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Michel Broué, Ruth Corran, Jean Michel. Cyclotomic root systems and bad primes. 2017. hal-01506943v2

HAL Id: hal-01506943
https://hal.science/hal-01506943v2

Preprint submitted on 13 Apr 2017

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CYCLOTOMIC ROOT SYSTEMS AND BAD PRIMES

MICHEL BROUÉ, RUTH CORRAN, JEAN MICHEL

Abstract. We generalize the definition and properties of root systems to complex reflection groups — roots become rank one projective modules over the ring of integers of a number field k.

In the irreducible case, we provide a classification of root systems over the field of definition k of the reflection representation.

In the case of spetsial reflection groups, we generalize as well the definition and properties of bad primes.

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Date: April 13, 2017.

1991 *Mathematics Subject Classification.* Primary : 20F55 ; Secondary: 20C08.

Key words and phrases. Complex reflection groups, root systems, cyclotomic Hecke algebras, spetses.

We thank Gabriele Nebe for her previous seminal work on this topic.

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1. Introduction

The spirit of the Spetses program ([BMM], [BMM2]) is to consider (at least some of) the complex reflections groups as Weyl groups for some mysterious object which looks like a "generic finite reductive group" — and which is yet unknown.

Some of the data attached to finite reductive groups, such as the parameterization of unipotent characters, their generic degrees, Frobenius eigenvalues, and also the families and their Fourier matrices, turn out to depend only on the $\mathbb{Q}$-representation of the Weyl group. But supplementary data, such as the parameterization of unipotent classes, the values of unipotent characters on unipotent elements, depend on the entire root datum.

For a complex reflection group, not necessarily defined over $\mathbb{Q}$, but defined over a number field k, it thus seems both necessary and natural to study "$\mathbb{Z}_k$-root data", where $\mathbb{Z}_k$ denotes the ring of integers of k.

Some related work has already been done in that direction, in particular by Nebe [Ne], who classified $\mathbb{Z}_k$-lattices invariant under the reflection group – and whose work inspired us. More recently, motivated by some work on p-compact groups, Grodal and others [AnGr], [Gr] defined root data for reflection groups defined over finite fields.

Here we define and classify $\mathbb{Z}_k$-root systems[1], as well as root lattices and coroot lattices, for all complex reflection groups. Of course most of the rings $\mathbb{Z}_k$ are not principal ideal domains (although — quite a remarkable fact — they are P.I.D. for the 34 exceptional irreducible complex reflection groups) and one has naturally to replace *elements of* $\mathbb{Z}_k$ by *ideals of* $\mathbb{Z}_k$. Taking this into account, our definition of root system mimics Bourbaki's definition [BouLie] and of course working with "ideal numbers" is much more appropriate, on general Dedekind domains, than working with "numbers", as suggested by what follows.

- A complex reflection group may occur as a parabolic subgroup of another reflection group whose field of definition is larger. A first problem with considering vectors (as in the usual approach of root systems) instead of one-dimensional $\mathbb{Z}_k$-modules (as we do here) is that we would have "too many" of them when restricting to a parabolic subgroup.
- In the case of the group of type B_2, over a field where the ideal generated by 2 has a square root, such as $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-1})$, not only do we find the usual system of type B_2 but we also find a system which affords the exterior automorphism ${}^2B_2$. If we were considering numbers instead of ideals, this automorphism would exist only if the number 2 (rather than the ideal generated by 2) has a square root.

  The exceptional group denoted by G_{29}, defined over $\mathbb{Q}(\sqrt{-1})$, has a subgroup of type B_2 and the normaliser of this subgroup induces the automorphism ${}^2B_2$. Our corresponding root system has the same automorphism since $1+i$ and $1-i$ generate the same ideal and $(1+i)(1-i)=2$, thus 2 has an "ideal square root" in the ring $\mathbb{Z}[\sqrt{-1}]$.

Perhaps the most interesting and intriguing fact which comes out of the classification concerns the generalisation of the notions of *connection index* and *bad primes*: in the case of spetsial reflection groups, the order of the group is divisible by the factorial of the rank times the connection index, and the bad primes for the corresponding Spets make up the remainder, just as in the case of finite reductive groups and Weyl groups.[2]

2. Complex reflection groups

We denote $\lambda \mapsto \lambda^*$ the complex conjugation and we denote by k a subfield of $\mathbb{C}$ stable by complex conjugation.

[1]By analogy with the well established terminology *"cyclotomic Hecke algebras"* — a crucial notion in the Spetses program —, we propose to call these generalised root systems *"cyclotomic root systems"*.

[2]Notice though that, as shown by Nebe [Ne], the bad primes for spetsial groups (see Section 8) do *not* occur as divisors of the orders of the quotient of the root lattice by the root lattice of maximal reflection subgroups.

2.1. **Preliminary material about reflections.**

Let (V, W) be a pair of finite dimensional k-vector spaces with a given Hermitian pairing $V \times W \to k : (v, w) \mapsto \langle v, w\rangle$; that is:

- $\langle$-,-$\rangle$ is linear in V and semi-linear in W: for $\lambda, \mu \in k$, $v \in V$ and $w \in W$ we have $\langle \lambda v, \mu w\rangle = \lambda\mu^*\langle v, w\rangle$.
- $\langle v, w\rangle = 0$ for all $w \in W$ implies $v = 0$.
- $\langle v, w\rangle = 0$ for all $v \in V$ implies $w = 0$.

Similarly $(w, v) \mapsto \langle v, w\rangle^*$ defines a Hermitian pairing $W \times V \to k$ that we will also denote $(w, v) \mapsto \langle w, v\rangle$ when its meaning is clear from the context.

Any vector space can be naturally endowed with a Hermitian pairing with its twisted dual:

Definition 2.1. *The* twisted dual *of a k-vector space V, denoted ${}^*V$, is the k-vector space which is the conjugate under $*$ of the dual V^* of V. In other words,*

- *as an abelian group, ${}^*V = V^*$,*
- *an element $\lambda \in k$ acts on ${}^*V$ as λ^* acts on V^*.*

The pairing $V \times {}^*V \to k : (v, \phi) \mapsto \phi(v)$ is a Hermitian pairing, called the canonical pairing associated with V.

When V is a real vector space, the twisted dual is the usual dual.

Let $G_{(V,W)}$ be the subgroup of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ which preserves the pairing. The first (resp. second) projection gives an isomorphism $G_{(V,W)} \xrightarrow{\sim} \mathrm{GL}(V)$ (resp. $G_{(V,W)} \xrightarrow{\sim} \mathrm{GL}(W)$). Composing the second isomorphism with the inverse of the first we get an isomorphism $g \mapsto g^\vee : \mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}(W)$. The inverse morphism has the same definition reversing the roles of V and W, and we will still denote it $g \mapsto g^\vee$ so that $(g^\vee)^\vee = g$.

In the case of the canonical pairing associated with V, the isomorphism $g \mapsto g^\vee$ is just the contragredient ${}^tg^{-1}$.

Definition 2.2. *A* reflection *is an element $s \in \mathrm{GL}(V)$ of finite order such that $\ker(s-1)$ is an hyperplane. Define the*

- reflecting hyperplane *of s as $H_s := \ker(s-1)$,*
- reflecting line *of s as $L_s := \mathrm{im}\,(s-1)$,*
- dual reflecting line *M_s of s as the orthogonal (in W) of H_s,*
- dual reflecting hyperplane *K_s of s as the orthogonal (in W) of L_s.*

Denote by ζ_s the determinant of s, which is a root of unity.

It is clear that a reflection s is determined by H_s, L_s and ζ_s. In turn, H_s is determined by the dual reflecting line M_s. Note that M_s is not orthogonal to L_s since H_s does not contain L_s. Thus giving a reflection is equivalent to giving the following data:

Definition 2.3. *A* reflection triple *is a triple (L, M, ζ) where*

- *L is a line in V and M a line in W which are not orthogonal.*
- *$\zeta \in k^\times$ is a root of unity.*

Formulae for the reflections defined by a reflection triple are symmetric in V and W:

Proposition 2.4. *A reflection triple (L, M, ζ) defines a pair of reflections $(s, s^\vee) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ (which preserve the pairing) by the formulae:*

$$s(v) = v - \frac{\langle v, y\rangle}{\langle x, y\rangle}(1-\zeta)x,$$

$$s^\vee(w) = w - \frac{\langle w, x\rangle}{\langle y, x\rangle}(1-\zeta)y,$$

for any non-zero $x \in L$ and $y \in M$.

Proof. An easy computation shows that the pair $(s, s^\vee)$ of reflections preserves the pairing $\langle\text{-},\text{-}\rangle$. Furthermore, the reflection s defined by this formula determines the triple (L, M, ζ), and the reflection $s^\vee$ (in W) determines the triple (M, L, ζ). □

As long as $\zeta^m \neq 1$, the pair $(s^m, s^{\vee m})$ is precisely the pair of reflections defined by the triple (L, M, ζ^m): the order of s is the order of the element $\zeta \in k^\times$.

To summarize, we have:

Proposition 2.5. *Reflection triples are in bijection with*

- *reflections in $\mathrm{GL}(V)$,*
- *reflections in $\mathrm{GL}(W)$,*
- *pairs of reflections $(s, s^\vee)$ in $\mathrm{GL}(V) \times \mathrm{GL}(W)$ which preserve the pairing $\langle\text{-},\text{-}\rangle$.*

An element $g \in \mathrm{GL}(V)$ acts naturally on reflection triples (L, M, ζ) through the action of $(g, g^\vee)$ on pairs (L, M). It follows from the previous proposition that g commutes with a reflection s if and only if $(g, g^\vee)$ stabilizes (L_s, M_s).

Notation 2.6. Let $t = (L, M, \zeta)$ be a reflection triple; denote by s_t the corresponding reflection, and write $s_t^\vee$ for the reflection corresponding to (M, L, ζ). We will also write L_t, H_t, M_t, K_t for $L_{s_t}, H_{s_t}, M_{s_t}, K_{s_t}$.

Stable subspaces.

A reflection is diagonalisable, hence so is its restriction to a stable subspace. The next lemma follows directly.

Lemma 2.7. *Let V_1 be a subspace of V stable by a reflection s. Then*

- *either V_1 is fixed by s (i.e., $V_1 \subseteq H_s$),*
- *or V_1 contains L_s, and then $V_1 = L_s \oplus (H_s \cap V_1)$, in which case the restriction of s to V_1 is a reflection.*

In particular, the restriction of a reflection to a stable subspace is either trivial or a reflection.

Commuting reflections.

Lemma 2.8. *Let $t_1 = (L_1, M_1, \zeta_1)$ and $t_2 = (L_2, M_2, \zeta_2)$ be two reflection triples. We have the following three sets of equivalent assertions.*

- *(I)* *(i)* $(s_{t_1}, s_{t_1}^\vee)$ *acts trivially on* (L_2, M_2).
  - *(ii)* $(s_{t_2}, s_{t_2}^\vee)$ *acts trivially on* (L_1, M_1).
  - *(iii)* $L_1 \subseteq H_{t_2}$ *and* $L_2 \subseteq H_{t_1}$, *in which case we say that* t_1 *and* t_2 *are* orthogonal.
- *(II)* *(i)* $(s_{t_1}, s_{t_1}^\vee)$ *acts by* ζ_1 *on* (L_2, M_2).
  - *(ii)* $(s_{t_2}, s_{t_2}^\vee)$ *acts by* ζ_2 *on* (L_1, M_1).
  - *(iii)* $L_1 = L_2$ *and* $H_{t_1} = H_{t_2}$, *in which case we say that* t_1 *and* t_2 *are* parallel.
- *(III)* *(i)* $s_{t_1} s_{t_2} = s_{t_2} s_{t_1}$,
  - *(ii)* s_{t_1} *stabilizes* t_2.
  - *(iii)* s_{t_2} *stabilizes* t_1.
  - *(iv)* t_1 *and* t_2 *are either orthogonal or parallel.*

Proof. The proof of both (I) and (II) proceeds by showing the equivalence of (i) and (iii), from which the equivalence between (ii) and (iii) follows by symmetry. (III) then follows from (I), (II), and the definitions. □

From now on and until the end of this section, we shall work "on the V" side. Nevertheless, we shall go on using notions previously introduced in connection with an Hermitian pairing $V \times W \to k$.

2.2. **Reflection groups.**

Definition 2.9. *A* reflection group *on k is a pair (V, G), where V is a finite dimensional k-vector space and G is a subgroup of* $\mathrm{GL}(V)$ *generated by reflections. A reflection group is said to be* finite *if G is finite, and* complex *if $k \subseteq \mathbb{C}$.*

Throughout this subsection, (V, G) denotes a reflection group.

Whenever a set of reflections S generates G, then $\bigcap_{s \in S} H_s = V^G$, the set of elements fixed by G.

Definition 2.10. *A reflection group (V, G) is* essential *if $V^G = \{0\}$.*

Definition 2.11. *A set of reflections is* saturated *if it is closed under conjugation by the group it generates.*

When S is saturated, G is a (normal) subgroup of the subgroup of $\mathrm{GL}(V)$ which stabilizes S, and the subspace V_S, defined by

$$V_S := \sum_{s \in S} L_s \,,$$

is stable by the action of G.

Orthogonal decomposition.

Definition 2.12.

(1) *Define an equivalence relation $\sim$ on S as the transitive closure of: $s_t \sim s_{t'}$ whenever t is not orthogonal to t'.*
(2) *The set of reflections S is said to be* irreducible *if it consists of a unique $\sim$-equivalence class.*

Lemma 2.13. *Let S be a saturated set of reflections which generates the reflection group (V, G). Then the $\sim$-equivalence classes of S are stable under G-conjugacy.*

Proof. This results from the stability under G-conjugacy of S and of the relation "*being orthogonal*". □

When a group G acts on a set V (which could be G itself on which G acts by conjugation), for $X \subset V$ we denote by $N_G(X)$ the normalizer (stabilizer) of X in G, *i.e.*, the set of $g \in G$ such that $g(X) = X$. We denote by $C_G(X)$ the centralizer (fixator) of X, *i.e.*, the set of $g \in G$ such that, for all $x \in X$, $g(x) = x$. Notice that $C_G(X) \triangleleft N_G(X)$.

Lemma 2.14. *Let S be a set of reflections on V.*

(1) *The action of $N_{\mathrm{GL}(V)}(S)$ on S induces an injection:*
$$N_{\mathrm{GL}(V)}(S)/C_{\mathrm{GL}(V)}(S) \hookrightarrow \mathfrak{S}(S),$$
into the symmetric group on S.
(2) *If S is saturated and irreducible, then $C_{\mathrm{GL}(V)}(S)$ acts by scalars on V_S.*

Proof. Item (1) is trivial. Let us prove (2).

Take any $g \in C_{\mathrm{GL}(V)}(S)$. The pair $(g, g^\vee)$ stabilizes the reflection triples of all reflections in S. Take two non-orthogonal reflections s, s' in S with corresponding reflection triples (L, M, ζ), (L', M', ζ') and choose non-zero elements $x \in L, y \in M, v \in L', w \in M'$. Then $g(x) = \lambda x$ and $g(v) = \mu v$ for some non-zero scalars λ and μ.

Since S is saturated, it also contains the reflection corresponding to the triple $s \cdot (L', M', \zeta')$. But $s(v) = v - \frac{\langle v,y\rangle}{\langle x,y\rangle}(1-\zeta)x$, so $g(s(v)) = \mu v - \frac{\langle v,y\rangle}{\langle x,y\rangle}(1-\zeta)\lambda x = \alpha s(v)$ for some non-zero scalar α. In particular, this implies $\mu = \lambda$, and g acts by scalar multiplication on $L_s + L_{s'}$.

Since S is irreducible, this shows that g acts by scalar multiplication on $L_s + L_t$ for any pair $s, t \in S$, hence acts by scalar multiplication on V_S. □

Lemma 2.15. *The number of $\sim$-equivalence classes of reflections in S is bounded by the dimension of V. In particular, it is finite.*

Proof. Assume that $t_1 = (L_1, M_1, \zeta_1), \ldots, t_m = (L_m, M_m, \zeta_m)$ correspond to reflections which belong to distinct $\sim$-equivalence classes. So in particular all the t_i are mutually orthogonal. This implies that the

L_i are linearly independent: for let $x_i \in L_i, y_i \in M_i$ be non-zero elements and assume $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m = 0$. The Hermitian product with y_i yields $\lambda_i \langle x_i, y_i \rangle = 0$, hence $\lambda_i = 0$. □

The following lemma is straightforward to verify.

Lemma 2.16. *Let S be a set of reflections which generate the reflection group (V, G). Let $S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_m$ be the decomposition of S into $\sim$-equivalence classes. Denote by G_i the subgroup of G generated by the reflections s for $s \in S_i$ and by V_i the subspace of V generated by the lines L_s for $s \in S_i$.*

(1) The group G_i acts trivially on $\sum_{j \neq i} V_j$.
(2) For $1 \leq i \neq j \leq m$, the groups G_i and G_j commute.
(3) $G = G_1 G_2 \ldots G_m$.

The above result can be refined in the case where G acts completely reducibly on V – then the decomposition is direct (see 2.19 below).

Lemma 2.16 can be applied to obtain a criterion for finiteness of G.

Proposition 2.17. *Let G be the group generated by a saturated set of reflections S. If S is finite, then G is finite.*

Proof. Consider the case where S is irreducible. By the second part of Lemma 2.14, we know that the centralizer of S is contained in $k^\times$, so in particular, $G \cap C_{\mathrm{GL}(V)}(S)$ is contained in $k^\times$. Moreover, each reflection has finite order, so in fact $G \cap C_{\mathrm{GL}(V)}(S) \subset \boldsymbol{\mu}(k)$, the set of roots of unity of k. Now the determinants of the elements of G belong to the finite subgroup of $\boldsymbol{\mu}(k)$ generated by the determinants of the elements of S. Thus $G \cap C_{\mathrm{GL}(V)}(S)$ is finite.

As G is contained in $N_{\mathrm{GL}(V)}(S)$, by the first part of Lemma 2.14, it is an extension of $G \cap C_{\mathrm{GL}(V)}(S)$ by a subgroup of $\mathfrak{S}(S)$. By finiteness of S, this is a finite extension. Hence G is finite.

Finally, consider the general case $G = G_1 \cdots G_m$, where the groups G_i are generated by reflections in distinct equivalence classes, as in Lemma 2.16. By the preceding comments, each G_i is finite; so G must also be finite. □

The case when G is completely reducible.

Proposition 2.18. *Let S be a set of reflections generating the reflection group (V, G), and suppose that the action of G on V is completely reducible. Then*

(1) $V = V_S \oplus V^G$, and
(2) the restriction from V to V_S induces an isomorphism from G onto its image in $\mathrm{GL}(V_S)$, an essential reflection group on V_S.

Proof. The subspace V_S is G-stable, hence since G is completely reducible there is a complementary subspace V' which is G-stable. We

have $V_S \supset L_s$ for all $s \in S$, thus (by Lemma 2.7) V' is contained in H_s; it follows that $V' \subseteq \bigcap_{s\in S} H_s \subseteq V^G$. So it suffices to prove that $V_S \cap V^G = 0$.

Since V^G is stable by G, there exists a complementary subspace V'' which is stable by G. Whenever $s \in S$, we have $L_s \subseteq V''$ (otherwise, by Lemma 2.7, we have $V'' \subseteq H_r$, which implies that s is trivial since $V = V^G \oplus V''$, a contradiction). This shows that $V_S \subseteq V''$, and in particular that $V_S \cap V^G = 0$. □

Proposition 2.19. *Let S be a set of reflections generating the reflection group (V, G), and suppose that the action of G on V is completely reducible. Denote by $\{V_i\}_i$ the subspaces associated to an orthogonal decomposition as in Lemma 2.16.*

(1) For $1 \le i \le m$, the action of G_i on V_i is irreducible.
(2) $V_S = \bigoplus_{i=1}^{i=m} V_i$.
(3) $G = G_1 \times G_2 \times \cdots \times G_m$

Proof. (1) The subspace V_i is stable under G, and the action of G on a stable subspace is completely reducible. But the image of G in $\mathrm{GL}(V_i)$ is the same as the image of G_i. So the action of G_i on V_i is completely reducible.

So we write $V_i = V_i' \oplus V_i''$ where V_i' and V_i'' are stable by G_i. Define:

$$S_i' := \{s \in S_i \mid L_s \subseteq V_i'\} \quad \text{and} \quad S_i'' := \{s \in S_i \mid L_s \subseteq V_i''\}\,.$$

By Lemma 2.7, if $s \in S_i'$, then $V_i'' \subseteq H_s$, and if $s \in S_i''$, then $V_i' \subseteq H_s$. Hence any two elements of S_i' and S_i'' are mutually orthogonal. Thus one of them has to be all of S_i.

