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WEIGHTED AND BOUNDARY L^p ESTIMATES FOR SOLUTIONS OF THE $\bar{\partial}$ -EQUATION ON LINEALLY CONVEX DOMAINS OF FINITE TYPE AND APPLICATIONS

P. CHARPENTIER & Y. DUPAIN

ABSTRACT. We obtain sharp weighted estimates for solutions of the equation $\bar{\partial}u = f$ in a lineally convex domain of finite type. Precisely we obtain estimates in the spaces $L^p(\Omega, \delta^\gamma)$, δ being the distance to the boundary, with two different types of hypothesis on the form f : first, if the data f belongs to $L^p(\Omega, \delta^\gamma)$, $\gamma > -1$, we have a mixed gain on the index p and the exponent γ ; secondly we obtain a similar estimate when the data f satisfies an appropriate anisotropic L^p estimate with weight $\delta_\Omega^{\gamma+1}$. Moreover we extend those results to $\gamma = -1$ and obtain $L^p(\partial\Omega)$ and $BMO(\partial\Omega)$ estimates. These results allow us to extend the $L^p(\Omega, \delta^\gamma)$ -regularity results for weighted Bergman projection obtained in [CDM14b] for convex domains to more general weights.

1. INTRODUCTION

Sharp estimates for solutions of the $\bar{\partial}$ -equation are a fundamental tool to study various problems in complex analysis of several variables.

In this paper we consider the case of smoothly bounded lineally convex domain of finite type Ω in \mathbb{C}^n . Precisely, we obtain new weighted L^p estimates for solutions of the equation $\bar{\partial}u = f$, f being a $(0, r)$ -form, $1 \leq r \leq n-1$, in Ω .

Our first result is a weighted $L^p(\Omega, \delta_\Omega^\gamma)$ estimate (δ_Ω being the distance to the boundary of Ω) with a mixed gain on p and γ extending results obtained (without weights) for convex domains of finite type by various authors (for example, A. Cumenge in [Cum01a, Theorem 1.2] and B. Fisher in [Fis01, Theorem 1.1]) (see also T. Hefer [Hef02]).

The second one gives a weighted $L^p(\Omega, \delta_\Omega^\gamma)$, $\gamma > -1$, estimate with gain on γ , with a non isotropic hypothesis $\|f\|_{k,p}$ (formula (2.1)) on the form f generalizing the estimate obtained by A. Cumenge ([Cum01b, Theorem 1.3]) in convex domains of finite type for $p = 1$ with the “norm” $\|f\|_k$ defined in [BCD98]. As far as we know, the estimate presented here (for $p > 1$) was only stated for strictly pseudoconvex domains in [Cha80, Theorem 1.4 and Remark that follows].

The two last results are $L^p(\partial\Omega)$ estimates. The first is the limit case of the previous one when γ tends to -1 and the second one is a $BMO(\partial\Omega)$ and $L^p(\partial\Omega)$ estimate with an hypothesis based on Carleson measure for $(0, 1)$ -forms only.

To prove these results, we use the method introduced in [CDM14a], which overcomes the fact that the Diederich-Fornaess support function is only locally defined and that it is not possible to extend it to the whole domain (like W. Alexandre did it in the convex case in [Ale01]) using a division with good estimates, our domain being non convex.

These estimates are used next to generalize an estimate for weighted Bergman projections obtained in [CDM14b] for convex domains of finite type.

The study of the regularity of the Bergman projection onto holomorphic functions in a given Hilbert space is a very classical subject. When the Hilbert space is the standard

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Lebesgue L^2 space on a smoothly bounded pseudoconvex domain Ω in \mathbb{C}^n , many results are known and there is a very large bibliography.

When the Hilbert space is a weighted L^2 space on a smoothly bounded pseudoconvex domain Ω in \mathbb{C}^n , it is well known for a long time that the regularity of the Bergman projection depends strongly on the weight ([Koh73], [Bar92], [Chr96]). Until last years few results were known (see [FR75], [Lig89], [BG95], [CL97]) but recently some positive and negative results were obtained by several authors (see for example [Zey11], [Zey12], [Zey13b], [Zey13a], [CDM14b], [CDM15], [ČZ16], [Zey16] and references therein).

Let Ω be a convex domain of finite type m in \mathbb{C}^n . Let g be a gauge function for Ω and define $\rho_0 = g^4 e^{1-1/s} - 1$. Let P_{ω_0} be the Bergman projection of the space $L^2(\Omega, \omega_0)$, where $\omega_0 = (-\rho_0)^r$, $r \in \mathbb{Q}_+$. Then in [CDM14b, Theorem 2.1] we proved that P_{ω_0} maps continuously the spaces $L^p(\Omega, \delta_\Omega^\beta)$, $p \in]1, +\infty[$, $0 < \beta + 1 \leq p(r+1)$, into themselves. Here we consider a weight ω which is a non negative rational power of a \mathcal{C}^2 function in $\overline{\Omega}$ equivalent to the distance to the boundary and we prove that the Bergman projection P_ω of the Hilbert space $L^2(\Omega, \omega)$ maps continuously the spaces $L^p(\Omega, \delta_\Omega^\beta)$, $p \in]1, +\infty[$, $0 < \beta + 1 \leq r + 1$ into themselves and the lipschitz spaces $\Lambda_\alpha(\Omega)$, $0 < \alpha \leq 1/m$, into themselves.

This result is obtained comparing the operators P_{ω_0} and P_ω with the method described in [CDM15]. To do it, we use the weighted $L^p(\Omega, \delta_\Omega^\gamma)$ estimates with appropriate gains on the index p and on the power γ for solution of the $\bar{\partial}$ -equation obtained in the first part.

2. NOTATIONS AND MAIN RESULTS

Throughout this paper we will use the following general notations:

- Ω is a smoothly bounded linearly convex domain of finite type m in \mathbb{C}^n . Precisely (c.f. [CDM14a]) “linearly convex” means that, for all point in the boundary $\partial\Omega$ of Ω , there exists a neighborhood W of p such that, for all point $z \in \partial\Omega \cap W$,

$$(z + T_z^{1,0}) \cap (D \cap W) = \emptyset,$$

where $T_z^{1,0}$ is the holomorphic tangent space to $\partial\Omega$ at the point z . Furthermore, we can assume that there exists a smooth defining function ρ of Ω such that, for δ_0 sufficiently small, the domains $\Omega_t = \{\rho(z) < t\}$, $-\delta_0 \leq t \leq \delta_0$, are all linearly convex of finite type $\leq m$.

- δ_Ω denotes the distance to the boundary of Ω .
- For any real number $\gamma > -1$, we denote by $L^p(\Omega, \delta_\Omega^\gamma)$ the L^p -space on Ω for the measure $\delta_\Omega^\gamma(z) d\lambda(z)$, λ being the Lebesgue measure.
- $BMO(\Omega)$ denotes the standard BMO space on Ω and $\Lambda_\alpha(\Omega)$ the standard lipschitz space.

Our first results give sharp $L^q(\Omega, \delta_\Omega^\gamma)$, $BMO(\Omega)$ and $\Lambda_\alpha(\Omega)$ estimates for solutions of the $\bar{\partial}$ -equation in Ω with data in $L^p(\Omega, \delta_\Omega^\gamma)$:

Theorem 2.1. *Let N be a positive large integer. let γ and γ' be two real numbers such that $\gamma' > -1$ and $\gamma - 1/m \leq \gamma' \leq \gamma \leq N - 2$. Then there exists a linear operator T , depending on ρ and N , such that, for any $\bar{\partial}$ -closed $(0, r)$ -form $(1 \leq r \leq n - 1)$ with coefficients in $L^p(\Omega, \delta_\Omega^\gamma)$, $p \in [1, +\infty]$, Tf is a solution of the equation $\bar{\partial}(Tf) = f$ such that:*

- (1) *If $1 \leq p < \frac{m(\gamma'+n)+2-(m-2)(r-1)}{1-m(\gamma-\gamma')}$, then T maps continuously the space of $\bar{\partial}$ -closed forms with coefficients in $L^p(\Omega, \delta_\Omega^\gamma)$ into the space of forms whose coefficients are in $L^q(\Omega, \delta_\Omega^{\gamma'})$ with $\frac{1}{q} = \frac{1}{p} - \frac{1-m(\gamma-\gamma')}{m(\gamma'+n)+2-(m-2)(r-1)}$.*

- (2) If $p = m(\gamma + n) + 2 - (m - 2)(r - 1)$, then T maps continuously the space of $\bar{\partial}$ -closed forms with coefficients in $L^p(\Omega, \delta_\Omega^\gamma)$ into the space of forms whose coefficients are in $BMO(\Omega)$;
- (3) If $p \in]m(\gamma + n) + 2 - (m - 2)(r - 1), +\infty]$, then T maps continuously the space of $\bar{\partial}$ -closed forms with coefficients in $L^p(\Omega, \delta_\Omega^\gamma)$ into the space of forms whose coefficients are in the lipschitz space $\Lambda_\alpha(\Omega)$ with $\alpha = \frac{1}{m} \left[1 - \frac{m(\gamma + n) + 2 - (m - 2)(r - 1)}{p} \right]$.

Remark.

- Note that, if $\gamma' < \gamma$, then $\frac{m(\gamma' + n) + 2 - (m - 2)(r - 1)}{1 - m(\gamma - \gamma')} > m(\gamma + n) + 2 - (m - 2)(r - 1)$, and (3) is sharper than (1) for

$$p \in \left] m(\gamma + n) + 2 - (m - 2)(r - 1), \frac{m(\gamma' + n) + 2 - (m - 2)(r - 1)}{1 - m(\gamma - \gamma')} \right[.$$

- For $\gamma = \gamma' = 0$ and $r = 1$, these estimates are known to be sharp (see [CKM93]).
- For $m > 2$ and $r \geq 2$, the above result is strictly better than the one obtain for convex domains by A Cumenge in [Cum01a, Theorem 1.2] and B. Fisher in [Fis01, Theorem 1.1] and for complex ellipsoids in [CKM93].

The two next propositions, which are immediate corollaries of the theorem, will be used in the last section:

Proposition 2.1. *For all large integer N , there exists a linear operator T solving the $\bar{\partial}$ -equation in Ω such that, for all $p \in [1, +\infty[$ and all γ , $-1 < \gamma \leq N - 3$, there exists a constant $C_{N,p,\gamma} > 0$ such that, for all $\bar{\partial}$ -closed $(0, r)$ -form f , $1 \leq r \leq n - 1$, we have*

$$\int_{\Omega} |Tf|^p \delta_\Omega^\gamma d\lambda \leq C_{N,p,\gamma} \int_{\Omega} |f|^p \delta_\Omega^{\gamma+1/m} d\lambda.$$

Proposition 2.2. *For all large integer N , there exists a constant $\varepsilon_N > 0$ and a linear operator T solving the $\bar{\partial}$ -equation in Ω such that, for and all $p \in [1, +\infty[$ and all γ , $-1 < \gamma \leq N - 3$, there exists a constant $C_{N,p,\gamma} > 0$ such that for all $\bar{\partial}$ -closed $(0, r)$ -form f , $1 \leq r \leq n - 1$, we have*

$$\int_{\Omega} |Tf|^{p+\varepsilon_N} \delta_\Omega^\gamma d\lambda \leq C_{N,p,\gamma} \int_{\Omega} |f|^p \delta_\Omega^\gamma d\lambda.$$

In [Cum01b] A. Cumenge obtained a weighted L^1 -anisotropic estimate for solutions of the $\bar{\partial}$ -equation for convex domains of finite type, using the punctual norm $\|f\|_k$ introduced in [BCD98]:

Theorem (A. cumenge [Cum01b]). *Let D be a convex domain of finite type. There exists a constant $C > 0$ such that, for all $\alpha > 0$ and all smooth $\bar{\partial}$ -closed $(0, 1)$ -form f on \bar{D} , there exists a solution of the equation $\bar{\partial}u = f$, continuous on \bar{D} such that*

$$\int_D |u| \delta_D^{\alpha-1} d\lambda \leq C \frac{1}{\min\{\alpha, 1\}} \int_D \|f\|_k \delta_D^\alpha d\lambda.$$

The estimate given by Theorem 2.1 when $p = q = 1$ (and then $\gamma' = \gamma - 1/m$) is weaker than the one given above. Then it is natural to ask if the above estimate can be extended to weighted L^p norms. For example, if D is a smooth strictly pseudoconvex domain it is proved in [Cha80, Théorème 1.4] that: for $\alpha > 0$, if f is a smooth $\bar{\partial}$ -closed form such that the coefficients of f are in $L^p(D, \delta_D^\alpha)$ and the coefficients of $f \wedge \bar{\partial}\rho$ are in $L^p(D, \delta_D^{\alpha-1/2})$, $1 \leq p < +\infty$, $\alpha > 0$, then there exists a solution of the equation $\bar{\partial}u = f$ which is in $L^p(D, \delta_D^{\alpha-1})$.

