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Bismut-Elworthy-Li formulae for Bessel processes

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Abstract

In this article we are interested in the differentiability property of the Markovian semi-group corresponding to the Bessel processes of nonnegative dimension. More precisely, for all $\delta \geq 0$ and $T > 0$, we compute the derivative of the function $x \mapsto P_T^\delta F(x)$, where $(P_t^\delta)_{t \geq 0}$ is the transition semi-group associated to the δ -dimensional Bessel process, and F is any bounded Borel function on \mathbb{R}_+ . The obtained expression shows a nice interplay between the transition semi-groups of the δ - and the $(\delta+2)$ -dimensional Bessel processes. As a consequence, we deduce that the Bessel processes satisfy the strong Feller property, with a continuity modulus which is independent of the dimension. Moreover, we provide a probabilistic interpretation of this expression as a Bismut-Elworthy-Li formula.

1 Introduction

Bessel processes are a one-parameter family of nonnegative diffusion processes with a singular drift, which present a reflecting behavior when they hit the origin. The smaller the parameter (called dimension), the more intense the reflection. Hence, studying the dynamics of these processes is a non-trivial problem, especially when the dimension is small. Despite these apparent difficulties, Bessel processes have remarkably nice properties. Therefore they provide an instructive insight in the study of stochastic differential equations (SDEs) with a singular drift, as well as the study of reflected SDEs.

For all $x \geq 0$ and $\delta \geq 0$, the squared Bessel process of dimension δ started at x^2 is the unique strong solution of the equation:

$$X_t = x^2 + 2 \int_0^t \sqrt{X_s} dB_s + \delta t. \quad (1)$$

Such a process X is nonnegative, and the law of its square-root $\rho = \sqrt{X}$ is, by definition, the δ -dimensional Bessel process started at x (see [9], section XI, or Chapter 3 of [13] for

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an introduction to Bessel processes). The process ρ satisfies the following SDE before its first hitting time T_0 of 0:

$$\forall t \in [0, T_0), \quad \rho_t = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s} + B_t.$$

This is an SDE with non-Lipschitz continuous drift term given by the function $x \mapsto \frac{\delta-1}{2} \frac{1}{x}$ on $(0, +\infty)$. Note that when $\delta < 1$, this function is nondecreasing on \mathbb{R}_+ , and as δ decreases this "wrong" monotonicity becomes more and more acute. As a consequence, for δ small, the process ρ is not mean-square differentiable, so that classical criteria for the Bismut-Elworthy-Li formula to hold (see [4], Section 1.5, and Section 2 below) do not apply here. Hence, one would not even expect such a formula to hold for $\delta < 1$. For instance, even continuity of the flow is not known in this regime, see Remark 10 below.

The aim of the present paper is to study the derivative in space of the family of transition kernels $(P_T^\delta)_{T \geq 0}$ of the δ -dimensional Bessel process. In a first part, we show that this derivative can be expressed in terms of the transition kernels of the δ - and the $(\delta + 2)$ -dimensional Bessel processes. More precisely, we prove that, for all function $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ bounded and Borel, all $T > 0$ and all $x \geq 0$, we have:

$$\frac{d}{dx} P_T^\delta F(x) = \frac{x}{T} \left(P_T^{\delta+2} F(x) - P_T^\delta F(x) \right). \quad (2)$$

As a consequence, the Bessel processes satisfy the strong Feller property uniformly in δ . In a second part, we interpret the above result probabilistically as a Bismut-Elworthy-Li formula. More precisely, given a realization ρ of the Bessel process through the SDE (1), we introduce the derivative η_t of ρ_t with respect to the initial condition x , and show that when $\delta > 0$, the stochastic integral $\int_0^t \eta_s dB_s$ is well-defined as an L^p martingale, for some $p > 1$ depending on δ . Moreover, it turns out that $\int_0^T \eta_s dB_s$ is (up to a constant) the Radon-Nikodym derivative of the $(\delta + 2)$ -dimensional Bessel process over the interval $[0, T]$ w.r.t. the δ -dimensional one. As a consequence, we deduce that the above equation can be rewritten:

$$\frac{d}{dx} P_T^\delta F(x) = \frac{1}{T} \mathbb{E} \left[F(\rho_t(x)) \left(\int_0^t \eta_s(x) dB_s \right) \right] \quad (3)$$

which is an apparition, in an unexpected context, of the well-known Bismut-Elworthy-Li formula (see [7] for a precise statement and proof of the Bismut-Elworthy-Li formula in the case of diffusions with smooth coefficients).

One surprising feature is that, while (2) is very easy to prove whatever the value of $\delta \geq 0$, on the other hand, the process $(\int_0^t \eta_s(x) dB_s)_{t \geq 0}$ has less and less finite moments as δ decreases, which makes the proof of (3) more involved for small δ . In particular, this process is not in L^2 for $\delta < 2(\sqrt{2} - 1)$, and when $\delta = 0$, we do not even know whether the stochastic integral $\int_0^t \eta_s(x) dB_s$ is well-defined as a local martingale.

This article was originally motivated by the hope to prove the strong Feller property for some singular reflected SDEs or SPDEs. Recently, several works have brought about new techniques to prove the strong Feller property for singular SPDEs. Thus, in [10], the authors established this property for the $P(\Phi)_2$ equation, and in [8], the authors

established it for a large class of singular semilinear SPDEs. The fact, mentioned above, that blowup of η does not affect the strong Feller property of Bessel processes is reminiscent of the latter article, where the setting used to prove the strong Feller property allows blowup in finite time of the solution. Also, we hope that the techniques used in the present article might give inspiration to treat more general cases. Note that, even in the present context, where many computations can be performed explicitly, we still have an open problem concerning the Strong Feller bounds for Bessel processes of dimension $\delta \leq 2(\sqrt{2} - 1)$ (see Remark 14 below).

The plan of our paper is as follows. In Section 2 we recall the classical Bismut-Elworthy-Li formula for diffusions in \mathbb{R} with a dissipative drift, and show how this implies the strong Feller property. In Section 3 we recall the definition of Bessel processes and their basic properties. In Section 4 we compute the derivative of the Bessel semi-group. In section 5 we establish the differentiability of the Bessel flow at any given point in \mathbb{R}_+^* , and we give an expression for (some modification of) the derivative. In Section 6, we show that this derivative is not bounded in time when $\delta < 1$. We prove, however, that it is linked to an interesting martingale corresponding to the family of Radon-Nikodym derivatives of the $(\delta + 2)$ -dimensional Bessel process w.r.t.the δ -dimensional one. In Section 7 we prove the Bismut-Elworthy-Li formula for the Bessel processes of dimension $\delta > 0$.

2 Classical Bismut-Elworthy-Li formula for one-dimensional diffusions

In this section we recall very briefly the Bismut-Elworthy-Li formula in the case of one-dimensional diffusions, and the way this formula implies the strong Feller property.

Consider an SDE on \mathbb{R} of the form :

$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x \tag{4}$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is smooth and satisfies:

$$\begin{aligned} |b(x) - b(y)| &\leq C|x - y|, & x, y \in \mathbb{R} \\ b'(x) &\leq L, & x \in \mathbb{R} \end{aligned} \tag{5}$$

where $C > 0$, $L \in \mathbb{R}$ are some constants. By the classical theory of SDEs, for all $x \in \mathbb{R}$, there exists a unique continuous, square-integrable process $(X_t(x))_{t \geq 0}$ satisfying (4). Actually, by the Lipschitz assumption on b , there even exists a bi-continuous process $(X_t(x))_{t \geq 0, x \in \mathbb{R}}$ such that, for all $x \in \mathbb{R}$, $(X_t(x))_{t \geq 0}$ solves (4).

Let $x \in \mathbb{R}$. Consider the solution $(\eta_t(x))_{t \geq 0}$ to the variation equation obtained by formally differentiating (4) with respect to x :

$$d\eta_t(x) = b'(X_t)\eta_t(x)dt, \quad \eta_0(x) = 1$$

Note that this is a (random) linear ODE with explicit solution given by:

$$\eta_t(x) = \exp\left(\int_0^t b'(X_s)ds\right)$$

It is easy to prove that, for all $t \geq 0$ and $x \in \mathbb{R}$, the map $y \rightarrow X_t(y)$ is a.s. differentiable at x and:

$$\frac{dX_t}{dx} \stackrel{\text{a.s.}}{=} \eta_t(x) \quad (6)$$

Remark 1. Note that $\eta_t(x) > 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. This reflects the fact that, for all $x \leq y$, by a comparison theorem for SDEs (see Theorem 3.7 in Chapter IX in [9]), one has $X_t(x) \leq X_t(y)$.

Recall that a Markovian semi-group $(P_t)_{t \geq 0}$ on a Polish space E is said to satisfy the strong Feller property if, for all $t > 0$ and $\varphi : E \rightarrow \mathbb{R}$ bounded and Borel, the function $P_t\varphi : E \rightarrow \mathbb{R}$ defined by:

$$P_t\varphi(x) = \int \varphi(y)P_t(x, dy), \quad x \in \mathbb{R}$$

is continuous.

The strong Feller property is very useful in the study of SDEs and SPDEs, namely for the proof of ergodicity (see, e.g., the monographs [4], [5] and [13], as well as the recent articles [8] and [10], for applications of the strong Feller property in the context of SPDEs).

Let $(P_t)_{t \geq 0}$ be the Markovian semi-group associated to the SDE (9). We are interested in proving the strong Feller property for $(P_t)_{t \geq 0}$. Note that, by assumption (5), $\eta_t(x) \leq e^{Lt}$ for all $t \geq 0$ and $x \in \mathbb{R}$. Therefore, by (6) and the dominated convergence theorem, for all $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ differentiable with a bounded derivative, one has:

$$\frac{d}{dx}(P_t\varphi)(x) = \frac{d}{dx}\mathbb{E}[\varphi(X_t(x))] = \mathbb{E}[\varphi(X_t(x))\eta_t(x)]$$

As a consequence, for all $t \geq 0$, P_t preserves the space $C_b^1(\mathbb{R})$ of bounded, continuously differentiable functions on \mathbb{R} with a bounded derivative. It turns out that, actually, for all $t > 0$, P_t maps the space $C_b(\mathbb{R})$ of bounded and continuous functions into $C_b^1(\mathbb{R})$. This is a consequence of the following, nowadays well-known, result:

Theorem 1 (Bismut-Elworthy-Li formula). *For all $T > 0$ and $\varphi \in C_b(\mathbb{R})$, the function $P_T\varphi$ is differentiable and we have:*

$$\frac{d}{dx}P_T\varphi(x) = \frac{1}{T}\mathbb{E}\left[\varphi(X_T(x))\int_0^T \eta_s(x)dB_s\right] \quad (7)$$

Proof. See [7], Theorem 2.1, or [13], Lemma 5.17 for a proof. \square

Corollary 1. *The semi-group $(P_t)_{t \geq 0}$ satisfies the strong Feller property and, for all $T > 0$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ bounded and Borel, one has:*

$$\forall x, y \in \mathbb{R}, \quad |P_T\varphi(x) - P_T\varphi(y)| \leq e^L \frac{\|\varphi\|_\infty}{\sqrt{T \wedge 1}} |x - y|, \quad (8)$$

where $\|\cdot\|_\infty$ denotes the supremum norm.

The following remark is crucial.

Remark 2. Inequality (8) involves only the dissipativity constant L , not the Lipschitz constant C . This makes the Bismut-Elworthy-Li formula very useful in the study of SPDEs with a dissipative drift.