(2) By Lemma 2.16, we have $\sum_{j\neq i} V_j \subset V^{G_i}$. By (1), and by Proposition 2.18, we then get $V_i \cap \sum_{j\neq i} V_j = 0$.

(3) An element $g \in G_i$ which also belongs to $\prod_{j\neq i} G_j$ acts trivially on V_i. Since (by (1) and by Proposition 2.18) the representation of G_i on V_i is faithful, we see that $g = 1$. □

Reflecting pairs.

For H a reflecting hyperplane, notice that

$$C_G(H) = \{1\} \cup \{g \in G \mid \ker(g-1) = H\}\,.$$

For L a reflecting line, we have:

$$C_G(V/L) = \{1\} \cup \{g \in G \mid \mathrm{im}\,(g-1) = L\}\,.$$

So $C_G(V/L)$ is the group of all elements of G which stabilize L and which act trivially on V/L; a normal subgroup of $N_G(L)$.

Similarly, if M is a dual reflecting line in W, $C_G(W/M) \triangleleft N_G(M)$.

Remark 2.20. Recall that orthogonality between V and W induces a bijection between reflecting hyperplanes and dual reflecting lines, as well as between reflecting lines and dual reflecting hyperplanes.

Then if M is the orthogonal (in W) of the reflecting hyperplane H, we have

$$C_G(H) = C_G(W/M)\,.$$

We will be considering the following property.

Property 2.21. *The reflection group (V, G) is such that the representations of G and all its proper subgroups on V are completely reducible.*

Note that this is the case in particular when G is finite.

Proposition 2.22. *Let (V, G) be a reflection group with Property 2.21.*

(1) Let H be a reflecting hyperplane for G. There exists a unique reflecting line L such that $C_G(V/L) = C_G(H)$.
In other words:
Let M be a dual reflecting line for G. There exists a unique reflecting line L such that $C_G(V/L) = C_G(W/M)$.

(2) Let L be a reflecting line for G. There exists a unique reflecting hyperplane H such that $C_G(H) = C_G(V/L)$.
In other words: Let K be a dual reflecting hyperplane for G. There exists a unique reflecting hyperplane H such that $C_G(K) = C_G(H)$.

(3) If (L, H) (or (L, M), or (H, K)) is a pair as above, then
(a) $C_G(H)$ consists of the identity and of reflections s where $H_s = H$ and $L_s = L$,
(b) $C_G(H)$ is isomorphic to a subgroup of $k^\times$, and so is cyclic if G is finite, and
(c) $N_G(H) = N_G(L) = N_G(M) = N_G(K)$.

Proof of 2.22.

• Assume $C_G(H) \neq \{1\}$. Since the action of $C_G(H)$ on V is completely reducible, there is a line L which is stable by $C_G(H)$ and such that $H \oplus L = V$. Such a line is obviously the eigenspace (corresponding to an eigenvalue different from 1) for any non-trivial element of $C_G(H)$. This shows that L is uniquely determined, and that $C_G(H)$ consists of 1 and of reflections with hyperplane H and line L. It follows also that $C_G(H) \subseteq G(V/L)$. Notice that H and L are the isotypic components of V under the action of $C_G(H)$.

• Assume $C_G(V/L) \neq \{1\}$. Since the action of $C_G(V/L)$ on V is completely reducible, there is a hyperplane H which is stable by $C_G(V/L)$ and such that $L \oplus H = V$. Such a hyperplane is clearly the kernel of any nontrivial element of $C_G(V/L)$. This shows that H is uniquely determined, and that $C_G(V/L) \subseteq C_G(H)$.

We let the reader conclude the proof. □

Notice the following improvement to Lemma 2.8 due to complete reducibility.

Proposition 2.23. *Let t, t' be two reflection triples such that s_t and $s_{t'}$ belong to a group satisfying Property 2.21. Then*

(1) t and t' are orthogonal if $L_t \subseteq H_{t'}$ or $L_{t'} \subseteq H_t$.
(2) t and t' are parallel if $L_t = L_{t'}$ or $H_t = H_{t'}$,

2.3. **The Shephard–Todd classification.**

Definition 2.24. *Given (V, G) and (V', G') finite reflection groups on k, an isomorphism from (V, G) to (V', G') is a k-linear isomorphism $f : V \xrightarrow{\sim} V'$ which conjugates the group G onto the group G'.*

From now on in this subsection we assume that $k \subseteq \mathbb{C}$.

The family of finite complex reflection groups denoted $G(de, e, r)$.

Let d, e and r be three positive integers.

Let $D_r(de)$ be the set of diagonal complex matrices with diagonal entries in the group $\boldsymbol{\mu}_{de}$ of all de–th roots of unity. The d–th power of the determinant defines a surjective morphism

$$\det{}^d : D_r(de) \twoheadrightarrow \boldsymbol{\mu}_e \,.$$

Let $A(de, e, r)$ be the kernel of the above morphism. In particular we have $|A(de, e, r)| = (de)^r/e$. Identifying the symmetric group $\mathfrak{S}_r$ with the usual $r \times r$ permutation matrices, we define

$$G(de, e, r) := A(de, e, r) \rtimes \mathfrak{S}_r \,.$$

We have $|G(de, e, r)| = (de)^r r!/e$, and $G(de, e, r)$ is the group of all monomial $r \times r$ matrices, with entries in $\boldsymbol{\mu}_{de}$, and product of all non-zero entries in $\boldsymbol{\mu}_d$.

Examples 2.25.

- $G(e, e, 2)$ is the dihedral group of order $2e$.
- $G(d, 1, r)$ is isomorphic to the wreath product $\boldsymbol{\mu}_d \wr \mathfrak{S}_r$. For $d = 2$, it is isomorphic to the Weyl group of type B_r (or C_r).
- $G(2, 2, r)$ is isomorphic to the Weyl group of type D_r.

The following theorem, stated in terms of abstract groups, is the main result of [ShTo]. It is explicitly proved in [Co, 2.4, 3.4 and 5.12].

Theorem 2.26 (Shephard–Todd)). *Let (V, G) be a finite irreducible complex reflection group. Then one of the following assertions is true:*

- *$(V, G) \simeq (\mathbb{C}^r, G(de, e, r))$ for some integers d, e, r, with $de \geq 2$, $r \geq 1$*
- *$(V, G) \simeq (\mathbb{C}^{r-1}, \mathfrak{S}_r)$ for some an integer $r \geq 1$*
- *(V, G) is isomorphic to one of 34 exceptional reflection groups.*

The exceptional groups are traditionally denoted $G_4, \ldots, G_{37}$.

Remark 2.27. Conversely, any group $G(de, e, r)$ is irreducible on $\mathbb{C}^r$ except for $d = e = 1$ and $d = e = r = 2$.

Remark 2.28. Theorem 2.26 has the following consequence.

Assume that (V, G) is a complex finite reflection group where V is r-dimensional. Choose a basis of V so that G is identified with a subgroup of $\mathrm{GL}_r(\mathbb{C})$. Now, given an automorphism σ of the field $\mathbb{C}$, applying σ to all entries of the matrices of G defines another group ${}^\sigma G$ and so another complex finite reflection group $(V, {}^\sigma G)$.

Then it follows from Theorem 2.26 that there exists $\phi \in \mathrm{GL}(V)$ and $a \in \mathrm{Aut}(G)$ such that, for all $g \in G$,

$$\sigma(g) = \phi a(g) \phi^{-1} .$$

Definition 2.29. *A finite reflection group (V, G) is said to be* well-generated *if G may be generated by r reflections, where $r = \dim(V)$.*

The well-generated irreducible groups are $G(d, 1, r), G(e, e, r)$ and all the exceptional groups excepted $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$.

Field of definition.

The following theorem has been proved (using a case by case analysis) by Benard [Ben] (see also [Bes1]), and generalizes a well known result on Weyl groups.

Theorem–Definition 2.30. *Let (V, G) be a finite complex reflection group. Let $\mathbb{Q}_G$ be the field generated by the traces on V of all elements of G. Then all irreducible $\mathbb{Q}_G G$–representations are absolutely irreducible.*

The field $\mathbb{Q}_G$ is called the field of definition *of the reflection group (V, G).*

- If $\mathbb{Q}_G \subseteq \mathbb{R}$, then (V, G) is a (finite) *Coxeter group*.
- If $\mathbb{Q}_G = \mathbb{Q}$, then (V, G) is a Weyl group.

2.4. **Parabolic subgroups.**

Throughout this subsection we assume only that V is a k-vector space of finite dimension, and that G is a finite subgroup of $\mathrm{GL}(V)$.

We denote by $\mathrm{Ref}(G)$ the set of all reflections of G, and by $\mathrm{Arr}(G)$ the set of reflecting hyperplanes of elements of $\mathrm{Ref}(G)$.

Notice that, since G is finite and k of characteristic zero, the kG-module V is completely reducible.

Definition 2.31. *We denote by $\mathrm{Arr}_X(G)$ the set of reflecting hyperplanes containing X, and by F_X the* flat of X in $\mathrm{Arr}(G)$*:*

$$F_X := \bigcap_{H \in \mathrm{Arr}_X(G)} H .$$

The assertion (1) of the following theorem has first been proved by Steinberg [St]. A short proof may now be found in [Le].

Theorem 2.32. *Let X be a subset of V.*

(1) The fixator $C_G(X)$ of X is generated by those reflections whose reflecting hyperplane contains X.
(2) The flat F_X is the set of fixed points of $C_G(X)$ and there exists a unique $C_G(X)$–stable subspace V_X of V such that $V = F_X \oplus V_X$.
(3) $C_G(X) = C_G(F_X)$ and $N_G(X)/C_G(X)$ is naturally isomorphic to a subgroup of $\mathrm{GL}(F_X)$.

Proof of (2). Since $C_G(X)$ is generated by reflections whose reflecting hyperplanes contain F_X, we see that the flat F_X is fixed by $C_G(X)$. Conversely, if $x \in V$ is fixed under $C_G(X)$, it is fixed by all the reflections of $C_G(X)$, hence belongs to F_X.

If $F_X = 0$, the assertion (2) is obvious. Assume $F_X \neq 0$. Then $C_G(X) \neq 1$. Since F_X is the trivial isotypic component of $C_G(X)$, the space V_X is the sum of all nontrivial isotypic components. □

Definition 2.33. *The fixators of subsets of G in V are called* parabolic subgroups *of G.*

By Theorem 2.32 above, a parabolic subgroup $C_G(X)$ acts faithfully as an essential reflection group on the uniquely defined subspace V_X.

Corollary 2.34. *The map $F \mapsto C_G(F)$ is an order reversing bijection from the set of all flats of $\mathrm{Arr}(G)$ onto the set of parabolic subgroups of G (where both sets are ordered by inclusion).*

2.5. Linear characters of a finite reflection group.

Let (V, G) be a finite reflection group.

The following description of the linear characters of a reflection group, inspired by the results of [Co], may be found, for example, in [Bro2, Theorem 3.9].

Denote by G^{ab} the quotient of G by its derived group, so that $\mathrm{Hom}(G, \mathbb{C}^\times) = \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{C}^\times)$. Recall that $\mathrm{Arr}(G)$ denote the collection of reflecting hyperplanes of the reflections s for $s \in G$.

In what follows, the notation $H \in \mathrm{Arr}(G)/G$ means that H runs over a complete set of representatives of the orbits of G on the set $\mathrm{Arr}(G)$ of its reflecting hyperplanes.

Theorem 2.35.

(1) The restrictions from G to $C_G(H)$ define an isomorphism

$$\mathrm{Hom}(G, \mathbb{C}^\times) \xrightarrow{\sim} \prod_{H \in \mathrm{Arr}(G)/G} \mathrm{Hom}(C_G(H), \mathbb{C}^\times)\,.$$

(2) The composition $i_H : C_G(H) \to G \to G^{\mathrm{ab}}$ is injective, and

$$\prod_{H \in \mathrm{Arr}(G)/G} i_H : \prod_{H \in \mathrm{Arr}(G)/G} C_G(H) \to G^{\mathrm{ab}}$$

is an isomorphism.

Corollary 2.36. *Let S be a generating set of reflections for G and let $\mathcal{O}$ be the set of G-conjugates of the elements of S. Then for any $H \in \mathrm{Arr}(G)$ the set $\mathcal{O} \cap C_G(H)$ generates $C_G(H)$.*

Proof. The set $\mathcal{O} \cap C_G(H)$ has the same image in G^{ab} as the set S_H of elements of S which are conjugate to an element of $C_G(H)$. If we denote $x \mapsto x^{\mathrm{ab}}$ the quotient map $G \to G^{\mathrm{ab}}$, we have $S^{\mathrm{ab}} = \coprod_{H \in \mathrm{Arr}(G)/G} S_H^{\mathrm{ab}}$, where S_H^{ab} lies in the component $C_G(H)$ of G^{ab}. Since S generates G, S^{ab} generates G^{ab}, thus S_H^{ab} generates $C_G(H)$. □

Definition 2.37. *Let G be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. A reflection $s \in G$ is said to be* distinguished *with respect to G if $\det(s) = \exp\left(\frac{2\pi i}{d}\right)$ where $d = |C_G(H_s)|$.*

In particular, if H is a reflecting hyperplane for a reflection of G, every $C_G(H)$ is generated by a single distinguished reflection.

The next property has been noticed by Nebe ([Ne], §5), as a consequence of [Co, (1.8) & (1.9)].

Corollary 2.38. *Let S be a generating set of distinguished reflections for G. Then any distinguished reflection of G is conjugate to an element of S.*

Proof. It follows from 2.36 and from the fact that the conjugate of a distinguished reflection is still distinguished. □

3. Root Systems

Notation and conventions.

From now on, the following notation will be in force.

The field k is a number field, stable by the complex conjugation denoted $\lambda \mapsto \lambda^*$. Its ring of integers is $\mathbb{Z}_k$, a Dedekind domain. A *fractional ideal* is a finitely generated $\mathbb{Z}_k$-submodule of k. Denote by $\lambda\mathbb{Z}_k$ the (principal) fractional ideal generated by $\lambda \in k$.

For $\mathfrak{a}$ a fractional ideal, we set

$$\mathfrak{a}^{-1} := \{b \in k \mid b\mathfrak{a} \subset \mathbb{Z}_k\}, \text{ and } \mathfrak{a}^{-*} := (\mathfrak{a}^{-1})^* .$$

Since $\mathbb{Z}_k$ is Dedekind, $\mathfrak{a}\mathfrak{a}^{-1} = 1\mathbb{Z}_k$ and $(\lambda\mathbb{Z}_k)^{-1} = \lambda^{-1}\mathbb{Z}_k$ for $\lambda \in k$.

Throughout, (V, W) is a pair of finite dimensional k-vector spaces with a given Hermitian pairing (see Subsection 2.1)

$$V \times W \to k : (v, w) \mapsto \langle v, w\rangle .$$

For I a finitely generated $\mathbb{Z}_k$-submodule of V and J a finitely generated $\mathbb{Z}_k$-submodule of W, we denote by $\langle I, J\rangle$ the fractional ideal generated by all $\langle \alpha, \beta\rangle$ for $\alpha \in I$ and $\beta \in J$.

Let I be a rank one finitely generated $\mathbb{Z}_k$-submodule of V, generating the line kI in V. Then whenever v is a nonzero element of kI, there

is a fractional ideal $\mathfrak{a}$ of $\mathbb{Z}_k$ such that $I = \mathfrak{a}v$. If, similarly, $J = \mathfrak{b}w$ for some fractional ideal $\mathfrak{b}$ and some $w \in kJ$, then

$$\langle I, J\rangle = \mathfrak{a}\mathfrak{b}^*\langle v, w\rangle\,.$$

3.1. $\mathbb{Z}_k$-roots.

Definition 3.1.

(1) A $\mathbb{Z}_k$-root *(for (V, W)) is a triple $\mathfrak{r} = (I, J, \zeta)$ where*
- *I is a rank one finitely generated $\mathbb{Z}_k$-submodule of V,*
- *J is a rank one finitely generated $\mathbb{Z}_k$-submodule of W,*
- *ζ is a nontrivial root of unity in k,*

such that $\langle I, J\rangle = (1-\zeta)\mathbb{Z}_k$, the principal ideal generated by $1-\zeta$.

A $\mathbb{Z}_k$-root $\mathfrak{r} = (I, J, \zeta)$ is called a $(\zeta, \mathbb{Z}_k)$-root.

(2) If $\mathfrak{r} = (I, J, \zeta)$ is a $\mathbb{Z}_k$-root, and $\mathfrak{a}$ is a fractional ideal, we set

$$\mathfrak{a}\cdot\mathfrak{r} := (\mathfrak{a}I, \mathfrak{a}^{-*}J, \zeta)\,.$$

Two $\mathbb{Z}_k$-roots $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are said to be of the same genus *if there exists a fractional ideal $\mathfrak{a}$ such that*

$$\mathfrak{r}_2 = \mathfrak{a}\cdot\mathfrak{r}_1\,.$$

The group $\mathrm{GL}(V)$ acts on left on the set of $\mathbb{Z}_k$-roots, as follows: for $g \in \mathrm{GL}(V)$ and $\mathfrak{r} = (I, J, \zeta)$ a $\mathbb{Z}_k$-root, set

$$g\cdot\mathfrak{r} := (g(I), g^\vee(J), \zeta).$$

In particular $\lambda \in k^\times \subset Z\mathrm{GL}(V)$ acts by $\lambda\cdot\mathfrak{r} = (\lambda I, \lambda^{-*}J, \zeta)$. The action of $k^\times\mathrm{Id}_V = Z\mathrm{GL}(V)$ preserves genera.

Remark 3.2. The pair (I, J) does not determine ζ.

Indeed one may have an equality of ideals $(1-\zeta)\mathbb{Z}_k = (1-\xi)\mathbb{Z}_k$ without ζ and ξ having even the same order. For example, as soon as ζ has a composite order, $1-\zeta$ is invertible and so $(1-\zeta)\mathbb{Z}_k = \mathbb{Z}_k$ (see Lemma A.3 in Appendix A).

Given a $\mathbb{Z}_k$-root $\mathfrak{r} = (I, J, \zeta)$, choose $v \in kI$ and $w \in kJ$ such that $\langle v, w\rangle = 1-\zeta$. Then the formula

$$x \mapsto x - \langle x, w\rangle v$$

defines a reflection independent of the choice of v, since it is also the reflection attached to the reflection triple (kI, kJ, ζ). We will denote by $s_\mathfrak{r}$ this reflection.

Definition 3.3.

(1) If s is a reflection, an $(s, \mathbb{Z}_k)$-root *is a $\mathbb{Z}_k$-root (I, J, ζ) where $(kI, kJ, \zeta) = (L_s, M_s, \zeta_s)$.*

(2) If $\mathfrak{r} = (I, J, \zeta)$ is a $\mathbb{Z}_k$-root for (V, W), we call $\mathfrak{r}^\vee = (J, I, \zeta)$ — a $\mathbb{Z}_k$-root for (W, V) — the dual root.

Notice that the dual of an $(s, \mathbb{Z}_k)$-root is an $(s^\vee, \mathbb{Z}_k)$-root. Thus $s_{\mathfrak{r}^\vee} = s_{\mathfrak{r}}^\vee$.

Lemma 3.4.

(1) Given a $\mathbb{Z}_k$-root $\mathfrak{r} = (I, J, \zeta)$, given $v \in kI$ and $w \in kJ$ such that $\langle v, w\rangle = 1 - \zeta$, there exists a fractional ideal $\mathfrak{a}$ such that $I = \mathfrak{a}v$ and $J = \mathfrak{a}^{-*}w$.

(2) For any $\mathbb{Z}_k$-root $\mathfrak{r} = (I, J, \zeta)$, there exists a unique reflection s in $\mathrm{GL}(V)$ such that $\mathfrak{r}$ is an $(s, \mathbb{Z}_k)$-root.

(3) For any reflection s in $\mathrm{GL}(V)$, the set of $(s, \mathbb{Z}_k)$-roots form a single genus of roots.

Proof. (1) and (2) are clear. Let us prove (3). Let s be a reflection.

Choose $v \in L_s$ and $w \in M_s$ such that $\langle v, w\rangle = 1 - \zeta_s$. For $\mathfrak{a}$ any fractional ideal, define $I := \mathfrak{a}v, J := \mathfrak{a}^{-*}w$. Then $\mathfrak{r} = (I, J, \zeta_s)$ is an $(s, \mathbb{Z}_k)$-root.