Our next result extends A. Cumenge's theorem and [Cha80] theorem to $(0, r)$ -forms in lineally convex domains of finite type for $1 \leq p < +\infty$.

To state it, we need to extend the definition of the punctual anisotropic norm $\|\cdot\|_k$ given in [CDM14a] to the L^p context.

We first introduce new quantities associated to the geometry: for z close to the boundary of Ω , let us define

$$\sigma_1(z) = \delta_\Omega(z),$$

and, for $2 \leq k \leq n$

$$\sigma_k(z) = \frac{1}{\prod_{j=1}^{k-1} \sigma_j(z)} \int_{B_{\delta_\Omega(z)}(z)} \frac{d\lambda(\zeta)}{|\zeta - z|^{2(n-k)+1}},$$

where $B_{\delta_\Omega(z)}(z)$ is the pseudo-ball defined by (3.4).

let f be a $(0, r)$ -form whose coefficients are functions; for z close to the boundary and $p \in [1, +\infty[$, we define

$$(2.1) \quad \|f(z)\|_{k,p} = \sup_{\|v_i\|=1} |\langle f; \overline{v_1}, \dots, \overline{v_n} \rangle(z)|^p \frac{\prod_{i=1}^r \tau(z, v_i, \delta_\Omega(z))}{\sigma_1(z) \prod_{\substack{\{k \text{ such that } n-r+2 \leq k \\ \text{and } k \leq n\}}} \sigma_k(z)},$$

where $\tau(z, v_i, \delta_\Omega(z))$ is given by formula (3.3).

Note that, if the coefficients of f are continuous, then $\|f(z)\|_{k,p}$ is also continuous.

Remark.

- If (e_i) is a $(z, \delta(z))$ -extremal basis (see section 3, after (3.4)), property (3) of the geometry implies that (with the notation (3.5))

$$\begin{aligned} & |\langle f; \overline{v_1}, \dots, \overline{v_r} \rangle(z)|^p \prod_{i=1}^r \tau(z, v_i, \delta_\Omega(z)) \lesssim \\ & \sum_{\substack{|I|=r \\ I=(i_1, \dots, i_r)}}' |\langle f(z); \overline{e_{i_1}}, \dots, \overline{e_{i_r}} \rangle|^p \prod_{k=1}^r \tau_{i_k}(z, \delta_\Omega(z)), \end{aligned}$$

and (3.9) and Lemma 3.9 show that, for all k , $\sigma_k(z) \simeq \tau_k(z, \delta_\Omega(z))$ so

$$\|f(z)\|_{k,p} \simeq \sum_{\substack{|I|=r \\ I=(i_1, \dots, i_r)}}' |\langle f(z); \overline{e_{i_1}}, \dots, \overline{e_{i_r}} \rangle|^p \frac{\prod_{k=1}^r \tau_{i_k}(z, \delta_\Omega(z))}{\delta_\Omega(z) \prod_{\substack{\{k \text{ such that } \\ n-r+2 \leq k \\ \text{and } k \leq n\}}} \tau_k(z, \delta_\Omega(z))}.$$

But, of course, in general, the second member of this last equivalence is not a continuous function of z .

- In [CDM14a], for $p = 1$, we defined $\|f(z)\|_k = \sup_{\|v_i\|=1} \frac{|\langle f; \overline{v_1}, \dots, \overline{v_r} \rangle(z)|}{\sum_i \frac{\delta_\Omega(z)}{\tau(z, v_i, \delta_\Omega(z))}}$, and, as before, (3.9) and Lemma 3.9 show that this definition is equivalent to

$$\sup_{\substack{I=(i_1, \dots, i_r) \\ i_p < i_{p+1}}} |\langle f(z); \overline{e_{i_1}}, \dots, \overline{e_{i_r}} \rangle| \min_{i \in I} \frac{\tau_i(z, \delta_\Omega(z))}{\delta_\Omega(z)}.$$

Thus, when $p = 1$, the definition $\|f(z)\|_{k,p}$ is equal to $\|f(z)\|_k$ when $r = 1$ and gives a smaller quantity when $r \geq 2$. So, when $r = p = 1$ we will write indifferently $\|f(z)\|_k$ or $\|f(z)\|_{k,1}$.

- In the case of strictly pseudo-convex domains it is clear that the integrability of $\|f(z)\|_{k,p}$ is equivalent to the integrability of $|f|^p + \frac{|f \wedge \bar{\partial} \rho|^p}{\sqrt{-\rho}}$.

Theorem 2.2. *For all $\alpha > 0$ and all $p \in [1, +\infty[$, there exists a constant $C > 0$ such that, for all smooth $\bar{\partial}$ -closed $(0, r)$ -form f , $1 \leq r \leq n-1$, on $\bar{\Omega}$, there exists a solution of the equation $\bar{\partial}u = f$, continuous on $\bar{\Omega}$ such that*

$$\int_{\Omega} |u|^p \delta_{\Omega}^{\alpha-1} d\lambda \leq C \frac{1}{\min\{\alpha, 1\}} \int_{\Omega} \|f\|_{k,p} \delta_{\Omega}^{\alpha} d\lambda.$$

Remark. Related (but non comparable) estimates can be found in [AC00, Theorem 4.1] for strictly pseudoconvex domains and in [Ale11, Theorem 2.8] for convex domains of finite type.

This result can be extended to $\alpha = 0$ to get an $L^p(\partial\Omega)$ estimate (for $p = 1$ this was done in [CDM14a]):

Theorem 2.3. *For $p \in [1, +\infty[$ there exists a constant $C > 0$ such that, for all smooth $\bar{\partial}$ -closed $(0, r)$ -form f , $1 \leq r \leq n-1$, on $\bar{\Omega}$, there exists a solution of the equation $\bar{\partial}u = f$, continuous on $\bar{\Omega}$ such that*

$$\int_{\partial\Omega} |u|^p d\sigma \leq C \int_{\Omega} \|f\|_{k,p} d\lambda.$$

For $r = 1$ this result is weaker than the one which can be obtained using Carleson measure of order β . Before stating our last estimate let us recall these notions.

A measure μ in Ω is called a Carleson measure if

$$\|\mu\|_{W^1} := \sup_{z \in \partial\Omega, \varepsilon > 0} \frac{\mu(P_{\varepsilon}(z) \cap \Omega)}{\sigma(P_{\varepsilon}(z) \cap \partial\Omega)} < +\infty,$$

where $P_{\varepsilon}(z)$ is the extremal polydisk defined below in (3.6) and σ the surface measure on $\partial\Omega$. $W^1(\Omega)$ will denote the space of Carleson measures on Ω . Then, for $\beta \in]0, 1[$ the space $W^{\beta}(\Omega)$ is the complex interpolated space between the space of bounded measures on Ω , denoted usually $W^0(\Omega)$, and $W^1(\Omega)$. Moreover, we denote by $BMO(\partial\Omega)$ the BMO-space associated to the anisotropic geometry on $\partial\Omega$. Then:

Theorem 2.4. *There exists a constant $C > 0$ such that, for all $p \in [1, +\infty[$ and all smooth $\bar{\partial}$ -closed $(0, 1)$ -form f on $\bar{\Omega}$, there exists a solution of the equation $\bar{\partial}u = f$, continuous on $\bar{\Omega}$ such that:*

- $\|u\|_{BMO(\partial\Omega)} \leq C \| \|f(\zeta)\|_k d\lambda \|_{W^1};$
- $\|u\|_{L^p(\partial\Omega)} \leq C \| \|f(\zeta)\|_k d\lambda \|_{W^{1-1/p}}.$

This type of estimates were originally obtained, for strictly pseudoconvex domains, by E. Amar & A. Bonami in [AB79] and extended to convex domains of finite type by N. Nguyen in [Ngu01] and W. Alexandre in [Ale11].

A classical application of Theorem 2.2 (for $p = 1$ and $r = 1$) is the extension to lineally convex domains of the characterization of the zero sets of the weighted Nevanlinna classes (called Nevanlinna-Djrbachian classes in [Cum01b]) obtained by A. Cumenge for convex domains:

Theorem 2.5. *A divisor \mathcal{D} in Ω can be defined by a holomorphic function f satisfying $\int_{\Omega} \ln^+ |f| \delta_{\Omega}^{\alpha-1} d\lambda < +\infty$, $\alpha > 0$, if and only if it satisfies the generalized Blaschke condition $\int_{\mathcal{D}} \delta^{\alpha+1} d\lambda_{2n-2} < +\infty$.*

As the proof of such result using Theorem 2.2 is very classical we will not give any detail in this paper.

The two propositions 2.1 and 2.2 will be used to generalize some estimates obtained for weighted Bergman projections of convex domains of finite type in [CDM14b] using the method introduced in [CDM15]:

Theorem 2.6. *Let D be a smoothly bounded convex domain of finite type m in \mathbb{C}^n . Let χ be any \mathcal{C}^2 non negative function in \overline{D} which is equivalent to the distance δ_D to the boundary of D and let η be a strictly positive \mathcal{C}^1 function on \overline{D} . Let P_ω be the (weighted) Bergman projection of the Hilbert space $L^2(D, \omega)$ where $\omega = \eta \chi^r$ with r a non negative rational number. Then:*

- (1) *For $p \in]1, +\infty[$ and $-1 < \beta \leq r$, P_ω maps continuously $L^p(D, \delta_D^\beta)$ into itself.*
- (2) *For $0 < \alpha \leq 1/m$ P_ω maps continuously the Lipschitz space $\Lambda_\alpha(D)$ into itself.*

This theorem combined with theorems 2.1 and 2.2 extends to weighted situations the Corollary 1.3 of [Cum01a]

Corollary. *Let f a $\overline{\partial}$ -closed $(0, 1)$ -form on D . Under the assumptions and notations of Theorem 2.6, the solution u of the equation $\overline{\partial}u = f$ which is orthogonal to holomorphic functions in $L^2(D, \omega)$ ($\omega = \eta \chi^r$) satisfies the following estimates:*

- (1) *If the coefficients of f are in $L^p(D, \delta_D^\gamma)$, $-1 < \gamma$ then:*
 - (a) *$u \in L^q(D, \delta_D^{\gamma'})$, with $\frac{1}{q} = \frac{1}{p} - \frac{1-m(\gamma-\gamma')}{m(\gamma'+n)+2}$ and $-1 < \gamma' \leq r$, $\gamma - 1/m \leq \gamma' \leq \gamma$, if $1 \leq p < \frac{m(\gamma'+n)+2}{1-m(\gamma-\gamma')}$, and $q > 1$;*
 - (b) *$u \in \Lambda_\alpha(D)$, with $\alpha = \frac{1}{m} \left[1 - \frac{m(\gamma+n)+2}{p} \right]$, if $p \in]m(\gamma+n)+2, +\infty[$.*
- (2) *If $\|f\|_{k,p} \delta_D^{\gamma+1}$ is in $L^1(D)$, $-1 < \gamma \leq r$, then $u \in L^p(D, \delta_D^\gamma)$.*

3. PROOFS OF THEOREMS 2.1 TO 2.4

First, by standard regularization procedure, it suffices to prove theorems 2.1, and 2.2 for forms smooth in $\overline{\Omega}$.

To solve the $\overline{\partial}$ -equation on a lineally convex domain of finite type, we use exactly the method introduced in [CDM14a], except for the proof of Theorem 2.4 where a modification of the form $s(z, \zeta)$ is done. We now briefly recall the notations and main results from that work.

If f is a smooth $(0, r)$ -form $\overline{\partial}$ -closed, then

$$(3.1) \quad f(z) = (-1)^{r+1} \overline{\partial}_z \left(\int_{\Omega} f(\zeta) \wedge K_N^1(z, \zeta) \right) - \int_{\Omega} f(\zeta) \wedge P_N(z, \zeta),$$

where K_N^1 (resp. P_N) is the component of a kernel K_N (formula (2.7) of [CDM14a]) of bi-degree $(0, r)$ in z and $(n, n-r-1)$ in ζ (resp. $(0, r)$ in z and $(n, n-r)$ in ζ) constructed with the method of [AB82] using the Diederich-Fornaess support function constructed in [DF03] (see also Theorem 2.2 of [CDM14a]) and the function $G(\xi) = \frac{1}{\xi^N}$ with a sufficiently large number N (instead of $G(\xi) = \frac{1}{\xi}$ in formula (2.7) of [CDM14a]).

