mathcal{A}} \{\mathbb{E}_q[e^{\beta(T-t)} \xi | \mathcal{F}_t]\} + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \bar{Y}_t, \quad (3.6)$$

with

$$\bar{Y}_t = e^{\beta(T-t)} \left(\frac{1}{\sqrt{1 - \lambda \gamma^2 (T - t)}} + e^{\frac{2}{\lambda}} \mathbb{E}[\Psi_\lambda(\xi) | \mathcal{F}_t] \right) + \int_t^T e^{\beta(s-t)} \alpha_s ds,$$

and (2.2) admits a solution (Y, Z) such that

$$Y_t \leq \bar{Y}_t.$$

Proof. Applying the above two propositions, we deduce

$$\begin{aligned} \mathbb{E}_q[\xi | \mathcal{F}_t] &= \mathbb{E}[M_T^q (M_t^q)^{-1} \xi | \mathcal{F}_t] \leq \mathbb{E} \left[e^{\int_t^T q_s dW_s} \xi \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E}[\Phi_\lambda(e^{\int_t^T q_s dW_s}) | \mathcal{F}_t] + e^{\frac{2}{\lambda}} \mathbb{E}[\Psi_\lambda(\xi) | \mathcal{F}_t] \\ &\leq \frac{1}{\sqrt{1 - \lambda \gamma^2 (T - t)}} + e^{\frac{2}{\lambda}} \mathbb{E}[\Psi_\lambda(\xi) | \mathcal{F}_t]. \end{aligned}$$

Then we get (3.6) and the rest follows from Theorem 2.1. □

4 General Case

Consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad (4.7)$$

where f satisfies

$$|f(s, y, z)| \leq \alpha_s + \beta|y| + \gamma|z|, \quad (4.8)$$

with $\alpha \in L^1(0, T)$, $\beta \geq 0$ and $\gamma > 0$.

Theorem 4.1 *Let f be a generator which is continuous with respect to (y, z) and verifies (4.8), and ξ be a terminal condition. Let us suppose that there exists $\lambda \in (0, \frac{1}{\gamma^2 T})$ such that*

$$\mathbb{E}[\Psi_\lambda(|\xi|)] < +\infty.$$

Then BSDE (4.7) admits a solution (Y, Z) such that

$$|Y_t| \leq e^{\beta(T-t)} \left(\frac{1}{\sqrt{1 - \lambda\gamma^2(T-t)}} + e^{\frac{2}{\lambda}} \mathbb{E}[\Psi_\lambda(|\xi|)|\mathcal{F}_t] \right) + \int_t^T e^{\beta(s-t)} \alpha_s ds.$$

Proof. Let us fix $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$ and set $\xi^{n,p} = \xi^+ \wedge n - \xi^- \wedge p$. Let $(Y^{n,p}, Z^{n,p})$ be the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the BSDE $(|\xi^{n,p}|, f)$. Set

$$\bar{f}(s, y, z) = \alpha_s + \beta y + \gamma|z|,$$

and $(\bar{Y}^{n,p}, \bar{Z}^{n,p})$ be the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the BSDE $(|\xi^{n,p}|, \bar{f})$.

By comparison theorem,

$$|Y_t^{n,p}| \leq |\bar{Y}_t^{n,p}|.$$

Setting $q_s^{n,p} = \gamma \operatorname{sgn}(Z_s^{n,p})$, we obtain,

$$\begin{aligned} |Y_t^{n,p}| &\leq |\bar{Y}_t^{n,p}| \\ &= \mathbb{E}_{q^{n,p}} \left[e^{\beta(T-t)} |\xi^{n,p}| \middle| \mathcal{F}_t \right] + \int_t^T e^{\beta(s-t)} \alpha_s ds. \end{aligned}$$

From inequality (3.6),

$$|Y_t^{n,p}| \leq \bar{Y}_t,$$

with

$$\bar{Y}_t = e^{\beta(T-t)} \left(\frac{1}{\sqrt{1 - \lambda\gamma^2(T-t)}} + e^{\frac{2}{\lambda}} \mathbb{E} \left[\Psi_\lambda(|\xi|) \middle| \mathcal{F}_t \right] \right) + \int_t^T e^{\beta(s-t)} \alpha_s ds.$$

Moreover, $Y^{n,p}$ is nondecreasing with respect to n , and nonincreasing with respect to p . Once again, we apply the localization method as follows to conclude the existence of solution.

Set

$$\tau_k = T \wedge \inf\{t \geq 0 : \bar{Y}_t > k\},$$

and

$$Y_k^{n,p}(t) = Y_{t \wedge \tau_k}^{n,p}, \quad Z_k^{n,p}(t) = Z_t^{n,p} \mathbf{1}_{t \leq \tau_k}.$$

Then $(Y_k^{n,p}, Z_k^{n,p})$ satisfies

$$Y_k^{n,p}(t) = Y_k^{n,p}(T) + \int_t^T \mathbf{1}_{s \leq \tau_k} f(s, Y_k^{n,p}(s), Z_k^{n,p}(s)) ds - \int_t^T Z_k^{n,p}(s) dW_s. \quad (4.9)$$

For fixed k , $Y_k^{n,p}$ is nondecreasing with respect to n and nonincreasing with respect to p , and remains bounded by k . We can now apply the stability property of BSDEs with bounded terminal data. Setting $Y_k(t) = \inf_p \sup_n Y_k^{n,p}$, there exists Z_k in \mathcal{M}^2 such that $\lim_p \lim_n Z_k^{n,p} = Z_k$ in \mathcal{M}^2 and

$$Y_k(t) = \inf_p \sup_n Y_{\tau_k}^{n,p} + \int_t^{\tau_k} f(s, Y_k(s), Z_k(s)) ds - \int_t^{\tau_k} Z_k(s) dW_s. \quad (4.10)$$

Finally, noting that

$$Y_{k+1}(t \wedge \tau_k) = Y_k(t \wedge \tau_k), \quad Z_{k+1} \mathbf{1}_{t \leq \tau_k} = Z_k \mathbf{1}_{t \leq \tau_k},$$

we conclude the existence of solution (Y, Z) . □

References

- [1] P. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **108** (2003), no. 1, 109-129.
- [2] P. Briand and Y. Hu, BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields* **136** (2006), no. 4, 604-618.
- [3] F. Delbaen, Y. Hu and A. Richou, On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011), no. 2, 559-574.
- [4] N. El Karoui, S. Peng and M. C. Quenez, Backward stochastic differential equations in finance. *Math. Finance* **7** (1997), no. 1, 1-71.
- [5] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** (1990), no. 1, 55-61.
- [6] S. Tang, Dual representation as stochastic differential games of backward stochastic differential equations and dynamic evaluations. *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 10, 773-778.