Let now $\mathfrak{r}'$ be a root giving rise to the same reflection triple. Then $\mathfrak{r}' = (\mathfrak{b}v, \mathfrak{b}^{-*}w, \zeta)$ for some fractional ideal. We have $\mathfrak{r}' = \mathfrak{b}\mathfrak{a}^{-1} \cdot \mathfrak{r}$ thus $\mathfrak{r}$ and $\mathfrak{r}'$ are in the same genus. □

Remark 3.5. Given a reflection s and an $(s, \mathbb{Z}_k)$-root $\mathfrak{r} = (I, J, \zeta)$, Lemma 3.4, (1) ensures that J is determined by I (and similarly I is determined by J).

Pairing between $\mathbb{Z}_k$-roots.

Let $\mathfrak{r}_1 = (I_1, J_1, \zeta_1)$ and $\mathfrak{r}_2 = (I_2, J_2, \zeta_2)$ be two $\mathbb{Z}_k$-roots. There is a pairing on the set of $\mathbb{Z}_k$-roots, defined to be the fractional ideal:

$$\mathfrak{n}(\mathfrak{r}_1, \mathfrak{r}_2) := \langle I_1, J_2\rangle .$$

If $\mathfrak{r} = (I, J, \zeta)$, then by definition we have $\mathfrak{n}(\mathfrak{r}, \mathfrak{r}) = (1 - \zeta)\mathbb{Z}_k$.

Principal $\mathbb{Z}_k$-roots.

Let I be a rank one $\mathbb{Z}_k$-submodule of V. The reader will easily check that the following assertions are equivalent:

(i) I is a free $\mathbb{Z}_k$-module (hence of rank 1),

(ii) whenever $v \in kI$ and $\mathfrak{a}$ is a fractional ideal of k such that $I = \mathfrak{a}v$, then $\mathfrak{a}$ is a principal ideal.

This implies the following result:

Lemma–Definition 3.6. *Let $\mathfrak{r} = (I, J, \zeta)$ be a $\mathbb{Z}_k$-root. The following assertions are equivalent:*

(i) I is a free $\mathbb{Z}_k$-module (hence of rank 1),

(ii) J is a free $\mathbb{Z}_k$-module (hence of rank 1).

If the preceding properties are true, we say that the root $\mathfrak{r}$ is a principal $\mathbb{Z}_k$-root.

Remark 3.7. If $\mathfrak{r} = (I, J, \zeta)$ is a principal $\mathbb{Z}_k$-root, we may choose $\alpha \in kI$ and $\beta \in kJ$ such that $I = \mathbb{Z}_k\alpha$, $J = \mathbb{Z}_k\beta$ and $\langle \alpha, \beta \rangle = 1 - \zeta$. The vector α is then unique up to multiplication by a unit of $\mathbb{Z}_k$, and it determines β (and conversely).

3.2. $\mathbb{Z}_k$-root systems.

Definition and first properties.

The following definition is modeled on that of Bourbaki [BouLie, chap. VI, §1, *Définition*].

Definition 3.8. *Let $\mathfrak{R} = \{\mathfrak{r} = (I_\mathfrak{r}, J_\mathfrak{r}, \zeta_\mathfrak{r})\}$ be a set of $\mathbb{Z}_k$-roots. We say that $\mathfrak{R}$ is a $\mathbb{Z}_k$-*root system *if it satisfies the following conditions:*

(RS$_I$): *$\mathfrak{R}$ is finite, and the family $(I_\mathfrak{r})_{\mathfrak{r}\in\mathfrak{R}}$ generates V,*
(RS$_{II}$): *Whenever $\mathfrak{r} \in \mathfrak{R}$, we have $s_\mathfrak{r} \cdot \mathfrak{R} = \mathfrak{R}$,*
(RS$_{III}$): *Whenever $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathfrak{R}$, we have $\mathfrak{n}(\mathfrak{r}_1, \mathfrak{r}_2) \subseteq \mathbb{Z}_k$.*

In particular, in the case when $\mathbb{Z}_k = \mathbb{Z}$, the root datum above is equivalent to that required for a root system as defined in *loc.cit.* (see Remark 3.10).

If G is any of the 34 exceptional reflection groups of the classification of finite irreducible complex reflection groups, and $k = \mathbb{Q}_G$ is the field of definition of G, then $\mathbb{Z}_k$ is known to be a principal ideal domain [Ne].

Principal $\mathbb{Z}_k$-root systems.

If $\mathbb{Z}_k$ is a principal ideal domain, all $\mathbb{Z}_k$-roots are principal.

Definition 3.9. *A $\mathbb{Z}_k$-root system is* principal *if all its roots are principal.*

Remark 3.7 implies that a principal $\mathbb{Z}_k$-root may be viewed as a triple (A, B, ζ) where

- ζ is a root of unity,
- $A = \mathbb{Z}_k^\times\alpha$ and $B = \mathbb{Z}_k^\times\beta$, where α and β are nonzero elements of V and W respectively, and
- $\langle \alpha, \beta \rangle = 1 - \zeta$.

Such a triple r defines the unique reflection s_r with reflecting line kA and reflecting hyperplane the orthogonal of kB.

Thus a principal $\mathbb{Z}_k$-root system may be viewed as a set R of triples $r = (A_r, B_r, \zeta_r)_{r\in R}$ such that

(RS$_I$) R is finite and the family $(A_r)_{r\in R}$ generates V,
(RS$_{II}$) Whenever $r \in R$, we have $s_r \cdot R = R$,
(RS$_{III}$) Whenever $r_1 = (A_1, B_1, \zeta_1) \in R$ and $r_2 = (A_2, B_2, \zeta_2) \in R$, for $\alpha_1 \in A_1$ and $\beta_2 \in B_2$, we have $\langle \alpha_1, \beta_2 \rangle \in \mathbb{Z}_k$.

Remark 3.10. If $\mathbb{Z}_k = \mathbb{Z}$ (which implies that G is a Weyl group), the previous definition coincides with the usual definition of root system attached to G: let $\mathfrak{R}_0$ be a root system in the Bourbaki sense, then

$$\mathfrak{R} := \{(\mathbb{Z}\alpha, \mathbb{Z}\alpha^\vee, -1) \mid \alpha \in \mathfrak{R}_0\}$$

is a $\mathbb{Z}$-root system in our sense. Notice that the cardinality of $\mathfrak{R}_0$ is twice that of $\mathfrak{R}$ as Bourbaki has distinct roots $\pm\alpha$, which give rise to a single $\mathbb{Z}$-root.

Remark 3.11. Nebe's definition of a *reduced k-root system for G* (see [Ne, Def.19]) coincides with our definition of *distinguished principal $\mathbb{Z}_k$-root system for G* (see Definition 3.24 below).

Reflections and integrality results.

We return to the general case, where $\mathbb{Z}_k$ need not be a P.I.D.

Lemma 3.12. *Given $\mathbb{Z}_k$-roots $\mathfrak{r}_1 = (I_1, J_1, \zeta_1)$ and $\mathfrak{r}_2 = (I_2, J_2, \zeta_2)$,*

(1) $(s_{\mathfrak{r}_1} - \mathrm{Id}_V)(I_2) \subset \mathfrak{n}(\mathfrak{r}_2, \mathfrak{r}_1) I_1$.
(2) If $\mathfrak{n}(\mathfrak{r}_2, \mathfrak{r}_1) \subset \mathbb{Z}_k$, then $(s_{\mathfrak{r}_1} - \mathrm{Id}_V)(I_2) \subset I_1$.
(3) Reciprocally, if $(s_{\mathfrak{r}_1} - \mathrm{Id}_V)(I_2) \subset I_1$, then $\mathfrak{n}(\mathfrak{r}_2, \mathfrak{r}_1) \subset \mathbb{Z}_k$.

Proof. Choose $(v_1, w_1) \in kI_{\mathfrak{r}_1} \times kJ_{\mathfrak{r}_1}$ such that $\langle v_1, w_1\rangle = 1 - \zeta_1$, and denote by $\mathfrak{a}_1$ the fractional ideal such that $I_{\mathfrak{r}_1} = \mathfrak{a}_1 v_1$ (and so $J_{\mathfrak{r}_1} = \mathfrak{a}_1^{-*} w_1$).

Similarly, choose $(v_2, w_2) \in kI_{\mathfrak{r}_2} \times kJ_{\mathfrak{r}_2}$ such that $\langle v_2, w_2\rangle = 1 - \zeta_2$, and denote by $\mathfrak{a}_2$ the fractional ideal such that $I_{\mathfrak{r}_2} = \mathfrak{a}_2 v_2$ (and so $J_{\mathfrak{r}_2} = \mathfrak{a}_2^{-*} w_2$).

Then, for all $a_2 \in \mathfrak{a}_2$,

$$(\star) \qquad s_{\mathfrak{r}_1}(a_2 v_2) = a_2 v_2 - \langle a_2 v_2, w_1\rangle v_1 \,.$$

In order to prove (1), write $1 = \sum_i y_i x_i$ for $x_i \in \mathfrak{a}_1$ and $y_i \in \mathfrak{a}_1^{-1}$. Then the above equality $(\star)$ may be rewritten

$$\begin{aligned} s_{\mathfrak{r}_1}(a_2 v_2) &= a_2 v_2 - \langle a_2 v_2, w_1\rangle (\sum_i y_i x_i) v_1 \\ &= a_2 v_2 - \sum_i \left(\langle a_2 v_2, y_i^* w_1\rangle x_i v_1\right) , \end{aligned}$$

and that last equality shows (1).

Part (2) follows from (1) and from the inclusion $\mathbb{Z}_k I_1 \subset I_1$.

Now assume that $(s_{\mathfrak{r}_1} - \mathrm{Id}_V)(I_{\mathfrak{r}_2}) \subset I_{\mathfrak{r}_1}$. Equality $(\star)$ shows that, for all $a_2 \in \mathfrak{a}_2$, $\langle a_2 v_2, w_1\rangle v_1 \in I_1$, *i.e.*, $\langle a_2 v_2, w_1\rangle = a_2 \langle v_2, w_1\rangle \in \mathfrak{a}_1$. This shows that $\mathfrak{a}_2 \langle v_2, w_1\rangle \subset \mathfrak{a}_1$, hence $\langle v_2, w_1\rangle \in \mathfrak{a}_2^{-1}\mathfrak{a}_1$ and $\langle \mathfrak{a}_2 v_2, \mathfrak{a}_1^{-*} w_1\rangle \subset \mathbb{Z}_k$, which is (3). □

Corollary 3.13. *Condition **(RS$_{III}$)** is equivalent to:*

$$\textbf{(RS}_{III}\textbf{)}' \text{ Whenever } \mathfrak{r}_1, \mathfrak{r}_2 \in \mathfrak{R}, \text{ we have } (s_{\mathfrak{r}_2} - \mathrm{Id}) I_{\mathfrak{r}_1} \subset I_{\mathfrak{r}_2}.$$

The group $G(\mathfrak{R})$.

Definition 3.14. *Given a set $\mathfrak{R}$ of $\mathbb{Z}_k$-roots, we denote by $G(\mathfrak{R})$ the subgroup of $\mathrm{GL}(V)$ generated by the family of reflections $(s_{\mathfrak{r}})_{\mathfrak{r}\in\mathfrak{R}}$.*

If $G(\mathfrak{R}) = G$, we say that $\mathfrak{R}$ is a $\mathbb{Z}_k$-root system with group G, or a $\mathbb{Z}_k$-root system for G.

Remark 3.15. By Definition 3.8, (RS_{II}), $g(\mathfrak{r}) \in \mathfrak{R}$ whenever $g \in G(\mathfrak{R})$ and $\mathfrak{r} \in \mathfrak{R}$.

Theorem 3.16.

(1) Let $\mathfrak{R}$ be a $\mathbb{Z}_k$-root system in V. Then $(V, G(\mathfrak{R}))$ is an essential finite reflection group – that is, $V^{G(\mathfrak{R})} = 0$.

(2) Let (V, G) be an essential finite reflection group. Then there exists a $\mathbb{Z}_k$-root system in V with group G.

Proof. (1) The set of reflections $S_{\mathfrak{R}} = \{s_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}\}$ is finite, by (RS_I); is saturated, by (RS_{II}); and generates $G(\mathfrak{R})$, by definition. Thus Proposition 2.17 applies, and $G(\mathfrak{R})$ is finite.

(2) If X is a finite spanning set of V, the (finite) set

$$\{g(x) \mid (x \in X)(g \in G)\}$$

generates a finitely generated $\mathbb{Z}_k$-submodule E of V, which generates V as a k-vector space, and which is G-stable. Since E is torsion free (and since $\mathbb{Z}_k$ is Dedekind), E is a lattice in V, namely there exists a family $E_1, \dots, E_r$ of rank one projective $\mathbb{Z}_k$-modules such that $E = E_1 \oplus \cdots \oplus E_r$. Notice that if L is any line in V, then $L \cap E \neq 0$. Indeed, since $kE = V$, we have $L \subset kE$, thus for each $x \in L, x \neq 0$, there is $m \in E$ and $\lambda, \mu \in \mathbb{Z}_k, \lambda\mu \neq 0$, such that $x = \frac{\lambda}{\mu} m$, so λm is a nonzero element of $L \cap E$.

Let W be a vector space with a Hermitian pairing with V (for example, the twisted dual ${}^*V$). For each $s \in \mathrm{Ref}(G)$, with reflecting line L_s, dual reflecting line M_s, and determinant ζ_s,

- set $I_s := L_s \cap E$ (so I_s is a rank one $\mathbb{Z}_k$-submodule of E), and
- denote by J_s the rank one $\mathbb{Z}_k$-submodule of W in M_s such that

$$\langle I_s, J_s \rangle = (1 - \zeta_s)\mathbb{Z}_k \,.$$

Then (I_s, J_s, ζ_s) is an $(s, \mathbb{Z}_k)$-root.

Denote by $\mathfrak{R}_E$ the set of all roots (I_s, J_s, ζ_s) for $s \in \mathrm{Ref}(G)$. It is clear that $G(\mathfrak{R}_E) = G$. It remains to show that $\mathfrak{R}_E$ is a $\mathbb{Z}_k$-root system.

($\mathbf{RS}_I$) : It is clear that $\mathfrak{R}_E$ is finite. Besides, since $V^G = 0$, we have $V = \sum_{s\in\mathrm{Ref}(G)} L_s$, hence the family $(I_s)_{s\in\mathrm{Ref}(G)}$ generates V.

($\mathbf{RS}_{II}$) : Let $g \in G$. Whenever $s \in \mathrm{Ref}(G)$, we have $g(L_s) = L_{{}^g s}$, hence $g(L_s \cap E) = L_{{}^g s} \cap E$, which shows that $g(I_s) = I_{{}^g s}$. It follows immediately that $\mathfrak{R}_E$ is stable under the action of G.

(RS$_{III}$) : Let $s_1, s_2 \in \mathrm{Ref}(G)$, and set $\mathfrak{r}_i := (I_i, J_i, \zeta_i)$ for $i = 1, 2$.

Since the image of $s_2 - \mathrm{Id}_v$ is the line L_{s_2}, and since $s_2 - \mathrm{Id}_V$ sends E to E, we see that

$$(s_2 - \mathrm{Id}_V)(I_1) \subset E \cap L_{s_2} = I_2 \,.$$

This condition is equivalent to **(RS$_{III}$)** by Corollary 3.13. □

Remark 3.17. If $\mathfrak{r}_1$ and $\mathfrak{r}_2$ belong to the same root system $\mathfrak{R}$ and $\mathfrak{n}(\mathfrak{r}_1, \mathfrak{r}_2) = 0$ then $s_{\mathfrak{r}_1} s_{\mathfrak{r}_2} = s_{\mathfrak{r}_2} s_{\mathfrak{r}_1}$ and $\mathfrak{n}(\mathfrak{r}_2, \mathfrak{r}_1) = 0$. Indeed, by item (1) of Proposition 2.23, which is applicable since $G(\mathfrak{R})$ is finite, the equality $\mathfrak{n}(\mathfrak{r}_1, \mathfrak{r}_2) = 0$ implies that the reflection triples defined by $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are orthogonal.

Let (V, G) be a finite reflection group on k, assumed to be essential (Definition 2.10). Let S be a set of reflections of G, and for each $s \in S$ let $\mathfrak{r}_s = (I_s, J_s, \zeta_s)$ be a $(s, \mathbb{Z}_k)$-root (see Definitions 3.3). We set $\mathcal{S} := \{\mathfrak{r}_s \mid s \in S\}$.

A root $\mathfrak{r}$ is said to be *distinguished* with respect to a $\mathbb{Z}_k$-root system $\mathfrak{R}$ if $s_{\mathfrak{r}}$ is distinguished with respect to $G(\mathfrak{R})$.

Proposition 3.18. *Assume that*

(a) the set S generates G,
(b) for each $s, t \in S$, the ideal $\langle I_s, J_t \rangle$ is integral.

Then

(1) the orbit $\mathfrak{R}$ of $\mathcal{S}$ under G is a $\mathbb{Z}_k$-root system, and $G = G(\mathfrak{R})$,
(2) if all elements of S are distinguished, then so are the elements of $\mathfrak{R}$, and the map $\mathfrak{R} \to \mathrm{Ref}(G)$, $\mathfrak{r} \mapsto s_{\mathfrak{r}}$ is a bijection onto the set of distinguished reflections of G, and
(3) if each element of $\mathcal{S}$ is principal, then $\mathfrak{R}$ is principal as well.

Proof. (1) The axiom (RS$_I$) follows from the fact that (V, G) is essential, and the axiom (RS$_{II}$) is trivial. Let us prove (RS$_{III}$).

Lemma 3.19. *Let s, t, u be reflections on V, with associated $\mathbb{Z}_k$-roots respectively $\mathfrak{r}_s = (I_s, J_s, \zeta_s)$, $\mathfrak{r}_t$, $\mathfrak{r}_u$. Then*

$$\langle I_{sts^{-1}}, J_u \rangle \subset \langle I_s, J_u \rangle + \langle I_t, J_s \rangle \langle I_s, J_u \rangle \,.$$

Proof of 3.19. By item (1) of Lemma 3.12, $s(I_t) \subset I_t + \langle I_t, J_s \rangle I_s$ hence $\langle I_{sts^{-1}}, J_u \rangle \subset \langle I_s, J_u \rangle + \langle I_t, J_s \rangle \langle I_s, J_u \rangle$. □

We return to the proof of Proposition 3.18. Say that a set T of reflections of G is *integral* if $s, t \in T$ implies $\langle I_s, J_t \rangle \in \mathbb{Z}_k$. The preceding lemma shows that, given any integral set T of reflections of G, then $T \cup \{sts^{-1} \mid s, t \in S\}$ is again an integral set of reflections of G. Thus the set of all conjugates of the elements of S is integral, which is axiom (RS$_{III}$).

(2) Since the elements of $\mathcal{S}$ are all distinguished, the assertion results from Corollary 2.36.

Part (3) is obvious. □

Case where $V = W$ and $\langle\text{-},\text{-}\rangle$ is positive.

Remark 3.20. If $V = W$ and $\langle\text{-},\text{-}\rangle$ is positive, a reflection $s \in G$ is determined by its root line and its determinant. Indeed, since s preserves $\langle\text{-},\text{-}\rangle$, we have $s = s^\vee$, so if s is associated to the reflection triple (L, M, ζ), we have $L = M$.

Lemma 3.21. *Assume $V = W$ and $\langle\text{-},\text{-}\rangle$ is positive.*

(1) Let $\mathfrak{r} = (I, J, \zeta)$ be a $\mathbb{Z}_k$-root. Then $J = (1-\zeta^*)\langle I, I\rangle^{-1} I$.
(2) Let $\mathfrak{r}, \mathfrak{r}'$ be two roots from a $\mathbb{Z}_k$-root system $\mathfrak{R}$. Then

$$\mathfrak{n}(\mathfrak{r}', \mathfrak{r})^* = \mathfrak{n}(\mathfrak{r}, \mathfrak{r})^* \mathfrak{n}(\mathfrak{r}', \mathfrak{r}')^{-1} \mathfrak{n}(\mathfrak{r}', \mathfrak{r}'^\vee) \mathfrak{n}(\mathfrak{r}, \mathfrak{r}^\vee)^{-1} \mathfrak{n}(\mathfrak{r}, \mathfrak{r}').$$

Proof.

