



**HAL**  
open science

# Time discretization of diffusion equations: a functional scheme.

Julien Baptiste, Emmanuel Lépinette

► **To cite this version:**

Julien Baptiste, Emmanuel Lépinette. Time discretization of diffusion equations: a functional scheme.. 2017. hal-01507267v1

**HAL Id: hal-01507267**

**<https://hal.science/hal-01507267v1>**

Preprint submitted on 12 Apr 2017 (v1), last revised 29 Dec 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Time discretization of diffusion equations: a functional scheme

Julien Baptiste,<sup>1</sup>  
Emmanuel LEPINETTE,<sup>1</sup>

<sup>1</sup> *Paris Dauphine university, PSL research university, Ceremade, CNRS, UMR, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France.*

*Email: emmanuel.lepinette@ceremade.dauphine.fr*

**Abstract:** We introduce a seemingly new functional scheme which approximates diffusion equations of type  $h_t(t, x) + \frac{x^2 \sigma^2(t, x)}{2} h_{xx}(t, x) = 0$ . Contrarily to classical numerical methods, in particular finite difference methods, the principle is only based on a discretization in time. We establish the uniform convergence in time of the scheme and provide the rate of convergence. We illustrate the convergence by numerical examples.

**Keywords and phrases:** Financial market models, Liquidation value, Transaction costs, European options.  
*2000 MSC: 60G44, G11-G13.*

## 1. Introduction

There are two major methods to simulate a diffusion equation of type

$$h_t(t, x) + \frac{x^2 \sigma^2(t, x)}{2} h_{xx}(t, x) = 0, \quad t \in [0, T],$$

with the boundary condition  $h(T, x) = g(x)$ . The first one is to use the Monte Carlo methods as the solution  $h$  admits a probabilistic representation, see [2]. The second one is the very well known finite differences technique based on approximations of the successive derivatives. It requires a discretization, both in time and in the space variable, of the compact subset  $[0, T] \times [a, b]$ ,  $a \leq b$ , of interest. Moreover, as we need to fix extra boundary conditions, when  $x = a$  or  $x = b$ , that are a priori approximations of  $h$ , a second type of approximation errors may appear. For instance, if  $b$  is large enough, we generally set the condition  $h(t, b) = \lim_{x \rightarrow \infty} h(t, x)$  if the latter may be estimated while the condition  $h(t, 0) = g(0)$  is chosen if  $a = 0$  under some

mild conditions on  $\sigma$ . We here present a very simple alternative method only based on a discretization in time of the interval  $[0, T]$  that allows to simulate  $t \mapsto h(t, x)$  for a fixed  $x$ . The advantage of this method is the ease of implementation of the functional scheme which is basically defined. The numerical scheme we propose is defined by two functions depending on the number  $n$  of discretization dates. We focus here on a particular choice of such functions which leads to a uniform approximation in time of the diffusion equation under some mild conditions. An open problem is to study the suggested functional scheme more generally.

## 2. Functional scheme for diffusion equations

Let us consider a bounded diffusion function  $\sigma : [0, T] \times \mathbb{R}_+$  and the associated backward parabolic equation

$$h_t(t, x) + \frac{x^2 \sigma^2(t, x)}{2} h_{xx}(t, x) = 0, \quad t \in [0, T], \quad h(T, x) = g(x), \quad (2.1)$$

where the terminal condition is defined by a Lipschitz function  $g$  with Lipschitz constant  $L_g$  and  $h_t$  and  $h_{xx}$  are respectively the first and the second derivatives of  $h$  with respect to time  $t$  and space variable  $x \in \mathbb{R}$ . This equation is very well known in physics but also in mathematical finance for models without friction when the risky asset  $S$  is driven by a standard Brownian motion  $W$  so that it satisfies the stochastic differential equation  $dS_t = \sigma(t, S_t) S_t dW_t$  under a risk neutral probability measure  $P$ <sup>1</sup>. In that case,  $h(t, S_t)$  is the value at time  $t$  of the unique self-financing portfolio process  $V$  composed of a fraction of the risk-less bond  $B = 1$  and the risky asset  $S$  such that it replicates the European option payoff  $g(S_T)$ , i.e.  $V_T = g(S_T)$ . The quantity  $\sigma(t, S_t)$  is then interpreted as a local volatility coefficient.