Then, the form $\int_{\Omega} f(\zeta) \wedge P_N(z, \zeta)$ is $\overline{\partial}$ -closed and the operator T solving the $\overline{\partial}$ -equation in theorems 2.1 and 2.2 is defined on smooth forms by

$$Tf(z) = (-1)^{r+1} \int_{\Omega} f(\zeta) \wedge K_N^1(z, \zeta) - \overline{\partial}^* \mathcal{N} \left(\int_{\Omega} f(\zeta) \wedge P_N(z, \zeta) \right),$$

where $\overline{\partial}^* \mathcal{N}$ is the canonical solution of the $\overline{\partial}$ -equation derived from the theory of the $\overline{\partial}$ -Neumann problem on pseudoconvex domains of finite type.

This formula is justified by the fact that, when the coefficients of f are in $L^1(\Omega, \delta_\Omega^\gamma)$ ($\gamma > -1$) then, given a large integer s , if N is chosen sufficiently large, the coefficients of the form $\int_{\Omega} f(\zeta) \wedge P_N(z, \zeta)$ are in the Sobolev space $L_s^2(\Omega)$. More precisely, it is clear that lemmas 2.2 and 2.3 of [CDM14a] remains true with weighted estimates depending on the choice of N :

Lemma 3.1. *For $r \geq 1$ and $\gamma \leq N$, all the z -derivatives of $P_N(z, \zeta)(-\rho(\zeta))^{-\gamma}$ are uniformly bounded in $\overline{\Omega} \times \overline{\Omega}$, and, for each positive integer s , there exists a constant $C_{s,N,\gamma}$*

such that, if f is $(0, r)$ -form with coefficients in $L^1(\Omega, \delta_\Omega^\gamma)$,

$$\left\| \int_{\Omega} f(\zeta) \wedge P_N(z, \zeta) \right\|_{L^2_s(\Omega)} \leq C_{s, N, \gamma} \|f\|_{L^1(\Omega, \delta_\Omega^\gamma)}.$$

As Ω is assumed to be smooth and of finite type, the regularity results of the $\bar{\partial}$ -Neumann problem ([KN65] and [Cat87]) imply:

Lemma 3.2. *For $r \geq 1$ and $-1 < \gamma \leq N$, for each positive integer s , if f is a $\bar{\partial}$ -closed $(0, r)$ -form with coefficients in $L^1(\Omega, \delta_\Omega^\gamma)$ and $g = \int_{\Omega} f(\zeta) \wedge P_N(z, \zeta)$, then $\bar{\partial}^* \mathcal{N}(g)$ is a solution of the equation $\bar{\partial}u = g$ satisfying $\left\| \bar{\partial}^* \mathcal{N}(g) \right\|_{L^2_s(\Omega)} \leq C_{s, N, \gamma} \|f\|_{L^1(\Omega, \delta_\Omega^\gamma)}$.*

Applying Sobolev lemma we immediately get:

Lemma 3.3. *For $r \geq 1$ and $-1 < \gamma \leq N$, if f is a $\bar{\partial}$ -closed $(0, r)$ -form with coefficients in $L^1(\Omega, \delta_\Omega^\gamma)$ and $g = \int_{\Omega} f(\zeta) \wedge P_N(z, \zeta)$, then $\bar{\partial}^* \mathcal{N}(g)$ is a solution of the equation $\bar{\partial}u = g$ satisfying $\left\| \bar{\partial}^* \mathcal{N}(g) \right\|_{C^1(\bar{\Omega})} \leq C \|f\|_{L^1(\Omega, \delta_\Omega^\gamma)}$.*

Finally the proofs of our theorems are reduced to the proofs of good estimates for the operator T_K defined by

$$(3.2) \quad T_K : f \mapsto \int_{\Omega} f(\zeta) \wedge K_N^1(z, \zeta).$$

To do it we need to recall the anisotropic geometry of Ω and the basic estimates given in [CDM14a].

For ζ close to $\partial\Omega$ and $\varepsilon \leq \varepsilon_0$, ε_0 small, define, for all unitary vector v ,

$$(3.3) \quad \tau(\zeta, v, \varepsilon) = \sup \{c \text{ such that } \rho(\zeta + \lambda v) - \rho(\zeta) < \varepsilon, \forall \lambda \in \mathbb{C}, |\lambda| < c\}.$$

Note that the lineal convexity hypothesis implies that the function $(\zeta, \varepsilon) \mapsto \tau(\zeta, v, \varepsilon)$ is smooth. In particular, $\zeta \mapsto \tau(\zeta, v, \delta_\Omega(\zeta))$ is a smooth function. The pseudo-balls $B_\varepsilon(\zeta)$ (for ζ close to the boundary of Ω) of the homogeneous space associated to the anisotropic geometry of Ω are

$$(3.4) \quad B_\varepsilon(\zeta) = \{\xi = \zeta + \lambda u \text{ with } |u| = 1 \text{ and } |\lambda| < c_0 \tau(\zeta, u, \varepsilon)\}$$

where c_0 is chosen sufficiently small depending only on the defining function ρ of Ω .

Let ζ and ε be fixed. Then, an orthonormal basis (v_1, v_2, \dots, v_n) is called (ζ, ε) -extremal (or ε -extremal, or simply *extremal*) if v_1 is the complex normal (to ρ) at ζ , and, for $i > 1$, v_i belongs to the orthogonal space of the vector space generated by (v_1, \dots, v_{i-1}) and minimizes $\tau(\zeta, v, \varepsilon)$ in the unit sphere of that space. In association to an extremal basis, we denote

$$(3.5) \quad \tau(\zeta, v_i, \varepsilon) = \tau_i(\zeta, \varepsilon).$$

Then we defined polydiscs $AP_\varepsilon(\zeta)$ by

$$(3.6) \quad AP_\varepsilon(\zeta) = \left\{ z = \zeta + \sum_{k=1}^n \lambda_k v_k \text{ such that } |\lambda_k| \leq c_0 A \tau_k(\zeta, \varepsilon) \right\}.$$

$P_\varepsilon(\zeta)$ being the corresponding polydisc with $A = 1$ and we also define

$$d(\zeta, z) = \inf \{ \varepsilon \text{ such that } z \in B_\varepsilon(\zeta) \},$$

so

$$d(\zeta, z) \simeq \inf \{ \varepsilon \text{ such that } z \in P_\varepsilon(\zeta) \}.$$

Remark. Note that there is neither unicity of the extremal basis (v_1, v_2, \dots, v_n) nor of associated polydisk $P_\varepsilon(\zeta)$. However the polydisks associated to two different (ζ, ε) -extremal basis are equivalent. Thus in all the paper $P_\varepsilon(\zeta)$ will denote a polydisk associated to any (ζ, ε) -extremal basis and $\tau_i(\zeta, \varepsilon)$ the radius of $P_\varepsilon(\zeta)$.

The fundamental result here is that d is a pseudo-distance which means that there exists a constant K and, $\forall \alpha > 0$, constants $c(\alpha)$ and $C(\alpha)$ such that

$$(3.7) \quad \text{for } \zeta \in P_\varepsilon(z), P_\varepsilon(z) \subset P_{K\varepsilon}(\zeta),$$

and

$$(3.8) \quad c(\alpha)P_\varepsilon(\zeta) \subset P_{\alpha\varepsilon}(\zeta) \subset C(\alpha)P_\varepsilon(\zeta) \text{ and } P_{c(\alpha)\varepsilon}(\zeta) \subset \alpha P_\varepsilon(\zeta) \subset P_{C(\alpha)\varepsilon}(\zeta).$$

Moreover the pseudo-balls B_ε and the polydiscs P_ε are equivalent in the sense that there exists a constant $K > 0$ depending only on Ω such that

$$(3.9) \quad \frac{1}{K}P_\varepsilon(\zeta) \subset B_\varepsilon(\zeta) \subset KP_\varepsilon(\zeta).$$

For ζ close to $\partial\Omega$ and $\varepsilon > 0$ small, the basic properties of this geometry are (see [Con02] and [CDM14a]):

- (1) Let $w = (w_1, \dots, w_n)$ be an orthonormal system of coordinates centered at ζ . Then

$$\left| \frac{\partial^{|\alpha+\beta|} \rho(\zeta)}{\partial w^\alpha \partial \bar{w}^\beta} \right| \lesssim \frac{\varepsilon}{\prod_i \tau(\zeta, w_i, \varepsilon)^{\alpha_i + \beta_i}}, \quad |\alpha + \beta| \geq 1.$$

- (2) Let v be a unit vector. Let $a_{\alpha\beta}^v(\zeta) = \frac{\partial^{\alpha+\beta} \rho}{\partial \lambda^\alpha \partial \bar{\lambda}^\beta}(\zeta + \lambda v)|_{\lambda=0}$. Then

$$\sum_{1 \leq |\alpha+\beta| \leq 2m} \left| a_{\alpha\beta}^v(\zeta) \right| \tau(\zeta, v, \varepsilon)^{\alpha+\beta} \simeq \varepsilon,$$

where m is the type of Ω .

- (3) If (v_1, \dots, v_n) is a (ζ, ε) -extremal basis and $\gamma = \sum_1^n a_j v_j \neq 0$, then

$$\frac{1}{\tau(\zeta, \gamma, \varepsilon)} \simeq \sum_{j=1}^n \frac{|a_j|}{\tau_j(\zeta, \varepsilon)}.$$

- (4) If v is a unit vector then:

- (a) $z = \zeta + \lambda v \in P_\varepsilon(\zeta)$ implies $|\lambda| \lesssim \tau(\zeta, v, \varepsilon)$,
(b) $z = \zeta + \lambda v$ with $|\lambda| \leq \tau(\zeta, v, \varepsilon)$ implies $z \in CP_\varepsilon(\zeta)$.

- (5) If v is the unit complex normal, then $\tau(\zeta, v, \varepsilon) = \varepsilon$ and if v is any unit vector and $\lambda \geq 1$,

$$(3.10) \quad \lambda^{1/m} \tau(\zeta, v, \varepsilon) \lesssim \tau(\zeta, v, \lambda \varepsilon) \lesssim \lambda \tau(\zeta, v, \varepsilon),$$

where m is the type of Ω .

Lemma 3.4. For z close to $\partial\Omega$, ε small and $\zeta \in P_\varepsilon(z)$ or $z \in P_\varepsilon(\zeta)$, we have, for all $1 \leq i \leq n$:

- (1) $\tau_i(z, \varepsilon) = \tau(z, v_i(z, \varepsilon), \varepsilon) \simeq \tau(\zeta, v_i(\zeta, \varepsilon), \varepsilon)$ where $(v_i(\zeta, \varepsilon))_i$ is the (ζ, ε) -extremal basis;
(2) $\tau_i(\zeta, \varepsilon) \simeq \tau_i(z, \varepsilon)$;
(3) In the coordinate system (z_i) associated to the (z, ε) -extremal basis, $\left| \frac{\partial \rho}{\partial z_i}(\zeta) \right| \lesssim \frac{\varepsilon}{\tau_i}$ where τ_i is either $\tau_i(z, \varepsilon)$ or $\tau_i(\zeta, \varepsilon)$.

Proof. (1) is proved in [Con02] (together with the properties of the geometry). (2) follows the properties of the geometry ((3.7) and (3.8)) and formula (3.14) of Lemma 3.9 and (3) is a consequence of (1), (2) and the first property of the geometry. \square

Remark. In (1) above $\tau(\zeta, v_i(z, \varepsilon), \varepsilon)$ is not $\tau_i(\zeta, \varepsilon)$ because the extremal basis at z and ζ are different but (2) implies that these quantities are equivalent.

We now recall the detailed expression of K_N^1 ([CDM14a] sections 2.2 and 2.3):

$$K_N^1(z, \zeta) = \sum_{k=n-r}^{n-1} C'_k \frac{\rho(\zeta)^{k+N} s \wedge (\partial_{\bar{\zeta}} Q)^{n-r} \wedge (\partial_{\bar{z}} Q)^{k+r-n} \wedge (\partial_{\bar{z}} s)^{n-k-1}}{\langle s(z, \zeta), \zeta - z \rangle^{2(n-k)} \left(\frac{1}{K_0} S(z, \zeta) + \rho(\zeta) \right)^{k+N}},$$

where s is a $(1, 0)$ -form satisfying

$$c|z - \zeta|^2 \leq |\langle s(z, \zeta), \zeta - z \rangle| \leq C|\zeta - z|$$

uniformly for $\zeta \in \bar{\Omega}$ and z in any compact subset of Ω , and

$$Q(z, \zeta) = \frac{1}{K_0 \rho(\zeta)} \sum_{i=1}^n Q_i(z, \zeta) d(\zeta_i - z_i)$$

with

$$(3.11) \quad S(z, \zeta) = \chi(z, \zeta) S_0(z, \zeta) - (1 - \chi(z, \zeta)) |z - \zeta|^2 = \sum_{i=1}^n Q_i(z, \zeta) (z_i - \zeta_i),$$

S_0 being the holomorphic support function of Diederich-Fornaess (see [DF03] or Theorem 2.2 of [CDM14a]) and χ a truncating function which is equal to 1 when both $|z - \zeta|$ and $\delta_{\Omega}(\zeta)$ are small and 0 if one of these expressions is large (see the beginning of Section 2.2 of [CDM14a] for a precise definition). Recall that K_0 is chosen so that

$$\Re \left(\rho(\zeta) + \frac{1}{K_0} S(z, \zeta) \right) < \frac{\rho(\zeta)}{2},$$

which implies

$$(3.12) \quad \left| \rho(\zeta) + \frac{1}{K_0} S(z, \zeta) \right| \gtrsim |\rho(\zeta)|.$$

The precise choice of the form s is

$$(3.13) \quad s(z, \zeta) = \sum_{i=1}^n (\bar{\zeta}_i - \bar{z}_i) d(\zeta_i - z_i)$$

for all the proofs except for the proof of Theorem 2.4 where a different choice is needed.