(1) Choose $v \in kI, w \in kJ$ such that $\langle v, w\rangle = 1 - \zeta$. There is a fractional ideal $\mathfrak{a}$ such that $I = \mathfrak{a}v$ and $J = \mathfrak{a}^{-*}w$. By Remark 3.20 there exists $\lambda \in k$ such that $w = \lambda v$ thus $1 - \zeta = \lambda^*\langle v, v\rangle$ and $w = \frac{(1-\zeta^*)}{\langle v,v\rangle} v$. Thus

$$J = \mathfrak{a}^{-*}w = (1-\zeta^*)\frac{v}{\mathfrak{a}^*\langle v, v\rangle} = (1-\zeta^*)\frac{\mathfrak{a}v}{\mathfrak{a}\mathfrak{a}^*\langle v, v\rangle} = (1-\zeta^*)\frac{I}{\langle I, I\rangle}.$$

Part (2) follows from (1) observing that for a root $\mathfrak{r} = (I, J, \zeta)$ we have $\mathfrak{n}(\mathfrak{r}, \mathfrak{r}) = (1-\zeta)\mathbb{Z}_k$ and $\mathfrak{n}(\mathfrak{r}, \mathfrak{r}^\vee) = \langle I, I\rangle$. □

Remark 3.22. In the case where $k \subset \mathbb{R}$, and $\mathfrak{r} = (I, J, -1)$, $\mathfrak{r}' = (I', J', -1)$, Lemma 3.21 (2) reduces to $\mathfrak{n}(\mathfrak{r}', \mathfrak{r}) = \langle I', I'\rangle\langle I, I\rangle^{-1}\mathfrak{n}(\mathfrak{r}, \mathfrak{r}')$ which generalizes the case of finite Coxeter groups [BouLie, Chap 6, §1, no. 1.1. formula (9)].

Some properties of a root system.

We return to the general case, where W need not be the same as V.

Let $\mathfrak{R}$ be a $\mathbb{Z}_k$-root system. Recall that $G(\mathfrak{R})$ (or simply G) denotes the group generated by the reflections defined by the elements of $\mathfrak{R}$.

Proposition 3.23. *Let $\mathfrak{R}$ be a $\mathbb{Z}_k$-root system. Then, for any reflecting hyperplane H of $G(\mathfrak{R})$, the fixator of H, $C_{G(\mathfrak{R})}(H)$, is generated by the set of reflections $s_\mathfrak{r}$ (where $\mathfrak{r} \in \mathfrak{R}$) with reflecting hyperplane H.*

Proof. It is a consequence of Corollary 2.36. Indeed, it suffices to notice (see Remark 3.15) that, for $\mathfrak{r} \in \mathfrak{R}$ and $g \in G(\mathfrak{R})$, then $g s_\mathfrak{r} g^{-1} = s_{g(\mathfrak{r})}$ and $g(\mathfrak{r}) \in \mathfrak{R}$. □

Definition 3.24. *Let $\mathfrak{R}$ be a $\mathbb{Z}_k$-root system.*

(1) We say that $\mathfrak{R}$ is reduced *if the map $\mathfrak{r} \mapsto s_\mathfrak{r}$ is injective.*
(2) We say that $\mathfrak{R}$ is complete *if the map $\mathfrak{r} \mapsto s_\mathfrak{r}$ is surjective onto* $\mathrm{Ref}(G(\mathfrak{R}))$.
(3) We say that $\mathfrak{R}$ is distinguished *if*
  (a) it consists of distinguished roots, and
  (b) it is reduced.

Remarks 3.25.

(1) If all $s_{\mathfrak{r}}$ have order 2 (for example, the real reflection groups and the infinite family $G(e,e,r)$) then:
   - every distinguished root system is complete (and reduced), and
   - every complete and reduced root system is distinguished.

(2) In a reduced root system, distinct roots have different genus.

Proposition 3.26. *Let $\mathfrak{R}$ be a distinguished $\mathbb{Z}_k$-root system. Then the map $\mathfrak{r} \mapsto s_{\mathfrak{r}}$ is a bijection from $\mathfrak{R}$ onto the set of distinguished reflections of $G(\mathfrak{R})$.*

Proof. It suffices to prove that, whenever H is a reflecting hyperplane of $G(\mathfrak{R})$, there exists $\mathfrak{r} \in \mathfrak{R}$ such that $s_{\mathfrak{r}}$ is the distinguished reflection of $C_{G(\mathfrak{R})}(H)$. This results from 3.23. □

The following lemma follows the lines of [Ne, Remark 20].

Proposition 3.27. *Let (V,G) be a k-reflection group. Let $\mathfrak{R}$ be a reduced and complete $\mathbb{Z}_k$-root system (resp. a distinguished $\mathbb{Z}_k$-root system) with respect to G. Assume that $\mathfrak{R}_1, \dots, \mathfrak{R}_m$ are the orbits of G on $\mathfrak{R}$.*

Then any other reduced and complete $\mathbb{Z}_k$-root system (resp. distinguished $\mathbb{Z}_k$-root system) with group G is of the form $\mathfrak{a}_1\mathfrak{R}_1 \cup \cdots \cup \mathfrak{a}_m\mathfrak{R}_m$ where $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ are some fractional ideals.

Proof. We give the proof for reduced and complete systems (the proof for the distinguished ones is similar).

Let $\mathfrak{R}'$ be another reduced and complete root system with respect to G. If s is a reflection of G, and if $\mathfrak{r}_s \in \mathfrak{R}$ and $\mathfrak{r}'_s \in \mathfrak{R}'$ are the roots associated with s, we know by Lemma 3.4 that there exists a fractional ideal $\mathfrak{a}_s \neq 0$ such that $\mathfrak{r}'_s = \mathfrak{a}_s \cdot \mathfrak{r}_s$.

Choose i such that $1 \leq i \leq m$ and choose a reflection s such that $\mathfrak{r}_s \in \mathfrak{R}_i$. For $g \in G$, $g(\mathfrak{r}_s)$ is a $\mathbb{Z}_k$-root attached to $g(s)$, hence by hypothesis we have $g(\mathfrak{r}_s) = \mathfrak{r}_{gsg^{-1}}$. Similarly, $g(\mathfrak{r}'_s) = \mathfrak{r}'_{gsg^{-1}}$. Hence we see that $\mathfrak{a}_s = \mathfrak{a}_{g(s)}$, thus $\mathfrak{a}_s$ depends only on i. We set $\mathfrak{a}_i := \mathfrak{a}_s$. This shows that if $\mathfrak{R}_1, \dots, \mathfrak{R}_m$ are the orbits of G on $\mathfrak{R}$, then for each $i = 1, \dots, m$, there exists a fractional ideal $\mathfrak{a}_i$ such that $\mathfrak{a}_1\mathfrak{R}_1, \dots, \mathfrak{a}_m\mathfrak{R}_m$ are the orbits of G on $\mathfrak{R}'$. □

Distinguishing and completing root systems.

Proposition 3.28. *Any $\mathbb{Z}_k$-root system $\mathfrak{R}$ contains a reduced subsystem with group $G(\mathfrak{R})$.*

Proof. Let $[\mathrm{Arr}(G(\mathfrak{R}))/G(\mathfrak{R})]$ be a complete set of representatives of orbits of $G(\mathfrak{R})$ on the set $\mathrm{Arr}(G(\mathfrak{R}))$ of reflecting hyperplanes of $G(\mathfrak{R})$. For each $H \in [\mathrm{Arr}(G(\mathfrak{R}))/G(\mathfrak{R})]$, let us set

$$\mathrm{Ref}_{\mathfrak{R}}(H) := \{s_{\mathfrak{r}} \mid (\mathfrak{r} \in \mathfrak{R})\,(s_{\mathfrak{r}} \in C_{G(\mathfrak{R})}(H))\}\,.$$

By Corollary 2.36, we know that $\mathrm{Ref}_{\mathfrak{R}}(H)$ generates $C_{G(\mathfrak{R})}(H)$. For each $H \in [\mathrm{Arr}(G(\mathfrak{R}))/G(\mathfrak{R})]$, choose a subset $\mathrm{Ref}^0_{\mathfrak{R}}(H)$ of $\mathrm{Ref}_{\mathfrak{R}}(H)$ which is minimal subject to being a generating subset of $C_{G(\mathfrak{R})}(H)$. For each element of $\mathrm{Ref}^0_{\mathfrak{R}}(H)$, we choose a corresponding $\mathfrak{r} \in \mathfrak{R}$, so we get a set $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}$ of elements of $\mathfrak{R}$.

Define $\mathfrak{R}'$ to be the union of the $G(\mathfrak{R})$-orbits of $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}$. We claim that $\mathfrak{R}'$ is a reduced root system with group $G(\mathfrak{R})$.

It is clear that $\mathfrak{R}'$ is a root system with group $G(\mathfrak{R})$. Let us prove that $\mathfrak{R}'$ is reduced. Suppose that $\mathfrak{r}$ and $g \cdot \mathfrak{r} = \mathfrak{a} \cdot \mathfrak{r}$ are in $\mathfrak{R}'$ for $g \in G(\mathfrak{R})$ for some fractional ideal $\mathfrak{a}$, then $g^n \cdot \mathfrak{r} = \mathfrak{a}^n \cdot \mathfrak{r} = \mathfrak{r}$ for some n which shows that $\mathfrak{a}^n = \mathbb{Z}_k$ which implies (since $\mathbb{Z}_k$ is a Dedekind domain, hence fractional ideals have a unique decomposition in prime ideals) that $\mathfrak{a} = \mathbb{Z}_k$. □

Proposition 3.29. *Let $\mathfrak{R}$ be a complete $\mathbb{Z}_k$-root system. Then $\mathfrak{R}$ contains a distinguished subsystem with group $G(\mathfrak{R})$.*

Proof. Denote by $\mathfrak{R}_0$ the set of distinguished roots of $\mathfrak{R}$. Since $\mathfrak{R}$ is complete, any distinguished reflection of $G(\mathfrak{R})$ is of the form $s_{\mathfrak{r}}$ for $\mathfrak{r} \in \mathfrak{R}_0$.

Thus the set of reflecting lines of $\mathfrak{R}_0$ is the same as the set of reflecting lines of $\mathfrak{R}$, and this proves that condition (RS_I) (see Definition 3.8) is satisfied for $\mathfrak{R}_0$.

Condition (RS_{III}) on $\mathfrak{R}_0$ is inherited from $\mathfrak{R}$.

It remains to check that $\mathfrak{R}_0$ is stable under $G(\mathfrak{R}_0)$. Notice that $G(\mathfrak{R}_0) = G(\mathfrak{R})$. Thus (RS_{II}) follows from the fact that the image of a distinguished root by an element of $G(\mathfrak{R}_0)$ is still distinguished.

Now apply Proposition 3.28 to get a reduced subsystem of $\mathfrak{R}_0$, and we get a distinguished root system. □

Let $\mathfrak{r} = (I_{\mathfrak{r}}, J_{\mathfrak{r}}, \zeta)$ be a $\mathbb{Z}_k$-root with $\zeta = \exp(\frac{2\pi i}{d})$ with $d > 2$. For $1 \leq i < d$, we denote by $\mathfrak{r}^i$ the root:

$$\mathfrak{r}^i := \left(\frac{1-\zeta^i}{1-\zeta} I_{\mathfrak{r}},\ J_{\mathfrak{r}},\ \zeta^i \right),$$

which has the property that $s^i_{\mathfrak{r}} = s_{\mathfrak{r}^i}$.

An incomplete root system can be augmented by adjoining all of the $\mathfrak{r}^i$ to obtain a complete root system.

Proposition 3.30. *Let $\mathfrak{R}$ be a distinguished $\mathbb{Z}_k$-root system for V. Denote by $\widehat{\mathfrak{R}}$ the set of roots obtained by adjoining all roots of the form $\mathfrak{r}^i$ to $\mathfrak{R}$. Then $\widehat{\mathfrak{R}}$ is a complete reduced $\mathbb{Z}_k$-root system for V, whose set of distinguished roots is $\mathfrak{R}$.*

Proof. Since $\mathfrak{R}$ is finite and only a finite number of roots are added to form $\widehat{\mathfrak{R}}$, the condition $(\mathbf{RS}_I)$ is immediately satisfied for $\widehat{\mathfrak{R}}$.

Let us check $(\mathbf{RS}_{II})$. Since all the reflections $s'_{\mathfrak{r}}$ for $\mathfrak{r}' \in \widehat{\mathfrak{R}}$ belong to $G(\mathfrak{R})$, it suffices to check that $\widehat{\mathfrak{R}}$ is stable under all $s_{\mathfrak{r}}$ for $\mathfrak{r} \in \mathfrak{R}$. This results from the fact that if $s(\mathfrak{r}_1) = \mathfrak{r}_2$ then $s(\mathfrak{r}_1^i) = \mathfrak{r}_2^i$.

Finally, consider two roots $\mathfrak{r}_1^i$ and $\mathfrak{r}_2^j$ in $\widehat{\mathfrak{R}}$, where $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are in $\mathfrak{R}$. Then

$$\mathfrak{n}(\mathfrak{r}_1^i, \mathfrak{r}_2^j) = \left\langle \left(\frac{1-\zeta_1^i}{1-\zeta_1}\right) I_{\mathfrak{r}_1}, J_{\mathfrak{r}_2} \right\rangle = \left(\frac{1-\zeta_1^i}{1-\zeta_1}\right) \langle I_{\mathfrak{r}_1}, J_{\mathfrak{r}_2} \rangle \ .$$

Since $\mathfrak{n}(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{Z}_k$ by $(\mathbf{RS}_{III})$ for $\mathfrak{R}$, and since $\frac{1-\zeta_1^i}{1-\zeta_1}$ is always integral, $\mathfrak{n}(\mathfrak{r}_1^i, \mathfrak{r}_2^j) \in \mathbb{Z}_k$ as well, verifying the last condition $(\mathbf{RS}_{III})$ for $\widehat{\mathfrak{R}}$.

The system $\widehat{\mathfrak{R}}$ is reduced since $s_{\mathfrak{r}^i} = s_{\mathfrak{r}}^i$. □

Dual root system, irreducible root systems.

Recall (see Definitions 3.3(2)) that if $\mathfrak{r} = (I, J, \zeta)$ is a $\mathbb{Z}_k$-root in V, its *dual root* is the $\mathbb{Z}_k$-root in W defined by $\mathfrak{r}^\vee := (J, I, \zeta)$.

Lemma–Definition 3.31. *If $\mathfrak{R}$ is a $\mathbb{Z}_k$-root system in V, the set*

$$\mathfrak{R}^\vee := \{\mathfrak{r}^\vee \mid \mathfrak{r} \in \mathfrak{R}\}$$

is a $\mathbb{Z}_k$-root system in W, called the dual root system *of $\mathfrak{R}$.*

Proof. The fact that $\mathfrak{R}^\vee$ is a $\mathbb{Z}_k$-root system follows directly from the definition and the equality:

$$\mathfrak{n}(\mathfrak{r}_1^\vee, \mathfrak{r}_2^\vee) = \langle J_1, I_2 \rangle = \langle I_2, J_1 \rangle^* = \mathfrak{n}(\mathfrak{r}_2, \mathfrak{r}_1)^* \ .$$

□

Recall (Definition 2.12 (2)) that a set of reflections is *irreducible* if it consists of a single equivalence class with respect to the closure of the "is not orthogonal" relation $\sim$.

Definition 3.32. *Let $\mathfrak{R}$ be a set of $\mathbb{Z}_k$-roots, and $S_{\mathfrak{R}}$ be the corresponding set of reflections. Then $\mathfrak{R}$ is said to be* irreducible *if $S_{\mathfrak{R}}$ is irreducible.*

So a set of roots $\mathfrak{R}$ is irreducible if for every pair of roots $\mathfrak{r}$ and $\mathfrak{r}'$, there is a sequence $\mathfrak{r} = \mathfrak{r}_{i_0}, \mathfrak{r}_{i_1}, \ldots, \mathfrak{r}_{i_p} = \mathfrak{r}'$ such that each adjacent pair of roots in the sequence is not orthogonal – that is, $\mathfrak{n}(\mathfrak{r}_{i_j}, \mathfrak{r}_{i_j+1}) \neq 0$.

3.3. **Root systems and parabolic subgroups.**

In this subsection, $\mathfrak{R}$ denotes a $\mathbb{Z}_k$-root system, and $G := G(\mathfrak{R})$.

Let F be a flat of G in V, that is, an intersection of reflecting hyperplanes of G in V. Recall that we denote by $\mathrm{Arr}_F(G)$ the family of all reflecting hyperplanes of G containing F, so that $F = \bigcap_{H \in \mathrm{Arr}_F(G)} H$.

For $H \in \mathrm{Arr}_F(G)$, we denote by L_H the reflecting line in V attached to H (see Proposition 2.22), by M_H the orthogonal of H in W (a dual reflecting line for G in W), and by K_H the corresponding dual reflecting hyperplane in W.

We set

$$V_F := \sum_{H \in \mathrm{Arr}_F(G)} L_H \text{ and } W_F := \sum_{H \in \mathrm{Arr}_F(G)} M_H .$$

The Hermitian pairing between V and W restricts to a Hermitian pairing between V_F and W_F.

Let $C_G(F)$ be the corresponding parabolic subgroup of G, the fixator of F. We recall (see Theorem 2.32 above) that

- $C_G(F)$ is generated by those reflections whose reflecting hyperplanes belong to $\mathrm{Arr}_F(G)$,
- F is the set of fixed points of $C_G(F)$ in V,

and $C_G(F)$ is naturally identified with a subgroup of $\mathrm{GL}(V_F)$ generated by reflections.

Let $\mathfrak{r} = (I, J, \zeta) \in \mathfrak{R}$ such that $s_\mathfrak{r} \in C_G(F)$. Then $s_\mathfrak{r}$ is a reflection in its action on V_F, and $\mathfrak{r}$ may be viewed as a $\mathbb{Z}_k$-root for (V_F, W_F), since $I \subset V_F$ and $J \subset W_F$.

Proposition 3.33. *Let $\mathfrak{R}$ be a $\mathbb{Z}_k$-root system in V, and let F be a flat of $G(\mathfrak{R})$ in V.*

(1) The set

$$\mathfrak{R}_F := \{\mathfrak{r} \mid s_\mathfrak{r} \in C_G(F)\}$$

is a $\mathbb{Z}_k$-root system for the parabolic subgroup $C_G(F)$ viewed as a reflection group acting on V_F.

(2) If $\mathfrak{R}$ is complete then $\mathfrak{R}_F$ is a complete root system for $C_G(F)$.

(3) If $\mathfrak{R}$ is distinguished then $\mathfrak{R}_F$ is a distinguished root system for $C_G(F)$.

Proof. (1) It suffices to check that $\mathfrak{R}_F$ is stable under the action of $C_G(F)$. It is enough to check that for $\mathfrak{r} \in \mathfrak{R}_F$ and $t \in C_G(F)$, we have $t \cdot \mathfrak{r} \in \mathfrak{R}_F$. But $s_{t\cdot\mathfrak{r}} = t s_\mathfrak{r} t^{-1}$, which fixes F.

Completeness and distinguishedness (items (2) and (3)) are inherited directly from $\mathfrak{R}$. □

3.4. **Root lattices, root bases.**

Definition 3.34. *Let $\mathfrak{R} = \{\, \mathfrak{r} = (I_\mathfrak{r}, J_\mathfrak{r}, \zeta_\mathfrak{r}) \,\}$ be a $\mathbb{Z}_k$-root system.*

(1) The root lattice *$Q_\mathfrak{R}$ and the* coroot lattice *$Q^\vee_\mathfrak{R}$ are defined by*

$$Q_\mathfrak{R} := \sum_{\mathfrak{r} \in \mathfrak{R}} I_\mathfrak{r} \quad \textit{and} \quad Q^\vee_\mathfrak{R} := \sum_{\mathfrak{r} \in \mathfrak{R}} J_\mathfrak{r} .$$

(2) The weight lattice *$P_\mathfrak{R}$ and the* coweight lattice *$P^\vee_\mathfrak{R}$ are the dual of $Q^\vee_\mathfrak{R}$ and $Q_\mathfrak{R}$ respectively, i.e.,*

$$P_\mathfrak{R} := \{\, x \in V \mid \langle x, Q^\vee_\mathfrak{R} \rangle \subseteq \mathbb{Z}_k \}$$
$$P^\vee_\mathfrak{R} := \{\, y \in W \mid \langle y, Q_\mathfrak{R} \rangle \subseteq \mathbb{Z}_k \}$$

The following properties are straightforward.

- $Q_\mathfrak{R} \subseteq P_\mathfrak{R}$ and $Q^\vee_\mathfrak{R} \subseteq P^\vee_\mathfrak{R}$.

- $Q_{\mathfrak{R}^\vee} = Q_{\mathfrak{R}}^\vee$, $P_{\mathfrak{R}^\vee} = P_{\mathfrak{R}}^\vee$.

