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ISOTOPES OF OCTONION ALGEBRAS AND TRIALITY

SEIDON ALSAODY AND PHILIPPE GILLE

Abstract. Octonion algebras over rings are, in contrast to those over fields, not determined by their norm forms. Octonion algebras whose norm is isometric to the norm q of a given algebra C are twisted forms of C by means of the $\mathbf{Aut}(C)$–torsor $\mathbf{O}(C) \to \mathbf{O}(q)/\mathbf{Aut}(C)$.

We show that, over any commutative unital ring, these twisted forms are precisely the isotopes $C^{a,b}$ of C, with multiplication given by $x * y = (xa)(by)$, for unit norm octonions $a, b \in C$. The link is provided by the triality phenomenon, which we study from new and classical perspectives. We then study these twisted forms using the interplay, thus obtained, between torsor geometry and isotope computations, thus obtaining new results on octonion algebras over e.g. rings of (Laurent) polynomials.

Keywords: Octonion algebras, isotopes, triality, homogeneous spaces, torsors.

MSC: 17D05, 14L30, 20G10.

1. Introduction

The aim of this paper is to give a concrete construction, over unital commutative rings, of all pairwise non-isomorphic octonion algebras having isometric norm forms. To this end we develop the framework of triality over rings. We begin by outlining our philosophy and intuition.

Let C be an octonion algebra over a unital, commutative ring R. In particular C is equipped with a multiplicative, regular quadratic form q. Then every automorphism of C is an isometry with respect to q, but the converse is far from being true. More precisely, the automorphism group of C is a 14-dimensional exceptional group scheme of type G_2, while the orthogonal group of C has dimension 28. Despite this, if R is a field or, more generally, a local ring, then every octonion algebra whose quadratic form is isometric to q is isomorphic, as an algebra, to C. This was proved false by the second author in [Gi2] over more general rings, using torsors and homotopy theory to study the homogeneous space $\mathbf{SO}(q)/\mathbf{Aut}(C)$.

This space measures, in a sense, the gap between the (special) orthogonal group of q and the automorphism group. Our initial observation in the current work is that any special orthogonal map with respect to q, while in general not an automorphism of C, is an isomorphism from C to a certain isotope $C^{a,b}$ with $a, b \in \mathbf{S}_C(R)$, the 7-dimensional unit sphere of C, that is the sphere for the octonionic norm on C. The algebra $C^{a,b}$ is simply the module C with multiplication $x * y = (xa)(by)$. As a and b run through $\mathbf{S}_C(R)$, we exhaust the special orthogonal group modulo the automorphism group. As the match in dimensions $(28 - 14 = 7 + 7)$ roughly indicates, these isotopes essentially account for the full gap between isometries and automorphisms.

The key to this observation is the phenomenon of *triality*. Well known over fields, it was extended to rings in [KPS]. We give another formulation of this phenomenon

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over rings, in a way more streamlined for our arguments. With this tool we are able to show that the isotopes above arise naturally as twists of C by torsors of the type considered in [Gi2]. Moreover we show that these isotopes account for *all* octonion algebras with isometric norms.

The paper is organized as follows. In Section 2, we introduce the isotopes of a given octonion R–algebra and recall background information on octonions and their isotopes. In Section 3, we prove using scheme theoretic arguments that the spin group of an octonion algebra over any unital commutative ring can be described in terms of triality and so called related triples. This enables us to give a precise description of the $\mathbf{Aut}(C)$–torsor $\Pi : \mathbf{Spin}(q) \to \mathbf{Spin}(q)/\mathbf{Aut}(C)$. For the sake of completeness we also give a concrete construction of the spin-triality correspondence by extending arguments valid over fields in a slightly different way than was done in [KPS].

Our main results are found in Sections 4 and 6. One the one hand, we show that the Π–twists of C correspond canonically to the isotopes $C^{a,b}$ (Theorem 4.6). On the other hand, in Theorem 6.6, we show that all octonion algebras with norm isometric to that of C are obtained in this way. This is done by showing that the torsor Π gives the same objects as the $\mathbf{O}(q) \to \mathbf{O}(q)/\mathbf{Aut}(C)$ (i.e. the corresponding cohomology kernels coincide). In the same theorem we obtain similar coincidences for a number of other torsors. We then relate our results to that of so-called compositions of quadratic forms.

Finally in Section 7 we discuss interesting special cases, such as rings over which all isotopes are isomorphic, and isotopes which are isomorphic over all rings.

We fix, once and for all, a unital commutative ring R. By an *R–ring* we mean a unital, commutative and associative R–algebra.

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2. Background

2.1. **Octonion algebras and isotopy.** Isotopy of algebras is a very general notion, which offers a way of deforming the multiplicative structure of an algebra. The notion goes back to Albert [Al]. We will outline it here in order to derive, from its generality, the proper context needed for our purposes in a self contained manner. The contents of this subsection can otherwise be found in [McC].

An *algebra* $A = (A, *) = (A, *_A)$ over R is an R–module A endowed with an R–bilinear multiplication $*$. In this generality, neither associativity nor commutativity nor the existence of a unity is required. For each $a \in A$ we write L_a or L_a^* for the linear map $x \mapsto a * x$ on A, and R_a or R_a^* for the linear map $x \mapsto x * a$. If A is alternative, then $L_a R_a = R_a L_a$ for any $a \in A$, and we denote this map by B_a.

An *octonion algebra* is an R-algebra whose underlying module is projective of constant rank 8, and which is endowed with a regular multiplicative quadratic form. (See also [LPR, §4] for further discussion.) Equivalently, it is a composition algebra of constant rank 8. (An algebra A is a *composition algebra* if the underlying module is finitely generated, faithfully projective, and if A is endowed with a non-singular multiplicative quadratic form $q = q_A$, referred to as the *norm* of A. *We moreover always require that composition algebras be unital.*) It is known that locally, composition algebras have rank 1, 2, 4 or 8.

Two arbitrary R–algebras $(A, *_A)$ and $(B, *_B)$ said to be *isotopic* or *isotopes of one another* if there exist invertible linear maps $f_i : A \to B$, $i = 1, 2, 3$, such that

$$f_1(x *_A y) = f_2(x) *_B f_3(y)$$

for all $x, y \in A$. The triple (f_1, f_2, f_3) is then called an *isotopy from A to B*. It is an easy known fact that isotopy is an equivalence relation. Then A and B are isomorphic as R–modules, but not necessarily as R–algebras. However, the map $f_1 : A \to B$ is an isomorphism of algebras between A and $(B, *'_B)$, where the new multiplication on B is defined by

$$x *'_B y = f_2 f_1^{-1}(x) *_B f_3 f_1^{-1}(y).$$

An algebra $(B, *')$ is said to be a *principal isotope* of $(B, *)$ if there exist invertible linear maps $g, h : B \to B$ such that $x *' y = g(x) * h(y)$ holds for all $x, y \in B$. Then we denote $(B, *')$ by $B_{g,h}$ and note that B and $B_{g,h}$ are isotopic. Conversely, the previous paragraph implies that two algebras A and B is isotopic if and only if A is isomorphic to a principal isotope of B. Thus restricting to principal isotopes induces no loss of generality

In this paper, we are mainly concerned with unital algebras. Let $(A, *)$ be an R–algebra and let g, h be invertible maps on A. Then $A_{g,h}$ is unital precisely when there exists $e \in A$ such that

$$g(x) * h(e) = g(e) * h(x) = x$$

for all $x \in A$. This is equivalent to the condition

$$g^{-1} = R^*_{h(e)} \quad \text{and} \quad h^{-1} = L^*_{g(e)}.$$

If A is moreover a unital alternative algebra, then by [McC, Proposition 2] the above implies that $g(e)$ and $h(e)$ are invertible, and denoting the inverses by b and a, respectively, the above condition becomes

$$g = R^*_a \quad \text{and} \quad h = L^*_b.$$

Conversely, for any $a, b \in A^*$, the algebra A_{R_a, L_b} is unital with unity $(ab)^{-1}$. In fact [McC] shows the following result.

Proposition 2.1. *Let C be a unital alternative algebra over R and let C' be isotopic to C. Then C' is unital if and only if $C' \simeq C_{R_a, L_b}$ for some $a, b \in C^*$.*

To lighten notation, we will henceforth write $A^{a,b}$ instead of A_{R_a, L_b}.

Remark 2.2. This construction is stable under base change, i.e. if $\phi : R \to S$ is a map of R–rings and $a, b \in \mathbf{S}_C(R)$, then $(C^{a,b})_S = (C_S)^{\phi(a), \phi(b)}$ follows from the bilinearity of the multiplication. Therefore, we may write $C_S^{a,b}$ without ambiguity.

Remark 2.3. If C is a composition algebra with quadratic form q, then $C' = C^{a,b}$ is a composition algebra as well, with norm $q' = \lambda q$, where $\lambda = q(a *_C b)$. Since $q \circ L_{a *_C b} = \lambda q = q'$, it follows that (C, q_C) and $(C', q_{C'})$ are isometric quadratic spaces. Note that this isometry maps 1_C to $1_{C'}$.

2.2. **Relations between isotopes.** As the previous subsection shows, if C is an octonion algebra, then any unital algebra isotopic to C is again an octonion algebra and is isomorphic to $C^{a,b}$ for some $a, b \in C^*$. We will simplify the presentation further by exhibiting some basic isomorphisms between certain isotopes. This is the content of the following result.

Proposition 2.4. *Let C be a unital alternative R–algebra and let $a, b \in C^*$.*

(1) *In the commutative diagram*

$$\begin{tikzcd} & C^{a,b} & \\ C^{1,aba} \arrow[ur, "L_a"] \arrow[rr, "R_{b^{-1}} L_a"'] & & C^{bab,1} \arrow[ul, "R_b"'] \end{tikzcd}$$

each arrow is an isomorphism of algebras.

(2) *In the commutative diagram*

$$\begin{tikzcd} & C^{a,a^{-1}} \arrow[dr, "R_a"] & \\ C^{1,a} \arrow[ur, "L_a"] \arrow[rr, "B_a"'] & & C^{a^{-1},1} \end{tikzcd}$$

each arrow is an isomorphism of algebras.

(3) *The maps* $B_a : C^{a,b} \to C^{1,ba^{-1}}$ *and* $B_b : C^{a,b} \to C^{b^{-1}a,1}$ *are isomorphisms of algebras. In particular* $C^{a,a} \simeq C$.

(4) *The maps* $B_{ba^{-1}}B_a : C^{a,b} \to C^{ab^{-1},1}$ *and* $B_{b^{-1}a}B_b : C^{a,b} \to C^{1,a^{-1}b}$ *are isomorphisms of algebras.*

Proof. All map are clearly invertible and (1) was established by McCrimmon [McC, prop. 6]. Assertion (2) is the special case $b = a^{-1}$ of (1). For, (3) We check the first fact, the second being similar. Given $x, y \in C$, we have

$$\begin{aligned} B_a(x *_{a,b} y) &= B_a((xa)(by)) \\ &= (a(xa))\,((by)a) \quad \text{[middle Moufang law]} \\ &= (a(xa))\,\big(((ba^{-1})a)y)a\big) \\ &= (a(xa))\,\big((ba^{-1})(a(ya))\big) \quad \text{[one-sided Moufang law]} \\ &= B_a(x) *_{1,ba^{-1}} B_a(y). \end{aligned}$$

Finally, (4) follows by combining (2) and (3). □

Remark 2.5. Knus–Parimala–Sridharan introduced another kind of isotope [KPS, Rem. 4.7]. For an R–alternative algebra C and an element $u \in C^*$, they defined the new multiplication $x \star_u y = (x(yu))u^{-1}$ on C. We leave it to the reader to check that the map $R_u : C \to C$ induces an isomorphism of R–algebras $R_u : (C, \star_u) \xrightarrow{\sim} (C, *_{u^{-1},1}) = C^{u^{-1},1} \cong C^{1,u}$.

Remark 2.6. The above proposition in fact implies that if C is a composition algebra, then any composition algebra C' isotopic to C is isomorphic to e.g. $C^{1,a}$ with $q(a) = 1$. Indeed, combining items (2) and (3) of the proposition, we have

$$C' \simeq C^{c^{-1},c} \simeq C^{1,c^2}.$$

As multiplication by $q(c)$ is an isomorphism $C^{1,c^2} \to C^{1,q(c)^{-1}c^2}$, the claim follows.

These relations will be explained partly by triality in Section 5.2.

2.3. **Direct summands of octonion algebras.** As for Azumaya algebras, we recall the following fact, which is of course obvious if 2 belongs to R^*.

Lemma 2.7. *The trace map* $\mathrm{tr} : C \to R$ *admits a section.*

Proof. By means of partition of unity, we can assume that R is local with maximal ideal $\mathfrak{M}$. We know that $\mathrm{tr}(c) = b_q(1, c)$ for each $c \in C$. Since b_q is a non-degenerate bilinear form, there exists $c_K \in K = R/\mathfrak{M}$ such that $\mathrm{tr}(c_K) \neq 0$. Let c be a lift of c_K in $C_{R/\mathfrak{M}}$. Then $\mathrm{tr}(c) \in R^*$ so that $\mathrm{tr} : C \to R$ is surjective. Thus $\mathrm{tr} : C \to R$ admits a section. □

Let us recall also the next fact.

Lemma 2.8. *We have*

$$R1 = \Big\{x \in C \mid xy = yx \ \ \forall y \in C\Big\} = \Big\{x \in C \mid y(xz) = (yx)z \ \ \forall y, z \in C\Big\}.$$

Proof. We prove it for the right hand side (denoted by M) and the middle term can be done analogously. Let $(c_i)_{i=1,\dots,n}$ be a generating family of the R–module C. By bilinearity of the product, we have

$$M = \Big\{ x \in C \mid c_i(xc_j) = (c_i x)c_j \ \ \forall i,j = 1,\dots,n \Big\}.$$

By using the classical trick to write R as the limit of its finitely generated $\mathbb{Z}$-algebras, we may assume that R is noetherian, so that M is finitely presented.

Let $\mathfrak{M}$ be a maximal ideal of R. By the corresponding result over fields [SV, Proposition 1.9.2], we have $R/\mathfrak{M} = M \otimes_R (R/\mathfrak{M}) \cong M/\mathfrak{M}M$. Nakayama's lemma shows that $R_{\mathfrak{M}} = M \otimes_R R_{\mathfrak{M}}$. Since it holds for all maximal ideals, we conclude that $R = M$ as desired. □

For the next lemma we recall the identity $b_q(x, ay) = b_q(\overline{a}x, y)$, which implies that

$$\{y \in C \mid b_q(ay, x) = 0\} = (\overline{a}x)^{\perp}$$

for all $a, x \in C$. Here $\perp$ denotes the orthogonal complement with respect to b_q.

Lemma 2.9. *Let $x \in C$ be an invertible element and let $a \in C$ be an element of trace 1.*[1] *Then*

(1) $C = Rx \oplus (\overline{ax})^{\perp}$, *and*
(2) $x^{\perp}$ *is a direct summand of C which is locally free of rank 7. Furthermore the restriction q_x of q to $x^{\perp}$ is a non-singular quadratic form.*

Proof. (1) For each $z \in C$, the element

$$z - q(x)^{-1} b_q(az, x)x$$

is orthogonal to $\overline{a}x$, whence $C = Rx + (\overline{a}x)^{\perp}$. The sum is direct since $b_q(x, \overline{a}x) = q(x) \in R^*$, so that $\lambda x \in (\overline{a}x)^{\perp}$ only holds with $\lambda = 0$.

(2) Again we can assume that R is local with maximal ideal $\mathfrak{M}$. Since b_q is non-degenerate, there exists $y_K \in K = C_{R/\mathfrak{M}}$ such that $b_q(x_K, y_K) \neq 0 \in K$. We lift y_K to an element $y \in C$ and we have $b_q(x, y) \in R^*$. It follows that $C = C_x \oplus Ry$.

To show that q_x is a non-singular quadratic form it suffices to consider the case when $R = k$ is an algebraically closed field. We consider the radical

$$\mathrm{rad}(q_x) = \big\{ z \in x^{\perp} \mid q(z) = b_q(z, x^{\perp}) = 0 \big\}.$$

Since b_q is regular, $(x^{\perp})^{\perp}$ is of dimension 1, whence it is equal to kx. But $q(x) \neq 0$, whence $\mathrm{rad}(q_x) = \{0\}$ in any characteristic, and q_x is non-singular. □

Remark 2.10. If $2 \in R^*$, then $a = \frac{1}{2}$ works, and $C = Rx \oplus (\frac{1}{2}x)^{\perp} = Rx \oplus x^{\perp}$.

We denote by $\mathbb{P}(C^{\vee})$ the projective space attached to $C^{\vee} = \mathrm{Hom}_R(C, R)$ using EGA's conventions. For each R–ring S, the set $\mathbb{P}(C^{\vee})(S)$ is that of invertible S–submodules of M_S which are locally direct summands of C_S. Each point a of $\mathbf{S}_q(S)$ defines the free S–submodule Sa of C_S. Since a is an invertible octonion, Sa is a direct summand of C by Lemma 2.9. We have thus defined an R–map $u : \mathbf{S}_C \to \mathbb{P}(C^{\vee})$. It is $\boldsymbol{\mu}_2$–invariant, thus it induces an R–map $\underline{u} : \underline{\mathbf{S}}_C \to \mathbb{P}(C^{\vee})$.[2]

Lemma 2.11. *The R–map $\underline{u} : \underline{\mathbf{S}}_C \to \mathbb{P}(C^{\vee})$ is an open R–immersion. The projective R–quadric $Q_C = \{q_C = 0\} \subset \mathbb{P}(C^{\vee})$ is a closed complement of the image of $\underline{u}$.*

[1] Such an element exists by Lemma 2.7.
[2] The quotient $\underline{\mathbf{S}}_C$ is defined and discussed in the appendix.