Proof of Corollary 1. By approximation, it suffices to prove (8) for $\varphi \in C_b(\mathbb{R})$. For such a φ and for all $T > 0$, by the Bismut-Elworthy-Li formula, one has:

$$\left| \frac{d}{dx} P_T \varphi(x) \right| \leq \frac{\|\varphi\|_\infty}{T} \mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right]$$

Remark that the process $(\eta_t(x))_{t \geq 0}$ is locally bounded since it is dominated by $(e^{Lt})_{t \geq 0}$, so that the stochastic integral $\left(\int_0^t \eta_s(x) dB_s \right)_{t \geq 0}$ is an L^2 martingale. Hence using Jensen's inequality as well as Itô's isometry formula, we obtain:

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right] &\leq \sqrt{\mathbb{E} \left[\int_0^T \eta_s(x)^2 ds \right]} \\ &\leq \sqrt{\int_0^T e^{2Ls} ds} \end{aligned}$$

and the last quantity is bounded by $\sqrt{e^{2LT}} = e^L \sqrt{T}$ for all $T \in (0, 1]$. Therefore, we deduce that:

$$\forall x \in \mathbb{R}, \quad \left| \frac{d}{dx} P_T \varphi(x) \right| \leq e^L \frac{\|\varphi\|_\infty}{\sqrt{T}}$$

so that:

$$\forall x, y \in \mathbb{R}, \quad |P_T \varphi(x) - P_T \varphi(y)| \leq e^L \frac{\|\varphi\|_\infty}{\sqrt{T}} |x - y|$$

for all $\varphi \in C_b(\mathbb{R})$ and $T \in (0, 1]$. The case $T > 1$ follows at once by using the semi-group property of $(P_t)_{t \geq 0}$:

$$\begin{aligned} |P_T \varphi(x) - P_T \varphi(y)| &= |P_1 (P_{T-1} \varphi)(x) - P_1 (P_{T-1} \varphi)(y)| \\ &\leq e^L \frac{\|P_{T-1} \varphi\|_\infty}{\sqrt{1}} |x - y| \\ &\leq e^L \|\varphi\|_\infty |x - y| \end{aligned}$$

The claim follows. \square

Remark 3 (A brief history of the Bismut-Elworthy-Li formula). A particular form of this formula had originally been derived by J.M. Bismut in [2] using Malliavin calculus in the framework of the study of the logarithmic derivative of the fundamental solution of the heat equation on a compact manifold. In [7], K.D. Elworthy and X.-M. Li generalized this formula to a large class of diffusion processes on manifolds with smooth coefficients, and gave also variants of this formula to higher-order derivatives.

The key property allowing the analysis performed in this section is the dissipativity property (5). Without this property being true, one would not even expect the Bismut-Elworthy-Li formula to hold. However, in the sequel, we shall prove that results such as Theorem 1 and Corollary 1 above can also be obtained for a family of diffusions with a non-dissipative drift (informally $L = +\infty$), namely for the Bessel processes of dimension smaller than 1.

3 Bessel processes: notations and basic facts

In the sequel, for any subinterval I of \mathbb{R}_+ , $C(I)$ will denote the set of continuous functions $I \rightarrow \mathbb{R}$. We shall consider this set endowed with the topology of uniform convergence on compact sets, and will denote by $\mathcal{B}(C(I))$ the corresponding Borel σ -algebra.

Consider the canonical measurable space $(C(\mathbb{R}_+), \mathcal{B}(C(I)))$ endowed with the canonical filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $(B_t)_{t \geq 0}$ be a standard linear $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion. For all $x \geq 0$ and $\delta \geq 0$, there exists a unique continuous, predictable, nonnegative process $(X_t^\delta(x))_{t \geq 0}$ satisfying:

$$X_t = x^2 + 2 \int_0^t \sqrt{X_s} dB_s + \delta t. \quad (9)$$

$(X_t^\delta(x))_{t \geq 0}$ is a squared Bessel process of dimension δ started at x^2 , and the process $\rho_t^\delta(x) := \sqrt{X_t^\delta(x)}$ is a δ -dimensional Bessel process started at x . In the sequel, we will also write the latter process as $(\rho_t(x))_{t \geq 0}$, or ρ , when there is no risk of ambiguity.

We recall the following monotonicity property of the family of Bessel processes:

Lemma 1. *For all couples $(\delta, \delta'), (x, x') \in \mathbb{R}_+$ such that $\delta \leq \delta'$ and $x \leq x'$, we have, a.s.:*

$$\forall t \geq 0, \quad \rho_t^\delta(x) \leq \rho_t^{\delta'}(x').$$

Proof. By Theorem (3.7) in [9], Section IX, applied to the equation (9), the following property holds a.s.:

$$\forall t \geq 0, \quad X_t^\delta(x) \leq X_t^{\delta'}(x').$$

Taking the square root on both sides above, we deduce the result. \square

For all $a \geq 0$, let $T_a(x)$ denote the $(\mathcal{F}_t)_{t \geq 0}$ stopping time defined by:

$$T_a(x) := \inf\{t > 0, \rho_t(x) \leq a\}$$

(we shall also write T_a). We recall the following fact, (see e.g. Proposition 3.6 of [13]):

Proposition 1. *The following dichotomy holds:*

- $T_0(x) = +\infty$ a.s., if $\delta \geq 2$,
- $T_0(x) < +\infty$ a.s., if $0 \leq \delta < 2$.

Applying Itô's lemma to $\rho_t = \sqrt{X_t^\delta(x)}$, we see that ρ satisfies the following relation on the interval $[0, T_0)$:

$$\forall t \in [0, T_0), \quad \rho_t = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s} + B_t. \quad (10)$$

4 Derivative in space of the Bessel semi-group

Let $\delta \geq 0$. We denote by P_x^δ the law, on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$, of the δ -dimensional Bessel process started at x , and we write E_x^δ for the corresponding expectation operator. We also denote by $(P_t^\delta)_{t \geq 0}$ the family of transition kernels associated with the δ -dimensional Bessel process, defined by

$$P_t^\delta F(x) := E_x^\delta(F(\rho_t))$$

for all $t \geq 0$ and all $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ bounded and Borel. The aim of this section is to prove the following:

Theorem 2. *For all $T > 0$ and all $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ bounded and Borel, the function $x \rightarrow P_t^\delta F(x)$ is differentiable on \mathbb{R}_+ , and for all $x \geq 0$:*

$$\frac{d}{dx} P_T^\delta F(x) = \frac{x}{T} \left(P_T^{\delta+2} F(x) - P_T^\delta F(x) \right). \quad (11)$$

In particular, the function $x \rightarrow P_t^\delta F(x)$ satisfies the Neumann boundary condition at 0:

$$\left. \frac{d}{dx} P_T^\delta F(x) \right|_{x=0} = 0.$$

Remark 4. By Theorem 2, the derivative of the function $x \mapsto P_T^\delta F(x)$ is a smooth function of $P_T^{\delta+2} F(x)$ and $P_T^\delta F(x)$. Hence, reasoning by induction, we deduce that the function $x \mapsto P_T^\delta F(x)$ is actually smooth on \mathbb{R}_+ .

Proof. The proof we propose here relies on the explicit formula for the transition semi-group of the Bessel processes. We first treat the case $\delta > 0$.

Given $\delta > 0$, let $\nu := \frac{\delta}{2} - 1$, and denote by I_ν the modified Bessel function of index ν . We have (see, e.g., Chap. XI.1 in [9]) :

$$P_t^\delta F(x) = \int_0^\infty p_T^\delta(x, y) F(y) dy$$

where, for all $y \geq 0$:

$$\begin{aligned} p_t^\delta(x, y) &= \frac{1}{T} \left(\frac{y}{x} \right)^\nu y \exp\left(-\frac{x^2 + y^2}{2T}\right) I_\nu\left(\frac{xy}{T}\right), \quad \text{if } x > 0, \\ p_t^\delta(0, y) &= \frac{2^{-\nu} T^{-(\nu+1)}}{\Gamma(\nu+1)} y^{2\nu+1} \exp\left(-\frac{y^2}{2T}\right) \end{aligned}$$

where Γ denotes the gamma function. By the power series expansion of the function I_ν we have, for all $x, y \geq 0$:

$$p_T^\delta(x, y) = \frac{1}{T} \exp\left(-\frac{x^2 + y^2}{2T}\right) \tilde{p}_T^\delta(x, y) \quad (12)$$

with:

$$\tilde{p}_T^\delta(x, y) := \sum_{k=0}^{\infty} \frac{y^{2k+2\nu+1} x^{2k} (1/2T)^{2k+\nu}}{k! \Gamma(k + \nu + 1)}.$$

Note that $\tilde{p}_T^\delta(x, y)$ is the sum of a series with infinite radius of convergence in x , hence we can compute its derivative by differentiating under the sum. We have:

$$\begin{aligned}\frac{\partial}{\partial x} \tilde{p}_T^\delta(x, y) &= \frac{\partial}{\partial x} \left(\sum_{k=0}^{\infty} \frac{x^{2k} y^{2k+2\nu+1} (1/2T)^{2k+\nu}}{k! \Gamma(k + \nu + 1)} \right) \\ &= \sum_{k=0}^{\infty} \frac{2k x^{2k-1} y^{2k+2\nu+1} (1/2T)^{2k+\nu}}{k! \Gamma(k + \nu + 1)} \\ &= \frac{x}{T} \sum_{k=1}^{\infty} \frac{x^{2k-2} y^{2k+2\nu+1} (1/2T)^{2k+\nu-1}}{(k-1)! \Gamma(k + \nu + 1)}.\end{aligned}$$

Hence, performing the change of variable $j = k - 1$, and remarking that $\nu + 1 = \frac{\delta+2}{2} - 1$, we obtain:

$$\begin{aligned}\frac{\partial}{\partial x} \tilde{p}_T^\delta(x, y) &= \frac{x}{T} \sum_{j=0}^{\infty} \frac{x^{2(j+1)-2} y^{2(j+1)+2\nu+1} (1/2T)^{2(j+1)+\nu-1}}{j! \Gamma((j+1) + \nu + 1)} \\ &= \frac{x}{T} \sum_{j=0}^{\infty} \frac{x^{2j} y^{2j+2(\nu+1)+1} (1/2T)^{2j+(\nu+1)}}{j! \Gamma(j + (\nu + 1) + 1)} \\ &= \frac{x}{T} \tilde{p}_T^{\delta+2}(x, y).\end{aligned}$$

As a consequence, differentiating equality (12) with respect to x , we obtain:

$$\begin{aligned}\frac{\partial}{\partial x} p_T^\delta(x, y) &= \left(-\frac{x}{T} \tilde{p}_T^\delta(x, y) + \frac{\partial}{\partial x} \tilde{p}_T^\delta(x, y) \right) \frac{1}{T} \exp\left(-\frac{x^2 + y^2}{2T}\right) \\ &= \frac{x}{T} \left(-p_T^\delta(x, y) + p_T^{\delta+2}(x, y) \right).\end{aligned}$$

Hence, we deduce that the function $x \mapsto P_T^\delta F(x)$ is differentiable, with a derivative given by (11).