Definition 3.35. *The group of automorphisms of a $\mathbb{Z}_k$-root system $\mathfrak{R}$ denoted* $\mathrm{Aut}(\mathfrak{R})$*, is the group of all $g \in \mathrm{GL}(V)$ such that $g(\mathfrak{R}) = \mathfrak{R}$.*

In other words,

$$\mathrm{Aut}(\mathfrak{R}) = \{g \in \mathrm{GL}(V) \mid (I, J, \zeta) \in \mathfrak{R} \Rightarrow (g(I), g^\vee(J), \zeta) \in \mathfrak{R}\}\,.$$

If $g \in \mathrm{Aut}(\mathfrak{R})$, g conjugates the reflection $s_{\mathfrak{r}}$ defined by a root $\mathfrak{r} \in \mathfrak{R}$ to the reflection $s_{g(\mathfrak{r})}$ defined by $g(\mathfrak{r})$, hence

$$G(\mathfrak{R}) \lhd \mathrm{Aut}(\mathfrak{R})\,.$$

Proposition 3.36.

(1) The lattices $Q_{\mathfrak{R}}$, $P_{\mathfrak{R}}$, $Q_{\mathfrak{R}}^\vee$, $P_{\mathfrak{R}}^\vee$ are all $\mathrm{Aut}(\mathfrak{R})$-stable finitely generated projective $\mathbb{Z}_k$-submodules of V and W respectively.

(2) The group $G(\mathfrak{R})$ acts trivially on $P_{\mathfrak{R}}/Q_{\mathfrak{R}}$ and on $P_{\mathfrak{R}}^\vee/Q_{\mathfrak{R}}^\vee$, hence the group $\mathrm{Aut}(\mathfrak{R})/G(\mathfrak{R})$ acts on these quotients.

(3) The Hermitian pairing $V \times W \to k$ induces a non-degenerate pairing of $\mathbb{Z}_k(\mathrm{Aut}(\mathfrak{R})/G(\mathfrak{R}))$–modules

$$P_{\mathfrak{R}}/Q_{\mathfrak{R}} \times P_{\mathfrak{R}}^\vee/Q_{\mathfrak{R}}^\vee \longrightarrow k/\mathbb{Z}_k\,.$$

Proof. Assertion (1) is clear. Assertion (2) results from the following lemma, which gives an alternative description of the reflection associated with a $\mathbb{Z}_k$-root.

Lemma 3.37. *Let $\mathfrak{r} = (I, J, \zeta)$ be a $\mathbb{Z}_k$-root. Assume that $(\alpha_i)_{i\in E}$ and $(\beta_i)_{i\in E}$ are finite families of elements of I and J respectively such that*

$$\sum_{i\in E} \langle \alpha_i, \beta_i\rangle = 1 - \zeta\,.$$

Then, for all $v \in V$,

$$s_{\mathfrak{r}}(v) = v - \sum_{i\in E} \langle v, \beta_i\rangle \alpha_i\,.$$

Proof. For all $i \in E$, we have (see Proposition 2.4)

$$s_{\mathfrak{r}}(v) = v - \frac{\langle v, \beta_i\rangle}{\langle \alpha_i, \beta_i\rangle}(1-\zeta)\alpha_i\,, \text{ hence}$$
$$\langle \alpha_i, \beta_i\rangle s_{\mathfrak{r}}(v) = \langle \alpha_i, \beta_i\rangle v - \langle v, \beta_i\rangle(1-\zeta)\alpha_i\,.$$

Summing the last equality over E, then simplifying by $(1-\zeta)$, gives the expected formula. □

Now (with the same notation as in the above lemma), for all $v \in P_{\mathfrak{R}}$, $\langle v, \beta_i\rangle \in \mathbb{Z}_k$, hence $\langle v, \beta_i\rangle\alpha_i \in Q_{\mathfrak{R}}$, which shows that $s_{\mathfrak{r}}$ acts trivially on $P_{\mathfrak{R}}/Q_{\mathfrak{R}}$.

Assertion (3) is immediate. □

Notice also that for $\mathfrak{a}$ a fractional ideal, we have

$$Q_{\mathfrak{a}\cdot\mathfrak{R}} = \mathfrak{a}Q_{\mathfrak{R}} \ , \ Q^{\vee}_{\mathfrak{a}\cdot\mathfrak{R}} = \mathfrak{a}^{-*}Q^{\vee}_{\mathfrak{R}}$$
$$P_{\mathfrak{a}\cdot\mathfrak{R}} = \mathfrak{a}P_{\mathfrak{R}} \ , \ P^{\vee}_{\mathfrak{a}\cdot\mathfrak{R}} = \mathfrak{a}^{-*}P^{\vee}_{\mathfrak{R}}.$$

Definition 3.38. *Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two $\mathbb{Z}_k$-root systems.*

(1) Say that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are of the same genus *if there exists a fractional ideal $\mathfrak{a}$ such that*

$$\mathfrak{R}_2 = \mathfrak{a}\cdot\mathfrak{R}_1 := \{\, \mathfrak{a}\cdot\mathfrak{r}_1 \mid (\mathfrak{r}_1 \in \mathfrak{R}_1)\,\}.$$

(2) Say that $\mathfrak{R}$ and $\mathfrak{R}'$ are lattice equivalent *if there exists a fractional ideal $\mathfrak{a}$ such that $Q_{\mathfrak{R}'} = \mathfrak{a}Q_{\mathfrak{R}}$ and $Q^{\vee}_{\mathfrak{R}'} = \mathfrak{a}^{-*}Q^{\vee}_{\mathfrak{R}}$.*

Notice that if two root systems have the same genus, then they are lattice equivalent. The converse is false, as seen in the following example.

Example 3.39. Consider $k := \mathbb{Q}(\zeta_3)$, hence $\mathbb{Z}_k = \mathbb{Z}[\zeta_3]$ (a principal ideal domain). Set $W = V = k$ and $G := \boldsymbol{\mu}_3$ acting on V by multiplication. Now $(1-\zeta_3^2)\mathbb{Z}_k = (1-\zeta_3)\mathbb{Z}_k$, since $1-\zeta_3^2 = -\zeta(1-\zeta_3)$.

It is easily checked (see Section 4 below for the general case of the cyclic groups) that there are exactly *three* genera of complete root systems for G as follows: let $\mathfrak{p} = (1-\zeta_3)\mathbb{Z}_k = (1-\zeta_3^2)\mathbb{Z}_k$, then

$$\mathfrak{R}_{1,1} := \left\{ \left(\mathbb{Z}_k, \mathfrak{p}, \zeta_3\right), \left(\mathbb{Z}_k, \mathfrak{p}, \zeta_3^2\right) \right\}$$

$$\mathfrak{R}_{1,\mathfrak{p}} := \left\{ \left(\mathbb{Z}_k, \mathfrak{p}, \zeta_3\right), \left(\mathfrak{p}, \mathbb{Z}_k, \zeta_3^2\right) \right\}$$

$$\mathfrak{R}_{\mathfrak{p},1} := \left\{ \left(\mathfrak{p}, \mathbb{Z}_k, \zeta_3\right), \left(\mathbb{Z}_k, \mathfrak{p}, \zeta_3^2\right) \right\}$$

Now $Q_{\mathfrak{R}_{1,\mathfrak{p}}} = Q_{\mathfrak{R}_{\mathfrak{p},1}} = \mathbb{Z}_k$, and $P_{\mathfrak{R}_{1,\mathfrak{p}}} = P_{\mathfrak{R}_{\mathfrak{p},1}} = \mathbb{Z}_k$, so these two distinct (with respect to genus) root systems are in fact lattice equivalent.

Definition 3.40. *A subset Π of elements of $\mathfrak{R}$ is said to be*

- *a* set of root generators *if $Q_{\mathfrak{R}} = \sum_{\mathfrak{r}\in\Pi} I_{\mathfrak{r}}$*
- *a* root lattice basis *if $Q_{\mathfrak{R}} = \bigoplus_{\mathfrak{r}\in\Pi} I_{\mathfrak{r}}$*
- *a* root basis *if*
  (1) $Q_{\mathfrak{R}} = \bigoplus_{\mathfrak{r}\in\Pi} I_{\mathfrak{r}}$ and
  (2) the family $(s_{\mathfrak{r}})_{\mathfrak{r}\in\Pi}$ generates $G(\mathfrak{R})$.

Coroot generators, coroot lattice bases, coroot bases *are defined analogously.*

Example 3.41. As above, let us choose $k := \mathbb{Q}(\zeta_3)$, hence $\mathbb{Z}_k = \mathbb{Z}[\zeta_3]$, $V = W = k$ and $G := \boldsymbol{\mu}_3$ acting on V by multiplication. Then each of the root systems $\mathfrak{R}_{1,1}$, $\mathfrak{R}_{1,\mathfrak{p}}$, $\mathfrak{R}_{\mathfrak{p},1}$ contains a root basis – for example, $\Pi = \left\{\left(\mathbb{Z}_k, (1-\zeta_3)\mathbb{Z}_k, \zeta_3\right)\right\}$. Indeed, Π is both a root basis and a coroot basis of $\mathfrak{R}_{1,1}$. However $\mathfrak{R}_{1,\mathfrak{p}}$ and $\mathfrak{R}_{\mathfrak{p},1}$ do not contain a subset which is *simultaneously* a root basis and a coroot basis.

On the other hand, a distinguished root system for a well generated group always contains a subset which is simultaneously a root basis and a coroot basis:

Proposition 3.42. *Let $\mathfrak{R}$ be a distinguished $\mathbb{Z}_k$-root system. Let Π be a subset of $\mathfrak{R}$ such that $\{s_\mathfrak{r} \mid \mathfrak{r} \in \Pi\}$ generates $G(\mathfrak{R})$. Then*

(1) whenever $\mathfrak{r} \in \mathfrak{R}$, there exist $\mathfrak{r}_0, \mathfrak{r}_1, \dots, \mathfrak{r}_m \in \Pi$ such that $\mathfrak{r} = (s_{\mathfrak{r}_m} \cdots s_{\mathfrak{r}_1}) \cdot \mathfrak{r}_0$,
(2) Π is a set of root generators and a set of coroot generators,
(3) if Π consists of principal $\mathbb{Z}_k$-roots, then $\mathfrak{R}$ is principal,
(4) if $|\Pi| = \dim V$, then Π is a root basis and a coroot basis

Proof. Write G for $G(\mathfrak{R})$. Notice that it is enough to prove the results concerning roots: the ones concerning coroots follow by considering the contragredient operation $g \mapsto g^\vee$ of G on W.

(1) Let $\mathfrak{r} \in \mathfrak{R}$. By Corollary 2.38, $s_\mathfrak{r}$ is conjugate to some $s_{\mathfrak{r}_0}$ for $\mathfrak{r}_0 \in \Pi$. Thus there exist $\mathfrak{r}_1, \dots, \mathfrak{r}_m \in \Pi$ with $s_\mathfrak{r} = s_{\mathfrak{r}_m} \cdots s_{\mathfrak{r}_1} s_{\mathfrak{r}_0} s_{\mathfrak{r}_1}^{-1} \cdots s_{\mathfrak{r}_m}^{-1}$. Since $\mathfrak{R}$ is stable under G, and since $w s_{\mathfrak{r}_0} w^{-1} = s_{w.s_{\mathfrak{r}_0}}$, the above equality implies that $\mathfrak{r} = (s_{\mathfrak{r}_m} \cdots s_{\mathfrak{r}_1}) \cdot \mathfrak{r}_0$.

(2) Suppose that for $\mathfrak{r}, \mathfrak{r}_0, \dots, \mathfrak{r}_m \in \Pi$ are as in (1). By Lemma 3.12,

$$s_{\mathfrak{r}_1}(I_{\mathfrak{r}_0}) \subset I_{\mathfrak{r}_0} + I_{\mathfrak{r}_1} ,$$

so

$$s_{\mathfrak{r}_2} s_{\mathfrak{r}_1}(I_{\mathfrak{r}_0}) \subset s_{\mathfrak{r}_2}(I_{\mathfrak{r}_0}) + s_{\mathfrak{r}_2}(I_{\mathfrak{r}_1}) \subset I_{\mathfrak{r}_0} + I_{\mathfrak{r}_1} + I_{\mathfrak{r}_2} ,$$

and an iteration shows that

$$I_\mathfrak{r} = (s_{\mathfrak{r}_m} \cdots s_{\mathfrak{r}_1})(I_{\mathfrak{r}_0}) \subset I_{\mathfrak{r}_0} + I_{\mathfrak{r}_1} + \cdots + I_{\mathfrak{r}_m} .$$

Part (3) results from assertion (1) and from the remark that, for any $g \in \mathrm{GL}(V)$ and $\mathfrak{r}$ a principal $\mathbb{Z}_k$-root, $g(\mathfrak{r})$ is still principal.

Item (4) is clear. □

Proposition 3.43. *Let $\mathfrak{R}$ be a distinguished $\mathbb{Z}_k$-root system. If Π is a subset of $\mathfrak{R}$ such that $|\Pi| = \dim V$ and the family $(s_\mathfrak{r})_{\mathfrak{r}\in\Pi}$ generates $G(\mathfrak{R})$, then Π is a root basis and coroot basis of $\mathfrak{R}$.*

Proof. This results from Proposition 3.42 (4). □

Note that root bases only exist when $G(\mathfrak{R})$ is well generated.

Remark 3.44. When $\mathbb{Z}_k$ is not a P.I.D., a root basis does not necessarily provide a basis of $Q_\mathfrak{R}$ as a $\mathbb{Z}_k$-module. Nevertheless, we shall see later (Theorem 6.6, see also [Ne, Corollary 13]) that every reflection group has at least one principal $\mathbb{Z}_k$-root system, and the root lattice of a principal $\mathbb{Z}_k$-root system is always a *free $\mathbb{Z}_k$-module.*

3.5. **Example: the Weyl group of type B_2.**

Let k be a number field. Set $V = k^2$ with canonical basis $\{e_1, e_2\}$ and $W = k^2$ with canonical dual basis $\{f_1, f_2\}$. The Weyl group of type B_2, denoted G, may be considered to be the subgroup of $\mathrm{GL}(V)$ generated by $S = \{s, t\}$ where s and t are the automorphisms of V corresponding respectively to the following matrices on the basis $\{e_1, e_2\}$:

$$s := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The corresponding reflecting lines are

$$\begin{cases} L_s = kv_s \text{ with } v_s = e_1 & \text{and} \quad L_t = kv_t \text{ with } v_t = e_2 - e_1 , \\ M_s = kv_s^\vee \text{ with } v_s^\vee = 2f_1 & \text{and} \quad M_t = kv_t^\vee \text{ with } v_t^\vee = f_2 - f_1 . \end{cases}$$

The orbits under G of the following root bases (corresponding to generators s and t in that order) are $\mathbb{Z}_k$-root systems corresponding to the types B_2 and C_2 respectively:

$$\Pi(B_2) := \left\{ \left(\mathbb{Z}_k v_s, \mathbb{Z}_k v_s^\vee, -1\right), \left(\mathbb{Z}_k v_t, \mathbb{Z}_k v_t^\vee, -1\right) \right\} ,$$

$$\Pi(C_2) := \left\{ \left(2\mathbb{Z}_k v_s, \frac{1}{2}\mathbb{Z}_k v_s^\vee, -1\right), \left(\mathbb{Z}_k v_t, \mathbb{Z}_k v_t^\vee, -1\right) \right\} .$$

Swapping V and W, and s and t, defines an isomorphism between the coroot system of type B_2 and the root system of type C_2, and vice versa. We say that they are mutually dual root systems.

It is immediate to check that the element $\phi \in \mathrm{GL}(V)$ defined by

$$\phi : \begin{cases} e_1 \mapsto -e_1 + e_2 , \\ e_2 \mapsto e_1 + e_2 , \end{cases}$$

that is, the automorphism of V with matrix $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ on the basis (e_1, e_2), has the following properties:

(1) $\phi^2 = 2\mathrm{Id}_V$,
(2) it swaps s and t (by conjugation),
(3) it sends $\Pi(B_2)$ onto $\Pi(C_2)$ and $\Pi(C_2)$ onto $2\Pi(B_2)$, hence swaps $\mathfrak{R}(B_2)$ and $\mathfrak{R}(C_2)$, up to genus.

That is, the automorphism denoted ${}^2B_2$ of G swaps, up to genus, $\mathfrak{R}(B_2)$ and $\mathfrak{R}(C_2)$, which are thus isomorphic (up to genus).

Lemma 3.45.

(1) The following assertions are equivalent.
  (i) There exists a $\mathbb{Z}_k$-root system with group G which is stable by the automorphism ϕ (up to genus),
  (ii) there exists a principal ideal $\mathfrak{a}$ of $\mathbb{Z}_k$ such that $\mathfrak{a}^2 = 2\mathbb{Z}_k$.

(2) If $\mathfrak{a} = (\mathbb{Z}_k a)$ is such that $a^2 = 2u$ with $u \in \mathbb{Z}_k^\times$, we set:

$$\Pi_{\mathfrak{a}} := \left\{ \left(\mathfrak{a}\mathbb{Z}_k v_s, \mathfrak{a}^{-*}\mathbb{Z}_k v_s^\vee, -1\right), \left(\mathbb{Z}_k v_t, \mathbb{Z}_k v_t^\vee, -1\right) \right\},$$

and denote by $\mathfrak{R}_{\mathfrak{a}}$ the orbit of $\Pi_{\mathfrak{a}}$ under the group generated by the reflections $s_{\mathfrak{r}}$ for $\mathfrak{r} \in \Pi_{\mathfrak{a}}$. Then

(a) $\mathfrak{R}_{\mathfrak{a}}$ is a $\mathbb{Z}_k$-root system with group G,

(b) the flips between V and W and between s and t define an isomorphism between $\mathfrak{R}_{\mathfrak{a}}$ and its coroot system (thus, $\mathfrak{R}_{\mathfrak{a}}$ is "self-dual"),

(c) $\phi(\mathfrak{R}_{\mathfrak{a}}) = \mathfrak{a}\mathfrak{R}_{\mathfrak{a}}$ (thus $\mathfrak{R}_{\mathfrak{a}}$ is stable by ϕ up to genus).

That is, the root system $\mathfrak{R}_{\mathfrak{a}}$ affords an automorphism corresponding to the automorphism ${}^2B_2$ of G.

Proof. The proof of (2) is easy and left to the reader. Moreover, (2) implies the implication (ii)⇒(i) of (1).

Let us prove (1), (i)⇒(ii). We may assume that $\mathfrak{R}$, the $\mathbb{Z}_k$-root system for G which is stable (up to genus) under ϕ, has root basis:

$$\Pi := \left\{ \left(\mathfrak{a}_s v_s, \mathfrak{a}_s^{-*} v_s^\vee, -1\right), \left(\mathfrak{a}_t v_t, \mathfrak{a}_t^{-*} v_t^\vee, -1\right) \right\}$$

for some fractional ideals $\mathfrak{a}_s, \mathfrak{a}_t$. Now $\phi(\mathfrak{R}) = a\mathfrak{R}$ for some $a \in k^\times$, so

$$\mathfrak{a}_s v_t = a\mathfrak{a}_t v_t \quad \text{and} \quad 2\mathfrak{a}_t v_s = a\mathfrak{a}_s v_s\,,$$

from which we deduce that $a^2\mathfrak{a}_t = a\mathfrak{a}_s = 2\mathfrak{a}_t$. Multiplication by $\mathfrak{a}_t^{-1}$ gives $a^2\mathbb{Z}_k = 2\mathbb{Z}_k$. □

For example, set $k = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{2})$. The ring $\mathbb{Z}_k$ is a principal ideal domain. Setting $\mathfrak{a} := \mathbb{Z}[i](1+i)$ or $\mathbb{Z}[\sqrt{2}]\sqrt{2}$, then $\mathfrak{a} = \mathfrak{a}^*$ and $2\mathbb{Z}_k = \mathfrak{a}^2$ is the decomposition of $2\mathbb{Z}_k$ in $\mathbb{Z}_k$.