In the following, we consider the uniform grid on  $[0, T]$  given by  $t_i^n = (T/n)i$ ,  $i = 0, \dots, n$  where  $n \geq 1$ . We then define the following functions:

$$\begin{aligned} k_t^{n+}(x) &:= 1 + \sigma(t, x) \sqrt{\frac{T}{n}}, & k_t^{n-}(x) &:= 1 - \sigma(t, x) \sqrt{\frac{T}{n}}, & (2.2) \\ \lambda_t^n(x) &:= \frac{1 - k_t^{n-}(x)}{k_t^{n+}(x) - k_t^{n-}(x)} = \frac{1}{2}, & \mu_t^n(x) &:= 1 - \lambda_t^n(x) = \frac{1}{2}. \end{aligned}$$

---

<sup>1</sup>If we suppose that the risk-free interest rate is  $r = 0$

Let  $h^n$  be the piecewise function defined on  $[0, T]$  by  $h^n(t, x) = h^n(t_{i-1}, x)$  if  $t \in [t_{i-1}^n, t_i^n]$ ,  $i \leq n$ , where the sequence  $(h^n(t_i, x))_{i \leq n}$  is defined recursively by the following functional scheme:

$$\begin{aligned} h^n(t_{i-1}, x) &= \lambda_{t_{i-1}}^n(x) h^n(t_i, k_t^{n-}(x)x) + \mu_{t_{i-1}}^n(x) h^n(t_i, k_t^{n+}(x)x), \\ h^n(T, x) &= g(x), \quad i \leq n. \end{aligned} \quad (2.3)$$

Our main result states that the sequence of functions  $(h^n)_{n \in \mathbb{N}}$  uniformly converges to  $h$  as  $n \rightarrow \infty$  under some mild conditions (called Condition D below) satisfied by the successive derivatives of  $h$ , solution to (2.1). Of course, the functional scheme (2.3) may also be considered for other choices of functions  $k^{n-}$  and  $k^{n+}$ . In this paper, we restrict ourselves to the functions defined by (2.2) but it should be worthwhile to study such a scheme in its full generality.

Recall that  $h(t, x) = \mathbb{E}g(S_{x,t}(T))$  where, for  $t \leq T$ ,  $S_{x,t}$  is the unique solution to the stochastic differential equation (5.10). We deduce that  $h(t, 0) = g(0)$ . In the following, we suppose that  $\sigma$  is locally Lipschitz and bounded so that existence and uniqueness holds for (5.10), see for instance [2, Theorem 2.2, p104]. The conditions below hold in particular when the payoff function  $g$  is continuous with linear growth at infinity, e.g.  $g(x) = (x - K)^+$ ,  $K \in \mathbb{R}$ , and the diffusion coefficient  $\sigma$  is smooth enough and positive as proven in [3, Theorem 4.1] for a more general case <sup>2</sup>, see also [4, Section 5].

#### Condition D:

There exists a constant  $C > 0$  independent of  $n$  such that:

$$\begin{aligned} |h_x(t, x)| &\leq C, \quad |h_{xx}(t, x)| \leq \frac{C}{|x|\sqrt{T-t}}, \\ |h_{xxx}(t, x)| &\leq \frac{C}{x^2(T-t)}, \quad |h_{xt}(t, x)| \leq \frac{C}{T-t}, \\ |h_{xxt}(t, x)| &\leq \frac{C}{|x|(T-t)^{3/2}}, \quad |h_{tt}(t, x)| \leq \frac{C|x|}{(T-t)^{\frac{3}{2}}}, \\ |h_{ttt}(t, x)| &\leq \frac{C|x|}{(T-t)^{\frac{5}{2}}}, \quad |h_{xtt}(t, x)| \leq \frac{C}{(T-t)^2}, \end{aligned}$$

---

<sup>2</sup>Here, the coefficient  $\gamma_n$  of [3] appears to be a constant independent of  $n$ .