Now suppose that $\delta = 0$. We have, for all $x \geq 0$:

$$P_T^0 F(x) = \exp\left(-\frac{x^2}{2T}\right) F(0) + \int_0^\infty p_T(x, y) F(y) dy \quad (13)$$

where, for all $y \geq 0$:

$$p_T(x, y) = \frac{1}{T} \exp\left(-\frac{x^2 + y^2}{2T}\right) \tilde{p}_T(x, y)$$

with:

$$\tilde{p}_T(x, y) := x I_1\left(\frac{xy}{T}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+2} (y/2T)^{2k+1}}{k!(k+1)!}.$$

Here again, we can differentiate the sum term by term, so that, for all $x, y \geq 0$:

$$\begin{aligned}\frac{\partial}{\partial x} \tilde{p}_T(x, y) &= \frac{x}{T} \sum_{k=0}^{\infty} \frac{x^{2k} y^{2k+1} (1/2T)^{2k}}{k!^2} \\ &= \frac{x}{T} \tilde{p}_T^2(x, y).\end{aligned}$$

Therefore, for all $x, y \geq 0$, we have:

$$\begin{aligned}\frac{\partial}{\partial x} p_T(x, y) &= \left(-\frac{x}{T} \tilde{p}_T(x, y) + \frac{\partial}{\partial x} \tilde{p}_T(x, y) \right) \frac{1}{T} \exp\left(-\frac{x^2 + y^2}{2T}\right) \\ &= \frac{x}{T} (-\tilde{p}_T(x, y) + \tilde{p}_T^2(x, y)) \frac{1}{T} \exp\left(-\frac{x^2 + y^2}{2T}\right) \\ &= \frac{x}{T} (-p_T(x, y) + p_T^2(x, y))\end{aligned}$$

Hence, differentiating (13) with respect to x , and using the dominated convergence theorem to differentiate inside the integral, we obtain:

$$\begin{aligned}\frac{\partial}{\partial x} P_T^0 F(x) &= -\frac{x}{T} \exp\left(-\frac{x^2}{2T}\right) F(0) + \frac{x}{T} \int_0^{\infty} (-p_T(x, y) + p_T^2(x, y)) F(y) dy \\ &= \frac{x}{T} (-P_T^0 F(x) + P_T^2 F(x)),\end{aligned}$$

which yields the claim. \square

Remark 5. Formula (11) can also be derived using the Laplace transform of the one-dimensional marginals of the squared Bessel processes. Indeed, denote by $(Q_t^\delta)_{t \geq 0}$ the family of transition kernels of the δ -dimensional squared Bessel process. Then for all $\delta \geq 0$, $x \geq 0$, $T > 0$, and all function f of the form $f(x) = \exp(-\lambda x)$ with $\lambda \geq 0$, one has:

$$Q_T^\delta f(x) = \exp\left(-\frac{\lambda x}{1 + 2\lambda T}\right) (1 + 2\lambda T)^{-\delta/2}$$

(see [9], Chapter XI, Cor. (1.3)). For such test functions f , we check at once that the following equality holds:

$$\frac{d}{dx} Q_T^\delta f(x) = \frac{1}{2T} \left(Q_T^{\delta+2} f(x) - Q_T^\delta f(x) \right).$$

By linearity and by the Stone-Weierstrass theorem, we deduce that this equality holds for all bounded, continuous functions f . Then an approximation argument enables to deduce the equality for all functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ Borel and bounded. Finally, remarking that for all bounded Borel function F on \mathbb{R}_+ we have

$$P_T^\delta F(x) = Q_T^\delta f(x^2)$$

with $f(x) := F(\sqrt{x})$, we deduce that:

$$\begin{aligned} \frac{d}{dx} P_T^\delta F(x) &= 2x \frac{d}{dx} (Q_T^\delta f)(x^2) \\ &= \frac{x}{T} \left(Q_T^{\delta+2} f(x^2) - Q_T^\delta f(x^2) \right) \\ &= \frac{x}{T} \left(P_T^{\delta+2} F(x) - P_T^\delta F(x) \right) \end{aligned}$$

which yields the equality (11).

Corollary 2. *The semi-group $(P_t^\delta)_{t \geq 0}$ has the strong Feller property. More precisely, for all $T > 0$, $R > 0$, $x, y \in [0, R]$ and $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ bounded and Borel, we have:*

$$|P_T^\delta F(x) - P_T^\delta F(y)| \leq \frac{2R \|F\|_\infty}{T} |y - x|. \quad (14)$$

Proof. By Theorem 2, for all $x, y \in [0, R]$ such that $x \leq y$, we have:

$$\begin{aligned} |P_t^\delta F(x) - P_t^\delta F(y)| &= \left| \int_x^y \frac{u}{T} \left(P_T^{\delta+2} F(u) - P_T^\delta F(u) \right) du \right| \\ &\leq \frac{2 \|F\|_\infty}{T} \int_x^y u \, du \\ &\leq \frac{2R \|F\|_\infty}{T} |y - x|. \end{aligned}$$

□

Remark 6. The bound (14) is in $1/T$, which is not very satisfactory for T small. However, in the sequel, we will improve this bound by getting a better exponent on T , at least for $\delta \geq 2(\sqrt{2} - 1)$ (see inequality (28) below).

5 Differentiability of the flow

In the following, we are interested in finding a probabilistic interpretation of Thm 2, in terms of the Bismut-Elworthy-Li formula. To do so we study, for all $\delta \geq 0$, and all couple $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, the differentiability at x of the function:

$$\begin{aligned} \rho_t : \mathbb{R}_+ &\rightarrow \mathbb{R}_+ \\ y &\mapsto \rho_t^\delta(y). \end{aligned}$$

In this endeavour, we first need to choose an appropriate modification of the process $(\rho_t(x))_{t \geq 0, x > 0}$. We have the following result:

Proposition 2. *Let $\delta \geq 0$ be fixed. There exists a modification $(\tilde{\rho}_t^\delta(x))_{x, t \geq 0}$ of the process $(\rho_t^\delta(x))_{x, t \geq 0}$ such that, a.s., for all $x, x' \in \mathbb{R}_+$ with $x \leq x'$, we have:*

$$\forall t \geq 0, \quad \tilde{\rho}_t^\delta(x) \leq \tilde{\rho}_t^\delta(x'). \quad (15)$$

Proof. For all $q, q' \in \mathbb{Q}_+$, such that $q \leq q'$, by Lemma 1, the following property holds a.s.:

$$\forall t \geq 0, \quad \rho_t^\delta(q) \leq \rho_t^\delta(q').$$

For all $x \in \mathbb{R}_+$, we define the process $\tilde{\rho}^\delta(x)$ by:

$$\forall t \geq 0, \quad \tilde{\rho}_t^\delta(x) := \inf_{q \in \mathbb{Q}_+, q \geq x} \rho_t^\delta(q).$$

Then $(\tilde{\rho}_t^\delta(x))_{x,t \geq 0}$ yields a modification of the process $(\rho_t^\delta(x))_{x,t \geq 0}$ with the requested property. \square

Remark 7. We may not have, almost-surely, joint continuity of all the functions $t \mapsto \tilde{\rho}_t(x)$, $x \geq 0$. Note however that, by definition, for all $x \geq 0, x \in \mathbb{Q}$, we have a.s.:

$$\forall t \geq 0, \quad \tilde{\rho}_t(x) = \rho_t(x),$$

so that, a.s., $t \mapsto \tilde{\rho}_t(x)$ is continuous and satisfies:

$$\forall t \in [0, T_0(x)), \quad \tilde{\rho}_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\tilde{\rho}_s} + B_t.$$

As a consequence, by countability of \mathbb{Q} , there exists an almost sure event $\mathcal{A} \in \mathcal{F}$ on which, for all $x \in \mathbb{Q}_+$, the function $t \mapsto \tilde{\rho}_t(x)$ is continuous and satisfies:

$$\forall t \in [0, T_0(x)), \quad \tilde{\rho}_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\tilde{\rho}_s} + B_t.$$

In this section, as well as the Appendix, we always work with the modification $\tilde{\rho}$. Similarly, we work with :

$$\tilde{T}_0(x) := \inf\{t > 0, \tilde{\rho}_t^\delta(x) = 0\}$$

instead of $T_0(x)$, for all $\delta, x \geq 0$. We will write again ρ and T_0 instead of $\tilde{\rho}$ and \tilde{T}_0 . Note that, a.s., the function $x \mapsto T_0(x)$ is non-decreasing on \mathbb{R}_+ .

Proposition 3. *Let $\delta \geq 0, t > 0$ and $x > 0$. Then, a.s., the function ρ_t is differentiable at x , and its derivative there is given by:*

$$\left. \frac{d\rho_t(y)}{dy} \right|_{y=x} \stackrel{\text{a.s.}}{=} \eta_t(x) := \mathbf{1}_{t < T_0(x)} \exp\left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right). \quad (16)$$

The proof of this proposition is quite technical. Since, moreover, the result will not be necessary in the sequel, we prefer to postpone the proof to the Appendix of the article.

Remark 8. In particular, when $\delta = 1$, the above formula reduces to:

$$\left. \frac{d\rho_t(y)}{dy} \right|_{y=x} \stackrel{\text{a.s.}}{=} \mathbf{1}_{t < T_0(x)} \quad (17)$$

a formula which was already well-known (see e.g. [1], Lemma A.1).

Remark 9. Note that the indicator function $\mathbf{1}_{t < T_0(x)}$ in the right-hand side of (16) is related to the behavior of the Bessel process at the boundary 0. It is reminiscent of Theorem 1 in [6], where a similar indicator function appears in the expression of the spatial derivative of the flow of vector-valued solutions to SDEs with reflection.

Remark 10. Proposition 3 shows that, for all $t, x > 0$, the function ρ_t is almost-surely differentiable at x . We may, however, ask if, a.s., the function ρ_t is differentiable on the whole of \mathbb{R}_+^* . The case where $\delta > 1$ was treated in detail in [11], where it was shown that, a.s., for all $t \geq 0$ the function $x \mapsto \rho_t(x)$ is differentiable on \mathbb{R}_+^* , and that the derivative $\frac{d\rho_t(x)}{dx}$ is continuous in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. However, as δ gets smaller than 1, the regularity of the process $(\rho_t(x))_{t \geq 0, x > 0}$ becomes much worse. Note that $\delta = 1$ corresponds to the case of the flow of reflected Brownian motion on the half-line; in that case the flow is no longer continuously differentiable as suggested by (17). Many works have been carried out on the study of the flow of reflected Brownian motion on domains in higher dimension (see e.g. [3] and [12]) or on manifolds with boundary (see e.g. [1]). By contrast, the regularity of Bessel flows of dimension $\delta < 1$ seems to be a very open problem.

In the remainder of the article, however, we shall not need any regularity results on the Bessel flow. Instead, for all fixed $x > 0$, we shall study the process $(\eta_t(x))_{t \geq 0}$ defined above as a process in itself.

6 Properties of η

In the sequel, for all $x \geq 0$, we shall consider the process $(\eta_t(x))_{t \geq 0}$ defined as above:

$$\eta_t(x) := \mathbf{1}_{t < T_0(x)} \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right). \quad (18)$$

When there is no ambiguity we shall drop the x from our notation and denote this process by η .

6.1 Regularity of the sample paths of η

We are interested in the continuity property of the process η . It turns out that, as δ decreases, η becomes more and more singular, as shown by the following result.

Proposition 4. *If $\delta > 1$, then a.s. η is bounded and continuous on \mathbb{R}_+ .*

If $\delta = 1$, then a.s. η is constant on $[0, T_0)$ and $[T_0, +\infty)$, but has a discontinuity at T_0 .