It is immediate to check that, if $2\mathbb{Z}_k = \mathfrak{a}^2$, there are at least three genera of reduced $\mathbb{Z}_k$-root systems for G described as the orbits under G of the following three pairs of roots:

$$\begin{array}{l} \{\,(\mathbb{Z}_k v_s, \mathbb{Z}_k v_s^\vee, -1), (\mathbb{Z}_k v_t, \mathbb{Z}_k v_t^\vee, -1)\} \\ \{\,(2\mathbb{Z}_k v_s, \frac{1}{2}\mathbb{Z}_k v_s^\vee, -1), (\mathbb{Z}_k v_t, \mathbb{Z}_k v_t^\vee, -1)\} \\ \{\,(\mathfrak{a} v_s, \mathfrak{a}^{-*} v_s^\vee, -1), (\mathbb{Z}_k v_t, \mathbb{Z}_k v_t^\vee, -1)\}. \end{array}$$

3.6. **Connection index.**

Let $\mathfrak{R} = \{\mathfrak{r} = (I_{\mathfrak{r}}, J_{\mathfrak{r}}, \zeta_{\mathfrak{r}})\}$ be a $\mathbb{Z}_k$-root system. The *characteristic ideal* (see for example [Bro3, 2.3.4.2]) of the torsion $\mathbb{Z}_k$-module $P_{\mathfrak{R}}/Q_{\mathfrak{R}}$ is defined by

$$\bigwedge^r Q_{\mathfrak{R}} = \mathrm{Ch}(P_{\mathfrak{R}}/Q_{\mathfrak{R}}) \bigwedge^r P_{\mathfrak{R}}\,,$$

where $r := \dim V$ (see [BouAlg, §4, n^{0}6]).

Remark 3.46. The characteristic ideal is the image in the group of fractional ideals of $\mathbb{Z}_k$ of the divisor called "contenu" in [BouAlg, §4, n^{0}5, Definition 4].

The next definition is inspired by the definition given in [BouLie, chap. 6, no 1.9].

Definition 3.47. *The characteristic ideal of the torsion $\mathbb{Z}_k$-module $P_{\mathfrak{R}}/Q_{\mathfrak{R}}$ is called the* connection index of the root system $\mathfrak{R}$.

Theorem–Definition 3.48. *Let (V, G) be an irreducible well generated reflection group. The connection index of a distinguished $\mathbb{Z}_k$-root system $\mathfrak{R}$ for G does not depend on the choice of $\mathfrak{R}$, and is called the* connection index *of (V, G).*

Proof of Theorem 3.48. Let $r := \dim V$. Let $\mathfrak{R}$ be a distinguished $\mathbb{Z}_k$-root system for G. By item (4) of Proposition 3.42, since (V, G) is well generated, by Proposition 3.43 there exists a set Π of r roots such that $Q_{\mathfrak{R}} = \bigoplus_{\mathfrak{r}\in\Pi} I_{\mathfrak{r}}$ and $Q^\vee_{\mathfrak{R}} = \bigoplus_{\mathfrak{r}\in\Pi} J_{\mathfrak{r}}$.

For all $\mathfrak{r} \in \Pi$ write $I_{\mathfrak{r}} = \mathfrak{a}_{\mathfrak{r}} v_{\mathfrak{r}}$ and $J_{\mathfrak{r}} = \mathfrak{a}_{\mathfrak{r}}^{-*} w_{\mathfrak{r}}$ for some vectors $v_{\mathfrak{r}}$ and $w_{\mathfrak{r}}$ with $\langle v_{\mathfrak{r}}, w_{\mathfrak{r}}\rangle = 1 - \zeta_{\mathfrak{r}}$ and some fractional ideal $\mathfrak{a}_{\mathfrak{r}}$. Let $w'_{\mathfrak{r}}$ be the dual basis of $w_{\mathfrak{r}}$ and set $J'_{\mathfrak{r}} = \mathfrak{a}_{\mathfrak{r}} w'_{\mathfrak{r}}$. Then $\mathcal{P}_{\mathfrak{R}} = \bigoplus_{\mathfrak{r}\in\Pi} J'_{\mathfrak{r}}$.

Assume given another distinguished $\mathbb{Z}_k$-root system $\mathfrak{R}'$ associated with the same set of reflections. For each $\mathfrak{r} \in \mathfrak{R}$, associated with the reflection $s_{\mathfrak{r}}$, let us denote by $\mathfrak{r}'$ the element of $\mathfrak{R}'$ associated with the same reflection $s_{\mathfrak{r}}$. Then, if $\mathfrak{r}' = (I_{\mathfrak{r}'}, J_{\mathfrak{r}'}, \zeta_{\mathfrak{r}})$, we have $I_{\mathfrak{r}'} = \mathfrak{b}_{\mathfrak{r}} I_{\mathfrak{r}}$ and $J_{\mathfrak{r}'} = \mathfrak{b}_{\mathfrak{r}}^{-*} J_{\mathfrak{r}}$ for a fractional ideal $\mathfrak{b}_{\mathfrak{r}}$.

Then

$$Q_{\mathfrak{R}'} = \bigoplus_{\mathfrak{r}\in\Pi} \mathfrak{b}_{\mathfrak{r}} I_{\mathfrak{r}} \quad \text{and} \quad P_{\mathfrak{R}'} = \bigoplus_{\mathfrak{r}\in\Pi} \mathfrak{b}_{\mathfrak{r}} J'_{\mathfrak{r}},$$

and

$$\bigwedge^r Q_{\mathfrak{R}'} = \Big(\prod_{\mathfrak{r}\in\Pi} \mathfrak{b}_{\mathfrak{r}}\Big) \bigwedge^r Q_{\mathfrak{R}},$$

$$\bigwedge^r P_{\mathfrak{R}'} = \Big(\prod_{\mathfrak{r}\in\Pi} \mathfrak{b}_{\mathfrak{r}}\Big) \bigwedge^r P_{\mathfrak{R}}.$$

This shows that

$$\mathrm{Ch}(P_{\mathfrak{R}'}/Q_{\mathfrak{R}'}) = \mathrm{Ch}(P_{\mathfrak{R}}/Q_{\mathfrak{R}}),$$

and ends the proof. □

Remark 3.49. Let V and W be as above, such that $\dim V = r$. Let (V, G) be a reflection group and let $s_1, \dots, s_r$ be a set of reflections such that $V = \bigoplus_{i=1}^r L_{s_i}$ and $W = \bigoplus_{i=1}^r M_{s_i}$. For each $i = 1, \dots r$, pick $v_i \in L_{s_i}$ and $v_i^\vee \in M_{s_i}$ such that $\langle v_i, v_i^\vee\rangle = 1 - \zeta_{s_i}$. Then the *Cartan matrix* $(\langle v_i, v_i^\vee\rangle)_{i,j}$ depends only (up to conjugation by a diagonal matrix) on the choice of the set $s_1, \dots, s_r$, hence its determinant depends only on such a choice.

We shall see later (Proposition 6.7) that if moreover $s_1, \dots, s_r$ generate G, that determinant generates (as an ideal) the connection index, hence in particular it does not depend (up to a unit) on the choice of the generators $s_1, \dots, s_r$.

4. The cyclic groups

4.1. Generalities.

As an introduction, let us consider the case of the cyclic group. Let k be a finite extension of $\mathbb{Q}$ which contains $\boldsymbol{\mu}_d$, also viewed as a one dimensional vector space V over itself; it is paired with $W = k$ by $\langle a, b\rangle := ab^*$.

Let $\zeta := \exp(2\pi i/d)$, and let $G := \boldsymbol{\mu}_d = \{1, \zeta, \dots, \zeta^{d-1}\}$ be the cyclic subgroup of $\mathbb{C}^\times$ of order d. We let G act on k by multiplication.

Proposition 4.1.

(1) Whenever $\mathfrak{F} := (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{d-1})$ is a family of ideals of $\mathbb{Z}_k$ such that, for each j $(1 \le j \le d-1)$, $\mathfrak{a}_j$ divides $(1-\zeta^j)\mathbb{Z}_k$, then the set

$$\mathfrak{R}(\mathfrak{F}) := \{(\mathfrak{a}_j, (1-\zeta^j)^*\mathfrak{a}_j^{-*}, \zeta^j) \mid (1 \le j \le d-1)\}$$

is a complete reduced $\mathbb{Z}_k$-root system for $G = \boldsymbol{\mu}_d$.

(2) The family $(\mathfrak{R}(\mathfrak{F}))$ where $\mathfrak{F} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{d-1})$ runs over the families as above such that $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{d-1}$ are relatively prime, is a complete set of representatives for the genera of complete reduced $\mathbb{Z}_k$-root systems for G.

Proof. The assertion (1) is trivial. Let us prove (2).

According to Lemma 3.4(1) a $\mathbb{Z}_k$-root for a reflection of G is a triple $(\mathfrak{a}v, \mathfrak{a}^{-*}w, \zeta^j)$ where $v \in V = \mathbb{C}$ and $w \in W = \mathbb{C}$ and $\langle v, w\rangle = 1 - \zeta^j$. We will write this $(\mathfrak{a}_j, \mathfrak{b}_j, \zeta^j)$ where we have set $\mathfrak{a}_j = \mathfrak{a}v$ and $\mathfrak{b}_j = \mathfrak{a}^{-*}w$ and the condition becomes $\langle \mathfrak{a}_j, \mathfrak{b}_j\rangle = \mathfrak{a}_j\mathfrak{b}_j^* = (1-\zeta^j)\mathbb{Z}_k$. This implies that $\mathfrak{b}_j = ((1-\zeta^j)\mathfrak{a}_j^{-1})^* = (1-\zeta^j)^*\mathfrak{a}_j^{-*}$.

According to definitions 3.8, 3.24.1 and 3.24.2 a complete and reduced $\mathbb{Z}_k$-root system for G is a set of roots as above

$$\mathfrak{R} = \{(\mathfrak{a}_j, \mathfrak{b}_j, \zeta^j) \mid (0 < j < d)\},$$

subject to the integrality condition that for $1 \le j, k \le d-1$, we have

$$\mathfrak{a}_j\mathfrak{b}_k^* \subseteq \mathbb{Z}_k\,.$$

In other words,

$$\mathfrak{R} = \{(\mathfrak{a}_j, (1-\zeta^{-j})\mathfrak{a}_j^{-*}, \zeta^j) \mid (1 \le j \le d-1)\},$$

where $(1-\zeta^k)\mathfrak{a}_j\mathfrak{a}_k^{-1} \subseteq \mathbb{Z}_k$; or equivalently:

$$(\mathfrak{R}_{j,k}) \qquad (1-\zeta^k)\mathfrak{a}_j \subseteq \mathfrak{a}_k\mathbb{Z}_k \subseteq \mathfrak{a}_k\,.$$

Since we want to know only the genus of $\mathfrak{R}$, we may multiply the family $(\mathfrak{a}_j)_{1\le j\le d-1}$ by an integral ideal so that the result is a family (still denoted by $(\mathfrak{a}_j)_{1\le j\le d-1}$) such that *all $\mathfrak{a}_j$ are integral, and they are relatively prime* (*i.e.*, their sum is $\mathbb{Z}_k$).

Letting j vary in the equality $\mathfrak{R}_{j,k}$ shows then that, for all k, $\mathfrak{a}_k$ divides $(1-\zeta^k)\mathbb{Z}_k$. □

Corollary 4.2.

(1) Each genus of complete root systems for the cyclic group contains a root system whose root lattice is $\mathbb{Z}_k$.
(2) All root lattices for complete root systems for the cyclic group are lattice equivalent.

Proof. Reasoning as in the proof of item (2) of Proposition 4.1, each genus contains $\mathfrak{R}(\mathfrak{F}) := \{(\mathfrak{a}_j, \mathfrak{b}_j, \zeta^{i_j})\}_j$ where $\sum_i \mathfrak{a}_i = \mathbb{Z}_k$, which shows item (1).

Item (2) is an immediate consequence of item (1). □

Every cyclic group has at least one complete root system which is principal and has a root basis. For example, for the family of ideals $\mathfrak{F}$ for which $\mathfrak{a}_1 = \cdots = \mathfrak{a}_{d-1} = \mathbb{Z}_k$, then the singleton $\{(\mathbb{Z}_k, (1-\zeta)^*\mathbb{Z}_k, \zeta)\}$ is a root basis.

The next easy remark will be useful later.

Proposition 4.3. *Let $G = \boldsymbol{\mu}_d$ be the cyclic group of order d and let $\zeta := \exp(2\pi i/d)$.*

(1) The singleton $\{(\mathbb{Z}_k, (1-\zeta)^*\mathbb{Z}_k, \zeta)\}$ is a complete principal $\mathbb{Z}_k$-root system, and provides a root basis.
(2) The connection index is the principal ideal $(1-\zeta)\mathbb{Z}_k$.

4.2. **The case of $G = \boldsymbol{\mu}_2$.**

In that case, whatever the decomposition of $2\mathbb{Z}_k$ into prime ideals of $\mathbb{Z}_k$ is, the following description results from 4.1.

Lemma 4.4. *Whatever the field k is, there is only one genus of reduced root system for $\boldsymbol{\mu}_2$, represented by the singleton*

$$\{(\mathbb{Z}_k, 2\mathbb{Z}_k, -1)\},$$

or by any singleton $\{(\mathfrak{a}, \mathfrak{b}^*, -1)\}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are two integral ideals such that $\mathfrak{a}\mathfrak{b} = 2\mathbb{Z}_k$.

Remark 4.5. Motivated by questions related to exceptional Spetses (see [BMM]), we consider the following cases for the field k:

$$\mathbb{Q}\,,\ \mathbb{Q}(i)\,,\ \mathbb{Q}(\zeta_3)\,,\ \mathbb{Q}(\zeta_{12})\,,\ \mathbb{Q}(\sqrt{-7})\,,\ \mathbb{Q}(\sqrt{5}, \zeta_3)\,,\ \mathbb{Q}(\sqrt{-2}, \zeta_3)\,.$$

The prime decomposition of the ideal $2\mathbb{Z}_k$ in these cases is as follows:

Lemma 4.6.

(1) For $k = \mathbb{Q}$ or $k = \mathbb{Q}(\zeta_3)$, the ideal $2\mathbb{Z}_k$ is prime in $\mathbb{Z}_k$.
(2) For $k = \mathbb{Q}(i)$, $\mathbb{Q}(\zeta_{12})$, or $\mathbb{Q}(\sqrt{-2}, \zeta_3)$, $2\mathbb{Z}_k = \mathfrak{p}^2$ for some integral ideal $\mathfrak{p}$.
(3) For $k = \mathbb{Q}(\sqrt{-7})$ and $k = \mathbb{Q}(\sqrt{5}, \zeta_3)$, we have $2\mathbb{Z}_k = \mathfrak{p}\mathfrak{p}^*$ where $\mathfrak{p}$ is a prime ideal in $\mathbb{Z}_k$ and $\mathfrak{p} \neq \mathfrak{p}^*$.

Proof. Note first that, for example by [Was, Theorem 11.1], the rings $\mathbb{Z}[\zeta_3]$, $\mathbb{Z}[i]$, $\mathbb{Z}[\zeta_{12}]$ considered below are principal ideal domains.

(1) For $k = \mathbb{Q}(\zeta_3)$, we have $\mathbb{Z}_k = \mathbb{Z}[\zeta_3] = \mathbb{Z}[X]/(X^2+X+1)$, and since (X^2+X+1) is irreducible over $\mathbb{F}_2$, $2\mathbb{Z}_k$ is prime in $\mathbb{Z}_k$.

(2) For $k = \mathbb{Q}(i)$, we have $\mathbb{Z}_k = \mathbb{Z}[i]$ and $2\mathbb{Z}_k = ((1+i)\mathbb{Z}_k)^2$.

For $k = \mathbb{Q}(\zeta_{12})$, we have $\mathbb{Z}_k = \mathbb{Z}[\zeta_{12}]$, and $2\mathbb{Z}_k = ((1+i)\mathbb{Z}_k)^2$.

For $k = \mathbb{Q}(\sqrt{-2}, \zeta_3)$, we have $\mathbb{Z}_k = \mathbb{Z}[\sqrt{-2}, \zeta_3]$ (see Exercise 4.5.13 in [MuEs]) and $2\mathbb{Z}_k = (\sqrt{-2}\mathbb{Z}_k)^2$.

(3) For $k = \mathbb{Q}(\sqrt{-7})$, $\mathbb{Z}_k = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$, and $2 = \frac{1+\sqrt{-7}}{2} \cdot \frac{1-\sqrt{-7}}{2}$, hence $2\mathbb{Z}_k$ is a product of two (conjugate) prime ideals: $2\mathbb{Z}_k = \mathfrak{p}\mathfrak{p}^*$, and $\mathfrak{p} \neq \mathfrak{p}^*$ since there is no solution in integers a and b to the equation

$$\left(a + b\frac{1+\sqrt{-7}}{2}\right)\frac{1-\sqrt{-7}}{2} = \frac{1+\sqrt{-7}}{2}\,.$$

Finally, consider the case $k = \mathbb{Q}(\sqrt{5}, \zeta_3)$, with $\mathbb{Z}_k = \mathbb{Z}[\phi, \zeta_3]$, where $\phi = \frac{1+\sqrt{5}}{2}$. Since the polynomial $X^2 - X - 1$ is irreducible over $\mathbb{F}_2$, the prime 2 remains prime in $\mathbb{Z}[\phi]$, and we know it remains prime over $\mathbb{Z}[\zeta_3]$. Now $2 = (1+\zeta_3\phi)(1+\zeta_3^2\phi)$, which shows that in $\mathbb{Z}_k$ we have again $2\mathbb{Z}_k = \mathfrak{p}\mathfrak{p}^*$, and $\mathfrak{p} \neq \mathfrak{p}^*$, since the only solution in rational numbers to

$$(a + b\phi + c\zeta_3 + d\zeta_3\phi)(1+\zeta_3\phi) = 1 + \zeta_3^2\phi$$

is $a = c = -d = \frac{1}{2}, b = -1$.

□

4.3. **A few particular cases for $G = \boldsymbol{\mu}_3$.**

We first remark that $\frac{1-\zeta_3}{1-\zeta_3^2} = -\zeta_3$ is a unit thus the ideals $(1-\zeta_3)\mathbb{Z}_k$ and $(1-\zeta_3^2)\mathbb{Z}_k$ are the same ideal $\mathfrak{j} = \mathfrak{j}^*$ and $3\mathbb{Z}_k = \mathfrak{j}^2$.

We want to consider the following cases for the field k:

$$\mathbb{Q}(\zeta_3)\,,\ \mathbb{Q}(\zeta_{12})\,,\ \mathbb{Q}(\sqrt{-2}, \zeta_3)\,.$$

The cases where $k = \mathbb{Q}(\zeta_3)$ and $k = \mathbb{Q}(\zeta_{12})$.

In these cases $\mathfrak{j}$ is prime (in the case $k = \mathbb{Q}(\zeta_{12})$, this follows from A.5 and the fact that 3 is a generator of $(\mathbb{Z}/4\mathbb{Z})^\times$). It follows that there are two genera of reduced complete root systems, represented by the two systems

$$\mathfrak{R}_1 := \{\, (\mathbb{Z}_k, \mathfrak{j}, \zeta_3)\,,\ (\mathbb{Z}_k, \mathfrak{j}, \zeta_3^2)\,\}$$
$$\mathfrak{R}_2 := \{\, (\mathfrak{j}, \mathbb{Z}_k, \zeta_3)\,,\ (\mathbb{Z}_k, \mathfrak{j}, \zeta_3^2)\,\}$$

The case $k = \mathbb{Q}(\sqrt{-2}, \zeta_3)$.

We first compute the factorisation of $\mathfrak{j}$ into prime ideals in $\mathbb{Z}_k$.