The following estimates of the expressions appearing in K_N^1 are basic (see [CDM14a]):

Lemma 3.5. *For $\zeta \in P_{2\varepsilon}(z) \setminus P_{\varepsilon}(z)$ or $z \in P_{2\varepsilon}(\zeta) \setminus P_{\varepsilon}(\zeta)$, we have:*

$$\left| \rho(\zeta) + \frac{1}{K_0} S(z, \zeta) \right| \gtrsim \varepsilon, (z, \zeta) \in \bar{\Omega} \times \bar{\Omega}.$$

Lemma 3.6. *For z_0 close to $\partial\Omega$, ε small and $z, \zeta \in P_{\varepsilon}(z_0)$, in the coordinate system (ζ_i) associated to the (z_0, ε) -extremal basis, we have:*

- (1) $|Q_i(z, \zeta)| + |Q_i(\zeta, z)| \lesssim \frac{\varepsilon}{\tau_i},$
- (2) $\left| \frac{\partial Q_i(z, \zeta)}{\partial \zeta_j} \right| + \left| \frac{\partial Q_i(z, \zeta)}{\partial z_j} \right| + \left| \frac{\partial Q_i(z, \zeta)}{\partial z_j} \right| \lesssim \frac{\varepsilon}{\tau_i \tau_j},$
- (3) $\left| \frac{\partial^2 Q_i(z, \zeta)}{\partial \zeta_j \partial z_k} \right| + \left| \frac{\partial^2 Q_i(z, \zeta)}{\partial \zeta_j \partial \bar{z}_k} \right| \lesssim \frac{\varepsilon}{\tau_i \tau_j \tau_k},$

where τ_i are either $\tau_i(z, \varepsilon)$, $\tau_i(\zeta, \varepsilon)$ or $\tau_i(z_0, \varepsilon)$.

Proof. This Lemma follows [DF06] and Lemma 3.4. □

The preceding lemmas and the properties of the geometry easily give the following estimates of the kernel K_N^1 :

Lemma 3.7. *For ε small enough and z sufficiently close to the boundary, with the choice (3.13) for s , we have:*

If $\zeta \in P_\varepsilon(z)$ or $z \in P_\varepsilon(\zeta)$,

$$\left| K_N^1(z, \zeta) \right| \lesssim \frac{|\rho(\zeta)|^{N-1} (|\rho(\zeta)| + \varepsilon) \varepsilon^{n-r}}{\prod_{i=1}^{n-r} \tau_i^2 \left| \frac{1}{K_0} S(z, \zeta) + \rho(\zeta) \right|^{N+n-r}} \frac{1}{|z - \zeta|^{2r-1}},$$

where τ_i is $\tau_i(z, \varepsilon)$ or $\tau_i(\zeta, \varepsilon)$.

Proof. Indeed, under the conditions of the lemma, $Q(z, \zeta) = \frac{1}{K_0 \rho(\zeta)} \sum Q_i(z, \zeta) d(\zeta_i - z_i)$ is holomorphic in z and K_N^1 is reduced to

$$K_N^1(z, \zeta) = c \frac{\rho(\zeta)^{n+N-r} \wedge (d_{\bar{\zeta}} Q)^{n-r} \wedge (\partial_{\bar{\zeta}} s)^{r-1}}{|z - \zeta|^{2r} \left(\frac{1}{K_0} S(z, \zeta) + \rho(\zeta) \right)^{N+n-r}}.$$

□

In particular:

Lemma 3.8. *For ε small enough and z sufficiently close to the boundary and s given by (3.13):*

(1) *If $\varepsilon \leq \delta_{\partial\Omega}(z)$, for $\zeta \in P_\varepsilon(z)$ or $z \in P_\varepsilon(\zeta)$,*

$$\left| K_N^1(z, \zeta) \right| \lesssim \frac{1}{\prod_{i=1}^{n-r} \tau_i^2} \frac{1}{|z - \zeta|^{2r-1}},$$

where τ_i is $\tau_i(z, \varepsilon)$ or $\tau_i(\zeta, \varepsilon)$.

For $1 + r - n \leq k \leq N + n - 1$ and $\zeta \in P_{2\varepsilon}(z) \setminus P_\varepsilon(z)$ or $z \in P_{2\varepsilon}(\zeta) \setminus P_\varepsilon(\zeta)$:

(2)

$$\left| K_N^1(z, \zeta) \right| \lesssim \frac{|\rho(\zeta)|^k}{\varepsilon^k} \frac{1}{\prod_{i=1}^{n-r} \tau_i^2} \frac{1}{|z - \zeta|^{2r-1}},$$

(3)

$$\left| \nabla_z K_N^1(z, \zeta) \right| \lesssim \frac{|\rho(\zeta)|^k}{\varepsilon^{k+1}} \frac{1}{\prod_{i=1}^{n-r} \tau_i^2} \frac{1}{|z - \zeta|^{2r-1}},$$

where τ_i is either $\tau_i(z, \varepsilon)$ or $\tau_i(\zeta, \varepsilon)$.

Lemma 3.9. *For $z \in \Omega$, close to $\partial\Omega$, δ small, $r \in \{1, \dots, n-1\}$ and $0 \leq \mu < 1$,*

$$(3.14) \quad \int_{P_\delta(z)} \frac{d\lambda(\zeta)}{|z - \zeta|^{2r-1+\mu}} \simeq \tau_{n-r+1}(z, \delta)^{1-\mu} \prod_{j=1}^{n-r} \tau_j^2(z, \delta),$$

and, for $\alpha > 0$,

(3.15)

$$\int_{P_\delta(\zeta)} \frac{\delta_\Omega(z)^{\alpha-1}}{|z - \zeta|^{2r-1}} d\lambda(z) \lesssim \begin{cases} \frac{\delta_\Omega(z)^{\alpha-1}}{\alpha} \tau_{n-r+1}(\zeta, \delta) \prod_{j=1}^{n-r} \tau_j^2(\zeta, \delta), & \text{if } \alpha \leq 1, \\ (\delta_\Omega(\zeta) + \delta)^{\alpha-1} \tau_{n-r+1}(\zeta, \delta) \prod_{j=1}^{n-r} \tau_j^2(\zeta, \delta), & \text{if } \alpha > 1. \end{cases}$$

Proof. Let us prove (3.14). Let

$$D = \{ \zeta \text{ such that } |\zeta_i - z_i| \leq \tau_i(z, \delta), i \leq n-r \text{ and } |\zeta_i - z_i| \lesssim \tau_{n-r+1}(z, \delta), i > n-r \}.$$

Then the volume of D is $V(D) \simeq \prod_{j=1}^{n-r} \tau_j(z, \delta)^2 \tau_{n-r+1}(z, \delta)^{2r}$ and, on D , $\frac{1}{|\zeta - z|^{2r-1+\mu}} \gtrsim \left(\frac{1}{\tau_{n-r+1}(z, \delta)} \right)^{2r-1+\mu}$. This proves the inequality \gtrsim .

To prove the converse inequality let us first consider the sets

$$E_l = \left\{ |\zeta_i - z_i| \leq \tau_i(z, \delta), i \leq n-r \text{ and } 2^{-l} \tau_{n-r+1}(z, \delta) \leq |\zeta' - z'| \lesssim 2^{-l+1} \tau_{n-r+1}(z, \delta) \right\}$$

where $\zeta' = (\zeta_{n-r+1}, \dots, \zeta_n)$ and $z' = (z_{n-r+1}, \dots, z_n)$. Then the volume of E_l is

$$V(E_l) \lesssim \prod_{j=1}^{n-r} \tau_j^2(z, \delta) \left(2^{-l} \tau_{n-r+1}(z, \delta)\right)^{2r},$$

and, for $\zeta \in E_l$,

$$\frac{1}{|\zeta - z|^{2r-1+\mu}} \lesssim \left(\frac{1}{2^{-l} \tau_{n-r+1}(z, \delta)}\right)^{2r-1+\mu}.$$

Thus

$$\int_{E_l} \frac{d\lambda(\zeta)}{|z - \zeta|^{2r-1+\mu}} \lesssim \prod_{i=1}^{n-r} \tau_i^2(z, \delta) \tau_{n-r+1}(z, \delta)^{1-\mu} (2^{-l})^{1-\mu},$$

proving the inequality if $r = 1$. If $r \geq 2$, consider now the sets

$$F_l = \left\{ |\zeta_i - z_i| \leq \tau_i(z, \delta), i \leq n-r+1 \text{ and } 2^l \tau_{n-r+1}(z, \delta) \leq |\zeta'', -z''| \lesssim 2^{l+1} \tau_{n-r+1}(z, \delta) \right\}$$

where $\zeta'' = (\zeta_{n-r+2}, \dots, \zeta_n)$ and $z'' = (z_{n-r+2}, \dots, z_n)$. Then the volume of F_l is

$$V(F_l) \lesssim \prod_{j=1}^{n-r+1} \tau_j(z, \delta)^2 \left(2^l \tau_{n-r+1}(z, \delta)\right)^{2r-2},$$

and, for $\zeta \in F_l$,

$$\frac{1}{|\zeta - z|^{2r-1+\mu}} \lesssim \left(\frac{1}{2^l \tau_{n-r+1}(z, \delta)}\right)^{2r-1+\mu}.$$

Thus

$$\int_{F_l} \frac{d\lambda(\zeta)}{|z - \zeta|^{2r-1+\mu}} \lesssim \prod_{i=1}^{n-r} \tau_i(z, \delta)^2 \tau_{n-r+1}(z, \delta)^{1-\mu} (2^l)^{-1+\mu},$$

finishing the proof of (3.14).

(3.15) is proved similarly considering the sets

$$E_l = P_\delta(\zeta) \cap \left\{ z \text{ such that } \delta_\Omega(z) \in \left[\frac{1}{2^{l+1}} \delta_\Omega(\zeta), \frac{1}{2^l} \delta_\Omega(\zeta) \right] \right\}.$$

□

We now detail the different proofs of the theorems in the next sub-sections. Recall that, by Lemma 3.3, we only have to prove the estimates for the operator T_K associated to the kernel K_N^1 .

3.1. Proof of Theorem 2.1.

In this proof the form $s(z, \zeta)$ is given by the formula (3.13).

Proof of (1) of Theorem 2.1. It is based on a version of a classical operator estimate which can be found, for example, in Appendix B of the book of M. Range [Ran86]:

Lemma 3.10. *Let Ω be a smoothly bounded domain in \mathbb{C}^n . Let μ and ν be two positive measures on Ω . Let K be a measurable function on $\Omega \times \Omega$. Assume that there exists a positive number $\varepsilon_0 > 0$, a positive constant C and a real number $s \geq 1$ such that:*

- (1) $\int_\Omega |K(z, \zeta)|^s \delta_\Omega(\zeta)^{-\varepsilon} d\mu(\zeta) \leq C \delta_\Omega(z)^{-\varepsilon},$
- (2) $\int_\Omega |K(z, \zeta)|^s \delta_\Omega(z)^{-\varepsilon} d\nu(z) \leq C \delta_\Omega(\zeta)^{-\varepsilon},$

for all $0 < \varepsilon \leq \varepsilon_0$, where δ_Ω denotes the distance to the boundary of Ω . Then the linear operator T defined by

$$Tf(z) = \int_\Omega K(z, \zeta) f(\zeta) d\mu(\zeta)$$

is bounded from $L^p(\Omega, \mu)$ to $L^q(\Omega, \nu)$ for all $1 \leq p, q < \infty$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$.