If $\delta \in [0, 1)$, then a.s. η is continuous away from T_0 , but it diverges to $+\infty$ as $t \uparrow T_0$.

Proof. When $\delta \geq 2$, $T_0 = \infty$ almost-surely, so that, by (18), the following equality of processes holds:

$$\eta_t = \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right).$$

Hence, a.s., η takes values in $[0, 1]$ and is continuous on \mathbb{R}_+ . To treat the case $\delta < 2$ we need a lemma:

Lemma 2. *Let $\delta < 2$ and $x > 0$. Then the integral:*

$$\int_0^{T_0} \frac{ds}{(\rho_s(x))^2}$$

is infinite a.s.

We admit this result for the moment. Then, when $\delta \in (1, 2)$, η takes values in $[0, 1]$, is continuous away from T_0 and, almost-surely, as $t \uparrow T_0$:

$$\eta_t = \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \rightarrow 0.$$

Since, $\eta_t = 0$ for all $t \geq T_0$, η is continuous and the claim follows. When $\delta = 1$,

$$\eta_t(x) := \mathbf{1}_{t < T_0(x)}$$

so the claim follows at once. Finally, if $\delta \in [0, 1)$, then η is continuous away from T_0 , but by the above lemma, a.s., as $t \uparrow T_0$:

$$\eta_t = \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \rightarrow +\infty$$

so the claim follows. □

We now prove Lemma 2

Proof of Lemma 2. The proof is in two steps. In a first step we prove the lemma when ρ is replaced with a Brownian motion started at some positive point, and in a second step we invoke a representation theorem of Bessel processes as time-changes of some power of the Brownian motion to conclude.

First step: Let (β_t) be a Brownian motion started from some $y > 0$, and let T_0 denote its hitting time of the origin. Then the integral :

$$\int_0^{T_0} \frac{ds}{(\beta_s(y))^2}$$

is a.s. infinite. Indeed, denote by $h : [0, \infty) \rightarrow \mathbb{R}_+$ the function given by:

$$h(t) := \begin{cases} \sqrt{t|\log(1/t)|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Let $A > 0$. By Levy's modulus of continuity (see Theorem (2.7), Chapter I, in [9]), there exists a $\kappa > 0$, such that the event

$$\mathcal{M} := \{ \forall s, t \in [0, 1], \quad |\beta_t - \beta_s| \leq \kappa h(|s - t|) \}$$

has probability one. Therefore, by scale invariance of Brownian motion, setting $\kappa_A := \sqrt{A}\kappa$, one deduces that the event

$$\mathcal{M}_A := \{ \forall s, t \in [0, A], \quad |\beta_t - \beta_s| \leq \kappa_A h(|s - t|) \}$$

also has probability one. Moreover, under the event $\{T_0 < A\} \cap \mathcal{M}_A$, we have, for small $h > 0$.

$$\beta_{T_0-h}^2 = |\beta_{T_0-h} - \beta_{T_0}|^2 \leq \kappa_A^2 h \log(1/h).$$

Since $\frac{1}{h \log(1/h)}$ is not integrable as $h \rightarrow 0^+$, we deduce that, under the event $\{T_0 < A\} \cap \mathcal{M}_A$, we have $\int_0^{T_0} \frac{ds}{(\beta_s)^2} = +\infty$. Therefore:

$$\mathbb{P}[T_0 < A] = \mathbb{P}[\{T_0 < A\} \cap \mathcal{M}_A] \leq \mathbb{P}\left(\int_0^{T_0} \frac{ds}{(\beta_s)^2} = +\infty\right).$$

Since $T_0 < +\infty$ a.s., we have $\lim_{A \rightarrow \infty} \mathbb{P}[T_0 < A] = 1$. Hence, letting $A \rightarrow \infty$ in the above, we deduce that:

$$\mathbb{P}\left(\int_0^{T_0} \frac{ds}{(\beta_s)^2} = +\infty\right) = 1$$

as claimed.

Second step: Now consider the original Bessel process $(\rho_t(x))_{t \geq 0}$. Suppose that $\delta \in (0, 2)$. Then, by Thm 3.5 in [13], the process $(\rho_t(x))_{t \geq 0}$ is equal in law to $(|\beta_{\gamma(t)}|^{\frac{1}{2-\delta}})_{t \geq 0}$, where β is a Brownian motion started from $y := x^{2-\delta}$, and $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the inverse of the increasing function $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by:

$$\forall u \geq 0, \quad A(u) = \frac{1}{(2-\delta)^2} \int_0^u |\beta_s|^{\frac{2(\delta-1)}{2-\delta}} ds.$$

Therefore, denoting by T_0^β the hitting time of 0 by the Brownian motion β , we have:

$$\begin{aligned} \int_0^{T_0} \frac{ds}{(\rho_s(x))^2} &\stackrel{(d)}{=} \int_0^{A(T_0^\beta)} \frac{ds}{|\beta_{\gamma(s)}|^{\frac{2}{2-\delta}}} \\ &= \int_0^{T_0^\beta} \frac{1}{|\beta_u|^{\frac{2}{2-\delta}}} \frac{1}{(2-\delta)^2} |\beta_u|^{\frac{2(\delta-1)}{2-\delta}} du \\ &= \frac{1}{(2-\delta)^2} \int_0^{T_0^\beta} \frac{du}{\beta_u^2} \end{aligned}$$

where we have used the change of variable $u = \gamma(s)$ to get from the first line to the second one. By the first step, the last integral is infinite a.s., so the claim follows.

There still remains to treat the case $\delta = 0$. By Thm 3.5 in [13], in that case, the process $(\rho_t(x))_{t \geq 0}$ is equal in law to $\left(\left(\beta_{\gamma(t) \wedge T_0^\beta}\right)^{1/2}\right)_{t \geq 0}$, where β is a Brownian motion

started from $y := x^2$, T_0^β is its hitting time of 0 and $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the inverse of the increasing function $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by:

$$\forall u \geq 0, \quad A(u) = \frac{1}{4} \int_0^{u \wedge T_0^\beta} \beta_s^{-1} ds.$$

Then, the same computations as above yield the equality in law:

$$\int_0^{T_0} \frac{ds}{(\rho_s(x))^2} \stackrel{(d)}{=} \frac{1}{4} \int_0^{T_0^\beta} \frac{du}{\beta_u^2}$$

so the result follows as well. \square

6.2 Study of a martingale related to η

Let $\delta \in [0, 2)$ and $x > 0$ be fixed. In the previous section, we have shown that, a.s. :

$$\int_0^t \frac{ds}{\rho_s(x)^2} \xrightarrow{t \rightarrow T_0(x)} +\infty$$

As a consequence, for $\delta \in [0, 1)$, a.s., the modification η_t of the derivative at x of the stochastic flow ρ_t diverges at $T_0(x)$:

$$\eta_t(x) = \mathbf{1}_{t < T_0(x)} \exp\left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \xrightarrow{t \uparrow T_0(x)} +\infty.$$

However, since $\rho_t(x) \rightarrow 0$ as $t \rightarrow T_0(x)$, this does not exclude the possibility that the product $\rho_t(x)\eta_t(x)$ converges as $t \rightarrow T_0(x)$. This motivates to study the process :

$$D_t := \rho_t(x)\eta_t(x) = \mathbf{1}_{t < T_0(x)} \rho_t(x) \exp\left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right). \quad (19)$$

As a matter of fact, we will show that $(D_t)_{t \geq 0}$ is an L^p continuous martingale for some $p \geq 1$.

Remark 11. The process $(D_t)_{t \geq 0}$ appears as (one half times) the derivative of the stochastic flow associated with the squared Bessel process $X_t(x) = (\rho_t(x))^2$. Indeed, by applying formally the chain rule, we have, for all $t \geq 0$ and $x > 0$:

$$\frac{dX_t(x)}{dx} = 2\rho_t(x)\eta_t(x).$$

6.3 Continuity of $(D_t)_{t \geq 0}$

In this subsection we show that the process $(D_t)_{t \geq 0}$ has a.s. continuous sample paths. By the expression (19), continuity holds as soon as $T_0(x) = \infty$ a.s., i.e. as soon as $\delta \geq 2$. On the other hand, if $\delta \in [0, 2)$ it suffices to prove that, a.s., $D_t \rightarrow 0$ as $t \uparrow T_0(x)$. This is the content of the following proposition.

Proposition 5. For all $\delta \in [0, 2)$ and $x > 0$, with probability one:

$$\rho_t(x) \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \xrightarrow{t \rightarrow T_0(x)} 0.$$

Proof. If $\delta \in [1, 2)$, then $\exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \leq 1$ for all $t \geq 0$. Since $\rho_t \rightarrow 0$ as $t \rightarrow T_0(x)$, the claim follows at once.

If $\delta \in [0, 1)$, on the other hand, $\exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \xrightarrow{t \uparrow T_0(x)} +\infty$ whereas $\rho_t \xrightarrow{t \rightarrow T_0(x)} 0$ so a finer analysis is needed. We have:

$$\log \left[\frac{\rho_t}{x} \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right) \right] = \log \frac{\rho_t}{x} + \frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s^2}$$

Now, recall that a.s., for all $t < T_0$, we have:

$$\rho_t = x + \frac{\delta-1}{2} \int_0^t \frac{ds}{\rho_s} + B_t$$

Hence, defining for all integer $n \geq 1$ the $(\mathcal{F}_t)_{t \geq 0}$ -stopping time τ_n as:

$$\tau_n := \inf\{t > 0, \rho_t \leq 1/n\} \wedge n,$$

we have:

$$\rho_{t \wedge \tau_n} = x + \frac{\delta-1}{2} \int_0^{t \wedge \tau_n} \frac{ds}{\rho_s} + B_{t \wedge \tau_n}.$$

Hence, by Itô's lemma, we deduce that:

$$\log \frac{\rho_{t \wedge \tau_n}}{x} = \frac{\delta-1}{2} \int_0^{t \wedge \tau_n} \frac{ds}{\rho_s^2} + \int_0^{t \wedge \tau_n} \frac{dB_s}{\rho_s} - \frac{1}{2} \int_0^{t \wedge \tau_n} \frac{ds}{\rho_s^2}$$

so that:

$$\log \frac{\rho_{t \wedge \tau_n}}{x} + \frac{1-\delta}{2} \int_0^{t \wedge \tau_n} \frac{ds}{\rho_s^2} = \int_0^{t \wedge \tau_n} \frac{dB_s}{\rho_s} - \frac{1}{2} \int_0^{t \wedge \tau_n} \frac{ds}{\rho_s^2}. \quad (20)$$

Consider now the random time change:

$$\begin{aligned} A: [0, T_0) &\rightarrow \mathbb{R}_+ \\ t &\mapsto A_t := \int_0^t \frac{ds}{\rho_s^2}. \end{aligned}$$

Note that A is differentiable with strictly positive derivative. Moreover, since $A_t \xrightarrow{t \rightarrow T_0} +\infty$ a.s. by Lemma 2, we deduce that A is a.s. onto. Hence, a.s., A is a diffeomorphism $[0, T_0) \rightarrow \mathbb{R}_+$, the inverse of which we denote by

$$\begin{aligned} C: \mathbb{R}_+ &\rightarrow [0, T_0) \\ u &\mapsto C_u. \end{aligned}$$