**Theorem 2.1.** *Suppose that  $\sigma$  is a bounded Lipschitz function and assume that the solution to the equation (2.1) satisfies Condition (D). Then, there exists a constant  $C(x) > 0$  depending on  $x$ ,  $g$  and  $\sigma$  such that*

$$\sup_{t \in [0, T]} |h^n(t, x) - h(t, x)| \leq \frac{C|x| \ln(n)}{\sqrt{n}}, \quad n > 0.$$

The proof is deduced from the lemmas and the corollary of Section 4. Note that by choosing specific discretization dates, see [1], we should improve the convergence rate of the functional scheme. This is left for future research.

### 3. Numerical examples

To be done.

### 4. Proof

From a line to the next one, we shall use the same notation  $C$  for some distinct constants independent of  $n$ . Also, notice that there is a constant  $M > 0$  such that  $|k_t^{n-}| + |k_t^{n+}| \leq M$  for all  $n$  and  $t \in [0, T]$ . We also use the notation  $\Delta t_i^n = t_i^n - t_i^{n-1} = T/n$ ,  $i \leq n$ . At last observe that the results below are trivial when  $x = 0$  hence we only provide the proofs when  $x \neq 0$ .

**Lemma 4.1.** *Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant  $C > 0$  which does not depend on  $x$  such that*

$$\sup_{t \in [t_{n-1}^n, T]} |h^n(t, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{n}}.$$

*Proof.* Let us define  $\delta_t^n = |h(t, x) - h^n(t, x)|$ . Recall that  $h(t, x) = \mathbb{E}g(S_{x,t}(T))$  where  $S_{x,t}$  is the solution to (5.10). We have:

$$\begin{aligned}
\delta_t^n &= |h(t, x) - \frac{1}{2}g(k_{t_{n-1}}^{n-} x) - \frac{1}{2}g(k_{t_{n-1}}^{n+} x)| \\
&\leq |h(t, x) - g(x)| + \frac{1}{2}|g(k_{t_{n-1}}^{n-} x) - g(x)| + \frac{1}{2}|g(k_{t_{n-1}}^{n+} x) - g(x)| \\
&\leq \sup_{t_{n-1} \leq t \leq T} |h(t, x) - g(x)| + \frac{\mathbb{L}g}{2}(1 - k_{t_{n-1}}^d) |x| + \frac{\mathbb{L}g}{2}(k_{t_{n-1}}^u - 1) |x| \\
&\leq \mathbb{L}g \sup_{t_{n-1} \leq t \leq T} \mathbb{E} |S_{x,t}(T) - x| + \frac{\mathbb{L}g}{2}\sigma(t, x)\sqrt{\frac{T}{n}} |x| + \frac{\mathbb{L}g}{2}\sigma(t, x)\sqrt{\frac{T}{n}} |x| \\
&\leq C|x|\sqrt{\frac{T}{n}} + \frac{\mathbb{L}g}{2}\sigma^* \sqrt{\frac{T}{n}} |x|,
\end{aligned}$$

where the last inequality is deduced from Lemma 5.1 and  $\sigma^* = \sup_{t \in [0, T], x \in \mathbb{R}} |\sigma(t, x)|$ .

□

**Lemma 4.2.** *Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant  $C > 0$  such that*

$$|h^n(t_{n-2}^n, x) - h(t_{n-2}^n, x)| \leq \frac{C|x|}{\sqrt{n}}.$$

*Proof.* Let us introduce  $\delta_{t-2}^n = |h^n(t_{n-2}^n, x) - h(t_{n-2}^n, x)|$ . We have

$$\begin{aligned}
\delta_{t-2}^n &= \left| \frac{1}{2}h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) + \frac{1}{2}h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-2}^n, x) \right| \\
&\leq \frac{1}{2}|h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-2}^n, x)| + \frac{1}{2}|h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-2}^n, x)| \\
&\leq \frac{1}{2}|h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x)| + \frac{1}{2}|h(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-2}^n, k_{t_{n-2}^n}^{n-} x)| \\
&\quad + \frac{1}{2}|h(t_{n-2}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-2}^n, x)| + \frac{1}{2}(|h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x)| \\
&\quad + \frac{1}{2}|h(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-2}^n, k_{t_{n-2}^n}^{n+} x)|) \\
&\quad + \frac{1}{2}|h(t_{n-2}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-2}^n, x)|.
\end{aligned}$$