Short proof. This is exactly the proof given by M. Range in his book: let ε be sufficiently small. Writing

$$|Kf| = \left(|K|^s |f|^p \delta_\Omega(\zeta)^{\varepsilon \frac{p-1}{p} q} \right)^{1/q} \left(|K|^{1-\frac{s}{q}} \delta_\Omega(\zeta)^{-\varepsilon \frac{p-1}{p}} \right) |f|^{1-\frac{p}{q}},$$

Hölder's inequality (with $\frac{1}{q} + \frac{p-1}{p} + \frac{s-1}{s} = 1$) gives

$$|Tf(z)| \leq \left(\int_\Omega |K(z, \zeta)|^s \delta_\Omega(\zeta)^{\varepsilon \frac{p-1}{p} q} |f|^p(\zeta) d\mu(\zeta) \right)^{1/q} \left(\int_\Omega |K(z, \zeta)|^s \delta_\Omega(\zeta)^{-\varepsilon} \right)^{\frac{p-1}{p}} \left(\int_\Omega |f(\zeta)|^p d\mu(\zeta) \right)^{\frac{s-1}{s}}.$$

The first hypothesis of the lemma gives (for $\varepsilon \leq \varepsilon_0$)

$$|Tf(z)|^q \leq C \left(\int_\Omega |K(z, \zeta)|^s \delta_\Omega(\zeta)^{\varepsilon \frac{p-1}{p} q} \delta_\Omega(z)^{-\varepsilon \frac{p-1}{p} q} |f|^p(\zeta) d\mu(\zeta) \right) \left(\int_\Omega |f(\zeta)|^p d\mu(\zeta) \right)^{q \frac{s-1}{s}}.$$

Integration with respect to the measure $d\nu(z)$ gives (using the second hypothesis of the lemma with $\varepsilon \frac{p-1}{p} q \leq \varepsilon_0$)

$$\int_\Omega |Tf(z)|^q d\nu(z) \leq C^2 \left(\int_\Omega |f|^p d\mu \right)^{q/p}.$$

□

Applying this lemma to the operator T_K (formula (3.2)) with the measures $\mu = \delta_\Omega^\gamma d\lambda$ and $\nu = \delta_\Omega^\gamma d\lambda$, the required estimates on K_N^1 are summarized in the following Lemma:

Lemma 3.11. Let $\mu_0 = \frac{1-m(\gamma-\gamma')}{m(\gamma+n+1-r)+2r-1}$.

- (1) For N such that $-1 < \gamma' \leq \gamma < N-1$ and $\varepsilon > 0$ sufficiently small,

$$\int_\Omega \left| K_N^1(z, \zeta) \right|^{1+\mu_0} \delta_\Omega(\zeta)^{-\mu_0 \gamma - \varepsilon} d\lambda(\zeta) \lesssim \delta_\Omega(z)^{-\varepsilon}.$$

- (a) For $-1 < \gamma'$ and N such that $\max\{-1, \gamma - 1/m\} \leq \gamma' < \gamma < N-1$ and $\varepsilon > 0$ sufficiently small,

$$\int_\Omega \left| K_N^1(z, \zeta) \right|^{1+\mu_0} \frac{\delta_\Omega(z)^{\gamma' - \varepsilon}}{\delta_\Omega(\zeta)^{(1+\mu_0)\gamma}} d\lambda(\zeta) \lesssim \delta_\Omega(z)^{-\varepsilon}.$$

We now prove this last lemma.

Proof of (1) of Lemma 3.11. K_N^1 being bounded, uniformly in (z, ζ) , for ζ outside $P_{\varepsilon_0}(z)$, it is enough to prove that

$$\int_{P_{\varepsilon_0}(z)} \left| K_N^1(z, \zeta) \right|^{1+\mu_0} \delta_\Omega^{-\gamma \mu_0 - \varepsilon}(\zeta) d\lambda(\zeta) \lesssim \delta_\Omega^{-\varepsilon}(z)$$

for ε sufficiently small. As this is trivial if z is far from the boundary, we assume that z is sufficiently close to $\partial\Omega$.

Let $A(z, \zeta) = K_N^1(z, \zeta) |z - \zeta|^{2r-1}$. If $\zeta \in P_{\delta_\Omega(z)}(z)$ then $\delta_\Omega(z) \simeq \delta_\Omega(\zeta)$ and, by (2) of Lemma 3.8,

$$(3.16) \quad |A(z, \zeta)|^{1+\mu_0} \delta_\Omega(\zeta)^{-\gamma \mu_0 - \varepsilon} \lesssim \frac{\delta_\Omega(z)^{-\mu_0(\gamma+n+1-r)-\varepsilon}}{\prod_{j=1}^{n-r} \tau_j^2(z, \delta_\Omega(z))}.$$

Thus, by (3.14), we get

$$\begin{aligned} \int_{P_{\delta_{\Omega}(z)}(z)} \left| K_N^1(z, \zeta) \right|^{1+\mu_0} \delta_{\Omega}(\zeta)^{-\gamma\mu_0-\varepsilon} d\lambda(\zeta) &\lesssim \delta_{\Omega}(z)^{-\mu_0(\gamma+n+1-r+(2r-1)/m)-\varepsilon+\frac{1}{m}} \\ &= \delta_{\Omega}(z)^{-\varepsilon+(\gamma-\gamma')} \leq \delta_{\Omega}(z)^{-\varepsilon}. \end{aligned}$$

Now, let $\zeta \in P_i(z) = P_{2^i\delta_{\Omega}(z)}(z) \setminus P_{2^{i-1}\delta_{\Omega}(z)}(z)$, if N is sufficiently large ($N \geq \gamma+n+1$), by (2) of Lemma 3.8, we have

$$|A(z, \zeta)|^{1+\mu_0} \delta_{\Omega}(\zeta)^{-\gamma\mu_0-\varepsilon} \lesssim \frac{(2^i\delta_{\Omega}(z))^{-\mu_0(\gamma+n)-\varepsilon}}{\prod_{j=1}^{n-r} \tau_j^2(z, 2^i\delta_{\Omega}(z))}$$

which gives, by (3.14),

$$\begin{aligned} \int_{P^i(z)} \left| K_N^1(z, \zeta) \right|^{1+\mu_0} \delta_{\Omega}(\zeta)^{-\gamma\mu_0-\varepsilon} d\lambda(\zeta) &\lesssim (2^i\delta_{\Omega}(z))^{-\mu_0(\gamma+n+1-r)-\varepsilon+\frac{1-(2r-1)\mu_0}{m}} \\ &\leq \delta_{\Omega}(z)^{-\varepsilon} (2^i)^{-\varepsilon}, \end{aligned}$$

finishing the proof. \square

Proof of (2) of Lemma 3.11. As in the preceding proof we have to show that

$$\int_{P_{\delta_{\Omega}(\zeta)}(\zeta)} \left| \frac{K_N^1(z, \zeta)}{\delta_{\Omega}(\zeta)^{\gamma}} \right|^{1+\mu_0} \delta_{\Omega}(z)^{\gamma'-\varepsilon} d\lambda(z) \lesssim \delta_{\Omega}(\zeta)^{-\varepsilon}.$$

If $z \in P_{\delta_{\Omega}(\zeta)}(\zeta)$ then $\delta_{\Omega}(\zeta) \simeq \delta_{\Omega}(z)$, the estimate (3.16), which is still valid replacing $\tau_j(z, \delta_{\Omega}(z))$ by $\tau_j(\zeta, \delta_{\Omega}(\zeta))$ (Lemma 3.4 and (3.14)), we immediately get

$$\begin{aligned} \int_{P_{\delta_{\Omega}(\zeta)}(\zeta)} \left| \frac{K_N^1(z, \zeta)}{\delta_{\Omega}(\zeta)^{\gamma}} \right|^{1+\mu_0} \delta_{\Omega}(z)^{\gamma'-\varepsilon} d\lambda(z) &\lesssim \delta_{\Omega}(\zeta)^{-\mu_0(\gamma+n+1-r)-(\gamma-\gamma')+\frac{1-(2r-1)\mu_0}{m}-\varepsilon} \\ &= \delta_{\Omega}(\zeta)^{-\mu_0(\gamma+n+1-r+(2r-1)/m)+\frac{1}{m}-\gamma-\gamma'-\varepsilon} \\ &= \delta_{\Omega}(\zeta)^{-\varepsilon}. \end{aligned}$$

Assume now $z \in P_i(\zeta) = P_{2^i\delta_{\Omega}(\zeta)}(\zeta) \setminus P_{2^{i-1}\delta_{\Omega}(\zeta)}(\zeta)$.

If $\gamma' - \varepsilon \geq 0$, using $\delta_{\Omega}(z) \lesssim 2^i\delta_{\Omega}(\zeta)$, (2) of Lemma 3.8 and (3.14) give

$$\begin{aligned} \int_{P_i(\zeta)} \left| \frac{K_N^1(z, \zeta)}{\delta_{\Omega}(\zeta)^{\gamma}} \right|^{1+\mu_0} \delta_{\Omega}(z)^{\gamma'-\varepsilon} d\lambda(z) &\lesssim (2^i\delta_{\Omega}(\zeta))^{-\mu_0(\gamma+n+1-r)-(\gamma-\gamma')+\frac{1-(2r-1)\mu_0}{m}-\varepsilon} \\ &= (2^i\delta_{\Omega}(\zeta))^{-\varepsilon}, \end{aligned}$$

finishing the proof in that case.

If $-1 < \gamma' - \varepsilon \leq 0$, as

$$\begin{aligned} \int_{P_i(\zeta)} \left| \frac{\delta_{\Omega}(z)}{\delta_{\Omega}(\zeta)} \right|^{\gamma'-\varepsilon} \frac{d\lambda(z)}{(|z-\zeta|^{2r-1})^{1+\mu_0}} &\lesssim^{\gamma'-\varepsilon, \mu_0} \\ &\tau_{n-r+1}(\zeta, 2^i\delta_{\Omega}(\zeta))^{1-(2r-1)\mu_0} \prod_{j=1}^{n-r} \tau_j^2(\zeta, 2^i\delta_{\Omega}(\zeta)), \end{aligned}$$

the proof is done as before using (2) of Lemma 3.8. \square

The proof of (1) of Theorem 2.1 is now complete. \square

Proof of (2) and (3) of Theorem 2.1. By the Hardy-Littlewood lemma we have to prove the two following inequalities:

- if $p = m(\gamma+n+1-r) + 2r$, $|\nabla_z(\int_{\Omega} f(\zeta) \wedge K_N^1(z, \zeta))| \lesssim \delta_{\Omega}(z)^{-1}$,

- if $p > m(\gamma + n + 1 - r) + 2r$, $|\nabla_z (\int_{\Omega} f(\zeta) \wedge K_N^1(z, \zeta))| \lesssim \delta_{\Omega}(z)^{\alpha-1}$.

Then, using Hölder's inequality these two estimates are consequences of the following lemma:

Lemma 3.12. *Let $p \geq m(\gamma + n + 1 - r) + 2r$, p' the conjugate of p (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$) and let $\alpha = \frac{1}{m} \left[1 - \frac{m(\gamma + n + 1 - r) + 2r}{p} \right]$. Then*

$$\int_{\Omega} |\nabla_z K_N^1(z, \zeta)|^{p'} \delta_{\Omega}(\zeta)^{-\gamma p'/p} d\lambda(\zeta) \lesssim \delta_{\Omega}(z)^{p'(\alpha-1)}.$$

Proof of the lemma. Denote $p' = 1 + \eta$ so that $p'/p = \eta$ and $1/p = \frac{\eta}{1+\eta}$. Note that $\eta = \frac{1}{p-1} < \frac{1}{2r-1}$. By the basic estimates of K_N^1 (and the fact that $-\frac{\gamma p'}{p} > -1$) it suffices to estimate the above integral when the domain of integration is reduced to $P(z, \varepsilon_0)$.