Proof. Let us first show that $\underline{u}$ is a monomorphism. We are given an R–ring S and $\underline{x}, \underline{y} \in \underline{\mathbf{S}}_C(S)$ such that $\underline{u}(\underline{x}) = \underline{u}(\underline{y})$ in $\mathbb{P}(C^\vee)(S)$. Let S' be a flat R–cover of S such that $\underline{x}_{S'}$ (resp. $\underline{y}_{S'}$) lifts to an S'–point $x' \in \underline{\mathbf{S}}_C(S')$ (resp. $y' \in \underline{\mathbf{S}}_C(S')$. We then have $S'x' = S'y'$. Then $x' = \lambda y'$ for $\lambda \in (S')^*$. By taking the norms we get $1 = \lambda^2$, i.e. $\lambda \in \boldsymbol{\mu}_2(S')$. It follows that $\underline{x}_{S'} = \underline{y}_{S'} \in \underline{\mathbf{S}}_C(S')$ and thus $\underline{x} = \underline{y} \in \underline{\mathbf{S}}_C(S)$.

The R–morphism $\underline{u}$ is thus a monomorphism from a flat R–scheme to a smooth R–scheme and such that fibres over S are of same dimension 7. According to [EGAIV, $_4$.18.10.5], $\underline{u}$ is an open immersion.

For the last fact it is enough to check that $Q_C(k) \sqcup \underline{\mathbf{S}}_C(k) = \mathbb{P}(C^\vee)(k)$ for each R–field k which is algebraically closed. In this case $\underline{\mathbf{S}}_C(k) = \mathbf{S}_C(k)/\boldsymbol{\mu}_2(k)$ and it is clear that $\emptyset = Q_C(k) \cap \underline{\mathbf{S}}_C(k)$. Further, a point of $\mathbb{P}(C^\vee)(k)$ is a line $L = kx$ in $C \otimes_R k$. If $q_C(x) = 0$, it follows that $[kx] \in Q_C(k)$. If $q_C(x) \neq 0$, we may write $q_C(x) = a^{-2}$ with $a \in k^*$, and then $L = [k(ax)] \in \underline{\mathbf{S}}_C(k)$. □

2.4. **A result of Knus–Parimala–Sridharan.** As in Remark 2.3, given a composition algebra C' whose norm is isometric to λq_C, we can always find an isometry $(C, q_C) \cong (C', q_{C'})$ mapping 1_C to $1_{C'}$. In the following special case, Knus, Parimala and Sridharan noticed that another kind of isotopes describes such objects.

Proposition 2.12. [KPS, Remark 4.7] *Assume that R is a connected ring and that* $\mathrm{Pic}(R) = 0$. *If C, C' are octonion R–algebras such that there exists an isometry $f : (C, q_C) \xrightarrow{\sim} (C', q_{C'})$ with $f(1_C) = 1_{C'}$, then there exists $u \in C^*$ such that $f(x) *_{C'} f(y) = f\Big((x(yu))u^{-1}\Big)$.*

According to Remark 2.5, this algebra C' is isomorphic to $C^{u^{-1},u} \cong C^{1,u}$, hence is an isotope. Their work, as ours, involves the triality phenomenon, and this statement is in some sense the starting point of the present paper. However, as will be seen below, our approach is independent of theirs, and our point of view via torsors is of a different flavour. Moreover, our take on triality differs as well. A consequence is the avoidance of discriminant modules, which lightens the burden of necessary technicalities and enables us to formulate all results over general rings and without reference to the Picard group.

3. Triality

3.1. **Framework and properties.** Let henceforth C always denote an octonion R–algebra. In particular, C is endowed with a regular, multiplicative quadratic form $q = q_C$ and a natural involution $\kappa : x \mapsto \overline{x}$. We denote by $\mathbf{Aut}(C)$ its automorphism group scheme, which is a semisimple R–group scheme of type G_2 and is a closed R–subgroup scheme of the linear R–group scheme $\mathbf{GL}(C)$. Let further $\mathbf{SO}(q_C)$ be the special orthogonal group scheme [CF, §8.4] associated to q_C. This is a semisimple R–group scheme of type D_4, and the embedding $\mathbf{Aut}(C) \to \mathbf{GL}(C)$ induces a closed immersion $\mathbf{Aut}(C) \to \mathbf{SO}(q_C)$.

We denote by $\mathbf{RT}(C)$ the closed R–subscheme of $\mathbf{SO}(q_C)^3$ such that for each R–algebra S, we have

$$\mathbf{RT}(C)(S) = \Big\{(t_1, t_2, t_3) \in \mathbf{SO}(q)(S)^3 \,|\, t_{1T}(xy) = \overline{t_{2T}(\overline{x})}\ \overline{t_{3T}(\overline{y})} \\ \text{for any } S\text{–ring } T \text{ and any } x, y \in C_T\Big\}.$$

Remark 3.1. Since the multiplication in C_T is T–bilinear, it follows that

$$\mathbf{RT}(C)(S) = \Big\{(t_1, t_2, t_3) \in \mathbf{SO}(q)(S)^3 \,|\, \forall x, y \in C_S : t_1(xy) = \overline{t_2(\overline{x})}\ \overline{t_3(\overline{y})}\Big\}.$$

Moreover, if $(t_1, t_2, t_3) \in \mathbf{O}(q)(S)^3$ satisfies the above identity, then one can show that $(t_1, t_2, t_3) \in \mathbf{SO}(q)(S)^3$ and thus $(t_1, t_2, t_3) \in \mathbf{RT}(q)(S)$. We will use and show this later on.

An element of $\mathbf{RT}(C)(S)$ for an R–ring S is called a *related triple*. In fact, $\mathbf{RT}(C)$ is a closed R–subgroup scheme of $\mathbf{SO}(q_C)^3$ (i.e. related triples can be multiplied component-wise).

The definition of related triples here differs from that of [SV] and [KPS], where instead the condition is

$$t_1(xy) = t_2(x)t_3(y).$$

(I.e. (t_1, t_2, t_3) is an isotopy from C_S to itself, in the literature sometimes called an *autotopy*.) As defined here, a triple is related if and only if it satisfies

$$t_1(x \bullet y) = t_2(x) \bullet t_3(y),$$

i.e. it is an isotopy with respect to the *para-octonion* multiplication $\bullet$ on C, defined by $x \bullet y = \overline{x}\ \overline{y}$. The (non-unital) composition algebra $(C, \bullet)$ is an instance of what is over fields known as a *symmetric composition algebra*. This is the approach taken in [KMRT]. The following lemma, extended from [E], shows that the subgroup A_3 of S_3 acts on $\mathbf{RT}(C)$ by permuting the t_i's. The fact that the action of A_3 takes this simple form is the main advantage of using this version of the definition.

Lemma 3.2. *Let S be an R–ring. For $(t_1, t_2, t_3) \in \mathbf{SO}(q_C)(S)^3$, the following are equivalent.*

(1) $(t_1, t_2, t_3) \in \mathbf{RT}(C)(S)$.
(2) $(t_2, t_3, t_1) \in \mathbf{RT}(C)(S)$.
(3) $(t_3, t_1, t_2) \in \mathbf{RT}(C)(S)$.
(4) *The trilinear form Δ_q defined on C_S by $\Delta_q(w, x, y) = b_q(w, \overline{x}\ \overline{y})$ satisfies*

$$\Delta_q(t_1(w), t_2(x), t_3(y)) = \Delta_q(w, x, y)$$

for all $w, x, y \in C_S$.

Proof. It is readily verified that $\Delta_q(w, x, y) = \Delta_q(x, y, w) = \Delta_q(y, w, x)$. Thus Δ_q is invariant under cyclic permutations, and it suffices to prove the equivalence of the first and last statement. Since t_1 is orthogonal, the last statement is equivalent to

$$\forall w, x, y \in C : b_q(t_1(w), \overline{t_2(x)}\ \overline{t_3(y)}) = b_q(t_1(w), t_1(\overline{x}\ \overline{y})),$$

which by regularity of q is equivalent to the first statement. □

Example 3.3. If $c \in C$ satisfies $q(c) = 1$, then the middle Moufang identity $c(xy)c = (cx)(yc)$ implies that $(B_c, R_{\overline{c}}, L_{\overline{c}}) \in \mathbf{RT}(C)(R)$. By Lemma 3.2, the same holds for $(R_{\overline{c}}, L_{\overline{c}}, B_c)$ and $(L_{\overline{c}}, B_c, R_{\overline{c}})$. These, in a sense, basic triples will be useful in the sequel.

The following lemma provides other examples of related triples.

Lemma 3.4. *Let S be an R–ring and $t \in \mathbf{SO}(q_C)(S)$. Then $(t, t, t) \in \mathbf{RT}(C)(S)$ if and only if $t \in \mathbf{Aut}(C)(S)$.*

Proof. If $t \in \mathbf{Aut}(C)(S)$, then t commutes with the involution of C_S and satisfies $t(xy) = t(x)t(y)$ for all $x, y \in C_S$. From this it follows that $(t, t, t) \in \mathbf{RT}(C)(S)$.

Conversely, assume that $(t, t, t) \in \mathbf{RT}(C)(S)$. To show that $t \in \mathbf{Aut}(C)(S)$ it suffices to show that t commutes with the involution. By definition of a related triple we have

$$t(\overline{x}) = \overline{t(x)}\ \overline{t(1)} = \overline{t(1)}\ \overline{t(x)}$$

for all $x \in C_S$. Surjectivity of t implies that $\overline{t(1)}$ belongs to the centre $S1$ of C_S, whereby $\overline{t(1)} = \alpha 1 = t(1)$ for some $\alpha \in S$. Since t is an isometry we get $\alpha^2 = 1$. But then

$$\alpha = t(1) = t(1)t(1) = \alpha^2 = 1.$$

Thus t fixes 1 and by the above we get $t(\overline{x}) = \overline{t(x)}$ for all $x \in C_S$, as required. □

Thus $\mathbf{Aut}(C)$ embeds in $\mathbf{RT}(C)$ by means of the diagonal mapping, and we denote by $i : \mathbf{Aut}(C) \to \mathbf{RT}(C)$ that closed embedding, which we will henceforth view as an inclusion. We also define $f_i : \mathbf{RT}(C) \to \mathbf{SO}(C)$ as the projection on the ith component, for $i = 1, 2, 3$.

Proposition 3.5. $\mathbf{Aut}(C) \simeq \mathbf{RT}(C)^{A_3} \simeq \mathbf{RT}(C)^{S_3}$.

Proof. By the above lemma, the embedding i maps $\mathbf{Aut}(C)$ into $\mathbf{RT}(C)^{A_3}$, and the projection f_1 maps $\mathbf{RT}(C)^{A_3}$ into $\mathbf{Aut}(C)$ and satisfies $f_1 \circ i = \mathrm{Id}_{\mathbf{Aut}(C)}$ and $i \circ f_1 = \mathrm{Id}_{\mathbf{RT}(C)^{A_3}}$. The first isomorphism follows as both these maps are functorial, and the second holds as all elements of S_3 act trivially on the image of i. □

The centre of $\mathbf{SO}(q_C)^3$ is $\boldsymbol{\mu}_2^3$ and its intersection with $\mathbf{RT}(C)$ is

$$\boldsymbol{\mu} := \ker(\boldsymbol{\mu}_2^3 \xrightarrow{\text{mult}} \boldsymbol{\mu}_2).$$

As the following proposition shows, related triples are determined, modulo $\boldsymbol{\mu}_2$, by any of their components.

Lemma 3.6. *(1) Let $(t_1, t_2, t_3) \in \mathbf{RT}(C)(R)$ with $t_1 = \mathrm{Id}$. Then $t_2 = t_3 = \eta\mathrm{Id}$ for $\eta \in \boldsymbol{\mu}_2(R)$.*
(2) The kernel of f_1 is isomorphic to $\boldsymbol{\mu}_2$ embedded in $\boldsymbol{\mu}$ by $\eta \mapsto (1, \eta, \eta)$ and similarly for f_2 and f_3.

Proof. (1) The proof is inspired by [E]. Let $(t_1, t_2, t_3) \in \mathbf{RT}(R)(S)$ with $t_1 = \mathrm{Id}$. Then

$$\overline{t_2(x)}\ \overline{t_3(y)} = \overline{x}\ \overline{y},$$

or equivalently

$$t_3(y)t_2(x) = yx$$

for any $x, y \in C_S$, whereby

$$zt_2(x) = t_3^{-1}(z)x$$

for any $x, z \in C_S$. Setting $z = 1$ gives $t_2 = L_c$ with $c = t_3^{-1}(1)$, and setting $x = 1$ gives $t_3^{-1} = R_d$ with $d = t_2(1)$. Thus in general

$$z(cx) = (zd)x$$

for any $x, z \in C$. In particular with $x = z = 1$ we conclude that $d = c$, and then $z(cx) = (zc)x$ for all $x, z \in C$. But such an associativity relation only holds if $c = \eta$ for some $\eta \in R1$ (Lemma 2.8). Thus $t_2 = t_3^{-1} = \eta\mathrm{Id}$. Since e.g. t_2 is an isometry, this gives

$$1 = q(t_2(1)) = q(\eta 1) = \eta^2,$$

whence the claim.
(2) The first statement shows that the kernel is contained in $\boldsymbol{\mu}_2$. Conversely, if $t_2 = \eta\mathrm{Id}$ and $t_3 = \eta\mathrm{Id}$ with $\eta^2 = 1$, then for all $x \in C$,

$$t_1(x) = t_1(x1) = \overline{t_2(\overline{x})}\ \overline{t_3(1)} = \eta^2 x1 = x,$$

whereby the proof is complete. □

Given this, we observe the following addition to the above discussion on automorphisms.

Proposition 3.7. *A related triple* $(t_1, t_2, t_3) \in \mathbf{RT}(C)(S)$ *satisfies* $t_3(1) = t_2(1) \in \boldsymbol{\mu}_2(S)1$ *if and only if* $t_1 \in \mathbf{Aut}(C_S)$. *Moreover* $(t_1, t_2, t_3) \in \mathbf{RT}(C)(S)$ *satisfies* $t_3(1) = t_2(1) = 1$ *if and only if* $(t_1, t_2, t_3) \in i_S(\mathbf{Aut}(C)(S))$.

Proof. Assume that t_1 is an automorphism. Then by Lemma 3.4, (t_1, t_1, t_1) is a related triple, and by Lemma 3.6 we have $t_2 = t_3 = \eta t_1$ for some $\eta \in \boldsymbol{\mu}_2(S)$, whereby $t_3(1) = t_2(1) \in \boldsymbol{\mu}_2(S)1$. Conversely assume that $t_3(1) = t_2(1) = \eta 1$ with $\eta \in \boldsymbol{\mu}_2(S)$. Then for all $x \in C_S$,

$$t_1(x) = t_1(x1) = \overline{t_2(\overline{x})}\ \overline{t_3(1)} = \eta\overline{t_2(\overline{x})},$$

whereby $t_1 = \eta\kappa t_2\kappa$. Similarly one concludes that $t_1 = \eta\kappa t_3\kappa$. Thus for all $x, y \in C_S$,

$$t_1(x)t_1(y) = \eta^2\kappa t_2\kappa(x)\kappa t_3\kappa(y) = \overline{t_2(\overline{x})}\ \overline{t_3(\overline{y})} = t_1(xy),$$

Thus t_1 is an automorphism and $t_2 = t_3 = \eta t_1$. This proves the first statement and, with $\eta = 1$, the only if-direction of the second. The other direction is clear. □

Theorem 3.8. *The* R*–group scheme* $\mathbf{RT}(C)$ *is semisimple simply connected of type* D_4. *Moreover, for any* $i = 1, 2, 3$, *the mapping* $f_i : \mathbf{RT}(C) \to \mathbf{SO}(q_C)$ *is the universal cover of* $\mathbf{SO}(q_C)$.

The term *universal cover* is understood in the sense of SGA3; see [Co, 6.5.2].

Proof. We start with the first statement. Let C_0 be the Zorn octonion algebra over $\mathbb{Z}$. We know that there exists a flat cover (that is a finitely presented faithfully flat R-ring S) such that $C \otimes_R S \simeq C_0 \otimes_{\mathbb{Z}} S$ [LPR, Cor. 4.1]. It follows that $\mathbf{RT}(C) \times_R S \simeq \mathbf{RT}(C_0) \times_{\mathbb{Z}} S$ and the statement boils down to the case of C_0 over $\mathbb{Z}$.

According to [SV, Prop. 3.6.3], $\mathbf{RT}(C_0) \times_{\mathbb{Z}} \mathbb{Q}$ and $\mathbf{RT}(C_0) \times_{\mathbb{Z}} \mathbb{F}_p$ (p prime) are semisimple simply connected algebraic groups of type D_4, in particular smooth and connected. Lemma B.1 then shows that $\mathbf{RT}(C_0)$ is smooth over $\mathbb{Z}$. The affine smooth $\mathbb{Z}$–group scheme $\mathbf{RT}(C_0)$ thus has geometric fibres which are semisimple simply connected, whence by definition, $\mathbf{RT}(C_0)$ is a semisimple simply connected $\mathbb{Z}$–group scheme.

For the second statement, it is enough to deal with f_1. According to Lemma 3.6, $\boldsymbol{\mu}_2 = \ker(f_1)$ and is a central subgroup of $\mathbf{RT}(C)$. We denote by $\underline{\mathbf{RT}}(C) = \mathbf{RT}(C)/\boldsymbol{\mu}_2$ the quotient [SGA3, XXII.4.3.1]; it is a semisimple R–group scheme as well. Then the induced map $\underline{f_1} : \underline{\mathbf{RT}}(C) \to \mathbf{SO}(q_C)$ has trivial kernel. For each point $s \in \mathrm{Spec}(R)$, $\underline{f_1}_s : \underline{\mathbf{RT}}(C)_s \to \mathbf{SO}(q_C)_s$ is a closed immersion between smooth connected algebraic $\kappa(s)$–groups of the same relative dimension 28. Hence $\underline{f_1}_s$ is an isomorphism. Since $\underline{\mathbf{RT}}(C)$ is flat over R, the fibre-wise isomorphism criterion [EGAIV, $_4$.17.9.5] enables us to conclude that $\underline{f_1}$ is an isomorphism. Then f_1 is a central isogeny and since $\mathbf{RT}(C)$ is semisimple simply connected, it follows that it is the simply connected cover of the semisimple R–group scheme $\mathbf{SO}(q_C)$. □

On the other hand, we have the R–cover $\chi : \mathbf{Spin}(q_C) \to \mathbf{SO}(q_C)$ [CF, 8.4.0.63]. It follows that there is a unique R–isomorphism $F : \mathbf{Spin}(q_C) \xrightarrow{\sim} \mathbf{RT}(C)$ such that $f_1 \circ F = \chi$. A construction of this isomorphism will be given in the next subsection.