Under Condition D, by the mean value theorem, we deduce that

$$\begin{aligned} |h(t_{n-2}^n, x) - h(t_{n-2}^n, k_{t_{n-2}^n}^{n-} x)| &\leq C |x - k_{t_{n-2}^n}^{n-} x| \leq C |x| \cdot |1 - k_{t_{n-2}^n}^{n-}| \leq C |x| \sqrt{\frac{T}{n}}, \\ |h(t_{n-2}^n, x) - h(t_{n-2}^n, k_{t_{n-2}^n}^{n+} x)| &\leq C |x - k_{t_{n-2}^n}^{n+} x| \leq C |x| \cdot |1 - k_{t_{n-2}^n}^{n+}| \leq C |x| \sqrt{\frac{T}{n}}. \end{aligned}$$

Note that, under Condition D, we have

$$|h_t(t, x)| = \frac{x^2 \sigma^2(t, x)}{2} |h_{xx}(t, x)| \leq \frac{C |x|}{\sqrt{T-t}}. \quad (4.4)$$

Therefore, by the mean value theorem, since  $k_{t_{n-2}^n}^{n-}, k_{t_{n-2}^n}^{n+} \leq M$ ,

$$\begin{aligned} |h(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-2}^n, k_{t_{n-2}^n}^{n-} x)| &\leq \frac{CM |x|}{\sqrt{T-t_{n-1}^n}} \Delta t_{n-1}^n \leq CM |x| \sqrt{\frac{T}{n}}, \\ |h(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-2}^n, k_{t_{n-2}^n}^{n+} x)| &\leq \frac{CM |x|}{\sqrt{T-t_{n-1}^n}} \Delta t_{n-1}^n \leq CM |x| \sqrt{\frac{T}{n}}. \end{aligned}$$

At last, by Lemma 4.1,

$$\begin{aligned} |h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x)| &\leq C k_{t_{n-2}^n}^{n-} |x| \sqrt{\frac{T}{n}}, \\ |h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x)| &\leq C k_{t_{n-2}^n}^{n+} |x| \sqrt{\frac{T}{n}}. \end{aligned}$$

As  $k_{t_{n-2}^n}^{n-} + k_{t_{n-2}^n}^{n+} = 2$ , we deduce that

$$\frac{1}{2} |h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x) - h(t_{n-1}^n, k_{t_{n-2}^n}^{n-} x)| + \frac{1}{2} |h^n(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x) - h(t_{n-1}^n, k_{t_{n-2}^n}^{n+} x)| \leq C |x| \sqrt{\frac{T}{n}}.$$

The conclusion follows.  $\square$

**Lemma 4.3.** *Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant  $C > 0$  such that*

$$|h^n(t_i^n, x) - h(t_i^n, x)| \leq \frac{C|x|}{\sqrt{n}}, \quad \text{for all } i \leq n-2.$$

*Proof.* We first prove the result for  $i = n - 2$  and, then, we generalize the result by induction. Let us introduce the function

$$F_i(x) = \frac{1}{2}h(t_i^n, k_{t_{i-1}^n}^{n-})x + \frac{1}{2}h(t_i^n, k_{t_{i-1}^n}^{n+}x).$$

By the Taylor formula, we get that

$$\begin{aligned} F_i(x) &= \frac{1}{2} [h(t_{i-1}^n, x) + R_i^{11}(x) + R_i^{12}(x) + R_i^{13}(x)] \\ &+ \frac{1}{2} [h(t_{i-1}^n, x) + R_i^{21}(x) + R_i^{22}(x) + R_i^{23}(x)]. \end{aligned}$$

where, for some constants  $\alpha_i$ ,  $i = 1, \dots, 4$ , and  $\tilde{t}_{i-1}^n \in [t_{i-1}^n, t_i^n]$ ,  $\tilde{x} = x + \tilde{\theta}x(k_{t_{i-1}^n}^{n-} - 1)$ ,  $\tilde{\theta} \in [0, 1]$ ,  $\hat{t}_{i-1}^n \in [t_{i-1}^n, t_i^n]$ ,  $\hat{x} = x + \hat{\theta}x(k_{t_{i-1}^n}^{n+} - 1)$ ,  $\hat{\theta} \in [0, 1]$ , we have