Let $\beta_u := \int_0^{C_u} \frac{dB_r}{\rho_r}$, $u \geq 0$. Then β is a local martingale started at 0 with quadratic variation $\langle \beta, \beta \rangle_u = u$, so by Lévy's theorem it is a Brownian motion. The equality (20) can now be rewritten:

$$\log \frac{\rho_{t \wedge \tau_n}}{x} + \frac{1 - \delta}{2} \int_0^{t \wedge \tau_n} \frac{ds}{\rho_s^2} = \beta_{A_{t \wedge \tau_n}} - \frac{1}{2} A_{t \wedge \tau_n}.$$

Letting $n \rightarrow \infty$, we obtain, for all $t < T_0$:

$$\log \frac{\rho_t}{x} + \frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s^2} = \beta_{A_t} - \frac{1}{2} A_t.$$

By the asymptotic properties of Brownian motion (see Corollary (1.12), Chapter II in [9]), we know that, a.s.:

$$\limsup_{s \rightarrow +\infty} \frac{\beta_s}{h(s)} = 1$$

where $h(s) := \sqrt{2s \log \log s}$. In particular, a.s., there exists $T > 0$ such that, for all $t \geq T$, we have $\beta_t \leq 2h(t)$. Since, a.s., $A_t \xrightarrow[t \rightarrow T_0]{} +\infty$, we deduce that:

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \left(\beta_{A_t} - \frac{1}{2} A_t \right) &\leq \limsup_{t \rightarrow +\infty} \left(2h(A_t) - \frac{1}{2} A_t \right) \\ &= -\infty. \end{aligned}$$

Hence, a.s. :

$$\log \left[\frac{\rho_t}{x} \exp \left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2} \right) \right] \xrightarrow[t \uparrow T_0(x)]{} -\infty$$

i.e.

$$\rho_t \exp \left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2} \right) \xrightarrow[t \uparrow T_0(x)]{} 0$$

as claimed. □

6.4 Martingale property of $(D_t)_{t \geq 0}$

Let $\delta \geq 0$ and $x > 0$ be fixed. We show in this section that $(D_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ martingale which, up to a positive constant, corresponds to a Girsanov-type change of probability measure.

Recall that, by definition:

$$D_t = \mathbf{1}_{t < T_0(x)} \rho_t(x) \exp \left(-\frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s^2} \right). \quad (21)$$

Notation 1. For all $a \geq 0$ and $t \geq 0$, we denote by $P_x^a|_{\mathcal{F}_t}$ the image of the probability measure P_x^a under the restriction map:

$$\begin{aligned} (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+))) &\rightarrow (C([0, t]), \mathcal{F}_t) \\ w &\mapsto w|_{[0, t]} \end{aligned}$$

Proposition 6. *Let $\delta \geq 0$ and $x > 0$. Then, for all $t \geq 0$, the law $P_x^{\delta+2}|_{\mathcal{F}_t}$ is absolutely continuous w.r.t. the law $P_x^\delta|_{\mathcal{F}_t}$, and the corresponding Radon-Nikodym derivative is given by:*

$$\frac{dP_x^{\delta+2}}{dP_x^\delta} \Big|_{\mathcal{F}_t} (\rho) \stackrel{\text{a.s.}}{=} \mathbf{1}_{t < T_0(x)} \frac{\rho_t(x)}{x} \exp\left(-\frac{\delta-1}{2} \int_0^t \frac{ds}{\rho_s^2}\right).$$

Proof. Fix $\epsilon > 0$. Under $P_x^\delta|_{\mathcal{F}_t}$, the canonical process ρ stopped at T_ϵ satisfies the following SDE on $[0, t]$:

$$\rho_{s \wedge T_\epsilon} = x + \frac{\delta-1}{2} \int_0^{s \wedge T_\epsilon} \frac{ds}{\rho_s} + B_{s \wedge T_\epsilon}.$$

Consider the process M^ϵ defined on $[0, t]$ by:

$$M_s^\epsilon := \int_0^{s \wedge T_\epsilon} \frac{dB_u}{\rho_u}$$

M^ϵ is an L^2 martingale on $[0, t]$. The exponential local martingale thereto associated is:

$$\mathcal{E}(M^\epsilon)_s = \exp\left(\int_0^{s \wedge T_\epsilon} \frac{dB_u}{\rho_u} - \frac{1}{2} \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u^2}\right).$$

Since, by Itô's lemma:

$$\log\left(\frac{\rho_{s \wedge T_\epsilon}}{x}\right) = \int_0^{s \wedge T_\epsilon} \frac{dB_u}{\rho_u} + \left(\frac{\delta}{2} - 1\right) \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u^2},$$

we have:

$$\begin{aligned} \mathcal{E}(M^\epsilon)_s &= \exp\left[\log\left(\frac{\rho_{s \wedge T_\epsilon}}{x}\right) - \frac{\delta-1}{2} \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u^2}\right] \\ &= \frac{\rho_{s \wedge T_\epsilon}}{x} \exp\left[-\frac{\delta-1}{2} \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u^2}\right]. \end{aligned}$$

Note that

$$\begin{aligned} \mathbb{E}\left[\exp\left(\frac{1}{2}\langle M^\epsilon, M^\epsilon \rangle_t\right)\right] &\leq \exp\left(\frac{t}{2\epsilon}\right) \\ &< \infty \end{aligned}$$

so that, by Novikov's criterion, $\mathcal{E}(M^\epsilon)$ is a uniformly integrable martingale on $[0, t]$. So we may consider the probability measure $\mathcal{E}(M^\epsilon)P_x^\delta|_{\mathcal{F}_t}$.

Note also that:

$$\langle M^\epsilon, B \rangle_t = \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u}.$$

Hence, by Girsanov's theorem, under the probability measure $\mathcal{E}(M^\epsilon)P_x^\delta|_{\mathcal{F}_t}$, the process :

$$\rho_{s \wedge T_\epsilon} - x - \frac{\delta+1}{2} \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u}$$

is a local martingale, with quadratic variation given by $s \wedge T_\epsilon$. Therefore, by Theorem (1.7) in Chapter V of [9], there exists, on some enlarged probability space, a Brownian motion β such that, a.s.:

$$\forall s \in [0, t], \rho_{s \wedge T_\epsilon} = x + \frac{\delta + 1}{2} \int_0^{s \wedge T_\epsilon} \frac{du}{\rho_u} + \beta_{s \wedge T_\epsilon}.$$

Denote by $\bar{\rho}$ the unique strong solution on $[0, t]$ of the SDE:

$$\bar{\rho}_s = x + \frac{\delta + 1}{2} \int_0^s \frac{du}{\bar{\rho}_u} + \beta_s.$$

Then, by strong uniqueness of the solution to this SDE, we deduce that, under $\mathcal{E}(M^\epsilon)P_x^\delta|_{\mathcal{F}_t}$, a.s.:

$$\forall s \in [0, t], s < T_\epsilon \implies \rho_s = \bar{\rho}_s.$$

Since $\bar{\rho}$ has the law of a $\delta + 2$ -dimensional Bessel process started at x , we deduce that, for all $F : C([0, T], \mathbb{R}_+) \rightarrow \mathbb{R}_+$ Borel, we have:

$$E_x^\delta [\mathcal{E}(M^\epsilon)F(\rho)\mathbf{1}_{t < T_\epsilon}] = E_x^{\delta+2}[F(\rho)\mathbf{1}_{t < T_\epsilon}]$$

i.e.:

$$E_x^\delta \left[\frac{\rho_t}{x} \exp \left(-\frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s^2} \right) F(\rho)\mathbf{1}_{t < T_\epsilon} \right] = E_x^{\delta+2}[F(\rho)\mathbf{1}_{t < T_\epsilon}].$$

Letting $\epsilon \rightarrow 0$, by the monotone convergence theorem, we obtain:

$$E_x^\delta \left[\frac{\rho_t}{x} \exp \left(-\frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s^2} \right) F(\rho)\mathbf{1}_{t < T_0} \right] = E_x^{\delta+2}[F(\rho)\mathbf{1}_{t < T_0}].$$

But, since $P_x^{\delta+2}[T_0 < +\infty] = 0$, this yields:

$$E_x^\delta \left[\frac{\rho_t}{x} \exp \left(-\frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s^2} \right) F(\rho)\mathbf{1}_{t < T_0} \right] = E_x^{\delta+2}[F(\rho)]$$

as stated. □

Remark 12. Proposition 6 is actually a particular case of a more general result. Indeed, for all $x > 0$, $t \geq 0$, and $\delta' \geq \delta \geq 0$, such that $\delta' \geq 2$, $P_x^{\delta'}|_{\mathcal{F}_t}$ is absolutely continuous w.r.t. the law $P_x^\delta|_{\mathcal{F}_t}$, and the corresponding Radon-Nikodym derivative is given by:

$$\frac{dP_x^{\delta'}}{dP_x^\delta} \Big|_{\mathcal{F}_t} (\rho) \stackrel{\text{a.s.}}{=} \mathbf{1}_{t < T_0(x)} \left(\frac{\rho_t(x)}{x} \right)^{\frac{\delta' - \delta}{2}} \exp \left[-\frac{\delta' - \delta}{2} \left(\frac{\delta' + \delta}{4} - 1 \right) \int_0^t \frac{ds}{\rho_s^2} \right]. \quad (22)$$

The proof of this fact is in all respect similar to that of Proposition 6 above.

Corollary 3. $(D_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ continuous martingale

Proof. The process $(D_t)_{t \geq 0}$ is continuous. Moreover, for all $t \geq 0$, $\frac{1}{x}D_t$ is the Radon-Nikodym derivative of $P_x^{\delta+2}|_{\mathcal{F}_t}$ w.r.t. $P_x^\delta|_{\mathcal{F}_t}$. Therefore $(\frac{1}{x}D_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ martingale, so $(D_t)_{t \geq 0}$ is a martingale as well, and the claim follows. □

6.5 Moment estimates for the martingale $(D_t)_{t \geq 0}$

In this section, we prove that the martingale $(D_t)_{t \geq 0}$ is actually in L^p for some $p \geq 1$. We first recall the following fact:

Lemma 3. *For all $a \geq 0$, $t \geq 0$, and $m \geq 0$, we have:*

$$E_x^a(\rho_t^m) < \infty.$$

Proof. Denote by d any integer such that $d \geq a$. By Lemma 1, we have:

$$E_x^a(\rho_t^m) \leq E_x^d(\rho_t^m)$$

Since P_x^d is the law of $(\|B_s\|)_{s \geq 0}$, where $(B_s)_{s \geq 0}$ is a d -dimensional Brownian motion and $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^d (see [9], Chapter 11), this inequality can be rewritten as:

$$E_x^a(\rho_t^m) \leq \mathbb{E}(\|B_t\|^m)$$

Since B_t is a Gaussian random variable, $\mathbb{E}(\|B_t\|^m)$ is finite, and the result follows. \square

Proposition 7. *$(D_t)_{t \geq 0}$ is an L^p martingale for all finite positive number p such that $p \leq p(\delta)$, where $p(\delta) \in [1, +\infty]$ is given by:*

$$p(\delta) := \begin{cases} \frac{(2-\delta)^2}{4(1-\delta)} & \text{if } \delta < 1, \\ +\infty & \text{if } \delta \geq 1. \end{cases} \quad (23)$$

Moreover the above statement is sharp: for $\delta < 1$ and $t > 0$, the random variable D_t is not in L^p for $p > p(\delta)$.

Remark 13. We emphasize that p is finite in the above result. Indeed D_t is never in L^∞ even if $\delta \geq 1$; for example, when $\delta = 1$, $D_t = \rho_t \mathbf{1}_{t < T_0(x)}$ which is clearly not bounded a.s.