3.2. **Explicit construction.** For coherence, and in order to relate our work to its predecessors, we will give a constructive proof that the spin group of (C, q) is isomorphic to the group of related triples on C. A closely related version can be found in [KPS]. We closely follow the approach of [KMRT] (who work over fields with $2 \neq 0$), and [E] (who considers fields of any characteristic). It is independent of the proof given in the previous section. (Conversely, the remainder of the paper is independent of this section.)

To this end, we write $\mathrm{Cl}(C,q)$ for the Clifford algebra of the quadratic space (C,q), and $\mathrm{Cl}_0(C,q)$ for its even part. Multiplication in $\mathrm{Cl}(C,q)$ will always be denoted by $\cdot$. Consider the R–group scheme $\mathbf{Spin}(q)$ defined, for each R–ring S, by[3]

$$\mathbf{Spin}(q)(S) = \{u \in \mathrm{Cl}_0(C,q)^*_S \,|\, u \cdot C_S \cdot u^{-1} = C_S \text{ and } u\sigma(u) = 1\}.$$

We also introduce the R–group scheme $\mathbf{RT}'(C)$, defined by

$$\mathbf{RT}'(C)(S) = \left\{(t_1,t_2,t_3) \in \mathbf{O}(q)(S)^3 \,|\, \forall x,y \in C_S : t_1(xy) = \overline{t_2(\overline{x})}\ \overline{t_3(\overline{y})}\right\}.$$

Similarly to $\mathbf{RT}(C)(S)$, this is a subgroup scheme of $\mathbf{O}(q)^3$. The reason for introducing $\mathbf{RT}'(C)$ is merely for technical convenience; we will later show that in fact $\mathbf{RT}'(C) = \mathbf{RT}(C)$.

Recall from [K, IV.1.2.3] that for any R–ring S, the map $\mathrm{incl} \otimes \mathrm{Id} : C \otimes S \to \mathrm{Cl}(C,q) \otimes S$ induces an isomorphism of S–algebras

$$\mathrm{Cl}(C \otimes S, q \otimes 1) \simeq \mathrm{Cl}(C,q) \otimes S,$$

which obviously preserves the grading and which we shall view as an identification.

Proposition 3.9. *Let S be a unital commutative ring and C be an octonion algebra over S with quadratic form q. The map $\alpha : C \to \mathrm{End}_S(C \oplus C)$ defined by*

$$x \mapsto \left(\begin{smallmatrix} 0 & L_x \\ L_{\overline{x}} & 0 \end{smallmatrix}\right)$$

induces isomorphisms

$$\alpha' : (\mathrm{Cl}(C,q),\sigma) \to (\mathrm{End}_S(C \oplus C), \sigma_{q\perp q})$$

and

$$\alpha'' : (\mathrm{Cl}_0(C,q),\sigma) \to (\mathrm{End}_S(C),\sigma_q) \times (\mathrm{End}_R(C),\sigma_q)$$

of algebras with involution. Moreover α' is a morphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras with respect to the Clifford and chequerboard gradings.

Proof. The map α is linear and satisfies

$$\alpha(x)^2 = \left(\begin{smallmatrix} L_x L_{\overline{x}} & 0 \\ 0 & L_{\overline{x}} L_x \end{smallmatrix}\right) = q(x)\mathrm{Id},$$

whereby it induces a homomorphism α' as above by the universal property of Clifford algebras. The source of α' is, by [K, IV.2.2.3] and [IV.1.5.2], an Azumaya algebra of constant rank 2^8. Since C is faithfully projective, the target is an Azumaya algebra of constant rank $16^2 = 2^8$. Thus by [K, III.5.1.18], the map α' is an isomorphism of algebras. It respects the grading by construction, from which follows that α'' is an isomorphism as well. To show that α' and α'' are isomorphisms of algebras with involution, it suffices to show that $\alpha'(\sigma(x)) = \sigma_{q\perp q}(\alpha'(x))$ for all $x \in C$. But then $\sigma(x) = x$, and, setting $A = \alpha'(x)$, we have, for any $a,b,c,d \in C$,

$$b_{q\perp q}\left(A\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)\right) = b_q(xb,c) + b_q(\overline{x}a,d) = b_q(b,\overline{x}c) + b_q(a,xd) = b_{q\perp q}\left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), A\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)\right),$$

where the middle equality follows from the identity $b_q(zv,w) = b_q(v,\overline{z}w)$. Thus $A = \sigma_{q\perp q}(A)$ as required. □

Let now C be an octonion algebra over R. The morphism

$$\chi : \mathbf{Spin}(q_C) \to \mathbf{O}(q_C)$$

given, for any R–ring S and $u \in \mathbf{Spin}(q_C)(S)$, by $u \mapsto u_1$, where for all $x \in C_S$, $u_1 : x \mapsto u \cdot x \cdot u^{-1}$, maps $\mathbf{Spin}(q_C)$ inside $\mathbf{SO}(q_C)$ [K, IV.6]. We moreover define u_2, u_3 by

$$\alpha'(u) = \left(\begin{smallmatrix} u_2 & 0 \\ 0 & u_3 \end{smallmatrix}\right),$$

[3]The condition $u \cdot C_S \cdot u^{-1} = C_S$ implies, as an easy calculation shows, that the map $x \mapsto u \cdot x \cdot u^{-1}$ is an orthogonal operator on C_S. In [K] this latter property is put into the definition.

noting that they are invertible linear operators on C_S as well. We will write $u_i(S)$ for u_i, $i = 1, 2, 3$, whenever the ring S needs to be emphasized.

Theorem 3.10. *Let C be an octonion algebra over R. The map*

$$F : \mathbf{Spin}(q_C) \to \mathbf{RT}'(C),$$

defined, for each R–ring S and for each $u \in \mathbf{Spin}(q_C)(S)$, by

$$F_S(u) = (u_1(S), \kappa u_2(S)\kappa, u_3(S)),$$

is an isomorphism of affine group schemes.

The following basic lemma, extended from [E], will be useful.

Lemma 3.11. *Let (M, q) be a regular quadratic module over a ring S, and assume that f is a similitude with multiplier $\lambda \in R$, i.e. $q(f(x)) = \lambda q(x)$ for all $x \in M$, and moreover satisfies $b_q(f(x), f(y)) = b_q(x, y)$ for all $x, y \in M$. Then $\lambda = 1$.*

Proof. We have

$$b_q(x, y) = b_q(f(x), f(y)) = q(f(x) + f(y)) - q(f(x)) - q(f(y))$$

which, by the hypothesis on f, equals

$$\lambda q(x + y) - \lambda q(x) - \lambda q(y) = \lambda b_q(x, y)$$

for all $x, y \in M$. Thus $b_q^\vee(x) = \lambda b_q^\vee(x) \in M^*$ for all $x \in M$, whereby $b_q^\vee = \lambda b_q^\vee$, and, being invertible

$$\mathrm{Id}_{M^*} = \lambda b_q^\vee (b_q^\vee)^{-1} = \lambda \mathrm{Id},$$

whereby $\lambda = 1$. □

Proof of Theorem 3.10. Fix an R–ring S. We begin by showing that the map F_S is well defined. To begin with, $u_i(S) \in \mathbf{O}(q)(S)$ for each i. For $i = 1$, this holds by construction. For $i = 2, 3$, the fact that α'' is a morphism of algebras of involution, together with $\sigma(u) = u^{-1}$, implies that

$$b_q(u_i(x), u_i(y)) = b_q(\sigma(u)_i u_i(x), y) = b_q((u_i)^{-1} u_i(x), y) = b_q(x, y). \tag{3.1}$$

Arguing as in [E], we note that for any $x \in C$,

$$\alpha'(u)\alpha'(x) = \alpha'(u \cdot x \cdot u^{-1} \cdot u) = \alpha'(u_1 x)\alpha'(u).$$

Inserting the definition of α' into this equality yields

$$u_2 L_x = L_{u_1(x)} u_3 \quad \text{and} \quad u_3 L_{\overline{x}} = L_{\overline{u_1(x)}} u_2, \tag{3.2}$$

and applying both sides of both equations to $1 \in C$ one obtains (substituting x for $\overline{x}$ in the second)

$$u_2(x) = u_1(x) u_3(1) \quad \text{and} \quad u_3(x) = \overline{u_1(\overline{x})} u_2(1).$$

Now using the fact that u_1 and $y \mapsto \overline{y}$ are isometries of the multiplicative quadratic form q on C, this gives

$$q(u_2(x)) = q(u_3(1))q(x) \quad \text{and} \quad q(u_3(x)) = q(u_2(1))q(x).$$

With (3.1) and the above lemma we conclude that u_2 and u_3 are isometries. Since κ is an isometry, we have $F_S(u) \in \mathbf{O}(q)(S)^3$. To show that $F_S(u) \in \mathbf{RT}'(C)(S)$, we apply the second equation of (3.2) to $\overline{y}$ for an arbitrary $y \in C$. This gives

$$u_3(\overline{x}\ \overline{y}) = \overline{u_1(x)} u_2(\overline{y}) = \overline{u_1(x)}\ \overline{\kappa u_2 \kappa(y)},$$

whereby $(u_3, u_1, \kappa u_2 \kappa)$ is a related triple, and, by Lemma 3.2, so is $F_S(u)$. Thus the map F_S is well-defined; it is a group homomorphism since χ and α' preserve the multiplication, and injective since so is α'.

To show that it is onto, let $(t_1, t_2, t_3) \in \mathbf{RT}'(C)(S)$. Then by Proposition 3.9 $(\kappa t_2 \kappa, t_3)$ is the image under α'' of a unique $u \in \mathrm{Cl}_0(q)$. The element u satisfies, for all $x \in C_S$,

$$\alpha'(u \cdot x \cdot u^{-1}) = \left(\begin{smallmatrix} \kappa t_2 \kappa & 0 \\ 0 & t_3 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & L_x \\ L_{\overline{x}} & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} \kappa t_2^{-1} \kappa & 0 \\ 0 & t_3^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \kappa t_2 \kappa L_x t_3^{-1} \\ t_3 L_{\overline{x}} \kappa t_2^{-1} \kappa & 0 \end{smallmatrix}\right).$$

Using the fact that (t_1, t_2, t_3) is related together with Lemma 3.2, and the definition of a related triple, the right hand side simplifies to give

$$\alpha'(u \cdot x \cdot u^{-1}) = \left(\begin{smallmatrix} 0 & L_{t_1(x)} \\ L_{\overline{t_1(x)}} & 0 \end{smallmatrix}\right) = \alpha'(t_1(x)).$$

Thus by injectivity, $u \cdot x \cdot u^{-1} = t_1(x) \in C_S$. Moreover, since α'' is a morphism of algebras with involution, and since κ, t_2 and t_3 are isometries, we have

$$\alpha''(\sigma(u)) = (\sigma_q(\kappa t_2 \kappa), \sigma_q(t_3)) = (\kappa t_2^{-1} \kappa, t_3^{-1}),$$

whereby

$$\alpha(u\sigma(u)) = \alpha(u)\alpha(\sigma(u)) = (\mathrm{Id}, \mathrm{Id}),$$

which by injectivity implies $u\sigma(u) = 1$. Thus $u \in \mathbf{Spin}(q)(S)$ with $u_1 = t_1$, $u_2 = \kappa t_2 \kappa$ and $u_3 = t_3$. Then $F_S(u) = (t_1, t_2, t_3)$. This proves that F_S is surjective, hence a group isomorphism. Functoriality is clear, and the theorem is proved. □

Corollary 3.12. *The group schemes* $\mathbf{RT}'(C)$ *and* $\mathbf{RT}(C)$ *coincide.*

Proof. It suffices to show that $\mathbf{RT}'(C) \subseteq \mathbf{SO}(q)^3$, i.e. for each R–ring S and each $(t_1, t_2, t_3) \in \mathbf{RT}'(C)(S)$ we have $t_i \in \mathbf{SO}(C)(S)$. Now Lemma 3.2 holds, with the same proof, if $\mathbf{RT}(C)$ is replaced by $\mathbf{RT}'(C)$ and $\mathbf{SO}(q_C)$ by $\mathbf{O}(q_C)$. Thus it suffices to show that $t_1 \in \mathbf{SO}(q)(S)$. But the theorem provides $u \in \mathbf{Spin}(q)(S)$ such that $t_1 = u_1$, whereby $t_1 \in \mathbf{SO}(q)(S)$. □

Proposition 3.13. *In the following diagram of affine group schemes, the rows are exact, the vertical arrows are natural isomorphisms, and the squares commute.*

$$\begin{tikzcd}
\mathbf{1} \arrow[r] & \boldsymbol{\mu}_2 \arrow[r, "\iota"] \arrow[d, equal] & \mathbf{Spin}(q) \arrow[r, "\chi"] \arrow[d, "F"'] & \mathbf{SO}(q) \arrow[r] \arrow[d, equal] & \mathbf{1}. \\
\mathbf{1} \arrow[r] & \boldsymbol{\mu}_2 \arrow[r, "j"] & \mathbf{RT}(C) \arrow[r, "f_1"] & \mathbf{SO}(q) \arrow[r] & \mathbf{1}
\end{tikzcd}$$

Here, i and j are defined by setting, for each R–ring S,

- $\iota_S(\eta) = \eta 1$ *and*
- $j_S(\eta) = (\mathrm{Id}, \eta\mathrm{Id}, \eta\mathrm{Id})$,

while F and χ are defined above.

Proof. Let S be an R–ring and consider the diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r] & \boldsymbol{\mu}_2(S) \arrow[r, "\iota"] \arrow[d, equal] & \mathbf{Spin}(q)(S) \arrow[r, "\chi"] \arrow[d, "F"'] & \mathbf{SO}(q)(S) \arrow[r, "\mathrm{SN}"] \arrow[d, equal] & \mathbf{Disc}(S), \arrow[d, equal] \\
\mathbf{1} \arrow[r] & \boldsymbol{\mu}_2(S) \arrow[r, "j"] & \mathbf{RT}(C)(S) \arrow[r, "f_1"] & \mathbf{SO}(q)(S) \arrow[r, "\mathrm{SN}"] & \mathbf{Disc}(S)
\end{tikzcd}$$

where SN denotes the spinorial norm and $\mathbf{Disc}(S)$ the discriminant module (see the appendix). The upper row is exact by [K, IV.6.2.6]. If $u \in \mathbf{Spin}(q)(S)$, then by the above theorem

$$f_1 \circ F(u) = f_1(u_1, \kappa u_2 \kappa, u_3) = u_1 = \chi(u),$$

whereby the middle square commutes. If $\eta \in \boldsymbol{\mu}_2(S)$, then

$$F \circ \iota(\eta) = F(\eta 1) = ((\eta 1)_1, \kappa(\eta 1)_2 \kappa, (\eta 1)_3) = (\mathrm{Id}, (\eta 1)_2, (\eta 1)_3).$$

The left square commutes, since

$$((\eta 1)_2, (\eta 1)_3)) = \alpha''(\eta \mathrm{Id}) = (\eta \mathrm{Id}, \eta \mathrm{Id}) = \eta(\mathrm{Id}, \mathrm{Id}),$$

as α'' is a morphism of algebras. The statement follows since all maps are functoriality is clear, and since $\mathbf{Disc}(S)$ is trivial for a faithfully flat R–ring S. □

4. Twisting

4.1. **Torsors.** We consider the usual action of $\mathbf{SO}(q_C)$ on the unit sphere $\mathbf{S}_C = \mathbf{S}_{q_C}$. Recall that

$$\mathbf{S}_C(S) = \{c \in C_S \mid q_{C_S}(c) = 1\}$$

for any R–ring S. An important point is that $\mathbf{S}_C$ is smooth over R (Lemma A.1). We have three actions of $\mathbf{RT}(C)$ on $\mathbf{S}_C$ and will, throughout, consider the action of $\mathbf{RT}(C)$ on $\mathbf{S}_C \times \mathbf{S}_C$ defined by

$$\mathbf{t}.(u, v) = (t_3(u), t_2(v))$$

for any $\mathbf{t} = (t_1, t_2, t_3) \in \mathbf{RT}(C)(S)$ and $u, v \in \mathbf{S}_C(S)$.

We establish now in the ring setting a result by Jacobson over fields of nice characteristic [J, page 93], see also [Hu, th. 14.69] in the real case. Our proof is self-contained.

Theorem 4.1. (1) *The stabilizer* $\mathrm{Stab}_{\mathbf{RT}(C)}(1,1)$ *is* $i(\mathbf{Aut}(C))$.

(2) *The fppf quotient sheaf* $\mathbf{RT}(C)/\mathbf{Aut}(C)$ *is representable by an affine* R*–scheme and the induced map* $\mathbf{RT}(C)/\mathbf{Aut}(C) \xrightarrow{\sim} \mathbf{S}^2_C$ *is an* R*–isomorphism.*

Proof. (1) This follows from Proposition 3.7.

(2) Since $\mathbf{Aut}(C)$ is flat over R, it follows that the fppf quotient $\mathbf{RT}(C)/\mathbf{Aut}(C)$ is representable by an R–scheme and the induced map $h : \mathbf{RT}(C)/\mathbf{Aut}(C) \to \mathbf{S}^2_C$ is a monomorphism [SGA3, XVI.2.2] . Furthermore $\mathbf{RT}(C)/\mathbf{Aut}(C)$ is smooth since $\mathbf{RT}(C)$ is smooth [SGA3, VI_B.9.2.(xii)]. Since $\mathbf{RT}(C)/\mathbf{Aut}(C)$ and $\mathbf{S}^2_C$ are smooth over R of the same relative dimension 14, we know that h is an open immersion by [EGAIV, $_4$.18.10.5]. The proof that h is an isomorphism boils down, by the trick used in the proof of Theorem 3.8, to the case of $R = \mathbb{Z}$ and $C = C_0$.

Claim. For each prime p, $h_p = h \times_{\mathbb{Z}} \mathbb{F}_p$ is an isomorphism.