- We have $3 = (1+\sqrt{-2})(1-\sqrt{-2})$, and since $\boldsymbol{\mu}(\mathbb{Q}(\sqrt{-2})) = \{\pm 1\}$ (see for example [Sa], 4.5) it follows that the ideal $3\mathbb{Z}_k$ decomposes in a product of two different prime ideals in $\mathbb{Z}[\sqrt{-2}]$:
$$3\mathbb{Z}_k = (1+\sqrt{-2})\mathbb{Z}_k \cdot (1-\sqrt{-2})\mathbb{Z}_k\,.$$
These ideals are different since the equation $(a+b\sqrt{-2})(1+\sqrt{-2}) = 1-\sqrt{-2}$ has no solution in integers a, b.
- We know that $3\mathbb{Z}_k = \mathfrak{j}^2$ in $\mathbb{Z}[\zeta_3]$ (see remarks above).
- We also have:
$$\begin{cases} 1+\sqrt{-2} = (1+\zeta_3\sqrt{-2})(-1-\zeta_3^2\sqrt{-2}) \\ 1-\sqrt{-2} = (1-\zeta_3\sqrt{-2})(-1+\zeta_3^2\sqrt{-2})\,. \end{cases}$$
Moreover we have the products of ideals
$$(1+\zeta_3\sqrt{-2})\mathbb{Z}_k \cdot (1-\zeta_3\sqrt{-2})\mathbb{Z}_k = (-1-\zeta_3^2\sqrt{-2})\mathbb{Z}_k \cdot (-1+\zeta_3^2\sqrt{-2})\mathbb{Z}_k = \mathfrak{j}$$
- Notice that
$$\begin{cases} 1-\zeta_3^2\sqrt{-2} = (1-\zeta_3\sqrt{-2})(\zeta_3-\zeta_3^2-\sqrt{-2}) \\ 1-\zeta_3\sqrt{-2} = (1-\zeta_3^2\sqrt{-2})(\zeta_3^2-\zeta_3-\sqrt{-2}) \end{cases}$$
which shows the equality of ideals
$$\mathfrak{p} := (1-\zeta_3\sqrt{-2})\mathbb{Z}_k = (1-\zeta_3^2\sqrt{-2})\mathbb{Z}_k\,.$$
Similarly we have the equality of ideals
$$\mathfrak{q} := \mathfrak{p}^* = (1+\zeta_3\sqrt{-2})\mathbb{Z}_k = (1+\zeta_3^2\sqrt{-2})\mathbb{Z}_k\,.$$
hence one has the following decomposition of $3\mathbb{Z}_k$ into products of prime ideals in $\mathbb{Z}[\zeta_3, \sqrt{-2}]$:
$$3\mathbb{Z}_k = \underbrace{\mathfrak{p}\mathfrak{q}}_{=\mathfrak{j}}\ \underbrace{\mathfrak{p}\mathfrak{q}}_{=\mathfrak{j}}\,.$$
and $\mathfrak{p} \neq \mathfrak{q}$ since $\mathfrak{q}^2 = (1+\sqrt{-2})\mathbb{Z}_k$ and $\mathfrak{p}^2 = (1-\sqrt{-2})\mathbb{Z}_k$ are different ideals.

Now that we have the factorization of (3) we can determine the reduced complete root systems. Up to genus, they are of the form
$$\mathfrak{R}_{\mathfrak{a}_1,\mathfrak{a}_2} := \{\, (\mathfrak{a}_1, \mathfrak{j}\mathfrak{a}_1^{-*}, \zeta_3)\,,\ (\mathfrak{a}_2, \mathfrak{j}\mathfrak{a}_2^{-*}, \zeta_3^2)\,\}\,,$$
where the ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are integral, coprime, and divide $\mathfrak{j} = \mathfrak{p}\mathfrak{q}$. It is an elementary arithmetic exercise to deduce now the following classification.

Proposition 4.7. *There are 9 genera of reduced complete $\mathbb{Z}[\zeta_3, \sqrt{-2}]$-root systems for $\boldsymbol{\mu}_3$, represented by the following list:*

$$\begin{array}{ll}
\mathfrak{R}_{1,1} = \{\, (1, \mathfrak{p}\mathfrak{q}, \zeta_3)\,,\ (1, \mathfrak{p}\mathfrak{q}, \zeta_3^2)\,\}\,, & \\
\mathfrak{R}_{\mathfrak{p},\mathfrak{q}} = \{\, (\mathfrak{p}, \mathfrak{p}, \zeta_3)\,,\ (\mathfrak{q}, \mathfrak{q}, \zeta_3^2)\,\}\,, & \mathfrak{R}_{\mathfrak{q},\mathfrak{p}} = \{\, (\mathfrak{q}, \mathfrak{q}, \zeta_3)\,,\ (\mathfrak{p}, \mathfrak{p}, \zeta_3^2)\,\}\,, \\
\mathfrak{R}_{1,\mathfrak{q}} = \{\, (1, \mathfrak{p}\mathfrak{q}, \zeta_3)\,,\ (\mathfrak{q}, \mathfrak{q}, \zeta_3^2)\,\}\,, & \mathfrak{R}_{\mathfrak{q},1} = \{\, (\mathfrak{q}, \mathfrak{q}, \zeta_3)\,,\ (1, \mathfrak{p}\mathfrak{q}, \zeta_3^2)\,\}\,, \\
\mathfrak{R}_{1,\mathfrak{p}} = \{\, (1, \mathfrak{p}\mathfrak{q}, \zeta_3)\,,\ (\mathfrak{p}, \mathfrak{p}, \zeta_3^2)\,\}\,, & \mathfrak{R}_{\mathfrak{p},1} = \{\, (\mathfrak{p}, \mathfrak{p}, \zeta_3)\,,\ (1, \mathfrak{p}\mathfrak{q}, \zeta_3^2)\,\}\,, \\
\mathfrak{R}_{1,\mathfrak{p}\mathfrak{q}} = \{\, (1, \mathfrak{p}\mathfrak{q}, \zeta_3)\,,\ (\mathfrak{p}\mathfrak{q}, 1, \zeta_3^2)\,\}\,, & \mathfrak{R}_{\mathfrak{p}\mathfrak{q},1} = \{\, (\mathfrak{p}\mathfrak{q}, 1, \zeta_3)\,,\ (1, \mathfrak{p}\mathfrak{q}, \zeta_3^2)\,\}\,.
\end{array}$$

5. The group $G(de, e, r)$

Let d and e be two natural integers such that $de > 1$. Let $k := \mathbb{Q}(\zeta_{de})$, and let V be a k–vector space with basis $(e_1, e_2, \ldots, e_r)$. We denote by $(e'_1, e'_2, \ldots, e'_r)$ the dual basis of W with respect to the Hermitian pairing between V and W.

We denote by $G(de, 1, r)$ the group consisting of all monomial matrices with coefficients in $\boldsymbol{\mu}_{de}$, isomorphic to $(\boldsymbol{\mu}_{de})^r \rtimes \mathfrak{S}_r$.

We denote by $G(de, e, r)$ the normal subgroup of index e in $G(de, 1, r)$ consisting in those matrices in $G(de, 1, r)$ whose product of nonzero entries lies in $\boldsymbol{\mu}_d$. When $r = 1$ then $d \neq 1$ is not allowed.

Remark 5.1. Unless $r = 2$ and $d = 1$, the field of definition of $G(de, e, r)$ is $\mathbb{Q}(\zeta_{de})$. We will classify root systems over that field. The field of definition of $G(e, e, 2)$, the dihedral group of order $2e$, is the maximal real subfield of $\mathbb{Q}(\zeta_e)$, namely the field $\mathbb{Q}(\zeta_e + \zeta_e^{-1})$. The root systems of $G(e, e, 2)$ over that field will be treated in Subsection 5.4.

5.1. The reflections of $G(de, e, r)$.

It is well known (see for example [Ne], proof of Lemma 5) that the reflections $\mathrm{Ref}_1(d, r)$ and $\mathrm{Ref}_2(de, r)$ defined below exhaust the collection of reflections of $G(de, e, r)$. The set of reflections of $G(e, e, r)$ is precisely $\mathrm{Ref}_2(e, r)$.

$\mathrm{Ref}_1(d, r)$**:** elements s_k^i $(1 \leq i \leq d-1\,,\ 1 \leq k \leq r)$ defined by

$$s_k^i : \begin{cases} e_k \mapsto \zeta_d^i e_k \\ e_l \mapsto e_l \quad \text{if } l \neq k \end{cases}$$

Note that if $d = 1$ the set $\mathrm{Ref}_1(d, r)$ is empty, while if $d > 1$ it consists of reflections.

$\mathrm{Ref}_2(de, r)$**:** the involutive reflections $s_{k,l}^{(j)}$ (with $0 \leq j \leq de-1\,,\ 1 \leq k < l \leq r$) defined by

$$s_{k,l}^{(j)} : \begin{cases} e_k \mapsto \zeta_{de}^j e_l \\ e_l \mapsto \zeta_{de}^{-j} e_k \\ e_m \mapsto e_m \quad \text{if } m \neq k, l \end{cases}$$

It is also well known (see for instance [BMR, §3]) that the following set of $r+1$ reflections generates $G(de, e, r)$:

$$\left\{ s_{12}^{(0)}, s_{23}^{(0)}, \ldots, s_{(r-1),r}^{(0)}, s_{(r-1),r}^{(1)}, s_r^1 \right\}. \tag{5.2}$$

5.2. The complete reduced $\mathbb{Z}_k$-root system $\mathfrak{R}(de, e, r)$.

For each reflection above we define a $\mathbb{Z}_k$-root – here again, i, j, k, l, m denote natural integers such that $1 \leq k < l \leq r\,,\ 1 \leq i \leq d-1\,,\ 0 \leq$

$j \leq de-1$ (if $d=1$ some of our objects do not exist):

$$\begin{aligned} \mathfrak{r}_k^i &:= \big(\mathbb{Z}_k e_k, (1-\zeta_d^{-i})\mathbb{Z}_k e'_k, \zeta_d^i\big), && \text{an } s_k^i\text{-root (if } d>1), \\ \mathfrak{r}_{k,l}^{(j)} &:= \big(\mathbb{Z}_k(e_k - \zeta_{de}^j e_l), \mathbb{Z}_k(e'_k - \zeta_{de}^j e'_l), -1\big)\,, && \text{an } s_{k,l}^{(j)}\text{-root.} \end{aligned}$$

The actions of the reflections on these roots and the Cartan pairings of roots are described below. The results follow from direct calculation.

Lemma 5.3. *Action of the reflections on roots:*

$$s_{\mathfrak{r}_k^\alpha}(\mathfrak{r}_l^\beta) = \mathfrak{r}_l^\beta \qquad\qquad s_{\mathfrak{r}_k^\alpha}(\mathfrak{r}_{l,m}^{(\beta)}) = \begin{cases} \mathfrak{r}_{l,m}^{(\beta)} & \text{if } k \neq l,m \\ \mathfrak{r}_{l,m}^{(\beta-e\alpha)} & \text{if } k = l \\ \mathfrak{r}_{l,m}^{(\beta+e\alpha)} & \text{if } k = m \end{cases}$$

$$s_{\mathfrak{r}_{l,m}^{(\beta)}}(\mathfrak{r}_k^\alpha) = \begin{cases} \mathfrak{r}_k^\alpha & \text{if } k \neq l,m \\ \mathfrak{r}_m^\alpha & \text{if } k = l \\ \mathfrak{r}_l^\alpha & \text{if } k = m \end{cases} \qquad s_{\mathfrak{r}_{j,k}^\alpha}(\mathfrak{r}_{l,m}^{(\beta)}) = \begin{cases} \mathfrak{r}_{l,m}^{(\beta)} & \text{if } \{j,k\}\cap\{l,m\}=\emptyset \\ \mathfrak{r}_{k,m}^{(\beta-\alpha)} & \text{if } j=l, k<m \\ \mathfrak{r}_{m,k}^{(\alpha-\beta)} & \text{if } j=l, m<k \\ \mathfrak{r}_{l,k}^{(\beta+\alpha)} & \text{if } l<j=m<k \\ \mathfrak{r}_{j,m}^{(\beta+\alpha)} & \text{if } j<k=l<m \\ \mathfrak{r}_{j,l}^{(\alpha-\beta)} & \text{if } j<l, k=m \\ \mathfrak{r}_{l,j}^{(\beta-\alpha)} & \text{if } l<j, k=m \\ \mathfrak{r}_{j,k}^{(2\alpha-\beta)} & \text{if } j=l, k=m \end{cases}$$

Lemma 5.4. *Cartan pairings of roots:*

$$\mathfrak{n}(\mathfrak{r}_k^\alpha, \mathfrak{r}_l^\beta) = \begin{cases} 0 & \text{if } k \neq l \\ (1-\zeta_d^\beta)\mathbb{Z}_k & \text{if } k = l \end{cases}$$

$$\mathfrak{n}(\mathfrak{r}_k^\alpha, \mathfrak{r}_{l,m}^{(\beta)}) = \begin{cases} 0 & \text{if } k \neq l,m \\ \mathbb{Z}_k & \text{if } k = l \text{ or if } k = m \end{cases}$$

$$\mathfrak{n}(\mathfrak{r}_{l,m}^{(\beta)}, \mathfrak{r}_k^\alpha) = \begin{cases} 0 & \text{if } k \neq l,m \\ (1-\zeta_d^\alpha)\,\mathbb{Z}_k & \text{if } k = l \text{ or if } k = m \end{cases}$$

$$\mathfrak{n}(\mathfrak{r}_{j,k}^{(\alpha)}, \mathfrak{r}_{l,m}^{(\beta)}) = \begin{cases} 0 & \text{if } \{j,k\}\cap\{l,m\} = \emptyset \\ \mathbb{Z}_k & \text{if } |\{j,k\}\cap\{l,m\}| = 1 \\ \left(1+\zeta_{de}^{\alpha-\beta}\right)\mathbb{Z}_k & \text{if } \{j,k\} = \{l,m\} \end{cases}$$

The preceding calculations ensure that

$$\mathfrak{R}(de,e,r) := \mathfrak{R}_1(d,r) \cup \mathfrak{R}_2(de,r)$$

is a complete reduced root system for $G(de,e,r)$, where:

$$\mathfrak{R}_1(d,r) := \{\, \mathfrak{r}_k^i \,\}_{(1\leq k\leq r)(1\leq i\leq d-1)}$$

$$\mathfrak{R}_2(de,r) := \{\, \mathfrak{r}_{k,l}^{(j)} \,\}_{(1\leq k<l\leq r)(0\leq j\leq de-1)}$$

For a given i, we write $\mathfrak{R}_1^i(d,r) := \{\, \mathfrak{r}_k^i \,\}_{(1\leq k\leq r)}$ and for a given j, $\mathfrak{R}_2^{(j)}(de,r) := \{\, \mathfrak{r}_{k,l}^{(j)} \,\}_{(1\leq k<l\leq r)}$ so that

$$\mathfrak{R}_1 = \bigcup_{1\leq i\leq d-1} \mathfrak{R}_1^i \quad \text{and} \quad \mathfrak{R}_2(de,r) = \bigcup_{0\leq j\leq de-1} \mathfrak{R}_2^{(j)}(de,r)\,.$$

The "even" and "odd" parts of $\mathfrak{R}_2$ are defined to be:

$$\mathfrak{R}_2^0(de,r) := \bigcup_{j=0,2\ldots} \mathfrak{R}_2^{(j)}(de,r) \quad \text{and} \quad \mathfrak{R}_2^1(de,r) := \bigcup_{j=1,3\ldots} \mathfrak{R}_2^{(j)}(de,r)\,.$$

Lemma 5.5.

(1) The set $\mathfrak{R}(de,e,r)$ is a complete reduced $\mathbb{Z}_k$-root system for $G(de,e,r)$.

(2) The orbits of $G(de,e,r)$ on $\mathfrak{R}(de,e,r)$ are

- *when e is even and $r=2$:*

$$\left\{\mathfrak{R}_1^1(d,2), \mathfrak{R}_1^2(d,2), \ldots, \mathfrak{R}_1^{d-1}(d,2), \mathfrak{R}_2^0(de,2), \mathfrak{R}_2^1(de,2)\right\}\,,$$

- *in other cases:*

$$\{\,\mathfrak{R}_1^1(d,r), \mathfrak{R}_1^2(d,r), \ldots, \mathfrak{R}_1^{d-1}(d,r), \mathfrak{R}_2(de,r)\,\}$$

where the sets $\mathfrak{R}_1^j(d,2)$ are empty if $d=1$ and $\mathfrak{R}_2(de,r)$ is empty if $r=1$.

Proof. The results follow directly from Lemmas 5.3 and 5.4. □

5.3. **Classifying complete reduced root systems for $G(de,e,r)$.**

While $\mathfrak{R}(de,e,r)$ is a representative of one genus of complete reduced root system for $G(de,e,r)$ over $\mathbb{Q}(\zeta_{de})$, there may be others, as described by the following theorem.

The group $G(de,e,r)$ is a reflection group over k if and only if $\zeta_{de} \in k$, except for $G(e,e,2)$ where the field of definition is $\mathbb{Q}(\zeta_e + \zeta_e^{-1})$. Thus the following theorem fulfills our aim stated in the introduction: classify $\mathbb{Z}_k$-root systems for reflection groups over k, except for the case of $G(e,e,2)$ over its field of definition which will be considered in Subsection 5.4.

Theorem 5.6. *Given a family $\mathcal{F} = \{\,\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{d-1}, \mathfrak{b}_0, \mathfrak{b}_1\,\}$ of fractional ideals where, unless $r=2$ and $de = 2p^k$, p prime and $k>1$, we have $\mathfrak{b}_0 = \mathfrak{b}_1 = \mathbb{Z}_k$, we define the set*

$$\mathfrak{R}_{\mathcal{F}}(de,e,r) := \left(\bigcup_{1\le i\le d-1} \mathfrak{a}_i \cdot \mathfrak{R}_1^i(d,r)\right) \cup \mathfrak{b}_0 \cdot \mathfrak{R}_2^0(de,r) \cup \mathfrak{b}_1 \cdot \mathfrak{R}_2^1(de,r)\,.$$

Then every genus of complete reduced $\mathbb{Z}_k$-root system for $G(de,e,r)$ when $\zeta_{de} \in k$ contains exactly one root system $\mathfrak{R}_{\mathcal{F}}(de,e,r)$ where $\mathcal{F}$ satisfies the additional conditions

- *The $\mathfrak{a}_j$ and the $\mathfrak{b}_j$ are integral,*
- *$\mathfrak{b}_0$ and $\mathfrak{b}_1$ are relatively prime divisors of $(1+\zeta_{de})\mathbb{Z}_k$,*
- *for all $i = 1,2,\ldots,d-1$, $\mathfrak{a}_i$ divides $(1-\zeta_d^i)\mathbb{Z}_k$, and*
- *for all $i = 1,2,\ldots,d-1$, $\mathfrak{b}_0$ and $\mathfrak{b}_1$ divide $\mathfrak{a}_i$.*

Proof. By the description of the orbits of $G(de,e,r)$ over $\mathfrak{R}(de,e,r)$ (see Lemma 5.5 above) and 3.27, we see that any complete reduced

$\mathbb{Z}_k$-root system for $G(de,e,r)$ over a field containing ζ_{de} is of the form $\mathfrak{R}_{\mathcal{F}}(de,e,r)$ for some fractional ideals $\mathfrak{a}_j$ and $\mathfrak{b}_j$.

Without changing the genus, we may assume that the elements of $\mathcal{F}$ are integral and relatively prime.

Now computations of the Cartan pairings (see Lemma 5.4 above) show that the following ideals must be integral (for $1 \le \alpha, \beta \le d-1$ and $j,k = 0,1$):

$$\left(1-\zeta_d^{\beta}\right)\mathfrak{a}_\alpha\mathfrak{a}_\beta^{-1}\,,\ \mathfrak{a}_\alpha\mathfrak{b}_j^{-1}\,,\ (1-\zeta_d^{\alpha})\,\mathfrak{b}_j\mathfrak{a}_\alpha^{-1}\,,\ \left(1+\zeta_{de}^{-(j+k)}\right)\mathfrak{b}_j\mathfrak{b}_k^{-1}$$

We see that $\mathfrak{b}_0$ and $\mathfrak{b}_1$ divide all the $\mathfrak{a}_i$ and since the whole family is relatively prime, it follows that $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are relatively prime. Since $\mathfrak{a}_\alpha$ divides both $(1-\zeta_d^{\alpha})\mathfrak{b}_0$ and $(1-\zeta_d^{\alpha})\mathfrak{b}_1$, we see then that $\mathfrak{a}_\alpha$ divides $(1-\zeta_d^{\alpha})\mathbb{Z}_k$.

Finally we also see that $\mathfrak{b}_0$ and $\mathfrak{b}_1$ divide $(1+\zeta_{de})\mathbb{Z}_k$.

When $r>2$, $\mathfrak{b}_0 = \mathfrak{b}_1$ since $\mathfrak{R}_2^0(de,r)$ and $\mathfrak{R}_2^1(de,r)$ are in the same orbit. Since they are relatively prime, they must be trivial.

When $r=2$, we use that $1+\zeta_{de}$ is a unit unless de is of the form $2p^k$, see A.2. □

We remark that when $k = \mathbb{Q}(\zeta_{de})$, the ideal $(1+\zeta_{de})\mathbb{Z}_k$ is $\mathbb{Z}_k$ or prime, see A.4.