Assume first that $\zeta \in P_i(z) = P_{2^i \delta_{\Omega}(z)}(z) \setminus P_{2^{i-1} \delta_{\Omega}(z)}(z)$. Then, by (3) of Lemma 3.8, we have (note that $\gamma/p < 1$)

$$|\nabla_z K_N^1(z, \zeta)| \lesssim \frac{|\delta_{\Omega}(\zeta)|^{\gamma/p}}{(2^i \delta_{\Omega}(z))^{1+\eta/p}} \frac{1}{\prod_{j=1}^{n-r} \tau_j^2(z, 2^i \delta_{\Omega}(z))} \frac{1}{|z - \zeta|^{2r-1}},$$

and by (3.14), we get

$$\begin{aligned} \int_{P_i(z)} |\nabla_z K_N^1(z, \zeta)|^{p'} \delta_{\Omega}(\zeta)^{-\gamma p'/p} d\lambda(\zeta) &\lesssim (2^i \delta_{\Omega}(z))^{-(n-r+1)\eta + \frac{1-(2r-1)\eta}{m} - p' - \gamma\eta} \\ &= (2^i \delta_{\Omega}(z))^{p'(\alpha-1)}. \end{aligned}$$

Assume now that $\zeta \in P_{-i}(z) = P_{2^{-(i-1)} \delta_{\Omega}(z)}(z) \setminus P_{2^{-i} \delta_{\Omega}(z)}(z)$. Then, by (3) of Lemma 3.8 (for $k = -1$) and the fact that $\delta_{\Omega}(z) \simeq \delta_{\Omega}(\zeta)$, we have

$$|\nabla_z K_N^1(z, \zeta)| \lesssim \frac{1}{\prod_{j=1}^{n-r} \tau_j^2(z, 2^{-i} \delta_{\Omega}(z))} \frac{1}{|z - \zeta|^{2r-1}} \frac{1}{2^{-i} \delta_{\Omega}(z)},$$

and, as before, by (3.14), we have

$$\begin{aligned} \int_{P_{-i}(z)} |\nabla_z K_N^1(z, \zeta)|^{p'} \delta_{\Omega}(\zeta)^{-\gamma p'/p} d\lambda(\zeta) &\lesssim (2^{-i} \delta_{\Omega}(z))^{p'(\alpha-1)} (2^{-i})^{p'} \\ &= (\delta_{\Omega}(z))^{p'(\alpha-1)} (2^{-i})^{p'\alpha}, \end{aligned}$$

finishing the proof of the lemma. □

The proofs of (2) and (3) of Theorem 2.1 are complete. □

The proof of Theorem 2.1 is now complete.

3.2. Proof of Theorem 2.2.

In this proof the form $s(z, \zeta)$ is already given by the formula (3.13).

For condensing, we introduce a new notation. If $U = \{u_i\}_{1 \leq i \leq n}$ is a set of n vectors in \mathbb{C}^n , for $I = \{i_1, \dots, i_r\} \subset I_n = \{1, 2, \dots, n\}$, $i_j \neq i_k$ for $j \neq k$, we denote by U_I the set

$$U_I = \{u_{i_1}, \dots, u_{i_r}\}.$$

If I is ordered increasingly, we denote by \widehat{I} the subset of I_n , ordered increasingly, such that $I_n = I \cup \widehat{I}$.

Writing

$$K_N^1(z, \zeta) = \sum_{|J|=r-1}^{\prime} K_N^{1,J}(z, \zeta) d\bar{z}^J,$$

the symbol \sum_J' meaning that the summation is taken over increasing sets of integers J , if $V = \{v_1, \dots, v_n\}$ is an orthonormal basis, we write

$$K_N^{1,J}(z, \zeta) \wedge f(\zeta) = \sum_{|I|=r}' \left\langle K_N^{1,J}(z, \zeta); \left(V, \overline{V_I} \right) \right\rangle \langle f(\zeta); \overline{V_I} \rangle,$$

where, denoting by $(\zeta_i^V)_i$ the coordinate system associated to V , $\left\langle K_N^{1,J}(z, \zeta); \left(V, \overline{V_I} \right) \right\rangle$ is defined by

$$K_N^{1,J}(z, \zeta) \wedge d\overline{\zeta^V_I} = \left\langle K_N^{1,J}(z, \zeta); \left(V, \overline{V_I} \right) \right\rangle d\lambda(\zeta).$$

Thus, as we are able to estimate the kernel only when writing it in suitable extremal coordinates, to obtain the wanted estimate, we have to write

$$K_N^1(z, \zeta) \wedge f(\zeta) = \sum_{|J|=r-1}' \left\{ \sum_{|I|=r}' \left\langle K_N^{1,J}(z, \zeta); \left(L, \overline{L_I} \right) \right\rangle \langle f(\zeta); \overline{L_I} \rangle \right\} d\overline{z}^J$$

with a basis $L = (L_i(\zeta))_{1 \leq i \leq n} = (L_i)_{1 \leq i \leq n}$, extremal at ζ , and apply Hölder's inequality. But this cannot be done directly since it is well known that it is not possible to choose continuously the bases $(L_i(\zeta))_i$ and there is no guaranty that the functions

$$\left\langle K_N^{1,J}(z, \zeta); \left(L, \overline{L_I} \right) \right\rangle \langle f(\zeta); \overline{L_I} \rangle$$

are measurable. To circumvent this difficulty we will choose bases $(L_i(\zeta))_i$ which are locally constant (thus *not extremal* at ζ) and suitably close to an extremal basis at ζ . Then the properties of the geometry and the estimates of the kernel will allow us to conclude.

Let us consider a minimal covering of Ω by polydisks $P_{\delta_\Omega(z)}(z)$, so that it is locally finite. Let F be the union of the boundaries of these polydisks (F is closed in Ω) and let us denote by $\{\Omega_i\}_i$ the countable family of the connected components of $\Omega \setminus F$. For each $i \in \mathbb{N}$ we fix an arbitrary point Z_i in Ω_i and we define the coronas R_k^i by

$$R_0^i = \widehat{P_{\delta_\Omega(Z_i)}(Z_i)} \text{ and } R_k^i = \widehat{P_{2^k \delta_\Omega(Z_i)}(Z_i)} \setminus P_{2^{k-1} \delta_\Omega(Z_i)}(Z_i), k \geq 1.$$

Finally we denote Γ_i the union of the boundaries of the polydisks R_k^i and $\Gamma = \cup_i \Gamma_i$.

Note that both sets F and Γ are measurable of Lebesgue measure zero.

We now define our bases $(L_j(\zeta))_j$ for $(z, \zeta) \in (\Omega \setminus F) \times (\Omega \setminus \Gamma)$ as follows:

let $z \in \Omega \setminus F$; then there exists a unique i such that $z \in \Omega_i$ and a unique k such that $\zeta \in R_k^i$; then we define the vector field $L_j(\zeta) = L_{z,j}(\zeta)$ to be the j^{th} vector of a $2^k \delta_\Omega(Z_i)$ -extremal basis at the point Z_i used for the polydisk $P_{2^k \delta_\Omega(Z_i)}(Z_i)$ and set

$$L(\zeta) = (L_j(\zeta))_{1 \leq j \leq n}.$$

With this definition the expressions

$$\sum_{|I|=r}' \left\langle K_N^{1,J}(z, \zeta); \left(L(\zeta), \overline{L_I(\zeta)} \right) \right\rangle \langle f(\zeta); \overline{L_I(\zeta)} \rangle$$

is a measurable function on $\Omega \times \Omega$ (because it is the restriction to a set of total measure of a locally smooth function) and we can write, for $z \in \Omega \setminus F$,

$$\begin{aligned} u(z) &:= \int_{\Omega} K_N^1(z, \zeta) \wedge f(\zeta) \\ &= \sum_{|J|=r-1}' \left\{ \sum_{|I|=r}' \int_{\Omega \setminus \Gamma} \left\langle K_N^{1,J}(z, \zeta); \left(L(\zeta), \overline{L_I(\zeta)} \right) \right\rangle \langle f(\zeta); \overline{L_I(\zeta)} \rangle d\lambda(\zeta) \right\} d\overline{z}^J. \end{aligned}$$

The kernel $K_N^1(z, \zeta)$ being uniformly integrable in the variable ζ (c.f. [CDM14a]), Hölder's inequality gives

$$|u(z)|^p \lesssim \sum'_{|J|=r-1} \left\{ \sum'_{|I|=r} \int_{\Omega \setminus \Gamma} \left| \left\langle K_N^{1,J}(z, \zeta); \left(L(\zeta), \overline{L_I(\zeta)} \right) \right\rangle \right| \left| \left\langle f(\zeta), \overline{L_I(\zeta)} \right\rangle \right|^p d\lambda(\zeta) \right\}.$$

The measurability of the functions

$$(z, \zeta) \mapsto \left| \left\langle K_N^{1,J}(z, \zeta); \left(L(\zeta), \overline{L_I(\zeta)} \right) \right\rangle \right| \left| \left\langle f(\zeta), \overline{L_I(\zeta)} \right\rangle \right|^p,$$

the facts that F and Γ are of Lebesgue measure zero and Fubini's theorem finally give

$$\int_{\Omega} |u(z)|^p \delta_{\Omega}^{\alpha-1}(z) d\lambda(z) \lesssim \sum'_{|J|=r-1} \left\{ \int_{\Omega \setminus \Gamma} \left[\sum'_{|I|=r} \int_{\Omega \setminus F} \left| \left\langle K_N^{1,J}(z, \zeta); \left(L(\zeta), \overline{L_I(\zeta)} \right) \right\rangle \right| \left| \left\langle f(\zeta), \overline{L_I(\zeta)} \right\rangle \right|^p \delta_{\Omega}^{\alpha-1}(z) d\lambda(z) \right] d\lambda(\zeta) \right\}.$$

For $\zeta \in \Omega \setminus \Gamma$ let us consider the coronas $Q_0(\zeta) = P_{\delta_{\Omega}(\zeta)}(\zeta)$ and $Q_t(\zeta) = P_{2^t \delta_{\Omega}(\zeta)}(\zeta) \setminus P_{2^{t-1} \delta_{\Omega}(\zeta)}(\zeta)$, $t \geq 1$. We now evaluate the integrals

$$\int_{(\Omega \setminus \Gamma) \cap Q_t(\zeta)} \left| \left\langle K_N^{1,J}(z, \zeta); \left(L(\zeta), \overline{L_I(\zeta)} \right) \right\rangle \right| \left| \left\langle f(\zeta), \overline{L_I(\zeta)} \right\rangle \right|^p \delta_{\Omega}^{\alpha-1}(z) d\lambda(z).$$

Recall that if $z \in \Omega_s$ there exists a unique k such that $\zeta \in R_k^s$ and $L_j(\zeta)$ is the j^{th} vector of the $2^k \delta_{\Omega}(Z_s)$ -extremal basis at Z_s chosen before. To simplify the notations, we denote by $(\zeta_l)_l$ the coordinate system associated to that basis so that $L_l(\zeta) = \frac{\partial}{\partial \zeta_l}$. Writing everything in those coordinate systems, we have to integrate $\left| K_N^{1,J}(z, \zeta) \wedge d\overline{\zeta}_I \right| |f_I(\zeta)|^p$ over $(\Omega \setminus \Gamma) \cap Q_t(\zeta)$ where $f_I(\zeta) = \left\langle f(\zeta), \overline{L_I(\zeta)} \right\rangle$.

First, we remark that $K_N^{1,J}(z, \zeta) \wedge d\overline{\zeta}_I$ is a sum of expressions of the form $\frac{W}{D}$ where

$$D(\zeta, z) = |z - \zeta|^{2r} \left(\frac{1}{K_0} S(z, \zeta) + \rho(\zeta) \right)^{n+N-r},$$

and,

$$W = (\overline{\zeta}_m - \overline{z}_m) \rho(\zeta)^N \prod_{k=1}^{n-r} \frac{\partial Q_{i_k}(z, \zeta)}{\partial \overline{\zeta}_{j_k}} \bigwedge_{i=1}^n (d\zeta_i \wedge d\overline{\zeta}_i)$$

or

$$W = (\overline{\zeta}_m - \overline{z}_m) \rho(\zeta)^{N-1} \frac{\partial \rho(\zeta)}{\partial \overline{\zeta}_{j_{k_0}}} Q_{i_{k_0}}(\zeta, z) \prod_{\substack{1 \leq k \leq n-r \\ k \neq k_0}} \frac{\partial Q_{i_k}(z, \zeta)}{\partial \overline{\zeta}_{j_k}} \bigwedge_{i=1}^n (d\zeta_i \wedge d\overline{\zeta}_i),$$

with $\{j_1, \dots, j_{n-r}, I\} = \{1, \dots, n\}$, and, i_1, \dots, i_{n-r}, m all different.