Since h_p is an open immersion, it is enough to show that $\mathbf{RT}(C_0)(\overline{\mathbb{F}}_p)$ acts transitively on $\mathbf{S}^2_{C_0}(\overline{\mathbb{F}}_p)$ and a fortiori to show that for each $q = p^n$, $\mathbf{RT}(C_0)(\mathbb{F}_q)$ acts transitively on $\mathbf{S}^2_{C_0}(\mathbb{F}_q)$. The coset $\Sigma := \mathbf{RT}(C_0)(\mathbb{F}_q)/\mathbf{Aut}(C_0)(\mathbb{F}_q)$ is the $\mathbf{RT}(C_0)(\mathbb{F}_q)$–orbit of $(1,1)$ in $\mathbf{S}^2_{C_0}(\mathbb{F}_q)$. Using [Hu, Table page 6], we have

$$\sharp\Sigma = \frac{q^{12}(q^2-1)(q^4-1)^2(q^6-1)}{q^6(q^2-1)(q^6-1)} = \big(q^3(q^4-1)\big)^2.$$

On the other hand, since C_0 is hyperbolic of dimension 8, we set $V = \mathbb{F}_q^4$ and view the $\mathbb{F}_q$–points of $\mathbf{S}_{C_0}$ as the points of $V \oplus V^\vee$ having hyperbolic norm 1, i.e. the pairs (v, ϕ) where $v \in V$ and $\phi \in V^\vee$ satisfy $\phi(v) = 1$. It follows that $\mathbf{S}_{C_0}(\mathbb{F}_q)$ has cardinality $\sharp(V \setminus \{0\}) \times \sharp(\mathbb{F}_q^3) = (q^4-1)q^3$. Thus $\sharp\Sigma = \sharp\big(\mathbf{S}^2_{C_0}(\mathbb{F}_q)\big)$, implying that $\Sigma = \mathbf{S}^2_{C_0}(\mathbb{F}_q)$, and a fortiori that $\mathbf{RT}(C_0)(\mathbb{F}_q)$ acts transitively on $\mathbf{S}^2_{C_0}(\mathbb{F}_q)$.

The Claim is proved. Now $\mathbf{S}^2_{C_0}$ is a Jacobson scheme, i.e. its closed points are dense [EGAIV, $_3$.10.4.6]. Furthermore the mapping $\mathbf{S}^2_{C_0} \to \mathrm{Spec}(\mathbb{Z})$ maps closed points to closed points (*ibid*, $_3$.10.4.7). The Claim implies that the image of h contains all closed points of $\mathbf{S}^2_{C_0}$; hence h is surjective. Thus h is a surjective open immersion and thereby an isomorphism. □

By the above, the map $\Pi : \mathbf{RT}(C) \to \mathbf{S}_C \times \mathbf{S}_C$ given, for each R–ring S and each $\mathbf{t} \in \mathbf{RT}(C)(S)$, by

$$\Pi_S : \mathbf{t} = (t_1, t_2, t_3) \mapsto (t_3(1), t_2(1)),$$

is a $\mathbf{G}$–torsor in the fppf-topology [DG, III.4.1.8], where $\mathbf{G} = \mathbf{Aut}(C)$. Since $\mathbf{G}$ is smooth over R, it is additionally a $\mathbf{G}$–torsor for the étale topology [SGA3, XXIV.8.1]. We can actually prove a stronger result.

Theorem 4.2. *(1) There is a natural bijection*

$$\mathbf{RT}(C)(R)\backslash\Big(\mathbf{S}_C(R) \times \mathbf{S}_C(R)\Big) \xrightarrow{\sim} \ker\Big(H^1_{\mathrm{Zar}}\big(R, \mathbf{Aut}(C)\big) \to H^1_{\mathrm{Zar}}\big(R, \mathbf{RT}(C)\big)\Big).$$

(2) The $\mathbf{Aut}(C)$*–torsor* $\Pi : \mathbf{RT}(C) \to \mathbf{S}_C \times \mathbf{S}_C$ *is trivial for the Zariski topology.*

The statement shows that the defect in transitivity for the action of $\mathbf{RT}(C)(R)$ over $\mathbf{S}_C(R) \times \mathbf{S}_C(R)$ is encoded in a nice subset of $H^1_{\mathrm{Zar}}(R, \mathbf{Aut}(C))$, namely a subset of isomorphism classes of Zariski twists of C.

We write $W(C)$ for the vector R–group scheme of C, so that $W(C)(S) = C_S (= S \otimes_R C)$ for any R–ring S. Let $\mathbf{E}$ be a $\mathbf{G} = \mathbf{Aut}(C)$–torsor and let $\mathbf{E} \wedge^{\mathbf{G}} W(C)$ be the fppf-twist of $W(C)$ by $\mathbf{E}$. This is the sheaf associated to the presheaf $\mathbf{E}(C)$ defined, for each R–ring S, by

$$\mathbf{E}(C)(S) = (\mathbf{E}(S) \times C_S)/\sim,$$

where

$$(u, x) \sim (u', x') \Longleftrightarrow \exists g \in \mathbf{G}(S) : (ug, g^{-1}x) = (u', x').$$

This is the twisted octonion R–algebra written ${}^{\mathbf{E}}C$ for short. We can proceed to the proof of Theorem 4.2.

Proof. (1) According to cohomological properties [Gi3, prop. 2.4.3], there is a natural bijection

$$\mathbf{RT}(C)(R)\backslash\Big(\mathbf{S}_C(R) \times \mathbf{S}_C(R)\Big) \xrightarrow{\sim} \ker\Big(H^1_{\mathrm{fppf}}\big(R, \mathbf{Aut}(C)\big) \xrightarrow{i^*} H^1_{\mathrm{fppf}}\big(R, \mathbf{RT}(C)\big)\Big)$$

induced by the characteristic map which to a point $(a, b) \in \mathbf{S}_C(R) \times \mathbf{S}_C(R)$ associates the class of the $\mathbf{G}$–torsor $\mathbf{E}^{a,b} = \Pi^{-1}(a, b)$. The main point is that the orthogonal representation $\mathbf{Aut}(C) \to \mathbf{O}(q_C)$ factors through $\mathbf{RT}(C)$. It follows that we have the inclusion

$$\ker(i^*) \subset \ker\Big(H^1_{\mathrm{fppf}}\big(R, \mathbf{Aut}(C)\big) \to H^1_{\mathrm{fppf}}\big(R, \mathbf{O}(q_C)\big)\Big).$$

To refine the statement for the Zariski topology, we consider an $\mathbf{Aut}(C)$–torsor $\mathbf{E}/R$ such that its class belongs to the second kernel and want to show that it is trivial locally for the Zariski topology. Let $C' = {}^{\mathbf{E}}C$ be the twisted octonion R–algebra. The map $H^1_{\mathrm{fppf}}\big(R, \mathbf{Aut}(C)\big) \to H^1_{\mathrm{fppf}}\big(R, \mathbf{O}(q_C)\big)$ sends $[C']$ to the isometry class of the quadratic form $q_{C'}$ (which is a twisted form of q_C). Since $[C']$ belongs to the second kernel, it follows that $q_{C'}$ is isometric to q_C. According to Bix's theorem [B, lemma 1.1], we have that $C' \otimes_R R_{\mathfrak{P}} \cong C \otimes_R R_{\mathfrak{P}}$ for each prime ideal $\mathfrak{P}$ of R. In other words, we have $\mathbf{E}(R_{\mathfrak{P}}) \neq \emptyset$. But $\mathbf{E}$ is of finite presentation over R, so that

$$\mathbf{E}(R_{\mathfrak{P}}) = \varinjlim_{f \notin \mathfrak{P}} \mathbf{E}(R_f).$$

For each prime ideal $\mathfrak{P}$ there thus exists an element $f_{\mathfrak{P}} \notin \mathfrak{P}$ such that $\mathbf{E}(R_{f_{\mathfrak{P}}}) \neq \emptyset$. The $f_{\mathfrak{P}}$'s generate R where $\mathfrak{P}$ runs on the maximal ideals of R. In particular there exists a partition of unity $1_R = f_1 + \cdots + f_r$ such that $\mathbf{E}(R_{f_i}) \neq \emptyset$ for $i = 1, ..., n$. Thus $\mathbf{E}$ is a $\mathbf{G}$–torsor which is Zariski locally trivial. We have shown (1).

(2) We apply (1) to the coordinate ring $R[\mathbf{S}^2_C]$ and to the universal point $\eta \in \mathbf{S}^2_C\big(R[\mathbf{S}^2_C]\big)$. Its image by the characteristic map is nothing but the class of the

$\mathbf{G}$–torsor $\Pi : \mathbf{RT}(C) \to \mathbf{S}^2_C$. Then (1) yields that Π admits sections locally with respect to the Zariski topology. □

A natural question to ask at this stage is whether this construction is trivial or not. We provide here a variant of [Gi2].

Example 4.3. Assume that $R = \mathbb{R}$ is the field of real numbers and that C is the division algebra of Cayley–Graves octonions. In this case, $\mathbf{G} = \mathbf{Aut}(C)$ is the real semisimple anisotropic algebraic group of type G_2 and S_C is the standard real sphere with equation $\sum_{i=1}^8 x_i^2 = 1$. We claim that the $\mathbf{G}$–torsor $\Pi : \mathbf{RT}(C) \to \mathbf{S}^2_C$ is non-trivial. If it is trivial, there is a $\mathbf{G}$–isomorphism $\mathbf{RT}(C) \xrightarrow{\sim} \mathbf{S}^2_C \times_{\mathbb{R}} \mathbf{G}$ and in particular the morphism $\mathbf{G} \to \mathbf{RT}(C) \cong \mathbf{Spin}_8$ admits an algebraic retraction. When taking the real points, we have that the inclusion of topological groups $\mathbf{G}(\mathbb{R}) \to \mathbf{Spin}_8(\mathbb{R})$ admits a continuous retraction. Therefore for each $j \geq 1$, the homotopy group $\pi_j(\mathbf{G}(\mathbb{R}), 1)$ is a direct summand of $\pi_j(\mathbf{Spin}_8(\mathbb{R}))$. According to the tables [Mi, p. 970], we have $\pi_6(\mathbf{G}(\mathbb{R}), 1) = \mathbb{Z}/3\mathbb{Z}$ and $\pi_6(\mathbf{Spin}_8(\mathbb{R}), 1) = 0$ whence a contradiction. Thus the $\mathbf{G}$–torsor $\Pi : \mathbf{RT}(C) \to \mathbf{S}^2_C$ is non-trivial. Note also that the same fibration extended to the field $\mathbb{C}$ of complex numbers is still non-trivial. Proposition 6.10 provides an algebraic proof of this fact.

4.2. **Isotopes as twisted algebras.** We shall now see how the isotopes of Section 2 appear naturally as the twists above. We are given $a, b \in \mathbf{S}_C(R)$ and consider the $\mathbf{G}$–torsor $\mathbf{E}^{a,b} = \Pi^{-1}(a, b)$. It gives rise to the sheaf of twisted octonion R–algebras $\mathbf{E}^{a,b}(C)$. This is the sheaf associated to the presheaf $\mathbf{E}^{a,b}(C)$ defined, for each R–ring S, by

$$\mathbf{E}^{a,b}(C)(S) = (\Pi_S^{-1}(a_S, b_S) \times C_S)/\sim,$$

where

$$(\mathbf{t}, x) \sim (\mathbf{t}', x') \Longleftrightarrow \exists g \in \mathbf{G}(S) : (\mathbf{t}g, x) = (\mathbf{t}, g(x')).$$

Remark 4.4. Note that if $\Pi_S(\mathbf{t}) = \Pi_S(\mathbf{t}')$, then $\mathbf{t}^{-1}\mathbf{t}' \in i_S(\mathbf{G}(S))$.

The linear structure on $\mathbf{E}^{a,b}(C)(S)$ is induced by (cf. [DG, III.4.3.3])

$$(\mathbf{t}, x) + (\mathbf{t}', x') = (\mathbf{t}, x + t_1^{-1}t_1'(x'))$$

and, for $s \in S$,

$$s(\mathbf{t}, x) = (\mathbf{t}, sx),$$

and the multiplication by

$$(\mathbf{t}, x) \cdot (\mathbf{t}', x') = (\mathbf{t}, xt_1^{-1}t_1'(x')).$$

The following observation will be useful.

Proposition 4.5. *Let $a, b \in \mathbf{S}_C(R)$ and $(t_1, t_2, t_3) \in \mathbf{RT}(C)(S)$ for an R–ring S. Then t_1 is an isomorphism $C_S \to C_S^{a,b}$ if and only if $t_2(1) = \eta b_S$ and $t_3(1) = \eta a_S$ for some $\eta \in \boldsymbol{\mu}_2(S)$.*

Abusing notation, we write a and b for a_S and b_S, respectively.

Proof. If $t_2(1) = b$ and $t_2(1) = a$, then

$$t_1(x) = t_1(1x) = \overline{t_2(1)}\ \overline{t_3(\overline{x})} = \overline{b}\ \overline{t_3(\overline{x})},$$

whereby $L_{\overline{b}}\kappa t_3\kappa = t_1$, and thus $\kappa t_3\kappa = L_b t_1$. Likewise $\kappa t_2\kappa = R_a t_1$. This implies, by the definition of a related triple, that for all $x, y \in C_S$,

$$t_1(xy) = \kappa t_2\kappa(x)\kappa t_3\kappa(y) = R_a t_1(x) L_b t_1(y) = t_1(x) *_{a,b} t_1(y).$$

Since $C^{\eta a, \eta b} = C^{a,b}$ for any $\eta \in \boldsymbol{\mu}_2(S)$, this concludes the if-part.

Conversely, if $t_1 : C_S \to C_S^{a,b}$ is an isomorphism, then $(t_1, \kappa R_a t_1 \kappa, \kappa L_b t_1 \kappa)$ is a related triple. Since $t_1(1) = (ab)^{-1}$ we get $\kappa R_a t_1 \kappa(1) = b$ and $\kappa L_b t_1 \kappa(1) = a$. Lemma 3.6 then implies that $t_2(1) = \eta b$ and $t_3(1) = \eta a$ for some $\eta \in \boldsymbol{\mu}_2(S)$. □

We can now establish the link and formulate the main theorem of this section.

Theorem 4.6. *Let $a, b \in \mathbf{S}_C(R)$. Consider, for each R–ring S, the map*

$$\Theta_S^{a,b} : \Pi_S^{-1}(a_S, b_S) \times C_S \to C_S^{a,b}, \quad (\mathbf{t}, x) \mapsto t_1(x).$$

Then these maps induce a natural isomorphism of sheaves of algebras

$$\widehat{\Theta}^{a,b} : \mathbf{E}^{a,b} \wedge^{\mathbf{G}} W(C) \xrightarrow{\sim} W(C^{a,b}).$$

Naturality means that for any morphism $\phi : S \to T$ of R–rings, the diagram

$$\begin{array}{ccc} \mathbf{E}^{a,b}(C)(S) & \xrightarrow{\widehat{\Theta}_S^{a,b}} & C_S^{a,b} \\ \downarrow{\scriptstyle \widehat{\phi}} & & \downarrow{\scriptstyle \widehat{\phi}} \\ \mathbf{E}^{a,b}(C)(T) & \xrightarrow{\widehat{\Theta}_T^{a,b}} & C_T^{a,b} \end{array}$$

commutes. Note that the maps are well-defined by Remark 2.2.

Proof. By the universality of the sheaf associated to a presheaf, it suffices to prove that for each S such that $\Pi_S^{-1}(a_S, b_S) \neq \emptyset$, the map $\widehat{\Theta}_S^{a,b}$ is a natural isomorphism of algebras. Consider such an S.

Firstly, the map $\widehat{\Theta}_S^{a,b}$ is well defined, since if $(\mathbf{t}, x) \sim (\mathbf{t}', x') \in \Pi_S^{-1}(a_S, b_S) \times C_S$, then $\mathbf{t}' = \mathbf{t}g$ and $x' = g^{-1}(x)$ for some $g \in \mathbf{G}(S)$, whereby

$$t_1'(x') = t_1 g g^{-1}(x) = t_1(x).$$

Secondly, $\widehat{\Theta}_S^{a,b}$ is an algebra homomorphism. Indeed, it is linear, since

$$\Theta_S^{a,b}(s(\mathbf{t}, x)) = \Theta_S^{a,b}(\mathbf{t}, sx) = t_1(sx) = st_1(x) = s\Theta_S^{a,b}((\mathbf{t}, x))$$

for all $s \in S$, and

$$\Theta_S^{a,b}((\mathbf{t}, x) + (\mathbf{t}', x')) = \Theta_S^{a,b}(\mathbf{t}, x + t_1^{-1}t_1'(x')) = t_1(x + t_1^{-1}t_1'(x')) = t_1(x) + t_1'(x'),$$

which by definition is $\Theta_S^{a,b}((\mathbf{t}, x)) + \Theta_S^{a,b}((\mathbf{t}', x'))$. With respect to the multiplication,

$$\Theta_S^{a,b}((\mathbf{t}, x)(\mathbf{t}', x')) = \Theta_S^{a,b}(\mathbf{t}, xt_1^{-1}t_1'(x')) = t_1(xt_1^{-1}t_1'(y)),$$

and since $\mathbf{t} \in \Pi_S^{-1}(a, b)$, by Proposition 4.5 this equals

$$t_1(x) *_{a,b} t_1 t_1^{-1} t_1'(x') = t_1(x) *_{a,b} t_1'(y) = \Theta_S^{a,b}(\mathbf{t}, x) *_{a,b} \Theta_S^{a,b}(\mathbf{t}', x').$$

Thirdly, $\widehat{\Theta}_S^{a,b}$ is injective, since if $\Theta_S^{a,b}((\mathbf{t}, x)) = t_1(x) = 0$, then $x = 0$ since t_1 is invertible, whereby the class of $(\mathbf{t}, x)$ is the zero element of $\mathbf{E}^{a,b}(C)(S)$; it is surjective since given $x \in C_S^{a,b}$, any $\mathbf{t} \in \Pi_S^{-1}(a, b) \neq \emptyset$ satisfies $\Theta_S^{a,b}(\mathbf{t}, t_1^{-1}(x)) = x$. Thus $\widehat{\Theta}_S^{a,b}$ is an isomorphism. It is natural since the commutativity of the square above is the statement that, for each $(t_1, t_2, t_3) \in \mathbf{RT}(C)(S)$, the equality $t_{1T}\widehat{\phi} = \widehat{\phi}t_1$ holds, which follows from the definition of t_{1T}. □

In view of Remark 2.6 and the discussion preceding it, this implies that any octonion algebra twisted form of C, where the twist is effected by the torsor above, is an isotope. By that remark and the proposition preceding it, one may, without loss of generality (up to isomorphism), assume that $b = 1$ or, equivalently, $b = a^{-1}$. This and other variants are discussed in Section 6, where we will also show that the twists above account for all octonion algebras isometric to C.