$$\begin{aligned} R_i^{11}(x) &= h_t(t_{i-1}^n, x)\frac{T}{n} - h_x(t_{i-1}^n, x)x\sigma(t_{i-1}^n, x)\sqrt{\frac{T}{n}}, \\ R_i^{12}(x) &= -h_{tx}(t_{i-1}^n, x)x\sigma(t_{i-1}^n, x)\left(\frac{T}{n}\right)^{\frac{3}{2}} + \frac{1}{2}h_{xx}(t_{i-1}^n, x)x^2\sigma^2(t_{i-1}^n, x)\frac{T}{n} + \frac{1}{2}h_{tt}(t_{i-1}^n, x)\left(\frac{T}{n}\right)^2, \\ R_i^{13}(x) &= \alpha_1h_{ttt}(\tilde{t}_{i-1}^n, \tilde{x})\left(\frac{T}{n}\right)^3 - \alpha_2h_{ttx}(\tilde{t}_{i-1}^n, \tilde{x})\sigma(t_{i-1}^n, x)x\left(\frac{T}{n}\right)^{5/2} \\ &+ \alpha_3h_{txx}(\tilde{t}_{i-1}^n, \tilde{x})\sigma^2(t_{i-1}^n, x)x^2\left(\frac{T}{n}\right)^2 - \alpha_4h_{xxx}(\tilde{t}_{i-1}^n, \tilde{x})\sigma^3(t_{i-1}^n, x)x^3\left(\frac{T}{n}\right)^{3/2}, \\ R_i^{21}(x) &= h_t(t_{i-1}^n, x)\frac{T}{n} + h_x(t_{i-1}^n, x)x\sigma(t_{i-1}^n, x)\sqrt{\frac{T}{n}}, \\ R_i^{22}(x) &= h_{tx}(t_{i-1}^n, x)x\sigma(t_{i-1}^n, x)\left(\frac{T}{n}\right)^{\frac{3}{2}} + \frac{1}{2}h_{xx}(t_{i-1}^n, x)x^2\sigma^2(t_{i-1}^n, x)\frac{T}{n} + \frac{1}{2}h_{tt}(t_{i-1}^n, x)\left(\frac{T}{n}\right)^2, \\ R_i^{23}(x) &= \alpha_1h_{ttt}(\hat{t}_{i-1}^n, \hat{x})\left(\frac{T}{n}\right)^3 + \alpha_2h_{ttx}(\hat{t}_{i-1}^n, \hat{x})\sigma(t_{i-1}^n, x)x\left(\frac{T}{n}\right)^{5/2} \\ &+ \alpha_3h_{txx}(\hat{t}_{i-1}^n, \hat{x})\sigma^2(t_{i-1}^n, x)x^2\left(\frac{T}{n}\right)^2 + \alpha_4h_{xxx}(\hat{t}_{i-1}^n, \hat{x})\sigma^3(t_{i-1}^n, x)x^3\left(\frac{T}{n}\right)^{3/2}. \end{aligned}$$

We deduce that  $F_i(x) = h(t_{i-1}^n, x) + R_i^1(x) + R_i^2(x) + R_i^3(x)$  where  $R_i^1(x) = \frac{1}{2}(R_i^{11}(x) + R_i^{21}(x))$ ,  $R_i^2(x) = \frac{1}{2}(R_i^{12}(x) + R_i^{22}(x))$  and  $R_i^3(x) = \frac{1}{2}(R_i^{13}(x) + R_i^{23}(x))$ .



As  $h$  is the solution to (2.1), we may simplify  $F_i(x)$  as

$$F_i(x) = h(t_{i-1}^n, x) + \frac{1}{2} h_{tt}(t_{i-1}^n, x) \left( \frac{T}{n} \right)^2 + R_i^3(x).$$

Under Condition D, there exists a constant  $C$  independent of  $i$  and  $n$  such that

$$\frac{1}{2} |h_{tt}(t_{i-1}^n, x)| \leq \frac{C |x|}{(T - t_{i-1}^n)^{\frac{3}{2}}}. \quad (4.5)$$