Then, using Lemma 3.6 and the properties of the geometry, we obtain the following estimates:

- For $z \in \Omega_s \cap Q_0(\zeta)$, using inequality (3.12), (3) of Lemma 3.4 and the fact that $\delta(\zeta) \simeq \delta(Z_s)$, $\left| K_N^{1,J}(z, \zeta) \wedge d\overline{\zeta}_I \right|$ is a sum of expressions bounded by

$$\frac{1}{\prod_{j=1}^{n-r} \tau_j^2(Z_s, \delta_{\Omega}(\zeta))} \frac{\prod_{i \in I} \tau_i(Z_s, \delta_{\Omega}(\zeta))}{\prod_{j=n-r+1}^n \tau_j(Z_s, \delta_{\Omega}(\zeta))} \frac{1}{|z - \zeta|^{2r-1}},$$

and, using (1) and (2) of Lemma 3.4, these expressions are bounded by

$$\frac{1}{\prod_{j=1}^{n-r} \tau_j^2(\zeta, \delta_{\Omega}(\zeta))} \frac{\prod_{i \in I} \tau_i(\zeta, L_i(\zeta), \delta_{\Omega}(\zeta))}{\prod_{j=n-r+1}^n \tau_j(\zeta, \delta_{\Omega}(\zeta))} \frac{1}{|z - \zeta|^{2r-1}},$$

thus

$$(3.17) \quad \frac{1}{\prod_{j=1}^{n-r} \tau_j^2(\zeta, \delta_\Omega(\zeta))} \frac{1}{\tau_{n+r-1}(\zeta, \delta_\Omega(\zeta))} \frac{1}{|z - \zeta|^{2r-1}} \delta_\Omega(\zeta) \|f(\zeta)\|_{k,p}.$$

This gives (using (3.15))

$$\int_{(\Omega \setminus F) \cap Q_0(\zeta)} |K_N^{1,J}(z, \zeta) \wedge d\bar{\zeta}_I| |f_I(\zeta)|^p \delta_\Omega^{\alpha-1}(z) d\lambda(z) \lesssim \frac{\delta_\Omega^\alpha(\zeta)}{\min\{\alpha, 1\}} \|f(\zeta)\|_{k,p}.$$

- Similarly, for $z \in \Omega_s \cap Q_t(\zeta)$, $t \geq 1$, then $\zeta \in R_k^s$ with $2^t \delta_\Omega(\zeta) \simeq 2^k \delta_\Omega(Z_s)$, and, using Lemma 3.5 instead of inequality (3.12) (as in Lemma 3.8 (2)) and (3.10), $|K_N^{1,J}(z, \zeta) \wedge d\bar{\zeta}_I| |f_I(\zeta)|^p$ is a sum of expressions bounded by

$$\left(\frac{\delta_\Omega(\zeta)}{2^t \delta_\Omega(\zeta)} \right)^{N-1} \frac{1}{\prod_{j=1}^{n-r} \tau_j^2(\zeta, 2^t \delta_\Omega(\zeta))} \frac{1}{\tau_{n+r-1}(\zeta, 2^t \delta_\Omega(\zeta))} \times \frac{2^{tr}}{|z - \zeta|^{2r-1}} \delta_\Omega(\zeta) \|f(\zeta)\|_{k,p},$$

giving, for $N \geq \alpha + r + 3$,

$$\int_{(\Omega \setminus F) \cap Q_t(\zeta)} |K_N^{1,J}(z, \zeta) \wedge d\bar{\zeta}_I| |f_I(\zeta)|^p \delta_\Omega^{\alpha-1}(z) d\lambda(z) \lesssim \frac{\delta_\Omega^\alpha(\zeta)}{\min\{\alpha, 1\}} \frac{\|f(\zeta)\|_{k,p}}{2^t}.$$

Indeed, if $\alpha \geq 1$, $z \in Q_t(\zeta)$ implies $\delta_\Omega(z) \lesssim 2^t \delta_\Omega(\zeta)$ and if $0 < \alpha < 1$, using the proof of Lemma 3.9.

This concludes the proof of Theorem 2.2.

3.3. Proof of Theorem 2.3.

As f is assumed to be smooth in $\bar{\Omega}$, Theorem 2.1 implies that the form $z \mapsto u(z) = \int_\Omega K_N^1(z, \zeta) \wedge f(\zeta)$ is continuous on $\bar{\Omega}$.

The proof starts, as in the previous section, considering the sets F and Γ and the vectors $(L_j(\zeta))_j$. Note that, as F is of Lebesgue measure zero, for almost all $\varepsilon > 0$, $\varepsilon \leq \varepsilon_0$, ε_0 small, $\sigma_\varepsilon(\{\rho = -\varepsilon\} \cap F) = 0$, σ_ε being the euclidean measure on $\{\rho = -\varepsilon\}$. Then, the proof is done showing that there exists a constant $C > 0$ such that, for such ε ,

$$\int_{\{\rho = -\varepsilon\} \setminus F} |u|^p d\sigma_\varepsilon \leq C \int_\Omega \|f\|_{k,p} d\lambda.$$

We do it using almost the same proof as in the previous section so we will not give details. Simply, note that using the same notations for Ω_s , $Q_i(\zeta)$ and writing $K_N^{1,J}(z, \zeta) \wedge f(\zeta)$ in the same extremal bases, (3.17) implies

$$\int_{(\{\rho = -\varepsilon\} \setminus F) \cap Q_0(\zeta)} |K_N^{1,J}(z, \zeta) \wedge d\bar{\zeta}_I| |f_I(\zeta)|^p d\sigma(z) \lesssim \|f(\zeta)\|_{k,p}$$

and

$$\int_{(\{\rho = -\varepsilon\} \setminus F) \cap Q_i(\zeta)} |K_N^{1,J}(z, \zeta) \wedge d\bar{\zeta}_I| |f_I(\zeta)|^p d\sigma(z) \lesssim \frac{1}{2^i} \|f(\zeta)\|_{k,p},$$

for $N \geq 5 + r$. Finally we obtain

$$\int_{\{\rho = -\varepsilon\}} |u|^p d\sigma \lesssim \int_\Omega \|f(\zeta)\|_{k,p} d\lambda(\zeta).$$

3.4. Proof of Theorem 2.4.

Our proof is very similar to the one given by N. Nguyen in [Ngu01] for convex domains. As this paper was not published, we give some details below.

A standard interpolation argument shows that we only have to prove the BMO estimate. The form s used in the previous proofs (formula (3.13)) is not well adapted to a BMO-estimate, so, we change it here and make a choice similar to the one made in [DM01]:

$$s(z, \zeta) = -\rho(z) \sum (\bar{\zeta}_i - \bar{z}_i) d(\zeta_i - z_i) + \chi(z) \overline{S(\zeta, z)} \sum Q_i(\zeta, z) d(\zeta_i - z_i),$$

where S and Q_i are given by equation (3.11) and χ is a smooth cut-off function equal to 0 when $\rho(z) < -\eta_0$ and to 1 when $\rho(z) > -\eta_0/2$, η_0 sufficiently small. Clearly, for z in a compact subset of Ω and ζ in Ω ,

$$|\langle s(z, \zeta), z - \zeta \rangle| \gtrsim |\zeta - z|^2$$

and, on $\Omega \times \Omega$,

$$|\langle s(z, \zeta), z - \zeta \rangle| \lesssim |\zeta - z|$$

and, as S and Q_i are unchanged, formula (3.1) is still valid. Moreover Lemma 3.3 remains unchanged and the proof of the theorem is reduced to the proof of the same estimate for the operator T_K (formula (3.2)).

To get the continuity up to the boundary of $z \mapsto \int_{\Omega} K_N^1(z, \zeta) \wedge f(\zeta)$ we need to prove the uniform integrability of our new kernel K_N^1 that is the following lemma:

Lemma 3.13. *For z sufficiently close to $\partial\Omega$ and ε sufficiently small, if K_N is the component of K_N^1 of bi-degree $(n, n-1)$ in ζ and $(0, 0)$ in z ,*

$$\int_{P_{\varepsilon}(z)} |K_N(z, \zeta)| d\lambda(\zeta) \lesssim \varepsilon^{\frac{1}{n+1}}.$$

Proof. Under the conditions of the lemma, the kernel K_N is reduced to

$$\left(\frac{\rho(\zeta)}{\rho(\zeta) + \frac{1}{K_0} S(z, \zeta)} \right)^{N+n-1} \frac{(\partial_{\bar{\zeta}} Q)^{n-1} \sum (\overline{S(\zeta, z)} Q_i(\zeta, z) - \rho(z) (\bar{\zeta}_i - \bar{z}_i)) d\zeta_i}{-\rho(z) |\zeta - z|^2 + |S(\zeta, z)|^2}.$$

Let us denote $P_l = P_{2^{-l}\varepsilon}(z) \setminus P_{2^{-l-1}\varepsilon}(z)$ and $\frac{W}{D} = \frac{\overline{S(\zeta, z)} Q_i(\zeta, z) - \rho(z) (\bar{\zeta}_i - \bar{z}_i)}{-\rho(z) |\zeta - z|^2 + |S(\zeta, z)|^2}$ expressed in the $(z, 2^{-l}\varepsilon)$ -extremal basis.

Let us estimate $\frac{W}{D}$ for $\zeta \in P_l$.

Clearly $\frac{|-\rho(z) (\bar{\zeta}_i - \bar{z}_i)|}{D} \lesssim \frac{1}{|\zeta - z|}.$

For the first term of W , suppose $|S(\zeta, z)| \gtrsim 2^{-l}\varepsilon$. As $|Q_i(\zeta, z)| \lesssim \frac{2^{-l}\varepsilon}{\tau_i(z, 2^{-l}\varepsilon)}$ (Lemma 3.6), we have

$$\frac{|\overline{S(\zeta, z)} Q_i(\zeta, z)|}{D} \lesssim \frac{1}{\tau_i(z, 2^{-l}\varepsilon)}.$$

Conversely, if $|S(\zeta, z)| \ll 2^{-l}\varepsilon$, by Lemma 3.5, $\rho(z) \gtrsim 2^{-l}\varepsilon$ and

$$\frac{|\overline{S(\zeta, z)} Q_i(\zeta, z)|}{D} \lesssim \frac{(2^{-l}\varepsilon)^2}{2^{-l}\varepsilon \tau_i(z, 2^{-l}\varepsilon) |\zeta - z|^2}.$$

Then, using lemmas 3.4, 3.5 and 3.6, on P_l , $|K_N|$ is bounded by a sum of expressions of the type

$$\frac{\tau_i(z, 2^{-l}\varepsilon) \tau_k(z, 2^{-l}\varepsilon)}{\prod_{j=1}^n \tau_j^2(z, 2^{-l}\varepsilon)} \left[\frac{1}{\tau_i(z, 2^{-l}\varepsilon)} + \frac{2^{-l}\varepsilon}{\tau_i(z, 2^{-l}\varepsilon) |\zeta - z|^2} + \frac{1}{|\zeta - z|} \right].$$

Now, by Lemma 3.9

$$\int_{P_{2^{-l}\varepsilon}(z)} \frac{d\lambda}{|\zeta - z|} \lesssim \frac{\prod_{j=1}^n \tau_j^2(z, 2^{-l}\varepsilon)}{\tau_n(z, 2^{-l}\varepsilon)},$$

and, if $n \geq 3$,

$$\begin{aligned} \int_{P_{2^{-l}\varepsilon}(z)} \frac{d\lambda}{|\zeta - z|^2} &\lesssim \tau_n(z, 2^{-l}\varepsilon) \int_{P_{2^{-l}\varepsilon}(z)} \frac{d\lambda}{|\zeta - z|^3} \\ &\lesssim \prod_{j=1}^{n-2} \tau_j^2(z, 2^{-l}\varepsilon) \tau_{n-1}(z, 2^{-l}\varepsilon) \tau_n(z, 2^{-l}\varepsilon), \end{aligned}$$

and, if $n = 2$,

$$\int_{P_{2^{-l}\varepsilon}(z)} \frac{d\lambda}{|\zeta - z|^2} \lesssim \tau_1(z, 2^{-l}\varepsilon)^2 \log\left(\frac{1}{\tau_2(z, 2^{-l}\varepsilon)}\right).$$

This finishes the proof of the lemma. \square

We now follow the (unpublished) method developed by N. Nguyen in [Ngu01] to prove the BMO-estimate for the function $u(z) = \int_{\Omega} K_N(z, \zeta) \wedge f(\zeta)$, $z \in \partial\Omega$: for $z_0 \in \partial\Omega$ and ε small, denoting $B_0 = P_{\varepsilon}(z_0) \cap \partial\Omega$, we have to estimate

$$\frac{1}{\sigma(B_0)} \int_{B_0} |u(z) - u_{B_0}| d\sigma(z)$$

where σ is the euclidean measure on $\partial\Omega$ and $u_{B_0} = \frac{1}{\sigma(B_0)} \int_{B_0} u(w) d\sigma(w)$.