5. Related constructions

5.1. **Twisted automorphism groups.** We will briefly discuss the automorphism groups of the twisted algebras. We set $\mathbf{G}^{a,b} = \mathbf{Aut}(C^{a,b})$ for $a, b \in \mathbf{S}_C(R)$. It is the twisted group scheme of $\mathbf{G}$ by the torsor $\mathbf{E}^{a,b}$. According to Demarche's compatibility [Gi3, §2.4.1], we have a natural identification

$$\mathbf{G}^{a,b} \xrightarrow{\sim} \Big\{ t \in \mathbf{RT}(C) \mid t.(a,b) = (a,b) \Big\}.$$

In other words, $\mathbf{G}^{a,b}$ is nothing but the stabilizer of (a,b) for the action of $\mathbf{RT}(C)$ on $\mathbf{S}_C^2$. From another point of view, it consists of the fixed points of a twisted A_3-action on $\mathbf{RT}(C)$, which is given explicitly in the following proposition.

Proposition 5.1. *Let $a, b \in \mathbf{S}_C(R)$. The map*

$$i^{a,b} : \mathbf{G}^{a,b} \to \mathbf{RT}(C)$$

defined, for each R–ring S and each $t \in \mathbf{G}^{a,b}(S)$, by

$$t \mapsto (t, \kappa R_a t R_{\overline{a}} \kappa, \kappa L_b t L_{\overline{b}} \kappa),$$

is a monomorphism of R–group schemes. Moreover, the R–map

$$\widehat{\sigma}^{a,b} : \mathbf{RT}(C) \to \mathbf{RT}(C)$$

defined on $\mathbf{RT}(C)(S)$ for each R–ring S by

$$(t_1, t_2, t_3) \mapsto (R_{\overline{a}} B_{\overline{b}}, L_{\overline{a}} R_b, B_a L_b)(t_2, t_3, t_1)(R_{\overline{a}} B_{\overline{b}}, L_{\overline{a}} R_b, B_a L_b)^{-1}$$

is an automorphism of $\mathbf{RT}(C)$ of order three, and $i^{a,b}(\mathbf{G}^{a,b})$ is the fixed locus of $\widehat{\sigma}^{a,b}$.

In other words, $\widehat{\sigma}^{a,b} = \mathrm{Int}(\mathbf{s}_{a,b})\widehat{\sigma}$ with $\mathbf{s}_{a,b} = (R_{\overline{a}} B_{\overline{b}}, L_{\overline{a}} R_b, B_a L_b) \in \mathbf{RT}(C)(R)$, and $\widehat{\sigma}$ being the standard action of A_3, induced by the element $\sigma = (123)$. The crucial step in the proof is the use of Lemmata 3.2 and 3.6 to deduce that

$$(t, \kappa R_a t R_{\overline{a}} \kappa, \kappa L_b t L_{\overline{b}} \kappa) = (t, B_b R_a t R_{\overline{a}} B_{\overline{b}}, B_a L_b t L_{\overline{b}} B_{\overline{a}})$$

if $t \in \mathbf{G}^{a,b}(S)$, and conversely that the expression of the right hand side is a related triple only if $t \in \mathbf{G}^{a,b}(S)$. We omit the technical details

5.2. **Trialitarian action on the torsor.** Since A_3 acts trivially on $\mathbf{G} = \mathbf{Aut}(C)$, the trialitarian action of A_3 on $\mathbf{RT}(C)$ induces an action of A_3 on the $\mathbf{G}$–torsor $\Pi : \mathbf{RT}(C) \to \mathbf{S}_C^2$.

Lemma 5.2. *The action of the cycle $\sigma = (123)$ on $\mathbf{S}_C^2(R)$ maps (x, y) to $(\overline{y}\,\overline{x}, x)$.*

Proof. We write $(x', y') = \sigma(x, y)$. We may assume that R is local. Then there exists $\mathbf{t} = (t_1, t_2, t_3) \in \mathbf{RT}(C)(R)$ such that $t_3(1) = x$ and $t_2(1) = y$. Since $\sigma(\mathbf{t}) = (t_2, t_3, t_1)$, we have $x' = t_1(1)$ and $y' = t_3(1) = x$. We finish by using the relation $x' = t_1(1) = \overline{t_2(\overline{1})}\,\overline{t_3(\overline{1})} = \overline{y}\,\overline{x}$. □

This gives rise to the following formulae for isotopes.

Corollary 5.3. *Let $a, b \in \mathbf{S}_C(R)$. Then the R–algebras $C^{a,b}$, $C^{b^{-1}a^{-1},a}$ and $C^{b,b^{-1}a^{-1}}$ are isomorphic.*

It is reassuring that these relations can be derived from the known relations of Section 2.2. Indeed, McCrimmon's relation provides an isomorphism $C^{b^{-1}a^{-1},a} \cong C^{ab^{-1},1}$ which is isomorphic to $C^{a,b}$ by Proposition 2.4.(4).

Remark 5.4. The above theorem implies that the orbits of the group action of $\mathbf{RT}(C)(R)$ correspond to the isomorphism classes of the isotopes. This explains the isomorphism relations between these. Consider, for instance, the relation $C^{a,a} \cong C$. In this case the element $\mathbf{t} = (B_{\overline{a}}, R_a, L_a)$ belongs to $\mathbf{RT}(C)(R)$ due to the middle Moufang identity. Then $\Pi(\mathbf{t}) = (a, a)$, whence $C^{a,a} \cong C$. We will exploit this point of view further in Section 7.

5.3. **Compositions of quadratic forms.** We remind the reader of the notion of a composition of quadratic forms, which is essentially a generalization of that of a composition algebra. Let (M_1, q_1), (M_2, q_2), (M_3, q_3) be non-singular quadratic R–forms of common rank. A *composition of quadratic forms* is a bilinear R–map $f : M_2 \times M_3 \to M_1$ such that $q_1(f(m_2, m_3)) = q_2(m_2)\, q_3(m_3)$ for each $m_2 \in M_2$, $m_3 \in M_3$. More precisely, it is the datum $\mathcal{M} = (M_1, q_1, M_2, q_2, M_3, q_3, f)$ for which the notion of isomorphism is clear.

Our favourite example is the composition $\mathcal{M}_C$ attached to the octonion algebra C, i.e. $M_1 = M_2 = M_3 = C$ and f is the octonionic multiplication. In this case we have an isomorphism $\mathbf{RT}'(C)(C) \xrightarrow{\sim} \mathbf{Aut}(\mathcal{M}_C)(S)$ by mapping a triple (t_1, t_2, t_3) to $({}^{\sigma_C}t_1, {}^{\sigma_C}t_2, {}^{\sigma_C}t_3)$ where σ_C stands for the octonionic involution on C. It follows that the R–functor $\mathbf{Aut}(\mathcal{M}_C)$ is representable by the R–group scheme $\mathbf{RT}'(C)$ which is nothing but $\mathbf{RT}(C)$ according to Corollary 3.12.

It follows that the set $H^1_{\mathrm{fppf}}(R, \mathbf{RT}(C))$ classifies the compositions of rank eight which are locally isomorphic for the flat topology (or even étale topology) to $\mathcal{M}_C$. The centre $\ker(\boldsymbol{\mu}_2^3 \to \boldsymbol{\mu}_2)$ of $\mathbf{RT}(C)$ gives an action of the commutative group $H^1_{\mathrm{fppf}}(R, \ker(\boldsymbol{\mu}_2^3 \to \boldsymbol{\mu}_2))$ on $H^1_{\mathrm{fppf}}(R, \mathbf{RT}(C))$. Now $H^1_{\mathrm{fppf}}(R, \boldsymbol{\mu}_2) \simeq \mathbf{Disc}(R)$ (see the appendix), and thus

$$H^1_{\mathrm{fppf}}(R, \ker(\boldsymbol{\mu}_2^3 \to \boldsymbol{\mu}_2)) = \ker\big(\mathbf{Disc}(R)^3 \to \mathbf{Disc}(R)\big). \tag{5.1}$$

We will use this to explain how that action can be understood at the level of R–forms of $\mathcal{M}_C$. Given discriminant modules $(\mathcal{L}_1, \theta_1), (\mathcal{L}_2, \theta_2), (\mathcal{L}_3, \theta_3)$ and an isomorphism $\epsilon : (\mathcal{L}_2 \otimes_R \mathcal{L}_3, \theta_2 \otimes \theta_3) \to (\mathcal{L}_1, \theta_1)$, we can modify an R–composition

$$\mathcal{M} = (M_1, q_1, M_2, q_2, M_3, q_3, f)$$

as

$$\big(\mathcal{L}_1 \otimes_R M_1, \theta_1 \otimes q_1, \mathcal{L}_2 \otimes_R M_2, \theta_2 \otimes q_2, \mathcal{L}_3 \otimes_R M_3, \theta_3 \otimes q_3, \epsilon \otimes f\big).$$

By (5.1), this gives the action of $H^1_{\mathrm{fppf}}(R, \ker(\boldsymbol{\mu}_2^3 \to \boldsymbol{\mu}_2))$ on $H^1_{\mathrm{fppf}}(R, \mathbf{RT}(C))$.

Remark 5.5. Let C' be an R–form of C such that the compositions $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ are isomorphic. Then there exists a triple (t_1, t_2, t_3) of isometries between (C, q) and (C', q') such that

$$t_1(x *_C y) = t_2(x) *_{C'} t_3(y)$$

for all $x, y \in C$. In view of the introductory part of Section 2, this implies that the kernel of $H^1_{\mathrm{fppf}}(R, \mathbf{G}) \to H^1_{\mathrm{fppf}}(R, \mathbf{RT}(C))$ consists of isotopes of C. This confirms a consequence of the more precise construction given in Theorem 4.6.

6. Variants

6.1. **Codiagonal variant.** We consider the $\mathbf{Aut}(C)$–torsor $\Pi : \mathbf{RT}(C) \to \mathbf{S}^2_C$ and the codiagonal map $\nabla : \mathbf{S}_C \to \mathbf{S}^2_C$, $a \mapsto \nabla(a) = (a, a^{-1})$. We denote by $\mathbf{RT}(C)^\nabla \to \mathbf{S}_C$ the pull-back of the $\mathbf{Aut}(C)$–torsor Π with respect to ∇. We observe that $\mathbf{RT}(C)^\nabla = \mathbf{RT}(C) \times_{\mathbf{S}^2_C} \mathbf{S}_C$ is a closed R–subscheme of $\mathbf{RT}(C)$. For each R–ring S, we have

$$\begin{aligned}\mathbf{RT}(C)^\nabla(S) &= \big\{(t_1, t_2, t_3) \in \mathbf{RT}(C)(S) \mid t_3(1)t_2(1) = 1\big\}\\ &= \big\{(t_1, t_2, t_3) \in \mathbf{RT}(C)(S) \mid t_1(1) = 1\big\}.\end{aligned}$$

In terms of the covering $f_1 : \mathbf{RT}(C) \to \mathbf{SO}(q_C)$, $\mathbf{RT}(C)^{\nabla}$ is the inverse image of the R–subgroup scheme $\{t_1 \in \mathbf{SO}(q_C), \mid t_1(1) = 1\}$. We put $C_1 = 1^{\perp_q} \subset C$. By Lemma 2.9, C_1 is a locally free R–submodule of rank 7 of C and the restriction q_{C_1} of q_C to C_1 is non-singular.

Lemma 6.1. (1) *The restriction map* $\{t_1 \in \mathbf{O}(q_C) \mid t_1(1) = 1\} \to \mathbf{O}(q_{C_1})$, $f \mapsto f_{|C_1}$ *induces an isomorphism of group schemes*

$$\mathbf{SO}(q_C)_1 := \{t_1 \in \mathbf{SO}(q_C), \mid t_1(1) = 1\} \xrightarrow{\sim} \mathbf{SO}(q_{C_1}).$$

(2) *There is a unique R–homomorphism $h : \mathbf{Spin}(q_{C_1}) \to \mathbf{RT}(C)$ such that the following exact diagram commutes*

$$\begin{tikzcd}
\mathbf{1} \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, "\cong"] & \mathbf{Spin}(q_{C_1}) \arrow[r] \arrow[d, "h"'] & \mathbf{SO}(q_{C_1}) \arrow[r] \arrow[d] & \mathbf{1} \\
\mathbf{1} \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{RT}(C) \arrow[r, "f_1"] & \mathbf{SO}(q_C) \arrow[r] & \mathbf{1}.
\end{tikzcd}$$

Furthermore h induces an R–isomorphism $\mathbf{Spin}(q_{C_1}) \xrightarrow{\sim} \mathbf{RT}(C)^{\nabla}$.

Proof. (1) All relevant R–group schemes are smooth. We consider the restriction R–homomorphism $\phi : \mathbf{SO}(q_C)_1 \to \mathbf{O}(q_{C_1})$. Since $\mathbf{SO}(q_C)$ is connected, it factorizes by the neutral component $\mathbf{SO}(q_{C_1})$ of $\mathbf{O}(q_{C_1})$. In other words, we have an R–homomorphism $\psi : \mathbf{SO}(q_C)_1 \to \mathbf{SO}(q_{C_1})$. By [SV, Proposition 2.2.2], this is an isomorphism on fibres. Since $\mathbf{SO}(q_C)_1$ is smooth, hence flat, we conclude with the fibre-wise isomorphism criterion [EGAIV, $_4$.17.9.5] that ψ is an isomorphism.

(2) We pull back the extension $\mathbf{1} \to \boldsymbol{\mu}_2 \to \mathbf{RT}(C) \xrightarrow{f_1} \mathbf{SO}(q_C) \to \mathbf{1}$ by the composite map $\mathbf{Spin}(q_{C_1}) \to \mathbf{SO}(q_{C_1}) \to \mathbf{SO}(q_C)$ and get a central extension of R–group schemes $\mathbf{1} \to \boldsymbol{\mu}_2 \to \mathbf{H} \to \mathbf{Spin}(q_{C_1}) \to \mathbf{1}$. We observe that $\mathbf{H}$ is a closed subgroup scheme of $\mathbf{RT}(C)$. Since $\mathbf{Spin}(q_{C_1})$ is simply connected, this sequence is (uniquely) split [Co, 6.5.2.(iii)], i.e. there exists a unique R–homomorphism $h_0 : \mathbf{Spin}(q_{C_1}) \to \mathbf{H}$ splitting the above sequence. By construction the following diagram

$$\begin{tikzcd}
\mathbf{Spin}(q_{C_1}) \arrow[r] \arrow[d, "h"'] & \mathbf{SO}(q_{C_1}) \arrow[r] \arrow[d] & \mathbf{1} \\
\mathbf{RT}(C) \arrow[r, "f_1"] & \mathbf{SO}(q_C) \arrow[r] & \mathbf{1}
\end{tikzcd}$$

commutes. As explained before, the homomorphism h satisfying that commutativity is already unique. It remains to be shown that the induced map $h^\sharp : \boldsymbol{\mu}_2 \to \boldsymbol{\mu}_2$ is the identity. Over a field of odd characteristic, this follows of [Ga, Example 17.1]. To handle the general case, one can assume again that $R = \mathbb{Z}$ and that C is the split octonion algebra. Then $h^\sharp$ is the identity or the trivial map. It cannot be trivial since it is not over $\mathbb{F}_3$. We conclude that $h^\sharp$ is the identity as desired. It follows that the diagram

$$\begin{tikzcd}
\mathbf{Spin}(q_{C_1}) \arrow[r] \arrow[d, "h"'] & \mathbf{SO}(q_{C_1}) \arrow[d] \\
\mathbf{RT}(C) \arrow[r, "f_1"] & \mathbf{SO}(q_C)
\end{tikzcd}$$

is Cartesian so that $\mathbf{Spin}(q_{C_1})$ is R–isomorphic to $\mathbf{RT}(C)^{\nabla}$. $\square$

Since $\mathbf{Aut}(C)$ is an R–subgroup scheme of $\mathbf{RT}(C)^{\nabla}$, we obtain an embedding $\mathbf{Aut}(C) \to \mathbf{Spin}(q_{C_1})$. Furthermore we have a Cartesian diagram

$$\begin{tikzcd} \mathbf{Spin}(q_{C_1}) \arrow[r, "h"'] \arrow[d, "\Pi_1"] & \mathbf{RT}(C) \arrow[d, "\Pi"] \\ \mathbf{S}_C \arrow[r, "\nabla"] & \mathbf{S}_C \times_R \mathbf{S}_C \end{tikzcd}$$

where the vertical maps are $\mathbf{Aut}(C)$–torsors which are both Zariski locally trivial according to Theorem 4.2.

Remark 6.2. The map Π_1 is defined by the identification $\mathbf{Spin}(q_{C_1}) \xrightarrow{\sim} \mathbf{RT}(C)^{\nabla}$. As far as we know, there is no easier way to define it.

The next result completes Theorem 4.6.

Theorem 6.3. *There is a natural bijection*

$$\mathbf{Spin}(q_{C_1})(R)\backslash\mathbf{S}_C(R) \xrightarrow{\sim} \ker\Big(H^1_{\mathrm{Zar}}\big(R, \mathbf{Aut}(C)\big) \to H^1_{\mathrm{Zar}}\big(R, \mathbf{Spin}(q_{C_1})\big)\Big).$$

It associates to a point $a \in \mathbf{S}_C(R)$ the isomorphism class of the isotope $C^{a,a^{-1}} \simeq C^{1,a}$.