On the other hand, since  $\tilde{t}_{i-1}^n, \hat{t}_{i-1}^n \in [t_{i-1}^n, t_i^n]$  and  $|\hat{x}| + |\tilde{x}| \leq C|x|$  where  $C > 0$  is independent of  $i$  and  $n$ , the residual error  $R_i^3(x)$  is the sum of terms which may be dominated under Condition D using the following inequalities:

$$|\alpha_1 h_{ttt}(\hat{t}_{i-1}^n, \hat{x})| \leq \frac{C |x|}{(T - t_i^n)^{\frac{5}{2}}}. \quad (4.6)$$

$$|\alpha_2 h_{xtt}(\hat{t}_{i-1}^n, \hat{x}) \sigma(t_{i-1}^n, x) x| \leq \frac{C |x|}{(T - t_i^n)^2}. \quad (4.7)$$

$$|\alpha_3 h_{txx}(\hat{t}_{i-1}^n, \hat{x}) \sigma^2(t_{i-1}^n, x) x^2| \leq \frac{C |x|}{(T - t_{i-1}^n)^{\frac{3}{2}}}. \quad (4.8)$$

$$|\alpha_4 h_{xxx}(\hat{t}_{i-1}^n, \hat{x}) \sigma^3(t_{i-1}^n, x) x^3| \leq \frac{C |x|}{T - t_i^n}. \quad (4.9)$$

The inequalities above are also satisfied if we replace  $\hat{t}_{i-1}^n$  by  $\tilde{t}_{i-1}^n$  and  $\hat{x}$  by  $\tilde{x}$ . Recall that the constant  $C$  may denote distinct constants that change for a line to the next one but this constant does not depend on  $i$  and  $n$ .

Let us introduce  $\delta_{t_{i-1}}^n(x) := |h^n(t_{i-1}^n, x) - h(t_{i-1}^n, x)|$ ,  $i \leq n - 2$ . From

above, we get that

$$\begin{aligned}
\delta_{t_{i-1}}^n(x) &= \left| \frac{1}{2}h^n(t_i^n, k_{t_{i-1}}^{n-}) + \frac{1}{2}h^n(t_i^n, k_{t_{i-1}}^{n+}) - h(t_{i-1}^n, x) \right| \\
&= \left| \frac{1}{2}h^n(t_i^n, k_{t_{i-1}}^{n-}) + \frac{1}{2}h^n(t_i^n, k_{t_{i-1}}^{n+}) - F_i(x) + \frac{1}{2}h_{tt}(t_{i-1}^n, x) \left(\frac{T}{n}\right)^2 + R_i^3(x) \right| \\
&\leq \frac{1}{2}\delta_{t_i}^n(k_{t_{i-1}}^{n-}) + \frac{1}{2}\delta_{t_i}^n(k_{t_{i-1}}^{n+}) + \frac{1}{2}\left(\frac{T}{n}\right)^2 |h_{tt}(t_{i-1}^n, x)| + |R_i^3(x)|.
\end{aligned}$$

Using the inequalities (4.5), (4.6),  $\dots$ , (4.9), we then deduce a constant  $C > 0$  independent of  $i$  and  $n$  such that

$$\begin{aligned}
\delta_{t_{i-1}}^n(x) &\leq \frac{1}{2}\delta_{t_i}^n(k_{t_{i-1}}^{n-}) + \frac{1}{2}\delta_{t_i}^n(k_{t_{i-1}}^{n+}) \\
&+ \frac{C|x|}{(T-t_i^n)^{\frac{1}{2}}}\left(\frac{T}{n}\right)^{\frac{3}{2}} + \frac{C|x|}{T-t_i^n}\left(\frac{T}{n}\right)^{\frac{3}{2}} + \frac{C|x|}{(T-t_i^n)^{\frac{3}{2}}}\left(\frac{T}{n}\right)^2 \\
&+ \frac{C|x|}{(T-t_i^n)^2}\left(\frac{T}{n}\right)^{\frac{5}{2}} + \frac{C|x|}{(T-t_i^n)^{\frac{5}{2}}}\left(\frac{T}{n}\right)^3.
\end{aligned}$$

Recall that, by Lemma 4.2,

$$\frac{1}{2}\left(\delta_{t_{n-2}}^n(k_{t_{n-3}}^{n-}) + \delta_{t_{n-2}}^n(k_{t_{n-3}}^{n+})\right) \leq \frac{1}{2}C|x|(k_{t_{n-3}}^{n-} + k_{t_{n-3}}^{n+})\sqrt{\frac{T}{n}} = C|x|\sqrt{\frac{T}{n}}.$$