Let $\Omega_1 = \Omega \cap P_{C\varepsilon}(z_0)$ and $\Omega_2 = \Omega \setminus \Omega_1$ where $C > 0$ is a sufficiently large number, independent of ε and z_0 , that will be fixed later. Then

$$\sigma(B_0) \int_{B_0} |u(z) - u_{B_0}| d\sigma(z) \leq I_1 + I_2$$

where, for $i = 1, 2$

$$I_i = \int_{B_0} \int_{B_0} \int_{\Omega_i} |(K_N(z, \zeta) - K_N(w, \zeta)) \wedge f(\zeta)| d\lambda(\zeta) d\sigma(z) d\sigma(w).$$

To estimate I_1 we simply write

$$I_1 \leq 2\sigma(B_0) \int_{B_0} \int_{\Omega_1} |K_N(z, \zeta) \wedge f(\zeta)| d\lambda(\zeta) d\sigma(z).$$

For $\zeta \in \Omega_1$ let $B_i(\zeta) = B_0 \cap (P_{2^i\delta_{\Omega}(\zeta)}(\zeta) \setminus P_{2^{i-1}\delta_{\Omega}(\zeta)}(\zeta))$, $i \geq 1$ (recall that, by the choice of c_0 in the definition (3.6) of the polydisks, $P_{\delta_{\Omega}(\zeta)}(\zeta) \cap \partial\Omega = \emptyset$). Then, if $z \in B_i(\zeta)$, writing $K_N(z, \zeta) \wedge f(\zeta)$ in the $(\zeta, 2^i\delta_{\Omega}(\zeta))$ -extremal basis and using lemmas 3.4, 3.5 and 3.6, $|Q_j(\zeta, z)|$ is bounded by $\frac{2^i\delta_{\Omega}(\zeta)}{\tau_j(\zeta, 2^i\delta_{\Omega}(\zeta))}$ and $|S(\zeta, z)|$ is bounded from below by $2^i\delta_{\Omega}(\zeta)$ (because $z \in \partial\Omega$), and we get

$$\begin{aligned} |K_N(z, \zeta) \wedge f_l d\zeta_l| &\lesssim \left| \frac{\delta_{\Omega}(\zeta)}{2^i\delta_{\Omega}(\zeta)} \right|^{N-1} \frac{2^i\delta_{\Omega}(\zeta)}{\prod_{j=1}^n \tau_j^2(\zeta, 2^i\delta_{\Omega}(\zeta))} \frac{\tau(\zeta, L_l(\zeta), 2^i\delta_{\Omega}(\zeta))}{2^i\delta_{\Omega}(\zeta)} |f_l(\zeta)| \\ &\lesssim \left(\frac{1}{2^i} \right)^{N-1} \frac{1}{\sigma(B_i(\zeta))} \|f(\zeta)\|_k, \end{aligned}$$

the last inequality coming from (3.10). Integrating over $B_i(\zeta)$ and summing up we obtain (C being fixed, $\sigma(B_0) \simeq \sigma(\Omega_1 \cap \partial\Omega)$)

$$\frac{I_1}{(\sigma(B_0))^2} \lesssim \frac{1}{\sigma(\Omega_1 \cap \partial\Omega)} \int_{\Omega_1} \|f(\zeta)\|_k \leq \| \|f(\zeta)\|_k d\lambda \|_{W^1}.$$

To estimate I_2 it is necessary to evaluate the difference $(K_N(z, \zeta) - K_N(w, \zeta))$. When Ω is convex, in [Ngu01] N. Nguyen estimates the derivatives of $K_N(\cdot, \zeta)$ on the segment $[z, w]$. Here, this segment is not necessarily contained in $\overline{\Omega}$ and we introduce two points Z and W which are $c_1\varepsilon$ -translations of z and w in the direction of the inward real normal at z_0 , c_1 being chosen sufficiently large (independent of z_0 and ε) so that the segment $[Z, W]$ is contained in Ω . Thus the segments $[z, Z]$, $[Z, W]$ and $[W, w]$ are contained in a polydisk $P_{c_2\varepsilon}(z_0)$ (c_2 independent of z_0 and ε) and C is chosen sufficiently large so that, if $\zeta \in \Omega_2$, for all $u \in P_{c_2\varepsilon}(z_0)$, the anisotropic distance from ζ to u is equivalent to the distance from ζ to z_0 .

The points z and w being on the boundary of Ω , we have to control the variations of

$$\left(\frac{\rho(\zeta)}{\frac{1}{K_0}S(z, \zeta) + \rho(\zeta)} \right)^{n-1+N} \frac{(\partial_{\bar{\zeta}} Q(z, \zeta))^{n-1} \sum Q_j(\zeta, z) d\zeta_j}{S(\zeta, z)} = K'_N(z, \zeta).$$

Let $Q_i = P_{2^i C \varepsilon}(z_0) \setminus P_{2^{i-1} C \varepsilon}(z_0)$, $i \geq 1$, and let us estimate the derivatives of $K'_N(u, \zeta)$ for $u \in P_{c_2\varepsilon}(z_0)$ (in particular in the three segments) and $\zeta \in Q_i$. Using lemmas 3.4, 3.5 and 3.6, and the fact that, if $u \in P_{c_2\varepsilon}(z_0)$, $|\rho(u)| \lesssim c_2\varepsilon$, enlarging C if necessary, $S(z, u) \gtrsim 2^i C \varepsilon$ (by Lemma 3.5), writing K'_N and f in the $(z_0, 2^i C \varepsilon)$ -extremal basis we get

$$\left| \frac{\partial}{\partial u_t} K'_N(u, \zeta) \wedge d\bar{\zeta}_l \right| + \left| \frac{\partial}{\partial \bar{u}_t} K'_N(u, \zeta) \wedge d\bar{\zeta}_l \right| \lesssim \frac{2^i C \varepsilon}{\prod_{j=1}^n \tau_j^2(z_0, 2^i C \varepsilon)} \frac{\tau(z_0, L_l, 2^i C \varepsilon)}{2^i C \varepsilon} \frac{1}{\tau(z_0, L_t, 2^i C \varepsilon)}.$$

On each of the three segments $[z, Z]$, $[Z, W]$ and $[W, w]$ the increase in the direction L_t is bounded, up to a multiplicative constant, by $\tau(z_0, L_t, c_2\varepsilon)$, and, by (5) of the geometry,

$$\tau(z_0, L_t, 2^i C \varepsilon) \gtrsim \left(\frac{2^i C}{c_2} \right)^{1/m} \tau(z_0, L_t, c_2\varepsilon),$$

so

$$\left| (K_N(z, \zeta) - K_N(w, \zeta)) \wedge f_l(\zeta) d\bar{\zeta}_l \right| \lesssim \frac{|f_l(\zeta)|}{\sigma(Q_i \cap \partial\Omega)} \frac{\tau(\zeta, L_l, \delta_\Omega(\zeta))}{\delta_\Omega(\zeta)} \left(\frac{1}{2^i} \right)^{1/m},$$

because $\tau(z_0, L_l, 2^i C \varepsilon) \simeq \tau(\zeta, L_l, 2^i C \varepsilon)$ (Lemma 3.4) and

$$\frac{\tau(\zeta, L_l, 2^i C \varepsilon)}{2^i C \varepsilon} \lesssim \frac{\tau(\zeta, L_l, \delta_\Omega(\zeta))}{\delta_\Omega(\zeta)}$$

because $\delta_\Omega(\zeta) \lesssim 2^i C \varepsilon$. Finally

$$\int_{Q_i} \left| (K_N(z, \zeta) - K_N(w, \zeta)) \wedge f_l(\zeta) d\bar{\zeta}_l \right| d\lambda(\zeta) \lesssim \left(\frac{1}{2^i} \right)^{1/m} \frac{1}{\sigma(Q_i \cap \partial\Omega)} \int_{Q_i} \|f(\zeta)\|_k,$$

giving $\frac{J_2}{\sigma(B_0)^2} \lesssim \| \|f(\zeta)\|_k d\lambda \|_{W^1}$, and the proof is complete.

4. PROOF OF THEOREM 2.6

We use the method developed for the proofs of theorems 2.1 and 2.3 of [CDM15].

In [CDM14b] we proved the following result: let g be a gauge of D and $\rho_0 = g^4 e^{1-1/g} - 1$ then:

Theorem 4.1 (Theorem 2.1 of [CDM14b]). *Let $\omega_0 = (-\rho_0)^r$, r being a non negative rational number, and let P_{ω_0} be the Bergman projection of the Hilbert space $L^2(D, \omega_0)$. Then, for $p \in]1, +\infty[$ and $1 \leq \beta \leq p(r+1) - 1$, P_{ω_0} maps continuously the space $L^p(D, \delta_D^\beta)$ into itself and, for $\alpha > 0$, P_{ω_0} maps continuously the lipschitz space $\Lambda_\alpha(D)$ into itself.*

If ω is as in Theorem 2.6 then there exists a strictly positive \mathcal{C}^1 function in \bar{D} , φ , such that $\omega = \varphi \omega_0$. Then we compare the regularity of P_{ω_0} and P_ω using the following formula (Proposition 3.1 of [CDM15]): for $u \in L^2(D, \omega)$,

$$\varphi P_\omega(u) = P_{\omega_0}(\varphi u) + (\text{Id} - P_{\omega_0}) \circ A(P_\omega(u) \wedge \bar{\partial} \varphi),$$

where A is any operator solving the $\bar{\partial}$ -equation for $\bar{\partial}$ -closed forms in $L^2(D, \omega)$.

We first show that P_ω maps continuously $L^p(D, \delta_\Omega^r)$ into itself. Let $f \in L^p(D, \delta_\Omega^r)$, $p \in [2, +\infty[$. For A we choose the operator T of Proposition 2.2 with $\gamma = r$, and we choose ε and an integer M such that $0 < \varepsilon \leq \varepsilon_N$, ε_N as in Proposition 2.2, and $p = 2 + M\varepsilon$. Let us prove, by induction, that $P_\omega(f) \in L^{2+k\varepsilon}(D, \delta_D^r)$ for $k = 0, \dots, M$.

Assume this is true for $0 \leq k < M$. Then by Proposition 2.2,

$$A(P_\omega(f) \wedge \bar{\partial} \varphi) \in L^{2+(k+1)\varepsilon}(D, \delta_D^r)$$

and, by Theorem 4.1,

$$(\text{Id} - P_{\omega_0}) \circ A(P_\omega(u) \wedge \bar{\partial} \varphi) \in L^{2+(k+1)\varepsilon}(D, \delta_D^r).$$

As φ is continuous and strictly positive we get $P_\omega(f) \in L^{2+(k+1)\varepsilon}(D, \delta_D^r)$.

Thus, P_ω maps $L^p(D, \delta_D^r)$ into itself for $p \in [2, +\infty[$. The same result for $p \in]1, 2]$ follows because P_ω is self-adjoint.

To prove that P_ω maps $L^p(D, \delta_D^\beta)$ into itself, for $-1 < \beta \leq r$, we use a similar induction argument using Proposition 2.1 instead of Proposition 2.2:

For A we choose now the operator T of Proposition 2.1 with $\gamma = r$, and ε such that $0 < \varepsilon \leq 1/m$, and for which there exists an integer L such that $\beta = r - L\varepsilon$. For $f \in L^p(D, \delta_D^\beta)$, assume $P_\omega(f) \in L^2(D, \delta_D^{r-l\varepsilon})$, $0 \leq l < L$. Then, Proposition 2.1 and Theorem 4.1 imply $(\text{Id} - P_{\omega_0}) \circ A(P_\omega(u) \wedge \bar{\partial} \varphi) \in L^p(D, \delta_D^{r-(l+1)\varepsilon})$ which gives $P_\omega(f) \in L^p(D, \delta_D^{r-(l+1)\varepsilon})$. By induction this gives $P_\omega(f) \in L^p(D, \delta_D^\beta)$, concluding the proof of (1) of the theorem.

The proof of (2) of the theorem is now easy: assume $u \in \Lambda_\alpha(D)$, $0 < \alpha \leq 1/m$. Let $p \leq +\infty$ such that $\alpha = \frac{1}{m} \left[1 - \frac{m(r+n)+2}{p} \right]$. By part (1), $P_\omega(u) \in L^p(D, \delta_D^r)$, by (3) of Theorem 2.1, $A(P_\omega(u) \wedge \bar{\partial} \varphi) \in \Lambda_\alpha(D)$ (A being the operator T), and, by Theorem 4.1, $(\text{Id} - P_{\omega_0}) \circ A(P_\omega(u) \wedge \bar{\partial} \varphi) \in \Lambda_\alpha(D)$ concluding the proof.

Remark.

- (1) The restriction $-1 < \beta \leq r$ in Theorem 2.6 (instead of $0 < \beta + 1 \leq p(r+1)$ in [CDM14b]) is due to the method because if $f \in L^p(D, \delta_D^\beta)$ with $\beta > r$, a priori $P_\omega(f)$ does not exist.
- (2) The restriction $r \in \mathbb{Q}_+$ is not natural and it is very probable that Theorem 2.6 is true with $r \in \mathbb{R}_+$. To get that with our method we should first prove the result of Theorem 4.1 for r a non negative real number. Looking at the proof in [CDM14b], this should be done proving point-wise estimates of the Bergman kernel of a domain \tilde{D} of the form

$$\tilde{D} = \left\{ (z, w) \in \mathbb{C}^{n+m} \text{ such that } \rho_0(z) + \sum |w_i|^{2q_i} < 0 \right\},$$

with q_i large *real* numbers such that $\sum 1/q_i = r$. The difficulty here being that \tilde{D} is no more \mathcal{C}^∞ -smooth and thus the machinery induced by the finite type cannot be used.

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