Proof. We proceed is a similar, but simpler, way in comparison to the proof of Theorem 4.2. There is a natural bijection [Gi3, prop. 2.4.3],

$$\mathbf{Spin}(q_{C_1})(R)\backslash\mathbf{S}_C(R) \xrightarrow{\sim} \ker\Big(H^1_{\mathrm{fppf}}\big(R, \mathbf{Aut}(C)\big) \to H^1_{\mathrm{fppf}}\big(R, \mathbf{Spin}(q_{C_1})\big)\Big).$$

It maps a point $a \in \mathbf{S}_C(R)$ to the class of the $\mathbf{G}$–torsor $\Pi_1^{-1}(a) = \Pi^{-1}(a, a^{-1})$ which represents the isomorphism class of the isotope $C^{a,a^{-1}}$ by Theorem 4.6. Since the torsor Π_1 is locally trivial for the Zariski topology, we can replace the flat cohomology by the Zariski topology on the right hand side. □

6.2. **Orthogonal and adjoint variants.** We have seen that the centre of $\mathbf{RT}(C)$ is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ by the map $(\eta_1, \eta_2) \mapsto (\eta_1\eta_2, \eta_1, \eta_2)$ (before Lemma 3.6). We consider now one copy $\boldsymbol{\mu}_2$ inside $\mathbf{RT}(C)$, namely the image of $\eta \mapsto (1, \eta, \eta)$. We will refer to this as the *diagonal* $\boldsymbol{\mu}_2$, as it induces a diagonal embedding of $\boldsymbol{\mu}_2$ in $\mathbf{S}_C^2$ via Π.

We mod out the mapping $\Pi : \mathbf{RT}(C) \to \mathbf{S}_C \times_R \mathbf{S}_C$ by the action of the diagonal $\boldsymbol{\mu}_2$. This gives rise to the orthogonal variant of Π, namely $\Pi_+ : \mathbf{RT}(C)_+ = \mathbf{RT}(C)/\boldsymbol{\mu}_2 \to \big(\mathbf{S}_C \times_R \mathbf{S}_C\big)/\boldsymbol{\mu}_2$. The R–map f_1

induces an R–isomorphism $\mathbf{RT}(C)_+ \xrightarrow{\sim} \mathbf{SO}(q_C)$. We thus have a Cartesian diagram

$$\begin{tikzcd} \mathbf{RT}(C) \arrow[r, "f_1"] \arrow[d, "\Pi"] & \mathbf{SO}(q_C) \arrow[d, "\Pi_+"] \\ \mathbf{S}_C^2 \arrow[r] & \big(\mathbf{S}_C \times_R \mathbf{S}_C\big)/\boldsymbol{\mu}_2. \end{tikzcd} \tag{6.1}$$

where the vertical maps are $\mathbf{G}$–torsors. Let $\underline{\mathbf{S}}_C = \mathbf{S}_C/\boldsymbol{\mu}_2$ (see the appendix for further discussion on quotients of spheres). The above provides us with two projections $p_1, p_2 : \big(\mathbf{S}_C \times_R \mathbf{S}_C\big)/\boldsymbol{\mu}_2 \to \underline{\mathbf{S}}_C$ and, correspondingly, two actions of $\mathbf{SO}(q_C)$ on $\underline{\mathbf{S}}_C$, which we denote by $g \bullet_i x$ for $i = 1, 2$. An important step is the following.

Lemma 6.4. (1) *For $i = 1, 2$, the orbit map $\mathbf{SO}(q_C) \to \underline{\mathbf{S}}_C$, $g \mapsto g\bullet_i[1]$ admits a splitting.*
(2) *Both actions of $\mathbf{SO}(q_C)(R)$ on $\underline{\mathbf{S}}_C(R)$ are transitive.*

Proof. We will only do the case $i = 1$, the case $i = 2$ being similar.

(1) We consider the R–map $\mathbf{S}_C \to \mathbf{SO}(q_C)$ which maps an element $a \in \mathbf{S}_C(R)$ to the orthogonal mapping B_a. Since this map is $\boldsymbol{\mu}_2$–invariant, it gives rise to an R–map $h : \underline{\mathbf{S}}_C \to \mathbf{SO}(q_C)$. We claim that $x \mapsto h(x)^{-1}$ is a section of the orbit map. Let $x \in \underline{\mathbf{S}}_C(R)$ and let S/R be a flat cover such that x_S lifts to an element $a \in \mathbf{S}_C(S)$. We consider $h(x) : C \to C$. Then $h(x)_S = B_a : C_S \to C_S$ which lifts (with respect to f_1) to the element $\mathbf{t} = (B_a, R_{\overline{a}}, L_{\overline{a}}) \in \mathbf{RT}(C)(S)$. We compute in $\underline{\mathbf{S}}_C(S)$

$$h(x) \bullet_1 x_S = p_1(\mathbf{t}.x_S) = [L_{\overline{a}}(a)] = [1]$$

whence $h(x)^{-1}.[1] = x$. Thus $x \mapsto h(x)^{-1}$ is a section of the orbit map.

(2) This is immediate from (1).

□

One can also go further and consider the quotient of $\mathbf{RT}(C)$ by the whole centre. We observe that the $\mathbf{Aut}(C)$–torsor $\Pi : \mathbf{RT}(C) \to \mathbf{S}_C^2$ is $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$–equivariant for the antipodal action of $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ on $\mathbf{S}_C^2$. Taking the quotient with respect to this action, we get an R–morphism

$$\underline{\Pi} : \underline{\mathbf{RT}}(C) \to \underline{\mathbf{S}}_C^2$$

where $\underline{\mathbf{RT}}(C) = \mathbf{RT}(C)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$ is the adjoint R–group scheme of $\mathbf{RT}(C)$, which is isomorphic to $\mathbf{PSO}(q_C)$. We consider now the R–homomorphism

$$\underline{i} : \mathbf{Aut}(C) \xrightarrow{i} \mathbf{RT}(C) \to \underline{\mathbf{RT}}(C).$$

Lemma 6.5. (1) *The R–homomorphism $\underline{i}$ is a closed immersion and the fppf quotient $\underline{\mathbf{RT}}(C)/\underline{i}(\mathbf{Aut}(C))$ is representable by an affine R–scheme.*

(2) *The R–morphism $\underline{\Pi} : \underline{\mathbf{RT}}(C) \to \underline{\mathbf{S}}_C^2$ is an $\mathbf{Aut}(C)$–torsor and the map $\underline{\mathbf{RT}}(C)/\underline{i}(\mathbf{Aut}(C)) \to \underline{\mathbf{S}}_C^2$ is an R–isomorphism.*

Proof. The R–map $\mathbf{S}_C^2 \to \underline{\mathbf{S}}_C^2$ is faithfully flat. The main point is that taking the quotient by $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ commutes with base change so that the left hand square of the following diagram

$$\begin{tikzcd}
\mathbf{RT}(C) \arrow[r, "P"] \arrow[d, "\Pi"] & \underline{\mathbf{RT}}(C) \arrow[r, "f_1", "\sim"'] \arrow[d, "\underline{\Pi}"] & \mathbf{PSO}(q_C) \arrow[dl, "\underline{\Pi}'"] \\
\mathbf{S}_C^2 \arrow[r] & \underline{\mathbf{S}}_C^2. &
\end{tikzcd} \tag{6.2}$$

is Cartesian. The R–group scheme $\mathbf{Aut}(C)$ acts on the map P. Since the property for that action to define an $\mathbf{Aut}(C)$–torsor is insensitive to a flat cover, Π is an $\mathbf{Aut}(C)$-torsor if and only if $\underline{\Pi}$ is. Since Π is an $\mathbf{Aut}(C)$–torsor, so is $\underline{\Pi}$, and this implies that $\underline{i}$ is an R–monomorphism and that the fppf-quotient sheaf $\underline{\mathbf{RT}}(C)/\underline{i}(\mathbf{Aut}(C))$ is representable by the affine smooth R–scheme $\underline{\mathbf{S}}_C^2$. Then $\mathbf{Aut}(C)$ occurs as the fibre of $\underline{\Pi}$ at $[(1,1)] \in \underline{\mathbf{S}}_C^2(R)$ so $\underline{i}$ is a closed R–immersion. □

6.3. **Codiagonal adjoint variant.** Finally, one may combine the above and consider the quotient of $\mathbf{RT}(C)^\nabla \simeq \mathbf{Spin}(q_{C_1})$ by its centre. The centre of $\mathbf{RT}(C)^\nabla$ is $\boldsymbol{\mu}_2$, embedded as $\eta \mapsto (1, \eta, \eta)$ for each R–ring S and $\eta \in \boldsymbol{\mu}_2(S)$, and Lemma 6.1 implies that $\mathbf{RT}(C)^\nabla/\boldsymbol{\mu}_2 \simeq \mathbf{SO}(q_{C_1})$. Reasoning as in the proof of as in the preceding cases, we take the quotient of the torsor $\Pi_1 : \mathbf{Spin}(q_{C_1}) \to \mathbf{S}_C$ by $\boldsymbol{\mu}_2$, thus obtaining the Cartesian diagram

$$\begin{tikzcd}
\mathbf{Spin}(q_{C_1}) \arrow[r] \arrow[d, "\Pi_1"] & \mathbf{SO}(q_{C_1}) \arrow[d, "\underline{\Pi}_1"] \\
\mathbf{S}_C \arrow[r] & \underline{\mathbf{S}}_C.
\end{tikzcd}$$

The vertical maps are $\mathbf{Aut}(C)$–torsors; indeed this has already been seen for Π_1, and for $\underline{\Pi}_1$ it follows from the corresponding statement for $\underline{\Pi}$ in the above lemma. In fact we have the commutative diagram

$$\begin{tikzcd}
\mathbf{Spin}(q_{C_1}) \arrow[r] \arrow[d, "\Pi_1"] & \mathbf{SO}(q_{C_1}) \arrow[r] \arrow[d, "\underline{\Pi}_1"] & \mathbf{SO}(q_C) \arrow[d, "\Pi_+"] \\
\mathbf{S}_C \arrow[r] & \underline{\mathbf{S}}_C \arrow[r, "\underline{\nabla}"] & (\mathbf{S}_C \times_R \mathbf{S}_C)/\boldsymbol{\mu}_2
\end{tikzcd}$$

where $\underline{\nabla}$ is induced by $\nabla : \mathbf{S}_C \to \mathbf{S}_C \times_R \mathbf{S}_C$ and is well-defined since $\nabla(\eta x) = \eta\nabla(x)$ for all R–rings S, $\eta \in \boldsymbol{\mu}_2(S)$ and $x \in \mathbf{S}_C(S)$.

6.4. **Equivalence of all variants.** The following theorem shows that all variants considered above are equivalent.

Theorem 6.6. *For $i = 1, \dots, 8$, let*

$$N_i = \mathrm{Ker}\big(H^1_{\mathrm{fppf}}(R, \mathbf{G}) \to H^1_{\mathrm{fppf}}(R, \mathbf{H}_i)\big),$$

where

$$\begin{array}{llll}
\mathbf{H}_1 = \mathbf{Spin}(q_{C_1}), & \mathbf{H}_2 = \mathbf{Spin}(q_C), & \mathbf{H}_3 = \mathbf{SO}(q_C), & \mathbf{H}_4 = \mathbf{O}(q_C), \\
\mathbf{H}_5 = \mathbf{SO}(q_{C_1}), & \mathbf{H}_6 = \mathbf{PSO}(q_C), & \mathbf{H}_7 = \mathbf{GO}^+(q_C), & \mathbf{H}_8 = \mathbf{GO}(q_C),
\end{array}$$

Then all the sets N_i coincide and are subsets of $H^1_{\mathrm{Zar}}(R, \mathbf{G})$.

Proof. We have the following inclusions of sets

$$\begin{tikzcd}
N_1 \arrow[r] \arrow[dr] & N_2 \arrow[r] & N_3 \arrow[r] \arrow[d] \arrow[dr] & N_4 \arrow[r] & N_8 \arrow[r] & H^1_{\mathrm{fppf}}(R, \mathbf{G}) \\
& N_5 \arrow[ur] & N_6 & N_7 \arrow[ur] & &
\end{tikzcd}$$

and will show the reverse inclusions in three steps.

The sets $N_1, \dots, N_5$. We have already seen along the proof of Theorem 4.2 that N_4 consists of Zariski classes. According to Theorem 4.2 (resp. Th. 6.3), N_2 (resp. N_1) consists of isomorphism classes of isotopes $C^{a,b}$ with $a, b \in \mathbf{S}_C(R)$ (resp. $C^{a,a^{-1}}$ with $a \in \mathbf{S}_C(R)$). Then Proposition 2.4 shows that $N_1 = N_2$. On the other hand, the octonionic involution σ_C provides a splitting of the Dickson homomorphism $\mathbf{O}(q_C) \to \mathbb{Z}/2\mathbb{Z}$. This implies that the map $H^1_{\mathrm{fppf}}(R, \mathbf{SO}(q_C)) \to H^1_{\mathrm{fppf}}(R, \mathbf{O}(q_C))$ has trivial kernel, whence $N_3 = N_4$. Next we show that the $N_3 \subseteq N_2$. The characteristic map φ of the $\mathbf{G}$–torsor Π_+ induces the bijection

$$\mathbf{SO}(q_C)(R)\backslash\big((\mathbf{S}_C \times_R \mathbf{S}_C)/\boldsymbol{\mu}_2\big)(R) \xrightarrow{\sim} N_3.$$

Let $[C'] \in N_3$ and let $x \in \big((\mathbf{S}_C \times_R \mathbf{S}_C)/\boldsymbol{\mu}_2\big)(R)$ be such that $\varphi(x) = [C']$. We denote by x_1 its first projection on $\underline{\mathbf{S}}_C(R)$. Lemma 6.4.(2) shows that the group $\mathbf{SO}(q_C)(R)$ acts transitively on $\mathbf{S}_C(R)$. Up to replacing x by a suitable $\mathbf{SO}(q_C)(R)$–conjugate, we may thus assume that $x_1 = [1] \in \underline{\mathbf{S}}_C(R)$. The commutative diagram of $\boldsymbol{\mu}_2$–torsors

$$\begin{tikzcd}
\mathbf{S}_C \times_R \mathbf{S}_C \arrow[r, "p_1"] \arrow[d] & \mathbf{S}_C \arrow[d] \\
(\mathbf{S}_C \times_R \mathbf{S}_C)/\boldsymbol{\mu}_2 \arrow[r, "\underline{p}_1"] & \underline{\mathbf{S}}_C.
\end{tikzcd}$$

shows that x lifts to an element $(a, b) \in \mathbf{S}^2_C(R)$. The diagram (6.1) shows that $\Pi^{-1}(a, b) = \Pi_+^{-1}(x)$ whence $C^{a,b} \cong C'$. Thus $[C'] \in N_2$ as desired. Finally, the statement for N_5 follows from the inclusions $N_1 \subseteq N_5 \subseteq N_3$.

The set N_6. To show that $N_6 \subseteq N_3$, we consider the bijection

$$\mathbf{PSO}(q_C)(R)\backslash\underline{\mathbf{S}}_C(R)^2 \xrightarrow{\sim} N_6$$

induced by the characteristic map φ' of the torsor $\underline{\Pi}'$ from (6.2). Let $[C'] \in N_6$ and let $(y,z) \in \underline{\mathbf{S}}_C^2$ be such that $\varphi'(y,z) = [C']$. We need show that the $\mathbf{PSO}(q_C)(R)$–orbit of (y,z) contains an element in the image of $\mathbf{S}_C(R)^2$ under the quotient map. From Lemma 6.4, this orbit contains $(x,1)$ for some $x \in \underline{\mathbf{S}}_C(R)$. We now consider the R–map $\mathbf{S}_C \to \mathbf{SO}(\mathbf{C_q})$ defined by $a \mapsto L_{\overline{a}}$ for any R–ring S and $a \in \mathbf{S}_C(S)$. This map commutes with the $\boldsymbol{\mu}_2$–action since for any $a \in \mathbf{S}_C(S)$ and any $\eta \in \boldsymbol{\mu}_2(S)$,

$$\eta a \mapsto L_{\overline{\eta a}} = L_{\eta \overline{a}} = \eta L_{\overline{a}},$$

and thus it induces an R–map $g : \underline{\mathbf{S}}_C \to \mathbf{PSO}(q_C)$. Following the strategy of Lemma 6.4, given $x \in \underline{\mathbf{S}}_C(R)$, we pick a flat cover S/R such that $x_S = p_S(a)$ for some $a \in \mathbf{S}_C(S)$, where p denotes the quotient map $\mathbf{S}_C \to \underline{\mathbf{S}}_C$. Then with the notation of the diagram (6.2),

$$g(x)_S = \underline{f_1} \circ P(L_{\overline{a}}, B_a, R_{\overline{a}}).$$

Computing the action of $g(x)$ on $(x,1)$ in $\underline{\mathbf{S}}_C(S)$, we have

$$(g(x).(x,1))_S = (p_S(R_{\overline{a}}(a), p_S(B_a(1))) = (p_S(1), p_S(a^2)) = (1, x_S^2).$$

Thus $g(x).(x,1) = (1, x^2)$. Finally, x^2 is in the image of $p_R : \mathbf{S}_C(R) \to \underline{\mathbf{S}}_C(R)$. To see this, consider the map $a \mapsto a^2$ on $\mathbf{S}_C$. Since it is the trivial map on $\boldsymbol{\mu}_2$, it induces a map $f : \underline{\mathbf{S}}_C \to \mathbf{S}_C$. Thus $f(x) \in \mathbf{S}_C(R)$. But then $x^2 = p \circ f(x)$ since

$$p_S \circ f_S(x_S) = p_S(a^2) = p_S(a)^2 = x^2.$$

Thus under the action of $\mathbf{PSO}(q_C)$, the element (y,z) is conjugate to the image of some $(a,b) \in \mathbf{S}_C(R)^2$, whence C' is isomorphic to $C^{a,b}$ and thus $[C'] \in N_2$ as desired.