We deduce that

$$\begin{aligned}
|\delta_{t_{n-3}}^n(x)| &\leq C|x|\sqrt{\frac{T}{n}} + \frac{C|x|}{(T-t_{n-2}^n)^{\frac{1}{2}}}\left(\frac{T}{n}\right)^{\frac{3}{2}} + \frac{C|x|}{T-t_{n-2}^n}\left(\frac{T}{n}\right)^{\frac{3}{2}} \\
&+ \frac{C|x|}{(T-t_{n-2}^n)^{\frac{3}{2}}}\left(\frac{T}{n}\right)^2 + \frac{C|x|}{(T-t_{n-2}^n)^2}\left(\frac{T}{n}\right)^{\frac{5}{2}} + \frac{C|x|}{(T-t_{n-2}^n)^{\frac{5}{2}}}\left(\frac{T}{n}\right)^3.
\end{aligned}$$

Repeating the reasoning, given that  $\frac{1}{2}(k_{t_{i-1}}^{n-} + k_{t_{i-1}}^{n+}) = 1$ , we deduce by induction that, for every  $i \leq n-3$ ,

$$|\delta_{t_i}^n(x)| \leq C|x|\sqrt{\frac{T}{n}} + S_{i,n}^1 + S_{i,n}^2 + S_{i,n}^3 + S_{i,n}^4 + S_{i,n}^5,$$

where, for  $i \leq n - 3$ ,

$$\begin{aligned} S_{i,n}^1 &:= \sum_{j=i+1}^{n-2} \frac{C|x|}{(T - t_j^n)^{\frac{1}{2}}} \left(\frac{T}{n}\right)^{\frac{3}{2}}, & S_{i,n}^2 &= \sum_{j=i+1}^{n-2} \frac{C|x|}{T - t_j^n} \left(\frac{T}{n}\right)^{\frac{3}{2}}, \\ S_{i,n}^3 &= \sum_{j=i+1}^{n-2} \frac{C|x|}{(T - t_j^n)^{\frac{3}{2}}} \left(\frac{T}{n}\right)^2, & S_{i,n}^4 &= \sum_{j=i+1}^{n-2} \frac{C|x|}{(T - t_j^n)^2} \left(\frac{T}{n}\right)^{\frac{5}{2}}, \\ S_{i,n}^5 &= \sum_{j=i+1}^{n-2} \frac{C|x|}{(T - t_{n-2}^n)^{\frac{5}{2}}} \left(\frac{T}{n}\right)^3. \end{aligned}$$

Since  $(T - u)^{-1} \geq (T - t_j^n)^{-1}$  if  $u \in [t_j^n, t_{j+1}^n]$ , we deduce that

$$\begin{aligned} S_{i,n}^1 &\leq C|x| \sqrt{\frac{T}{n}} \int_0^T \frac{1}{\sqrt{T-t}} dt, \\ S_{i,n}^2 &\leq C|x| \sqrt{\frac{T}{n}} \int_0^{T-T/n} \frac{1}{T-t} dt \leq C|x| \ln(n) \sqrt{\frac{T}{n}}, \\ S_{i,n}^3 &\leq C|x| \frac{T}{n} \int_0^{T-T/n} \frac{1}{(T-t)^{3/2}} dt \leq C|x| \sqrt{\frac{T}{n}}, \\ S_{i,n}^4 &\leq C|x| \left(\frac{T}{n}\right)^{3/2} \int_0^{T-T/n} \frac{1}{(T-t)^2} dt \leq C|x| \sqrt{\frac{T}{n}}, \\ S_{i,n}^5 &\leq C|x| \left(\frac{T}{n}\right)^2 \int_0^{T-T/n} \frac{1}{(T-t)^{5/2}} dt \leq C|x| \sqrt{\frac{T}{n}}. \end{aligned}$$

Both with Lemma 4.1 and 4.2 and the inequalities above, we may conclude.