Proof of Prop 7. If $\delta \geq 1$, then, for all $t \geq 0$, $D_t \leq \rho_t$. Hence, for all $p \in (0, +\infty)$:

$$\mathbb{E}(D_t^p) \leq E_x^\delta(\rho_t^p)$$

which is finite by Lemma 3.

On the other hand, if $\delta \in [0, 1)$, then, for all $t > 0$ and $p > 0$, we have:

$$\mathbb{E}(D_t^p) = E_x^\delta \left[\mathbf{1}_{t < T_0} \rho_t^p \exp \left(-p \frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s^2} \right) \right].$$

By the absolute continuity relation (22) applied with $\delta' := 2$, the latter equals:

$$E_x^2 \left[x^{\frac{2-\delta}{2}} \rho_t^{p+\frac{\delta-2}{2}} \exp \left(\underbrace{\left(-p \frac{\delta - 1}{2} - \frac{(\delta - 2)^2}{8} \right)}_{:= A(p)} \int_0^t \frac{ds}{\rho_s^2} \right) \right].$$

For $p = p(\delta)$, $A(p) = 0$, so that:

$$\begin{aligned}\mathbb{E} \left[D_t^{p(\delta)} \right] &= E_x^2 \left[x^{\frac{2-\delta}{2}} \rho_t^{p(\delta) + \frac{\delta-2}{2}} \right] \\ &= x^{1-\frac{\delta}{2}} E_x^2 \left[\rho_t^{p(\delta) + \frac{\delta}{2} - 1} \right].\end{aligned}$$

Since $\frac{\delta}{2} + p(\delta) - 1 \geq 0$, by Lemma 3, the last quantity is finite. Hence D_t is indeed in $L^{p(\delta)}$.

Suppose now that $p = p(\delta) + r$ for some $r > 0$. We show that $D_t \notin L^p$. We have:

$$\begin{aligned}\mathbb{E} [D_t^p] &= E_x^2 \left[x^{\frac{2-\delta}{2}} \rho_t^{p + \frac{\delta-2}{2}} \exp \left(\left(-p \frac{\delta-1}{2} - \frac{(\delta-2)^2}{8} \right) \int_0^t \frac{ds}{\rho_s^2} \right) \right] \\ &= x^{1-\frac{\delta}{2}} E_x^2 \left[\rho_t^{p + \frac{\delta}{2} - 1} \exp \left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{\rho_s^2} \right) \right]\end{aligned}$$

We claim that the last quantity is infinite. Indeed, first note that by Jensen's inequality and Fubini, for any $C > 0$ we have:

$$E_x^2 \left[\exp \left(C \int_0^t \frac{ds}{\rho_s^2} \right) \right] \geq \exp \left(C \int_0^t E_x^2 (\rho_s^{-2}) ds \right)$$

and the right-hand side is infinite since, for all $s > 0$, $E_x^2 (\rho_s^{-2}) = +\infty$ (indeed, by formula (12), the transition density $p_s^2(x, y)$ does not integrate y^{-2} as $y \rightarrow 0$). Therefore:

$$E_x^2 \left[\exp \left(C \int_0^t \frac{ds}{\rho_s^2} \right) \right] = +\infty \quad (24)$$

Consider now any $c > 0$ and $a, b > 0$ such that $\frac{1}{a} + \frac{1}{b} = 1$. By (24) and Hölder's inequality, we have:

$$\begin{aligned}+\infty &= E_x^2 \left[\exp \left(\frac{1-\delta}{2a} r \int_0^t \frac{ds}{\rho_s^2} \right) \right] \\ &\leq E_x^2 \left[\rho_t^{ac} \exp \left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{\rho_s^2} \right) \right]^{1/a} E_x^2 \left[\rho_t^{-bc} \right]^{1/b}\end{aligned}$$

Set $c = \frac{\frac{\delta}{2} + p - 1}{\frac{\delta}{2} + p}$, $a = \frac{\delta}{2} + p$, and $b = \frac{\frac{\delta}{2} + p}{\frac{\delta}{2} + p - 1}$. Remark that $\frac{\delta}{2} + p - 1 > 0$ since $p > p(\delta) \geq 1$, so that this choice for c , a , and b makes sense. We obtain:

$$E_x^2 \left[\rho_t^{\frac{\delta}{2} + p - 1} \exp \left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{\rho_s^2} \right) \right]^{\frac{1}{\frac{\delta}{2} + p}} E_x^2 \left[\rho_t^{-1} \right]^{\frac{\frac{\delta}{2} + p - 1}{\frac{\delta}{2} + p}} = +\infty$$

By the comparison lemma 1 and the expression (12) for the transition density of the Bessel process, we have

$$E_x^2 \left[\rho_t^{-1} \right] \leq E_0^2 \left[\rho_t^{-1} \right] = \int_0^\infty \frac{1}{t} \exp \left(-\frac{y^2}{2t} \right) dy$$

so that $E_x^2[\rho_t^{-1}] < +\infty$. Therefore, we deduce that :

$$E_x^2 \left[\rho_t^{\frac{\delta}{2}+p-1} \exp \left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{\rho_s^2} \right) \right] = +\infty$$

as claimed. Hence $D_t \notin L^p$ for $p > p(\delta)$. \square

7 A Bismut-Elworthy-Li formula for the Bessel processes

We are now in position to provide a probabilistic interpretation of the right-hand-side of equation (11) in Theorem 2.

Let $\delta > 0$, and $x > 0$. As we saw in the previous section, the process $(\eta_t(x))_{t \geq 0}$ may blow up at time T_0 , so that the stochastic integral $\int_0^t \eta_s(x) dB_s$ is a priori ill-defined, at least for $\delta \in (0, 1)$. However, it turns out that we can define the latter process rigorously as a local martingale.

Proposition 8. *Suppose that $\delta > 0$. Then the stochastic integral process $\int_0^t \eta_s dB_s$ is well-defined as a local martingale and is indistinguishable from the continuous martingale $D_t - x$.*

Proof. We first treat the case $\delta \geq 2$, which is much easier to handle. In that case, $\eta_t \in [0, 1]$ for all $t \geq 0$, so that the stochastic integral $\int_0^t \eta_s dB_s$ is clearly well-defined as an L^2 martingale. Moreover, since $T_0 = +\infty$ a.s., by Itô's lemma we have:

$$\begin{aligned} D_t &= \rho_t \eta_t = x + \int_0^t \eta_s d\rho_s + \int_0^t \rho_s d\eta_s \\ &= x + \int_0^t \eta_s \left(\frac{\delta-1}{2} \frac{ds}{\rho_s} + dB_s \right) - \int_0^t \rho_s \frac{\delta-1}{2} \frac{\eta_s}{\rho_s^2} ds \\ &= x + \int_0^t \eta_s dB_s \end{aligned}$$

so the claim follows.

Now suppose that $\delta \in (0, 2)$ and fix an $\epsilon > 0$. Recall that $T_\epsilon(x) := \inf\{t \geq 0, \rho_t(x) \leq \epsilon\}$ and note that, since $T_\epsilon < T_0$, the stopped process η^{T_ϵ} is continuous on \mathbb{R}_+ , so that the stochastic integral $\int_0^{t \wedge T_\epsilon(x)} \eta_s(x) dB_s$ is well-defined as a local martingale. Using as above Itô's lemma, but this time with the stopped processes ρ^{T_ϵ} and η^{T_ϵ} , we have:

$$\int_0^{t \wedge T_\epsilon} \eta_s dB_s = D_{t \wedge T_\epsilon} - x. \quad (25)$$

Our aim would be to pass to the limit $\epsilon \rightarrow 0$ in this equality. By continuity of D , as $\epsilon \rightarrow 0$, $D_{t \wedge T_\epsilon}$ converges to $D_{t \wedge T_0} = D_t$ almost-surely. So the right-hand side of (25) converges to $D_t - x$ almost-surely.

The convergence of the left-hand side to a stochastic integral is more involved, since we first have to prove that the stochastic integral $\int_0^t \eta_s dB_s$ is indeed well-defined as a local martingale. For this, it suffices to prove that, almost-surely:

$$\forall t \geq 0, \quad \int_0^t \eta_s^2 ds < \infty.$$

We actually prove the following stronger fact. For all $t \geq 0$

$$\mathbb{E} \left[\left(\int_0^t \eta_s^2 ds \right)^{p/2} \right] < \infty \quad (26)$$

for all finite positive number p such that $p \in (1, p(\delta)]$. Indeed, applying successively the Burkholder-Davis-Gundy (BDG) inequality and Doob's inequality to the martingale $\int_0^{T_\epsilon \wedge \cdot} \eta_s dB_s$, we have:

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^{t \wedge T_\epsilon} \eta_s^2 ds \right)^{p/2} \right] &\leq C_p \mathbb{E} \left[\sup_{s \leq t \wedge T_\epsilon} \left| \int_0^s \eta_u dB_u \right|^p \right] \\ &= C_p \mathbb{E} \left[\sup_{s \leq t \wedge T_\epsilon} |D_s - x|^p \right] \\ &\leq C_p \left(\frac{p}{p-1} \right)^p \mathbb{E} [|D_{t \wedge T_\epsilon} - x|^p] \end{aligned}$$

where C_p is a constant depending only on p . Now, since $(D_t - x)_{t \geq 0}$ is a continuous martingale, by the optional stopping theorem and Jensen's inequality, we have:

$$\mathbb{E} [|D_{t \wedge T_\epsilon} - x|^p] \leq \mathbb{E} (|D_t - x|^p)$$

and the right-hand side is finite because D_t is in L^p . Hence, letting $\epsilon \rightarrow 0$ in the above, by the monotone convergence theorem we deduce that :

$$\mathbb{E} \left[\left(\int_0^{t \wedge T_0} \eta_s^2 ds \right)^{p/2} \right] < \infty$$

But since $\eta_t = 0$ for all $t \geq T_0$, this implies the bound (26), and hence the stochastic integral $\int_0^t \eta_s dB_s$ is well-defined as a local martingale. Moreover, for all $t \geq 0$, by the BDG inequality, we have:

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t \eta_s dB_s - \int_0^{t \wedge T_\epsilon} \eta_s dB_s \right)^p \right] &= \mathbb{E} \left[\left(\int_{t \wedge T_\epsilon}^{t \wedge T_0} \eta_s dB_s \right)^p \right] \\ &\leq c_p \mathbb{E} \left[\left(\int_{t \wedge T_\epsilon}^{t \wedge T_0} \eta_s^2 ds \right)^{p/2} \right] \end{aligned}$$

where c_p is some constant depending only on p . Now, by the dominated convergence theorem, the last quantity above goes to 0 as $\epsilon \rightarrow 0$, and hence:

$$\int_0^{t \wedge T_\epsilon} \eta_s dB_s \xrightarrow{\epsilon \rightarrow 0} \int_0^t \eta_s dB_s$$

in L^p . Hence, the left-hand side of equality (25) converges in L^p to the stochastic integral $\int_0^t \eta_s dB_s$. Letting $\epsilon \rightarrow 0$ in that equality, we thus obtain :

$$\int_0^t \eta_s dB_s = D_t - x$$

as claimed. \square

Using the above proposition, Theorem 2 can now be interpreted probabilistically as a Bismut-Elworthy-Li formula.