The sets N_7 and N_8. We are given $[C'] \in N_8$. The exact sequence

$$\mathbf{1} \to \mathbf{O}(q_C) \to \mathbf{GO}(q_C) \to \mathbf{G}_m \to \mathbf{1}$$

gives rise to an exact sequence of pointed sets

$$R^* \to H^1_{\mathrm{fppf}}(R, \mathbf{O}(q)) \to H^1_{\mathrm{fppf}}(R, \mathbf{GO}(q)).$$

Since $[q_{C'}] \in H^1_{\mathrm{fppf}}(R, \mathbf{O}(q))$ maps to $1 \in H^1_{\mathrm{fppf}}(R, \mathbf{GO}(q))$, there exists $\lambda \in R^*$ and an isomorphism $f : (C', q_{C'}) \xrightarrow{\sim} (C, \lambda q_C)$. Since $q_{C'}$ represents 1, q_C represents λ, i.e. there exists $x \in C^*$ such that $\lambda = q(x)$. Up to replacing f by $L_x^{-1} \circ f$, we may assume that $\lambda = 1$ since q_C is a multiplicative R–form. It follows that the quadratic spaces $(C', q_{C'})$ and (C, q_C) are isomorphic. Thus $[C'] \in N_4$ and is in particular a Zariski class. It follows that $N_8 = N_4$, and since $N_4 \subseteq N_7 \subseteq N_8$ we get $N_7 = N_4$ as well. □

Corollary 6.7. *Let C' be an octonion algebra whose norm is similar to that of C. Then C' is isomorphic to an isotope of C. More precisely there exists $a \in \mathbf{S}_C(R)$ such that C' is isomorphic to $C^{1,a} \cong C^{a,a^{-1}}$.*

Proof. . The set N_8 classifies forms C' of C whose norm is similar to q_C. Theorem 4.2 shows that C' is isomorphic to some isotope $C^{a,b}$ for $a, b \in \mathbf{S}_C(R)$. Proposition 2.4 allows us to assume that C' is isomorphic to $C^{1,d} \cong C^{d,d^{-1}}$ for $d = b^{-1}a$. □

Remark 6.8. This statement generalizes the result of [KPS] quoted in Proposition 2.12 above. It also recovers Remark 2.6: given $a, b \in C^*$, there exists $c \in \mathbf{S}_C(R)$ such that $C^{a,b}$ is isomorphic to $C^{1,c} \cong C^{c,c^{-1}}$. Indeed, we have seen that the norm of the isotope $C^{a,b}$ is isometric to q_C, and the previous Corollary applies.

Corollary 6.9. *Let C' be a R-form of C such that there exists a discriminant R-module $(\mathcal{L},\theta)$ such that the quadratic R-form $(C',q_{C'})$ is isometric to the quadratic R-form $(\mathcal{L},\theta)\otimes(C,q_C)$. Then C' is an isotope of C. In particular $(C',q_{C'})$ is isometric to (C,q_C).*

Proof. We consider the exact sequence of R-group schemes

$$1\to\boldsymbol{\mu}_2\to\mathbf{SO}(q_C)\to\mathbf{PSO}(q_C)\to 1.$$

The composite map

$$\mathbf{Disc}(R)\cong H^1_{\mathrm{fppf}}(R,\boldsymbol{\mu}_2)\to H^1_{\mathrm{fppf}}(R,\mathbf{SO}(q_C))\to H^1_{\mathrm{fppf}}(R,\mathbf{O}(q_C))$$

maps the class of the discriminant R-module $(\mathcal{L},\theta)$ to the class of the quadratic form $(\mathcal{L},\theta)\otimes(C,q_C)$. The hypothesis that $(C',q_{C'})$ is similar to (C,q_C) implies that $[C']$ belongs to the kernel of the map

$$H^1_{\mathrm{fppf}}(R,\mathbf{G})\to H^1_{\mathrm{fppf}}(R,\mathbf{PSO}(q_C))$$

Theorem 6.6 states that $[C']$ belongs to all relevant kernels and therefore is an isotope. □

A complement is the following.

Proposition 6.10. *With the notations of Theorem 6.6, each flat quotient $\mathbf{H}_i/\mathbf{G}$ is representable by an affine R-scheme of finite presentation ($i=1,...,8$). Furthermore the quotient map $\mathbf{H}_i\to\mathbf{H}_i/\mathbf{G}$ is a $\mathbf{G}$–torsor which is non-trivial if R is a non-zero ring.*

Note that we have already seen the first part of the statement in several cases.

Proof. The representability item boils down to the case of $\mathbb{Z}$ and of the Zorn algebra. Representability then follows from Anantharaman's theorem [An, Th. 4C page 53]; the fact that the quotient is affine of finite presentation and also the fact that the quotient map is a $\mathbf{G}$-torsor are due to Colliot-Thélène and Sansuc [CTS, Cor. 6.12] for the connected $\mathbf{H}_i$'s. This generalizes easily to the non-connected case by embedding $\mathbf{H}_i$ in some $\mathbf{GL}_{n_i}$.

Assuming that R is a non-zero ring, we want to show that the $\mathbf{G}$-torsor $\mathbf{H}_i\to\mathbf{H}_i/\mathbf{G}$ is non-trivial. We can then assume that $R=k$ is an algebraically closed field and would like to find a k-ring S such that $H^1_{\mathrm{fppf}}(S,\mathbf{G})\to H^1_{\mathrm{fppf}}(S,\mathbf{H}_i)$ is non-trivial. Theorem 6.6 enables us to choose one of the $\mathbf{H}_i$'s and we consider then the case of $\mathbf{H}_5=\mathbf{SO}(q_{C_1})$. Using the beginning of Section 6.3, we know that $\mathbf{SO}(q_{C_1})/\mathbf{G}\xrightarrow{\sim}\underline{\mathbf{S}}_q$. We take $S=k[\underline{\mathbf{S}}_q]$, i.e. the coordinate ring of $\underline{\mathbf{S}}_q$, and it is enough to show, using the universal point $\eta\in\underline{\mathbf{S}}_q(S)$, that the $\mathbf{G}$-torsor $\mathbf{SO}(q_{C_1})\to\underline{\mathbf{S}}_q$ is non-trivial.

According to Lemma 2.11, $\underline{\mathbf{S}}_q$ is the complement in $\mathbb{P}(C^\vee)$ of the projective quadric Q of equation $q_C=0$. We now use the theory of cycles and Chow groups. We have a localization exact sequence [F, §1.8]

$$\mathrm{Pic}(Q)\xrightarrow{i_*}\mathrm{CH}^2(\mathbb{P}(C^\vee))\to\mathrm{CH}^2(\underline{\mathbf{S}}_q)\to 0.$$

Let h be a hyperplane section of $\mathbb{P}(C^\vee)$. We have $\mathrm{CH}^2(\mathbb{P}(C^\vee))=\mathbb{Z}h^2$ and we know that the hyperplane section $h\cap Q$ generates $\mathrm{Pic}(Q)$. But $i_*i^*(h)=h.[Q]$ where $[Q]\in\mathrm{Pic}(\mathbb{P}(C^\vee))$ (*ibid*, prop. 2.6.(b)). Finally we have $[Q]=2h$ (*ibid*, Example 1.9.4) which allows us to conclude that $\mathrm{CH}^2(\underline{\mathbf{S}}_q)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. On the other hand, Marlin showed that $\mathrm{CH}^2(\mathbf{SO}(q_{C_1}))=0$ [Ma] since $\mathbf{SO}(q_{C_1})$ is an adjoint semisimple group of type B_3. The functoriality of Chow groups yields that the map $\mathbf{SO}(q_{C_1})\to\underline{\mathbf{S}}_q$ cannot admit a splitting. □

Remark 6.11. The advantage of this proposition compared to Example 4.3 is that it is algebraic and works, therefore, in any characteristic. Also it shows that some 2-torsion phenomenon is involved while Example 4.3 was based on 3-torsion. In their paper [AHW], Asok–Hoyois–Wendt investigate the 2-torsion and 3-torsion phenomena in the split case by means of cohomological invariants.

7. Particular cases

We have seen above that given an octonion algebra C over R, any octonion algebra with quadratic form equivalent to that of C is isomorphic to $C^{1,a} \simeq C^{a,\overline{a}}$ for some $a \in \mathbf{S}_C(R)$. Two natural questions arise. On the one hand, one may ask for criteria on a, independent of the base ring, in order that $C^{1,a} \simeq C$. This supplements the relations among isotopes of Section 2 and Remark 5.4. On the other hand one may wish to understand the situation over rings of particular interest. The aim of this section is to discuss these questions.

7.1. **Trivial isotopes.** A question that arises in this context is whether one can characterize those a for which $C^{1,a} \simeq C$. In this section we give sufficient conditions for this to hold. They are consequences of the following proposition, where, for any $c \in C^*$, we define the map $\tau_c^+ : C \to C$ by $x \mapsto cxc^2$ and $\tau_c^- : C \to C$ by $x \mapsto c^{-2}xc^{-1}$.

Proposition 7.1. *Let $a \in \mathbf{S}_C(R)$.*

(1) *If there exist $c_1, \dots, c_r \in C^*$ such that $L_{c_r} \cdots L_{c_1}(1) = L_{\overline{c_r}} \cdots L_{\overline{c_1}}(1) = a$, then*
$$B_{\overline{c_r}} \cdots B_{\overline{c_1}} : C \to C^{a,\overline{a}}$$
is an isomorphism of algebras.

(2) *If there exist $c_1, \dots, c_r \in C^*$ such that $a = \tau_{c_r}^+ \cdots \tau_{c_1}^+(1)$, then*
$$L_{c_r} R_{c_r}^{-1} \cdots L_{c_1} R_{c_1}^{-1} : C \to C^{a,\overline{a}}$$
is an isomorphism of algebras.

Remark 7.2. In [E], Elduque proves that over a field K, the image of the map $f_1 : \mathbf{RT}(C)(K) \to \mathbf{SO}(q_C)(K)$ is precisely
$$\left\{ \prod_{i=1}^r B_{d_i} | r \in \mathbb{N}, d_i \in C^*, \prod_{i=1}^r q(d_i) = 1 \right\}.$$
When this holds, the first item above gives, together with Proposition 4.5, a precise condition for the existence of an isomorphism from C to $C^{a,\overline{a}}$. Note however that the proof of Elduque's result uses the Cartan–Dieudonné Theorem.

Proof. (1) The hypothesis implies that $R_{c_r} \cdots R_{c_1}(1) = \overline{a}$ and that $q(c_1) \cdots q(c_r) = 1$. Thus
$$(B_{\overline{c_r}} \cdots B_{\overline{c_1}}, R_{c_r} \cdots R_{c_1}, L_{c_r} \cdots L_{c_1}) \in \mathbf{RT}(C),$$
and the claim follows from Proposition 4.5.

(2) Using the Moufang laws, one verifies that for each $c \in C^*$ we have
$$L_c R_c^{-1}(\overline{x}\ \overline{y}) = \tau_c^+(\overline{x})\tau_c^-(\overline{y}) = \overline{\tau_c^-(x)}\ \overline{\tau_c^+(y)}. \tag{7.1}$$
If now $c_1, \dots, c_r \in C^*$ are such that $\tau_{c_r}^+ \cdots \tau_{c_1}^+(1) \in \mathbf{S}_C(R)$, then $q(c_1)^3 \cdots q(c_r)^3 = 1$, whence (7.1) implies that
$$(L_{c_r} R_{c_r}^{-1} \cdots L_{c_1} R_{c_1}^{-1}, \tau_{c_r}^- \cdots \tau_{c_1}^-, \tau_{c_r}^+ \cdots \tau_{c_1}^+) \in \mathbf{RT}(C)(R).$$
We then conclude with Proposition 4.5. □

This has the following consequences.

Corollary 7.3. *Let $a \in \mathbf{S}_C(R)$. If either*

(1) *there exists $c \in C^*$ with $a = c^3$, or*
(2) *there exists $u \in C^*$ such that $b_q(u,1) = b_q(u,a) = 0$, or*
(3) *$b_q(a,1) = 0$,*

then $C^{a,\overline{a}} \simeq C$.

Proof. The first item follows from item (2) of the above proposition upon setting $r = 1$ and $c_1 = c$. For the second, we set instead $r = 2$, $c_1 = u^{-1}$ and $c_2 = au$, and note that $u\overline{a} = au$. Then $c_2^2 = (u\overline{a})(au) = u^2$ implies that

$$\tau^+_{c_2}\tau^+_{c_1}(1) = (au)(u^{-3}(au)^2) = (au)(u^{-3}u^2) = (au)u^{-1} = a.$$

The third item follows from the first since if $b_q(a,1) = 0$, then $a^2 = -q(a)1 = -1$ and $a = (-a)^3$. □

As a consequence of the second item, we easily obtain the fact that over fields, octonion algebras with equivalent quadratic forms are isomorphic.

Corollary 7.4. *Let k be a field and let C and C' be two octonion algebras over k with isometric norms. Then $C \simeq C'$.*

Proof. It suffices to prove that $C^{a,a^{-1}} \simeq C$ for any $a \in \mathbf{S}_C(k)$. By the above corollary, it moreover suffices to show that there exists an invertible $u \in a^\perp \cap 1^\perp$. But Lemma 2.9 implies that $a^\perp \cap 1^\perp$ is a subspace of dimension at least 6. By Witt's theorem, the regularity of q_C implies the existence of $u \in a^\perp \cap 1^\perp$ with $q_C(x) \neq 0$, and the claim follows. □

Remark 7.5. It is possible to extend this argument to cover the case of octonion algebras over local rings. This requires some technical work, including a generalization of Corollary 7.3. We therefore refrain from carrying it out, referring instead to the proof found in [B].

7.2. **Isotopes of the Zorn algebra.** We shall now slightly generalize a result by Asok–Hoyois–Wendt [AHW, Prop. 4.3.4]. We assume in this subsection that C is the split octonion algebra. It admits a subalgebra $R \times R$ and we know that its centralizer in $\mathbf{G}$ is the standard R–subgroup $\mathbf{SL}_3 \subset \mathbf{G}$ (in case R is a field, this is in [KMRT, §8, ex. 6]).

Proposition 7.6. *Assume that C is the split octonion algebra. We have*

$$\ker\Big(H^1_{Zar}(R,\mathbf{G}) \to H^1_{Zar}(R,\mathbf{Spin}(q_C))\Big) \subseteq \mathrm{Im}\Big(H^1_{Zar}(R,\mathbf{SL}_3) \to H^1_{Zar}(R,\mathbf{G})\Big)$$

Proof. We reason in terms of octonion algebras. Let $[C']$ be an element of the leftmost kernel, i.e. the class of an isotope C' of C. There exists an isometry $f : (C, q_C) \xrightarrow{\sim} (C', q_{C'})$ mapping $1_C \to 1_{C'}$. We put $A' = f(R \times R)$. Then A' is a composition R–subalgebra of C'. We consider now the flat sheaf defined by

$$F(S) = \Big\{ g : C_S \xrightarrow{\sim} C'_S \mid u \text{ induces } f_S : (R \times R)_S \xrightarrow{\sim} A'_S \Big\}$$

for each R-ring S. It is clearly representable by an affine R–scheme $\mathbf{Y}$ and comes equipped with a right action of $\mathbf{SL}_3$ so that the R–map $\mathbf{Y} \times_R \mathbf{SL}_3 \to \mathbf{Y} \times_R \mathbf{Y}$, $(y,h) \mapsto (h, y.h)$ is an isomorphism. Inspection of the proof of Bix's theorem [B, lemma 1.1] shows that $\mathbf{Y}(R)$ is non empty when R is local. Altogether it follows that $\mathbf{Y}$ is a $\mathbf{SL}_3$-torsor locally trivial for the Zariski topology. The $\mathbf{G}$–torsor $\mathrm{Isom}(C, C')$ then arises as the contracted product of the $\mathbf{SL}_3$-torsor $\mathbf{Y}$ with respect to the embedding $\mathbf{SL}_3 \hookrightarrow \mathbf{G}$. □

The map $H^1_{\mathrm{fppf}}(R, \mathbf{SL}_3) \to H^1_{\mathrm{fppf}}(R, \mathbf{G})$ is well understood in terms of octonion algebras. The set $H^1_{\mathrm{fppf}}(R, \mathbf{SL}_3)$ classifies the pairs (P, θ) where P is a locally free R-module of rank 3 and $\theta : R \xrightarrow{\sim} \Lambda^3(P)$ a trivialization of its determinant. Petersson [Pe, §3] explicitly constructed the R–algebra $\mathrm{Zorn}(P, \theta)$ associated to (P, θ) and its norm form is the hyperbolic form attached to the R–module $R \oplus P$. In conclusion, we have shown that an isotope of C is an R–algebra $\mathrm{Zorn}(P, \theta)$ such that the hyperbolic form attached to $R \oplus P$ is isomorphic to that of R^4.

Remark 7.7. (a) The original statement [AHW, Prop. 4.3.4] deals with the case of a smooth ring over an infinite field and uses $\mathbf{A}^1$-homotopy theory. Our proof is then in a sense much simpler.

(b) The map $H^1_{\mathrm{fppf}}(R, \mathbf{SL}_3) \to H^1_{\mathrm{fppf}}(R, \mathbf{GL}_3)$ has trivial kernel. If P is free, it follows that (P, θ) encodes the trivial class of $H^1_{\mathrm{fppf}}(R, \mathbf{SL}_3)$ so that $\mathrm{Zorn}(P, \theta)$ is the split octonion R–algebra. In particular, if all projective modules of rank 3 are free, Proposition 7.6 implies that all isotopes of the Zorn algebra are trivial.

Using Swan's extension of Quillen–Suslin's theorem [S, Cor. 1.4], we get the following fact. (In the polynomial case, this was already proved in [KPS, Prop. 7.4].)

Corollary 7.8. *If k is a field and $R = k[t_1^{\pm 1}, ..., t_d^{\pm 1}, t_{d+1}, \dots, t_n]$, then the isotopes of the split octonion algebra are split.*

7.3. **Polynomial and Laurent polynomial rings.** We will now consider octonion algebras over polynomial and Laurent polynomial rings. We start with a preliminary fact about unit spheres.