□

**Corollary 4.4.** *Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant  $C > 0$  such that*

$$\sup_{t \in [0, T]} |h^n(t, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{n}}.$$

*Proof.* Let  $t \in [0, T]$  be such that  $t \in [t_{i-1}^n, t_i^n]$  for some  $i \geq 1$ . By Lemma 4.1, we may suppose that  $i \leq n - 2$ . Then,

$$\begin{aligned} |h^n(t, x) - h(t, x)| &= |h^n(t_{i-1}^n, x) - h(t, x)| \\ &\leq |h^n(t_{i-1}^n, x) - h(t_{i-1}^n, x)| + |h(t_{i-1}^n, x) - h(t, x)|. \end{aligned}$$

By the mean value theorem and Inequality (4.4), since  $T - t_i^n \geq T/n$ , we deduce that

$$|h(t_{i-1}^n, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{T - t_i^n}} \Delta t_i^n \leq \frac{C|x|}{\sqrt{n}}.$$

We then conclude by Lemma 4.3.  $\square$

## 5. Appendix

Let us consider the unique solution  $S_{x,t}$ ,  $t \in [0, T]$ , to the stochastic differential equation

$$dS_{t,x}(u) = S_{t,x}(u)\sigma(u, S_{t,x}(u))dW_u, \quad u \in [t, T], \quad S_{t,x}(t) = x \in \mathbb{R}, \quad (5.10)$$

where  $W$  is a standard Brownian motion.

**Lemma 5.1.** *Suppose that  $t \in [0, T]$ . Let  $S_{x,t}$  be the solution to the stochastic differential equation (5.10). If  $\sigma$  is bounded by a constant  $\sigma^* > 0$ , there exists a constant  $C$  such that*

$$\mathbb{E} \sup_{t \leq u \leq T} S_{t,x}^2(u) \leq Cx^2, \quad \mathbb{E}(S_{t,x}(T) - x)^2 \leq Cx^2(T - t).$$

*Proof.* By the Doob's inequality, we obtain that for every  $t \leq r \leq T$  :

$$\phi(r) := \mathbb{E} \left| \sup_{t \leq u \leq r} S_{x,t}(u) \right|^2 \leq 4\mathbb{E} |S_{x,t}(r)|^2$$

As  $\hat{S}_{x,t}(r) = x + \int_t^r \sigma(u, S_{x,t}(u))S_{x,t}(u)dW_u$ , using the inequality  $(a + b)^2 \leq 2(a^2 + b^2)$  and the Ito isometry, we get that :

$$\begin{aligned} \phi(r) &\leq 8x^2 + 8\mathbb{E} \left( \int_t^r \sigma(u, S_{x,t}(u))S_{x,t}(u)dW_u \right)^2 \\ &\leq 8x^2 + 8\mathbb{E} \left( \int_t^r \sigma^2(u, S_{x,t}(u))S_{x,t}^2(u)du \right) \\ &\leq 8x^2 + 8(\sigma^*)^2 \int_t^r \mathbb{E}S_{x,t}^2(u)du \\ &\leq 8x^2 + 8(\sigma^*)^2 \int_t^r \phi(u)du. \end{aligned}$$

Applying the Gronwall lemma, we deduce that:

$$\mathbb{E} | \hat{S}_{x,t}(r) |^2 \leq 8x^2 \exp(8\sigma^*(T-t)) \leq Cx^2,$$

where  $C$  does not depend on  $x$ . By the Ito isometry, we then deduce that

$$\mathbb{E}(\hat{S}_{x,t}(T) - x)^2 = \mathbb{E} \int_t^T \sigma^2(u, S_{x,t}(u)) S_{x,t}^2(u) du$$

Using the inequality above, we may conclude.  $\square$

## References

- [1] Darses S., Lépinette E. Limit theorem for a modified Leland hedging strategy under constant transaction costs rate. *Inspired by Finance, The Musiela Festschrift*, Eds. Yu. Kabanov, M. Rutkowski, T. Zariphopoulou. Springer, 159-199 (2014), Springer.
- [2] Friedman A. *Stochastic Differential Equations and Applications*. Volume 1. *Academic Press* (1975).
- [3] Lépinette E., Tran T. Approximate hedging in a local volatility model with proportional transaction costs. *Applied Mathematical Finance*, 21 (2014), 4, 313-341.
- [4] Denis E. Marchés avec coûts de transaction: approximation de Leland et arbitrage. <http://indexation.univ-fcomte.fr/nuxeo/site/esupversions/8d5694b7-6409-413b-882a-37f2d7295a07>.