Theorem 3 (Bismut-Elworthy-Li formula). *Let $\delta > 0$. Then, for all $T > 0$, and all $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ bounded and Borel, the function $x \rightarrow P_t^\delta F(x)$ is differentiable on \mathbb{R}_+ , and for all $x > 0$:*

$$\frac{d}{dx} P_T^\delta F(x) = \frac{1}{T} \mathbb{E} \left[F(\rho_t(x)) \left(\int_0^T \eta_s(x) dB_s \right) \right]. \quad (27)$$

Proof. By Theorem 2, the differentiability property holds, and we have:

$$\frac{d}{dx} P_T^\delta F(x) = \frac{x}{T} \left[P_T^{\delta+2} F(x) - P_T^\delta F(x) \right].$$

Moreover, by Proposition 6, for all $x > 0$:

$$P_T^{\delta+2} F(x) - P_T^\delta F(x) = E_x^\delta \left[F(\rho_T) \left(\frac{D_T}{x} - 1 \right) \right]$$

and, by Proposition 8, we have:

$$E_x^\delta \left[F(\rho_T) \left(\frac{D_T}{x} - 1 \right) \right] = \frac{1}{x} \mathbb{E} \left[F(\rho_T(x)) \left(\int_0^T \eta_s(x) dB_s \right) \right]$$

so equality (27) follows. \square

Using the Bismut-Elworthy-Li formula, we are now able to sharpen the Strong Feller estimate obtained in equation (14) above.

Corollary 4. *Let $T > 0$ and $\delta \geq 2(\sqrt{2} - 1)$. Then, for all $R > 0$, there exists a constant $C > 0$ such that, for all $x, y \in [0, R]$ and $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ bounded and Borel, we have:*

$$|P_T^\delta F(x) - P_T^\delta F(y)| \leq \frac{C \|F\|_\infty}{T^{\alpha(\delta)}} |y - x| \quad (28)$$

where the exponent $\alpha(\delta) \in [\frac{1}{2}, 1)$ is given by:

$$\alpha(\delta) := \begin{cases} \frac{1}{2} + \frac{1-\delta}{2-\delta} & \text{if } \delta \in [2(\sqrt{2} - 1), 1], \\ 1/2 & \text{if } \delta \geq 1. \end{cases}$$

Proof. Let $x > 0$. By Theorem 3, we have :

$$\frac{d}{dx} P_T^\delta F(x) = \frac{1}{T} \mathbb{E} \left[F(\rho_t(x)) \left(\int_0^T \eta_s(x) dB_s \right) \right].$$

so that:

$$\left| \frac{d}{dx} P_T^\delta F(x) \right| \leq \frac{\|F\|_\infty}{T} \mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right].$$

We now bound the quantity $\mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right]$. If $\delta \geq 1$, then the process $(\eta_s(x))_{s \geq 0}$ takes values in $[0, 1]$, so that, using the Cauchy-Schwarz inequality and Itô's isometry formula, we have :

$$\mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right] \leq \sqrt{\mathbb{E} \left(\int_0^T \eta_s(x)^2 ds \right)} \leq \sqrt{T}.$$

Therefore:

$$\left| \frac{d}{dx} P_T^\delta F(x) \right| \leq \frac{\|F\|_\infty}{\sqrt{T}}$$

and the claim follows with $C = 1$.

Suppose now that $\delta \in [2(\sqrt{2}-1), 1)$. Letting $p := p(\delta)$ as in (23), we have, by Jensen's inequality:

$$\mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right] \leq \left(\mathbb{E} \left| \int_0^T \eta_s(x) dB_s \right|^p \right)^{1/p}$$

Now, applying successively the BDG inequality, Jensen's inequality and the absolute continuity relation (22) between P_x^2 and P_x^δ , we have, for some constant c_p depending only on p :

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right|^p \right] &\leq c_p \mathbb{E} \left[\left(\int_0^T \eta_s(x)^2 ds \right)^{p/2} \right] \\ &\leq c_p T^{p/2-1} \mathbb{E} \left(\int_0^T \eta_s(x)^p ds \right) \\ &\leq c_p T^{p/2-1} \int_0^T E_x^\delta(\eta_s^p) ds \\ &= c_p T^{p/2-1} \int_0^T E_x^2 \left[\left(\frac{\rho_s}{x} \right)^{\frac{\delta-2}{2}} \exp \left(\left(\frac{1-\delta}{2} p - \frac{(2-\delta)^2}{8} \right) \int_0^s \frac{du}{\rho_u^2} \right) \right] ds \\ &= c_p T^{p/2-1} \int_0^T E_x^2 \left[\left(\frac{\rho_s}{x} \right)^{\frac{\delta-2}{2}} \right] ds \end{aligned}$$

where the last equality follows from the fact that $\frac{1-\delta}{2} p - \frac{(2-\delta)^2}{8} = 0$ for $p = p(\delta)$. Now, since $\frac{\delta-2}{2} \leq 0$, by the comparison lemma 1, as well as the scaling property of the Bessel processes (see, e.g., Remark 3.7 in [13]), for all $s \in [0, T]$, we have:

$$E_x^2 \left[\left(\frac{\rho_s}{x} \right)^{\frac{\delta-2}{2}} \right] \leq E_0^2 \left[\left(\frac{\rho_s}{s} \right)^{\frac{\delta-2}{2}} \right] = s^{\frac{\delta-2}{4}} E_0^2 \left[\left(\frac{\rho_1}{1} \right)^{\frac{\delta-2}{2}} \right].$$

Let $c := E_0^2 \left[\rho_1^{\frac{\delta-2}{2}} \right]$. Using formula (12), we have:

$$c = \int_0^\infty y^{\delta/2} \exp\left(-\frac{y^2}{2}\right) dy < \infty.$$

Hence:

$$\begin{aligned} \int_0^T E_x^2 \left[\left(\frac{\rho_s}{x} \right)^{\frac{\delta}{2}-1} \right] ds &\leq c x^{1-\frac{\delta}{2}} \int_0^T s^{\frac{\delta-2}{4}} ds \\ &\leq \frac{4c}{\delta+2} x^{1-\frac{\delta}{2}} T^{\frac{\delta+2}{4}}. \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right|^p \right] &\leq K x^{1-\frac{\delta}{2}} T^{\frac{p}{2}-1} T^{\frac{\delta+2}{4}} \\ &\leq K x^{1-\frac{\delta}{2}} T^{\frac{p}{2}+\frac{\delta-2}{4}} \end{aligned}$$

where K is a constant depending only on δ . Hence

$$\mathbb{E} \left[\left| \int_0^T \eta_s(x) dB_s \right| \right] \leq K^{1/p} x^{\frac{1}{p}(1-\frac{\delta}{2})} T^{\frac{1}{2}+\frac{\delta-2}{4p}}.$$

Note that, since $p = p(\delta)$, we have $\frac{1}{p}(1-\frac{\delta}{2}) = \frac{2(1-\delta)}{2-\delta}$, and $\frac{\delta-2}{4p} = -\frac{1-\delta}{2-\delta}$. Therefore, we obtain:

$$\left| \frac{d}{dx} P_T^\delta F(x) \right| \leq K^{1/p} x^{2\frac{1-\delta}{2-\delta}} \|F\|_\infty T^{-\frac{1}{2}-\frac{1-\delta}{2-\delta}}.$$

Therefore, given $R > 0$, one has for all $x \in [0, R]$:

$$\left| \frac{d}{dx} P_T^\delta F(x) \right| \leq C \frac{\|F\|_\infty}{T^{\alpha(\delta)}}$$

with $C := K^{1/p} R^{2\frac{1-\delta}{2-\delta}}$. This yields the claim. \square

Remark 14. In the above proposition, the value $2(\sqrt{2}-1)$ that appears is the smallest value of δ for which η is in L^2 . For $\delta < 2(\sqrt{2}-1)$, η is no longer in L^2 but only in L^p for $p = p(\delta) < 2$, so that we cannot apply Jensen's inequality to bound the quantity $\mathbb{E} \left(\int_0^T \eta_s(x)^2 ds \right)^{p/2}$ anymore. It seems reasonable to expect that the bound (28) holds also for $\delta < 2(\sqrt{2}-1)$, although we do not have a proof of this fact.

8 Appendix

In this Appendix, we prove Proposition 3. Recall that we still denote by $(\rho_t(x))_{t,x \geq 0}$ the process $(\tilde{\rho}_t^\delta(x))_{t,x \geq 0}$ constructed in Proposition 2.

Lemma 4. *For all rational numbers $\epsilon, \gamma > 0$, let:*

$$\mathcal{U}_\gamma^\epsilon := [0, T_\epsilon(\gamma)) \times (\gamma, +\infty)$$

and set:

$$\mathcal{U} := \bigcup_{\epsilon, \gamma \in \mathbb{Q}_+^*} \mathcal{U}_\gamma^\epsilon.$$

Then, a.s., the function $(t, x) \mapsto \rho_t(x)$ is continuous on the open set \mathcal{U} .

Proof. By patching, it suffices to prove that, a.s., the function $(t, x) \mapsto \rho_t(x)$ is continuous on each $\mathcal{U}_\gamma^\epsilon$, where $\epsilon, \gamma \in \mathbb{Q}_+^*$.

Fix $\epsilon, \gamma \in \mathbb{Q}_+^*$, and let $x, y \in (\gamma, +\infty) \cap \mathbb{Q}$. We proceed to show that, a.s., for all $t \leq s < T_\epsilon(\gamma)$ the following inequality holds:

$$|\rho_t(x) - \rho_s(y)| \leq |x - y| \exp\left(\frac{|\delta - 1|}{2\epsilon^2}t\right) + \frac{|\delta - 1|}{2\epsilon}|s - t| + |B_s - B_t|. \quad (29)$$

Since $T_\epsilon(\gamma) < T_0(\gamma)$, a.s., for all $t \leq s \leq T_\epsilon(\gamma)$, we have:

$$\forall \tau \in [0, t], \quad \rho_\tau(x) = x + \frac{\delta - 1}{2} \int_0^\tau \frac{du}{\rho_u(x)} + B_\tau$$

as well as

$$\forall \tau \in [0, s], \quad \rho_\tau(y) = y + \frac{\delta - 1}{2} \int_0^\tau \frac{du}{\rho_u(y)} + B_\tau$$

and hence :

$$\forall \tau \in [0, t], \quad |\rho_\tau(x) - \rho_\tau(y)| \leq |x - y| + \frac{|\delta - 1|}{2} \int_0^\tau \frac{|\rho_u(x) - \rho_u(y)|}{\rho_u(x)\rho_u(y)} du.$$

By the monotonicity property of ρ , we have, a.s., for all t, s as above and $u \in [0, s]$:

$$\rho_u(x) \wedge \rho_u(y) \geq \rho_u(\gamma) \geq \epsilon \quad (30)$$

so that:

$$\forall \tau \in [0, t], \quad |\rho_\tau(x) - \rho_\tau(y)| \leq |x - y| + \frac{|\delta - 1|}{2} \int_0^\tau \frac{|\rho_u(x) - \rho_u(y)|}{\epsilon^2} du,$$

which, by Grönwall's inequality, implies that:

$$|\rho_t(x) - \rho_t(y)| \leq |x - y| \exp\left(\frac{|\delta - 1|}{2\epsilon^2}t\right). \quad (31)$$

Moreover, we have:

$$\rho_s(y) - \rho_t(y) = \frac{\delta - 1}{2} \int_t^s \frac{du}{\rho_u(y)} + B_s - B_t$$

which, by (30), entails the inequality:

$$|\rho_s(y) - \rho_t(y)| \leq \frac{|\delta - 1|}{2\epsilon} |s - t| + |B_s - B_t|. \quad (32)$$

Putting inequalities (31) and (32) together yields the claimed inequality (29). Hence, we have, a.s., for all rationals $x, y > \gamma$ and all $t \leq s < T_\epsilon(\gamma)$:

$$|\rho_t(x) - \rho_s(y)| \leq |x - y| \exp\left(\frac{|\delta - 1|}{2\epsilon^2} t\right) + \frac{\delta - 1}{2} |s - t| + |B_s - B_t|$$

and, by density of $\mathbb{Q} \cap (\gamma, +\infty)$ in $(\gamma, +\infty)$, this inequality remains true for all $x, y > \gamma$. Since, a.s., $t \mapsto B_t$ is continuous on \mathbb{R}_+ , the continuity of ρ on $\mathcal{U}_\gamma^\epsilon$ is proved. \square

Corollary 5. *Almost-surely, we have:*

$$\forall x \geq 0, \quad \forall t \in [0, T_0(x)), \quad \rho_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(x)} + B_t. \quad (33)$$

Remark 15. We have already remarked in Section 3 that, for all fixed $x \geq 0$, the process $(\rho_t(x))_{t \geq 0}$ satisfies the SDE (10). By contrast, the above Corollary shows the stronger fact that, considering the modification $\tilde{\rho}$ of the Bessel flow constructed in Proposition 2 above, a.s., for *each* $x \geq 0$, the path $(\tilde{\rho}_t(x))_{t \geq 0}$ still satisfies relation (10).