Lemma 7.9. *Assume that R is an integral domain with fraction field K. Let q be a non-singular quadratic R–form such that q_K is anisotropic. Then*

$$\mathbf{S}_q(R) = \mathbf{S}_q(R[t]) = \mathbf{S}_q(R[t^{\pm 1}]).$$

Proof. The assumption implies that the semisimple K–group $\mathbf{SO}(q_K)$ is anisotropic. Thus $\mathbf{SO}(q)\big(K[[t^{-1}]]\big) = \mathbf{SO}(q)\big(K((t^{-1}))\big)$ by Bruhat–Tits–Rousseau's theorem (see [Gi1, Th. 1.2]). Since $\mathbf{SO}(q)\big(K((t^{-1}))\big)$ acts transitively on $\mathbf{S}_q\big(K((t^{-1}))\big)$, we obtain $\mathbf{S}_q\big(K[[t^{-1}]]\big) = \mathbf{S}_q\big(K((t^{-1}))\big)$.

We are given a point $x \in \mathbf{S}_q(R[t])$; its image in $\mathbf{S}_q\big(K((t^{-1}))\big)$ then belongs to $\mathbf{S}_q\big(K[[\frac{1}{t}]]\big)$, so that the image x_K of x in $\mathbf{S}_q(K[t])$ extends to a morphism $\mathbb{P}^1_K \to \mathbf{S}_q \times_R K$. It follows that x_K belongs to $\mathbf{S}_q(K)$. But $K \cap R[t] = R$, whence $x \in \mathbf{S}_q(R)$. Similarly, working with $K((t))$, we find $\mathbf{S}_q(R) = \mathbf{S}_\varphi(R[t^{\pm 1}])$. □

Lemma 7.10. *Assume that R is an integral domain with fraction field K. Let C be an octonion R–algebra such that C_K is not split. Let C' be a $R[t]$-isotope of $C \otimes_R R[t]$. Then $C' \cong B \otimes_R R[t]$ where B is an R-isotope of C. The same holds with $R[t^{\pm 1}]$.*

Proof. Our assumption is that C_K is not split, equivalently the quadratic form $q_C \otimes_R K$ is anisotropic. Since isotopes of $C \otimes_R R[t]$ are parametrized by $\mathbf{S}_C(R[t])$, the statement follows from Lemma 7.9. □

Since anisotropy is preserved by purely transcendental extensions, we get by induction the following fact which recovers and slightly extends a result by Knus–Parimala–Sridharan [KPS, Cor. 7.2].

Corollary 7.11. *Let k be a field and assume that $R = k[t_1^{\pm 1}, ..., t_d^{\pm 1}, t_{d+1}, \dots, t_n]$, Let C be a non-split octonion k–algebra. Then all R–isotopes of $C \otimes_k R$ are trivial.*

In the case of Laurent polynomial rings over fields, we can generalize this statement as follows.

Proposition 7.12. *Assume that $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is a Laurent polynomial ring over a field k of characteristic 0. Let C be an octonion R–algebra such that q_C is isometric to a Pfister quadratic form. Then all R–isotopes of C are trivial.*

The proof requires a preliminary statement of the same flavour as Lemma 7.9.

Lemma 7.13. *Let A be an integral domain with fraction field K. Assume that 2 is invertible in A. Let (M_1, q_1) and (M_2, q_2) be regular quadratic A–forms and consider the $A[t^{\pm 1}]$–quadratic form $q = q_1 \perp tq_2$ defined on $M_1 \otimes_A A[t^{\pm 1}] \oplus M_2 \otimes_A A[t^{\pm 1}]$. We assume that q_1 and q_2 are K–anisotropic. Then*

$$\mathbf{S}_{q_1}(K) = \mathbf{S}_q\big(K[t^{\pm 1}]\big) \ \text{ and } \ \mathbf{S}_{q_1}(A) = \mathbf{S}_q\big(A[t^{\pm 1}]\big).$$

Proof. We claim that we have

$$(I) \qquad \mathbf{S}_q\big(K((t))\big) \subseteq M_1 \otimes_A K[[t]] \oplus M_2 \otimes_A K[[t]]$$

and

$$(II) \qquad \mathbf{S}_q\big(K((t^{-1}))\big) \subseteq M_1 \otimes_A K[[t^{-1}]] \oplus M_2 \otimes_A K[[t^{-1}]].$$

We start with (I). Let $m = m_1 + m_2 \in M \otimes_A K((t))$ such that $q(m) = 1$. For $i = 1, 2$, we write $m_i = t^{n_i} \underline{m}_i$ with $\underline{m}_i \in M_i \otimes_A K[[t]]^*$. (Recall that $K[[t]]^* = K[[t]] \setminus tK[[t]]$.)

If $m_1 = 0$, then $1 = q(t^{n_2} \underline{m}_2) = t^{2n_2+1} q_2(\underline{m}_2)$. Since q_2 is K-anisotropic, $q_2(\underline{m}_2)$ is a unit and we get a contradiction. Thus m_1 is non-zero. If $m_2 = 0$, then $1 = q(t^{n_1} \underline{m}_1) = t^{2n_1} q_1(\underline{m}_1)$. Since q_1 is K-anisotropic, $q_1(\underline{m}_1)$ is a unit and we get $n_1 = 0$, whence $m_1 \in M_1 \otimes_A K[[t]]^*$. If m_1 and m_2 are both non-zero, then

$$1 = q(t^{n_1} m_1 + t^{n_2} m_2) = t^{2n_1} q_1(\underline{m}_1) + t^{2n_2+1} q_2(\underline{m}_2)$$

where $q_1(\underline{m}_1)$ and $q_2(\underline{m}_2)$ are both units (for the evaluation at $t = 0$). For evaluation reasons the only possibility is that $n_1 = 0$ and $n_2 \geq 0$. In all cases, the inclusion (I) holds.

To show (II), the case $m_1 = 0$ and $m_2 = 0$ are similar to the corresponding cases in (I). Assume that m_1, m_2 are both non-zero and write $m_i = t^{-n_i} \underline{m}_i$ with $\underline{m}_i \in M_i \otimes_A K[[t^{-1}]]^*$ for $i = 1, 2$. We have

$$1 = q(t^{-n_1} m_1 + t^{-n_2} m_2) = t^{-2n_1} q_1(\underline{m}_1) + t^{-2n_2+1} q_2(\underline{m}_2)$$

where $q_1(\underline{m}_1)$ and $q_2(\underline{m}_2)$ are both units (for the evaluation at ∞). Once again for evaluation reasons, we have $n_1 = 0$ and $2n_2 - 1 \geq 0$ so that $n_2 \geq 0$. The inclusion (II) holds.

Consider now an element $x = m_1 + m_2 \in \mathbf{S}_q\big(K[t^{\pm 1}]\big)$ where $m_i \in M_i \otimes_A K[t^{\pm 1}]$ for $i = 1, 2$. The inclusions (I) and (II) yield that $m_i \in M_i \otimes_A K$ for $i = 1, 2$. Furthermore the equation $1 = q_1(m_1) + tq_2(m_2)$ implies that $q_1(m_1) = 1$ and $q_2(m_2) = 0$. The anisotropy of $q_{2,K}$ ensures that $m_2 = 0$. Thus $m = m_1$ comes from $\mathbf{S}_{q_1}(K)$ as desired.

The first statement is then established and the second statement readily follows. □

Proof of Proposition 7.12. Our assumption is that q_C is a Pfister form with entries $\langle\langle u, v, w \rangle\rangle$ where $u, v, w \in R^* = k^* t_1^{\mathbb{Z}} \dots t_n^{\mathbb{Z}}$. Up to using the multiplicativity properties of q_C and the action of $\mathrm{GL}_n(\mathbb{Z})$ on R, we may assume that q_C is isometric to a quadratic form of one of the following shapes:

(i) $u, v, w \in k^*$.

(ii) $u, v \in k^*$ and $w = w_n t_n$ with $w_n \in k^*$.

(iii) $u \in k^*$, $v = v_{n-1} t_{n-1}$, $w = w_n t_n$ with $v_{n-1}, v_n \in k^*$.

(iv) $u = u_{n-2}t_{n-2}$, $v = v_{n-1}t_{n-1}$, $w = w_n t_n$ with $u_{n-2}, v_{n-1}, v_n \in k^*$.

Since (i) includes the hyperbolic case, we may assume in cases (ii), (iii) and (iv) that q_C is anisotropic over the field $k(t_1, \dots, t_n)$, and even over $k((t_1)) \dots ((t_n))$ by [GP, Cor. 7.4.3].

Case (i): Let p be a 3-Pfister form such that $p \otimes_k R \cong q_C$. Let C_p be an octonion k–algebra whose norm is p. Such an algebra is unique up to isomorphism. Then C is a an isotope of the octonion R–algebra $C_p \otimes_k R$. If p is split (resp. anisotropic), Corollary 7.8 (resp. Cor. 7.11) shows that C is isomorphic to $C_p \otimes_k R$.

Case (ii), (iii) and (iv) : We apply Lemma 7.13 with $A = k[t_1^{\pm 1}, \dots, t_{n-1}^{\pm 1}]$, $t = t_n$ and $q_1 = \langle\langle u, v\rangle\rangle$, $q_2 = w_n \langle\langle u, v\rangle\rangle$. It follows that

$$\mathbf{S}_{\langle\langle u,v\rangle\rangle}(k[t_1^{\pm 1}, \dots, t_{n-1}^{\pm 1}]) = \mathbf{S}_{q_C}(R).$$

Let $a \in \mathbf{S}_{\langle\langle u,v\rangle\rangle}(k[t_1^{\pm 1}, \dots, t_{n-1}^{\pm 1}])$. Then the isotope $C^{a,\overline{a}}$ is trivial by Corollary 7.3.(2). We conclude that all isotopes of C are trivial. □

Remark 7.14. (a) The characteristic zero hypothesis is probably superfluous, it should work in odd characteristic as well.

(b) Assume that the field k is algebraically closed of characteristic zero. In this case, the so-called "loop octonion algebras" (that is whose underlying torsor is a loop torsor) over $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ are classified [GP, Cor. 11.2] and their norms are Pfister forms. The statement shows that those octonion algebras have no non-trivial isotopes.

Appendix A. Review on quadratic forms

A.1. **Definition.** In this work a quadratic module is a pair (M, q) consisting of a finitely generated and projective R–module M and a quadratic form $q\colon M \to R$. For a quadratic module (M, q) we will use the following notation and terminology: $b_q\colon M \times M \to R$ is the polar form of q, defined by $b_q(m_1, m_2) = q(m_1 + m_2) - q(m_1) - q(m_2)$ and often abbreviated by $b = b_q$. It gives rise to the R–linear map

$$b_q^\vee\colon M \to M^\vee = \mathrm{Hom}_R(M, R), \quad b_q^\vee(m)\,(n) = b_q(m, n). \tag{A.1}$$

The radicals of b_q and q are

$$\mathrm{rad}(b_q) = \big\{m \in M : b_q(m, M) = 0\big\} = \mathrm{Ker}(b_q^\vee),$$
$$\mathrm{rad}(q) = \big\{m \in M : q(m) = 0 = b_q(m, M)\big\} \subseteq \mathrm{rad}(b_q).$$

For $S \in R$-**alg** the quadratic form $q_S\colon M_S \to S$ is the unique quadratic form satisfying $q_S(m \otimes 1_S) = q(m) \otimes 1_S$.

We call a quadratic form q *regular* if $b_q^\vee$ is an isomorphism of R–modules, and *non-singular* if $\mathrm{rad}(q_K) = \{0\}$ for all fields $K \in R$-**alg**. We list some useful facts.

- Let $a \in R$. The "1–dimensional" form $\langle a\rangle\colon R \to R$, defined by $r \mapsto ar^2$, is non-singular if and only if $a \in R^*$; it is regular if and only if $2a \in R^*$.
- A regular quadratic form is non-singular, and if $2 \cdot 1_R \in R^*$ then these two properties coincide.
- If q is regular resp. non-singular, then so is q_S for all $S \in R$-**alg**.
- The hyperbolic form on R^{2n}, defined by $q_{\mathrm{hyp}}(r_{-n}, \dots, r_n) = \sum_{i=1}^n r_i r_{-i}$, is regular and non-singular.
- If (M, q) is a quadratic module where M has constant even rank, then q is regular if and only if q is non-singular.

Following Knus [K, III.3], we denote by $\mathbf{Disc}(R)$ the (commutative) group of isometry classes of non-singular quadratic forms of rank one over R (also called discriminant modules) where the product arises from the tensor product. Since $\boldsymbol{\mu}_2$

is the orthogonal group of the form $x \to x^2$, the yoga of descent shows that the group $\mathbf{Disc}(R)$ is isomorphic to $H^1_{\mathrm{fppf}}(R, \boldsymbol{\mu}_2)$ (ibid, III.3.2).

A.2. **Quotients of unit spheres by the antipodal relation.** We consider the unit sphere $\mathbf{S}_q$ of q. This is an affine R–scheme such that for any R–ring S, $\mathbf{S}_q(S) = \{m \in M_S \mid q(m) = 1\}$. We are also interested in the affine R–scheme $\widehat{\mathbf{S}}_q$ defined by $\widehat{\mathbf{S}}_q(S) = \{m \in M_S \mid q(m) \in S^*\}$. We have a closed immersion $\mathbf{S}_q \subset \widehat{\mathbf{S}}_q$.

Lemma A.1. (1) *The sphere $\mathbf{S}_q$ is faithfully flat over R and $\widehat{\mathbf{S}}_q$ is smooth over R.*
(2) *If M is of constant even rank $2r \geq 2$, then $\mathbf{S}_q$ is smooth over R of relative dimension $2r - 1$.*

Proof. (1) The map $\mathbf{S}_q \to \mathrm{Spec}(R)$ admits a splitting and hence is faithfully flat. The R–scheme $\widehat{\mathbf{S}}_q$ clearly satisfies the lifting criterion of smoothness; it is then formally smooth, hence smooth since $\widehat{\mathbf{S}}_q$ is of finite presentation.
(2) Using (1), the fibre-wise smoothness criterion [EGAIV, $_4$.17.8.2] reduces the proof to the case of an algebraically closed field k. In this case $\mathbf{S}_q$ is isomorphic to the affine hypersurface $\sum_{i=1}^n x_i y_i = 1$, which is smooth by the Jacobian criterion. □

The R–group scheme $\boldsymbol{\mu}_2$ (resp. $\mathbb{G}_m$) acts on $\mathbf{S}_q$ (resp. $\widehat{\mathbf{S}}_q$) by homothety. We come to the quotient scheme $\underline{\mathbf{S}}_q = \mathbf{S}_q/\boldsymbol{\mu}_2 = \widehat{\mathbf{S}}_q/\mathbb{G}_m$, that is the quotient of the unit sphere by the antipodal relation [SGA3, VIII.5.1]. We remind the reader that the quotient map $f_q : \mathbf{S}_q \to \underline{\mathbf{S}}_q$ is finite, faithfully flat and that $\mathbf{S}_q$ is an affine R–scheme. Since the morphism $\widehat{f}_q : \widehat{\mathbf{S}}_q \to \underline{\mathbf{S}}_q$ is smooth, we see that $\underline{\mathbf{S}}_q$ is smooth over R. We are interested in the characteristic map $\varphi : \mathbf{S}_q(R) \to H^1_{\mathrm{fppf}}(R, \boldsymbol{\mu}_2) \cong \mathbf{Disc}(R)$. For each $x \in \underline{\mathbf{S}}_q(R)$, $\widehat{f}_q^{-1}(x)$ defines an invertible R–module $\mathcal{L}_x$ which is an R–submodule of M.

Lemma A.2. *(1) Let $x \in \underline{\mathbf{S}}_q(R)$. Let S/R be a flat cover such that x lifts to some element $m \in \mathbf{S}_q(S)$. Then $\mathcal{L}_x \otimes_R S = Sm$.*
(2) The restriction of q to $\mathcal{L}_x$ is non-singular and $\varphi(x) = [(\mathcal{L}_x, q)] \in \mathbf{Disc}(R)$.

Proof. (1) Obvious.
(2) We have to compare the $\boldsymbol{\mu}_2$–torsor $\mathbf{E}_x = f^{-1}(x)$ with the $\boldsymbol{\mu}_2$–torsor $\mathbf{E}$ defined by $\mathbf{E}(S) = \mathrm{Isom}_S\big((R;1)S, (\mathcal{L}_x, q)\big)$ for each R–ring S. Let S be a flat cover of R such that $\mathbf{E}_x(S) \neq \emptyset$. There exists $m \in \mathbf{S}_q(S)$ such that $f_q(m) = x_S$. By (1), we have $\mathcal{L}_x \otimes_R S = Sm$ and an isomorphism $u_m : (R,1)_S \xrightarrow{\sim} (\mathcal{L}_x, q)$, $1 \mapsto m$. This isomorphism is $\boldsymbol{\mu}_2$–equivariant; by faithfully flat descent it gives rise to an isomorphism $\mathbf{E}_x \xrightarrow{\sim} \mathbf{E}$ of $\boldsymbol{\mu}_2$–torsors over R. □

Appendix B. A Smoothness Condition

Lemma B.1. *Let R be a Dedekind ring. Let $\mathfrak{X}$ be an R–scheme of finite presentation, equidimensional with smooth connected fibres. Then $\mathfrak{X}$ is smooth.*

Proof. We denote by K field of fractions of R. Let $\widetilde{\mathfrak{X}}$ be the schematic closure of $\mathfrak{X}_K$ in $\mathfrak{X}$ [EGAIV, $_2$.2.8.1]. This is a closed R–subscheme of $\mathfrak{X}$ which is flat. Let d be the dimension of the fibres of $\mathfrak{X}/R$. Let $s \in \mathrm{Spec}(R)$. According to the Chevalley semi-continuity theorem [EGAIV, $_3$.13.1.3], the $\kappa(s)$–scheme $\widetilde{\mathfrak{X}}_s$ has dimension $\geq d$. On the other hand, $\widetilde{\mathfrak{X}}_s$ is a closed subscheme of $\mathfrak{X}_s$, whereby $\widetilde{\mathfrak{X}}_s$ is of dimension d. But $\mathfrak{X}_s$ is smooth connected so that $\widetilde{\mathfrak{X}}_s = \mathfrak{X}_s$. It follows that $\mathfrak{X} = \widetilde{\mathfrak{X}}$, as $\widetilde{\mathfrak{X}}$ is flat. The fibre-wise smoothness criterion [EGAIV, $_4$.17.8.2] enables us to conclude that $\mathfrak{X}$ is smooth over R. □

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