Proof. Consider an almost-sure event $\mathcal{A} \in \mathcal{F}$ as in Remark 7. On the event \mathcal{A} , for all $r \in \mathbb{Q}_+$, we have:

$$\forall t \in [0, T_0(r)), \quad \rho_t(r) = r + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(r)} + B_t.$$

Denote by $\mathcal{B} \in \mathcal{F}$ any almost-sure event on which ρ satisfies the monotonicity property (15). We show that, on the event $\mathcal{A} \cap \mathcal{B}$, the property (33) is satisfied.

Suppose $\mathcal{A} \cap \mathcal{B}$ is fulfilled, and let $x \geq 0$. Then for all $r \in \mathbb{Q}$ such that $r \geq x$, we have:

$$\forall t \geq 0, \quad \rho_t(x) \leq \rho_t(r)$$

so that $T_0(r) \geq T_0(x)$. Hence, for all $t \in [0, T_0(x))$, we have in particular $t \in [0, T_0(r))$, so that:

$$\rho_t(r) = r + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(r)} + B_t.$$

Since, for all $u \in [0, t]$, $\rho_u(r) \downarrow \rho_u(x)$ as $r \downarrow x$ with $r \in \mathbb{Q}$, by the monotone convergence theorem, we deduce that:

$$\int_0^t \frac{du}{\rho_u(r)} \longrightarrow \int_0^t \frac{du}{\rho_u(x)}$$

as $r \downarrow x$ with $r \in \mathbb{Q}$. Hence, letting $r \downarrow x$ with $r \in \mathbb{Q}$ in the above equation, we obtain:

$$\rho_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(x)} + B_t.$$

This yields the claim. \square

One of the main difficulties for proving Proposition 3 arises from the behavior of $\rho_t(x)$ at $t = T_0(x)$. However we will circumvent this problem by working away from the event $t = T_0(x)$. To do so, we will make use of the following property.

Lemma 5. *Let $\delta < 2$ and $x \geq 0$. Then the function $y \mapsto T_0(y)$ is a.s. continuous at x .*

Proof. The function $y \mapsto T_0(y)$ is nondecreasing over \mathbb{R}_+ . Hence, if $x > 0$, it has left- and right-sided limits at x , $T_0(x^-)$ and $T_0(x^+)$, satisfying:

$$T_0(x^-) \leq T_0(x) \leq T_0(x^+). \quad (34)$$

Similarly, if $x = 0$, there exists a right-sided limit $T_0(0^+)$ satisfying $T_0(0) \leq T_0(0^+)$. Suppose, e.g., that $x > 0$. Then we have:

$$\mathbb{E} \left(e^{-T_0(x^+)} \right) \leq \mathbb{E} \left(e^{-T_0(x)} \right) \leq \mathbb{E} \left(e^{-T_0(x^-)} \right). \quad (35)$$

Now, by the scaling property of the Bessel processes (see, e.g., Remark 3.7 in [13]), for all $y \geq 0$, the following holds:

$$(y\rho_t(1))_{t \geq 0} \stackrel{(d)}{=} (\rho_{y^2 t}(y))_{t \geq 0},$$

so that $T_0(y) \stackrel{(d)}{=} y^2 T_0(1)$. Therefore, using the dominated convergence theorem, we have:

$$\begin{aligned} \mathbb{E} \left(e^{-T_0(x^+)} \right) &= \lim_{y \downarrow x} \mathbb{E} \left(e^{-T_0(y)} \right) \\ &= \lim_{y \downarrow x} \mathbb{E} \left(e^{-y^2 T_0(1)} \right) \\ &= \mathbb{E} \left(e^{-x^2 T_0(1)} \right) \\ &= \mathbb{E} \left(e^{-T_0(x)} \right). \end{aligned}$$

Similarly, we have $\mathbb{E} \left(e^{-T_0(x^-)} \right) = \mathbb{E} \left(e^{-T_0(x)} \right)$. Hence the inequalities (35) are actually equalities; recalling the original inequality (34), we deduce that $T_0(x^-) = T_0(x) = T_0(x^+)$ a.s.. Similarly, if $x = 0$, we have $T_0(0) = T_0(0^+)$ a.s. \square

Before proving Proposition 3, we need a coalescence lemma, which will help us prove that the derivative of ρ_t at x is 0 if $t > T_0(x)$:

Lemma 6. *Let $x, y \geq 0$, and let τ be a nonnegative $(\mathcal{F}_t)_{t \geq 0}$ -stopping time. Then, almost-surely:*

$$\rho_\tau(x) = \rho_\tau(y) \quad \Rightarrow \quad \forall s \geq \tau, \rho_s(x) = \rho_s(y).$$

Proof. On the event $\{\rho_\tau(x) = \rho_\tau(y)\}$, the processes $(X_t^\delta(x))_{t \geq 0} := (\rho_t(x)^2)_{t \geq 0}$ and $(X_t^\delta(y))_{t \geq 0} := (\rho_t(y)^2)_{t \geq 0}$ both satisfy, on $[\tau, +\infty)$, the SDE:

$$X_t = \rho_\tau(x)^2 + 2 \int_\tau^t \sqrt{X_s} dB_s + \delta(t - \tau).$$

By pathwise uniqueness of this SDE (see [9], Theorem (3.5), Chapter IX), we deduce that, a.s. on the event $\{\rho_\tau(x) = \rho_\tau(y)\}$, $X_t(x) = X_t(y)$, hence $\rho_t(x) = \rho_t(y)$ for all $t \geq \tau$. \square

Now we are able to prove Prop. 3.

Proof of Proposition 3. Let $t > 0$ and $x > 0$ be fixed. First remark that:

$$\mathbb{P}(T_0(x) = t) = 0.$$

Indeed, if $\delta > 0$, then :

$$\mathbb{P}(T_0(x) = t) \leq \mathbb{P}(\rho_t(x) = 0)$$

and the RHS is zero since the law of $\rho_t(x)$ has no atom on \mathbb{R}_+ (it has density $p_t^\delta(x, \cdot)$ w.r.t. Lebesgue measure on \mathbb{R}_+ , where p_t^δ was defined in equation (12) above). On the other hand, if $\delta = 0$, then 0 is an absorbing state for the process ρ , so that, for all $s \geq 0$:

$$\mathbb{P}(T_0(x) \leq s) = \mathbb{P}(\rho_s(x) = 0)$$

and the RHS is continuous in s on \mathbb{R}_+ , since it is given by $\exp(-\frac{x^2}{2s})$ (see [9], Chapter XI, Corollary 1.4). Hence, also in the case $\delta = 0$ the law of $T_0(x)$ has no atom on \mathbb{R}_+ . Hence, a.s., either $t < T_0(x)$ or $t > T_0(x)$.

First suppose that $t < T_0(x)$. A.s., the function $y \mapsto T_0(y)$ is continuous at x , so there exists a rational number $y \in [0, x)$ such that $t < T_0(y)$; since, by Remark (7), $t \mapsto \rho_t(y)$ is continuous, there exists $\epsilon \in \mathbb{Q}_+^*$ such that $t < T_\epsilon(y)$. By monotonicity of $z \mapsto \rho(z)$, for all $s \in [0, t]$ and $z \geq y$, we have:

$$\rho_s(z) \geq \rho_s(y) \geq \epsilon.$$

Hence, recalling Corollary 5, for all $s \in [0, t]$ and $h \in \mathbb{R}$ such that $|h| < |x - y|$:

$$\rho_s(x+h) = x+h + \int_0^s \frac{\delta-1}{2} \frac{du}{\rho_u(x+h)} + B_s.$$

Hence, setting $\eta_s^h(x) := \frac{\rho_s(x+h) - \rho_s(x)}{h}$, we have:

$$\forall s \in [0, t], \quad \eta_s^h(x) = 1 - \frac{\delta-1}{2} \int_0^s \frac{\eta_u^h(x)}{\rho_u(x)\rho_u(x+h)} du$$

so that :

$$\eta_t^h(x) = \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)\rho_s(x+h)}\right).$$

Note that, for all $s \in [0, t]$ and $h \in \mathbb{R}$ such that $|h| < |x - y|$, we have $(s, x + h) \in [0, T_\epsilon(y)) \times (y, +\infty) \subset \mathcal{U}$. Hence, by Lemma 4, we have, for all $s \in [0, t]$

$$\rho_s(x + h) \xrightarrow{h \rightarrow 0} \rho_s(x)$$

with the domination property:

$$\frac{1}{\rho_s(x)\rho_s(x + h)} \leq \epsilon^{-2}$$

valid for all $|h| < |x - y|$. Hence, by the dominated convergence theorem, we deduce that:

$$\eta_t^h(x) \xrightarrow{h \rightarrow 0} \exp\left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right)$$

which yields the claimed differentiability of ρ_t at x .

We now suppose that $t > T_0(x)$. Since the function $y \mapsto T_0(y)$ is a.s. continuous at x , a.s. there exists $y > x$, $y \in \mathbb{Q}$, such that $t > T_0(y)$. By Remark (7), the function $t \mapsto \rho_t(y)$ is continuous, so that $\rho_{T_0(y)}(y) = 0$. By monotonicity of $z \mapsto \rho(z)$, we deduce that, for all $z \in [0, y]$, we have :

$$\rho_{T_0(y)}(z) = 0.$$

By Lemma 6, we deduce that, leaving aside some event of probability zero, all the trajectories $(\rho_t(z))_{t \geq 0}$ for $z \in [0, y] \cap \mathbb{Q}$ coincide from time $T_0(y)$ onwards. In particular, we have:

$$\forall z \in [0, y] \cap \mathbb{Q}, \quad \rho_t(z) = \rho_t(x).$$

Since, moreover, the function $z \mapsto \rho_t(z)$ is nondecreasing, we deduce that it is constant on the whole interval $[0, y]$:

$$\forall z \in [0, y], \quad \rho_t(z) = \rho_t(x).$$

In particular, the function $z \mapsto \rho_t(z)$ has derivative 0 at x . This concludes the proof. \square

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