Corollary 5.7 (Case of $G(d,1,r)$)**.** *The map*

$$\mathcal{F} \mapsto \mathfrak{R}_{\mathcal{F}}(d,1,r) = \bigcup_{1\le d-1} \mathfrak{a}_i \cdot \mathfrak{R}_1^i(d,r) \,\cup\, \mathfrak{R}_2(d,r)$$

induces a bijection between the set of families $\mathcal{F} := \{\,\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{d-1}\,\}$ *of integral ideals such that*

- *for all* i, $\mathfrak{a}_i$ *divides* $(1-\zeta_d^i)\mathbb{Z}_k$ *and*
- *the ideals* $\mathfrak{a}_i$ *are relatively prime,*

and the set of genera of complete reduced $\mathbb{Z}_k$*-root systems for* $G(d,1,r)$*.*

Corollary 5.8 (Case of $G(e,e,r)$)**.** *Assume that* $r>2$*. There is only one genus of* $\mathbb{Z}_k$*-root systems for* $G(e,e,r)$*, represented by the principal root system* $\mathfrak{R}_2(e,r)$*.*

Corollary 5.9 (The dihedral group)**.**

(1) If e *is odd there is only one genus of* $\mathbb{Z}[\zeta_e]$*-root system for* $G(e,e,2)$ *represented by the principal root system* $\mathfrak{R}_2(e,2)$*.*

(2) If e *is even, the map*

$$(\mathfrak{b}_0,\mathfrak{b}_1) \mapsto \mathfrak{b}_0 \cdot \mathfrak{R}_2^0(e,2) \cup \mathfrak{b}_1 \cdot \mathfrak{R}_2^1(e,2)$$

induces a bijection between the set of pairs of relatively prime integral ideals of $\mathbb{Z}[\zeta_e]$ *dividing* $\mathbb{Z}[\zeta_e](1+\zeta_e)$ *and the set of genera of* $\mathbb{Z}[\zeta_e]$*-root systems for* $G(e,e,2)$*.*

Remark 5.10. As a particular case of Corollary 5.9 above, we recover the results already cited in Subsection 3.5 about the $\mathbb{Z}[i]$-root systems for the Weyl group of type B_2.

5.4. **Root systems for $G(e,e,2)$ on its field of definition.**

The field of definition of the dihedral group $G = G(e,e,2)$ is $k = \mathbb{Q}(\zeta_e + \zeta_e^{-1})$, the largest real subfield of $\mathbb{Q}(\zeta_e)$; its ring of integers is $\mathbb{Z}[\zeta_e + \zeta_e^{-1}]$ by Proposition A.10 (2). In this section we classify root systems of G over its field of definition. From now on we write ζ for ζ_e.

Lemma 5.11. *The involutory reflections:*

$$s = \begin{pmatrix} -1 & 2+\zeta+\zeta^{-1} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

generate G, and satisfy the dihedral relation $(st)^e = 1$.

Moreover, when e is even, $(st)^{e/2} = -1$.

Proof. The elements of G of the form $(st)^n$ are rotations. These elements have matrices:

$$(st)^n = \frac{1}{\Im(\zeta)} \begin{pmatrix} \Im(\zeta^n + \zeta^{n+1}) & -\Im(\zeta^{n-1} + 2\zeta^n + \zeta^{n+1}) \\ \Im(\zeta^n) & -\Im(\zeta^n + \zeta^{n-1}) \end{pmatrix},$$

where $\Im(z)$ denotes the imaginary part of a complex number z. From this computation the lemma is obvious. □

The reflection lines of s and t respectively are in the directions of the standard vectors $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$ respectively.

Define principal roots $\mathfrak{r}_s = \mathbb{Z}_k \cdot (e_1, 2e_1' - (2+\zeta+\zeta^{-1})e_2', -1)$ and $\mathfrak{r}_t = \mathbb{Z}_k \cdot (e_2, -e_1' + 2e_2', -1)$.

Proposition 5.12. *If e is odd let $\mathfrak{R} := \{(st)^n \mathfrak{r}_s \mid 0 \le n < e\}$. If e is even, let $\mathfrak{R}_s := \{(st)^n \mathfrak{r}_s \mid 0 \le n < e/2\}$, $\mathfrak{R}_t := \{(st)^n \mathfrak{r}_t \mid 0 \le n < e/2\}$, and $\mathfrak{R} := \mathfrak{R}_s \cup \mathfrak{R}_t$.*

(1) In both cases $\mathfrak{R}$ is a complete, reduced, distinguished and principal $\mathbb{Z}_k$-root system for G with root basis $\{\mathfrak{r}_s, \mathfrak{r}_t\}$.

(2) There is a single genus of reduced $\mathbb{Z}_k$-root systems for G unless $e = 2p^k$, p prime, $k \ge 1$.

In this last case, there is another genus, represented by the principal root system given by $\mathfrak{R}_s \cup (2+\zeta+\zeta^{-1}) \cdot \mathfrak{R}_t$.

Proof. We have:

$$(st)^n e_1 = \frac{1}{\Im(\zeta)} \begin{pmatrix} \Im(\zeta^n + \zeta^{n+1}) \\ \Im(\zeta^n) \end{pmatrix}.$$

• If e is odd, then the dihedral relation ensures that all the reflections of G are conjugate: setting $q = (e-1)/2$, we have $\zeta^{q+1} = -\zeta_{2e}$ and $\Im(\zeta^{q+1}) = -\Im(\zeta^q)$. Thus $(st)^q e_1 = \frac{\Im(\zeta^q)}{\Im(\zeta)} e_2$ and $(st)^q \mathfrak{r}_s = \alpha^{-1} \mathfrak{r}_t$ with

$$\alpha = \frac{\Im(\zeta)}{\Im(\zeta^q)} = -\zeta^q(1+\zeta) = -2\Re(\zeta^q) = \Re(\zeta_{2e})$$

and $\alpha^2 = 2 + \zeta + \zeta^{-1} = (1+\zeta)(1+\zeta^{-1})$.

By A.2 in Appendix A, $1 + \zeta_e$ is a unit unless e is of the form $2p^k$, p prime, $k \geq 1$. In particular, if e is odd, α is a unit in $\mathbb{Z}[\zeta]$, which proves that actually $(st)^q \mathfrak{r}_s = \mathfrak{r}_t$.

Since $\mathfrak{n}(\mathfrak{r}_s, \mathfrak{r}_t) = -(2 + \zeta + \zeta^{-1})$ and $\mathfrak{n}(\mathfrak{r}_t, \mathfrak{r}_s) = -1$, it results from Proposition 3.18 that $\mathfrak{R}$ is a $\mathbb{Z}_k$-root system for G. Thus there is a single genus of reduced root systems by 3.27.

- If e is even, there are two conjugacy classes of reflections, of which s and t are representatives. Thus the orbits of $\mathfrak{r}_s$ and $\mathfrak{r}_t$ are distinct, and their size is $e/2$ since $(st)^{e/2} = -1$. Again the pairings are integral by Proposition 3.18. But, this time we can scale one of the orbits by some fractional ideal $\mathfrak{a}$ (see 3.27), and for the pairings to remain integral we need that $\mathfrak{a}$ be an integral divisor of $2 + \zeta + \zeta^{-1} = (1+\zeta)(1+\zeta^{-1})$. Thus by A.2, $\mathfrak{a}$ can be non trivial only if $e = 2p^k$. In this last case, by Lemma A.12, $2 + \zeta + \zeta^{-1}$ is prime, so the only possible values for $\mathfrak{a}$ are the principal ideals $\mathbb{Z}_k$ and $(2 + \zeta + \zeta^{-1})\mathbb{Z}_k$.

Proposition 3.42 ensures that $\{\mathfrak{r}_s, \mathfrak{r}_t\}$ is always a root basis. □

The symmetric dihedral root system.

A symmetric ("self-dual") root system for $G(e, e, 2)$ can be obtained by adjoining $2\cos(\frac{\pi}{e})$. This results in the "symmetric Cartan matrix":

$$C_{sym} = \begin{pmatrix} 2 & -2\cos(\frac{\pi}{e}) \\ -2\cos(\frac{\pi}{e}) & 2 \end{pmatrix}$$

This corresponds to a root system over $\mathbb{Z}_k$ only if e is odd (see item (2) of A.9).

Now $(2 + \zeta + \zeta^{-1}) = \big(2\cos(\frac{\pi}{e})\big)^2$; that is, adjoining $2\cos(\frac{\pi}{e})$ to $\mathbb{Z}_k$ makes $(2 + \zeta + \zeta^{-1})$ a square. Unless e is a power of 2, $(2 + \zeta + \zeta^{-1})$ is a unit, so we do not get any additional root systems. However, if e is a power of 2, we get a third, self-dual, root system. The three are:

$$\mathfrak{R}_s \cup \mathfrak{R}_t, \qquad \left(2\cos\left(\frac{\pi}{e}\right)\right)^2 \cdot \mathfrak{R}_s \cup \mathfrak{R}_t \qquad \text{and} \qquad 2\cos\left(\frac{\pi}{e}\right) \cdot \mathfrak{R}_s \cup \mathfrak{R}_t\,.$$

The last root system in this list (with symmetric Cartan matrix) is stable under the outer automorphism of $G(e, e, 2)$.

5.5. **Classifying distinguished root systems for $G(de, e, r)$.**

The complete reduced systems for $G(e, e, r)$ described above are also distinguished because all reflections, being involutive, are distinguished.

For $G(de, e, r)$ with $d > 1$, the distinguished roots (those corresponding to distinguished reflections) are those in $\mathfrak{R}_2(de, r)$ (corresponding to involutive reflections) and $\mathfrak{R}_1^1(d, r)$.

Applying Proposition 3.29 (every complete root system contains a distinguished root system) to Theorem 5.6 allows us to deduce the following result directly.

Theorem 5.13. *Given a family $\mathcal{F}_r = \{\, \mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1 \,\}$ of fractional ideals, where $\mathfrak{b}_0 = \mathfrak{b}_1 = \mathbb{Z}_k$ unless $r = 2$ and $de = 2p^l$, p prime, $l \geq 1$; define*

$$\mathfrak{R}_{\mathcal{F}_r}(de, e, r) := \mathfrak{a} \cdot \mathfrak{R}_1^1(d, r) \cup \mathfrak{b}_0 \cdot \mathfrak{R}_2^0(de, r) \cup \mathfrak{b}_1 \cdot \mathfrak{R}_2^1(de, r)\,.$$

Then every genus of distinguished $\mathbb{Z}_k$-root system for $G(de, e, r)$ over a field containing ζ_{de} contains exactly one root system $\mathfrak{R}_{\mathcal{F}_r}(de, e, r)$ where $\mathcal{F}_r$ satisfies the additional conditions

- *$\mathfrak{a}$, $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are integral,*
- *$\mathfrak{b}_0$ and $\mathfrak{b}_1$ are relatively prime divisors of $(1 + \zeta_{de})\mathbb{Z}_k$,*
- *$\mathfrak{a}$ divides $(1 - \zeta_d)\mathbb{Z}_k$,*
- *$\mathfrak{b}_0$ and $\mathfrak{b}_1$ divide $\mathfrak{a}$.*

Again, we remark that when $k = \mathbb{Q}(\zeta_{de})$, the ideal $(1 + \zeta_{de})\mathbb{Z}_k$ is $\mathbb{Z}_k$ or prime, see Lemma A.4.

The complete root systems for $G(e, e, r)$, including all the dihedrals $G(e, e, 2)$, as well as $G(2, 1, r)$ as described in Corollaries 5.7, 5.8 and 5.9 are all distinguished by definition.

Corollary 5.14. *Each genus of distinguished root system for $G(d, 1, r)$ contains a root system of the form:*

$$\mathfrak{R}_{\mathcal{F}_r}(d, 1, r) = \mathfrak{a} \cdot \mathfrak{R}_1^1(d, r) \cup \mathfrak{R}_2(d, r)\,,$$

where $\mathfrak{a} \in \{\mathbb{Z}_k, (1 - \zeta_d)\mathbb{Z}_k\}$. Thus there are two genera of distinguished root systems of $G(d, 1, r)$, each containing a principal root system.

Remark 5.15. It was assumed at the beginning of this section that $de > 1$, excluding the case $G(1, 1, r)$.

However it is well known that the reflections contained in the symmetric group $\mathfrak{S}_r$, that is, the Weyl group of type A_{r-1}, are precisely the reflections:

$$\mathrm{Ref}_2(1, r) = \{s_{k,l}^{(0)} \mid 1 \leq k < l \leq r\}$$

defined as above, with a single orbit of roots

$$\mathfrak{R}(1, 1, r) := \mathfrak{R}_2(1, r) = \{\mathfrak{r}_{k,l}^{(0)} \mid 1 \leq k < l \leq r\}$$

where

$$\mathfrak{r}_{k,l}^{(0)} = (\mathbb{Z}_k(e_k - e_l), \mathbb{Z}_k(e'_k - e'_l), -1)\,,$$

as defined above. The set $\mathfrak{R}(1, 1, r)$ forms a complete and reduced root system for $\mathfrak{S}_r$, and as the reflections are all involutive, it is also distinguished.

Since the action of $\mathfrak{S}_r$ on the roots consists of a single orbit, there is a unique genus of root system, with representative given by $\mathfrak{R}(1, 1, r)$ above.

One may notice that (see Remark 3.11), as far as we deal with well-generated reflection groups, the next result allows us to reduce our classification problem to a problem which has already been solved by Nebe [Ne].

Theorem 5.16. *All genera of distinguished root systems over the field of definition of a well-generated irreducible complex reflection group contain a principal root system.*

Proof. Let k be the field of definition of the well-generated irreducible complex reflection group G. If G is primitive, then $\mathbb{Z}_k$ is a P.I.D. and there is nothing to prove. If $G = G(d, 1, r)$ the statement has been given in Corollary 5.14. If $G = G(e, e, r)$ with $r > 2$ the statement has been given in Corollary 5.8. Finally, if $G = G(e, e, 2)$ the result is proved in Proposition 5.12. □

6. Principal root systems and Cartan matrices

6.1. Cartan matrices.

Notation 6.1. Given a distinguished $\mathbb{Z}_k$-root system $\mathfrak{R}$,

- for every distinguished reflection s of $G(\mathfrak{R})$, by Proposition 3.26 there is a corresponding root in $\mathfrak{R}$ which we denote by $\mathfrak{r}_s$
- given a set of distinguished reflections $S \subset \mathrm{Ref}(G(\mathfrak{R}))$, we set $\mathfrak{R}_S := \{\mathfrak{r}_s \mid s \in S\}$.

Definition 6.2. *Let $\mathfrak{R}$ be a principal distinguished $\mathbb{Z}_k$-root system. Let S be an ordered subset of distinguished reflections of* $\mathrm{Ref}(G(\mathfrak{R}))$. *For each $s \in S$, choose generators α_s and β_s of respectively I_{r_s} and J_{r_s} as in Remark 3.7. The matrix with entries $C_{s,t} = \langle \alpha_s, \beta_t \rangle$ for $s, t \in S$ is called a* Cartan matrix *for $\mathfrak{R}_S$.*

Note that all entries of such a Cartan matrix belong to $\mathbb{Z}_k$.

The proof of the following lemma is left to the reader.

Lemma 6.3.

(1) *Two Cartan matrices for the same subset of a given principal $\mathbb{Z}_k$-root system are conjugate by a diagonal matrix over $\mathbb{Z}_k^\times$.*
(2) *Let R and R' be ordered subsets of two principal $\mathbb{Z}_k$-root systems such that the ordered sets of reflections attached to them are equal. Then any Cartan matrix for R is conjugate to any Cartan matrix for R' by a diagonal matrix.*

Cartan matrices and genera of principal root systems.

Theorem 6.4. *Let (V, G) be an irreducible reflection group. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two principal distinguished $\mathbb{Z}_k$-root systems such that $G(\mathfrak{R}) = G(\mathfrak{R}') = G$. Let S be an ordered list of distinguished reflections which generates G. If a Cartan matrix for $\mathfrak{R}_S$ is conjugate to a Cartan matrix for $\mathfrak{R}'_S$ by a diagonal matrix of units of $\mathbb{Z}_k$, then $\mathfrak{R}$ and $\mathfrak{R}'$ belong to the same genus (that is, there is $\lambda \in k$ such that $\mathfrak{R}' = \lambda \cdot \mathfrak{R}$).*

Proof. For each $s \in S$, let $\mathfrak{r}_s \in \mathfrak{R}_S$ and $\mathfrak{r}'_s \in \mathfrak{R}'_S$ be the corresponding roots. As in in Remark 3.7, choose α_s a generator of $I_{\mathfrak{r}_s}$ and β_s a generator of $J_{\mathfrak{r}_s}$, such that $\langle \alpha_s, \beta_s \rangle = 1 - \zeta_s$; similarly choose α'_s a generator of $I_{\mathfrak{r}'_s}$ and β'_s a generator of $J_{\mathfrak{r}'_s}$, such that $\langle \alpha'_s, \beta'_s \rangle = 1 - \zeta_s$. Let $\lambda_s \in k^\times$ such that $\alpha'_s = \lambda_s \alpha_s$. Thus $\beta'_s = \lambda_s^{-*} \beta_s$.

Let C (resp. C') be the Cartan matrix determined by the above choices. Up to changing these choices by units, we may (and we do) assume $C = C'$. For any $s, t \in S$ we have

$$C_{s,t} = \langle \alpha_s, \beta_t \rangle = \langle \alpha'_s, \beta'_t \rangle = \lambda_s \lambda_t^{-1} \langle \alpha_s, \beta_t \rangle .$$

Hence, if $C_{s,t} \neq 0$, then $\lambda_s = \lambda_t$. Choose $s_0 \in S$. Since G is irreducible, and S generates G, for any $t \in S$, there is a sequence $s_0, \ldots, s_l = t$ such that for all j, $s_j \in S$ and $C_{s_{j+1}, s_j} \neq 0$ for all $j = 0, \ldots, l-1$, which shows that $\lambda_{s_0} = \lambda_t$. Let $\lambda := \lambda_{s_0}$. It follows that $\mathfrak{R}'_S = \lambda \cdot \mathfrak{R}_S$.

Finally, for any $\mathfrak{r} \in \mathfrak{R}$, by the assumption on orbits there is an expression $s_{\mathfrak{r}} = {}^{s_1 \cdots s_n} s_0$ for $s_{\mathfrak{r}}$ in terms of s_0. In particular, since $\mathfrak{R}$ is reduced, this ensures that $\mathfrak{r} = (s_1 \cdots s_n) \cdot \mathfrak{r}_{s_0}$. Thus the fact that $\mathfrak{R}_S$ and $\mathfrak{R}'_S$ belong to the same genus propagates to $\mathfrak{R}$ and $\mathfrak{R}'$. □

The proof of the following proposition results from the bijection given in its second part.

Proposition 6.5. *Let G be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. Assume given a family S of reflections which generates G.*

Let $\mathfrak{R} = \{\mathfrak{r} := (I_{\mathfrak{r}}, J_{\mathfrak{r}}, \zeta_{\mathfrak{r}})\}$ be a distinguished principal $\mathbb{Z}_k$-root system such that $G = G(\mathfrak{R})$. We denote by $\mathrm{Car}(\mathfrak{R}_S)$ the set of all matrices M satisfying the following conditions:

- *M is conjugate to C, a Cartan matrix of $\mathfrak{R}_S$, by a diagonal matrix (with diagonal entries in $k^\times$),*
- *the entries of M belong to $\mathbb{Z}_k$.*

There is a bijection between

- *the set of conjugacy classes of $\mathrm{Car}(\mathfrak{R}_S)$ under the action of the group of diagonal matrices with entries in $\mathbb{Z}_k^\times$,*
- *the genera of distinguished principal $\mathbb{Z}_k$-root systems for G,*

defined as follows.

→ *For $M = DCD^{-1} \in \mathrm{Car}(\mathfrak{R}_S)$, where $D = (\lambda_s)_{s \in S}$ is an invertible diagonal matrix, and for $s \in S$ corresponding to the root $\mathfrak{r}_s \in \mathfrak{R}$, we define $\mathfrak{r}_s^{(M)} := \lambda_s \cdot \mathfrak{r}_s$ and we denote by $\mathfrak{R}^{(M)}$ the orbit under G of the family $(\mathfrak{r}_s^{(M)})_{s \in S}$. Then $\mathfrak{R}^{(M)}$ is a distinguished principal $\mathbb{Z}_k$-root system for G, and M is a Cartan matrix associated with $\mathfrak{R}_S^{(M)}$.*

← *Let $\mathfrak{R}'$ be a distinguished principal root system for G. By item (1) of Theorem 6.4, we know that $C(\mathfrak{R}'_S)$ is conjugate to $C(\mathfrak{R}_S)$ by a diagonal matrix, and this shows that a Cartan matrix for $\mathfrak{R}'_S$ does belong to $\mathrm{Car}(\mathfrak{R}_S)$.*

The above result is at the heart of the classification of distinguished root systems given in Section 9.

6.2. **Free root lattices.**

The third assertion of the following theorem has been proved in [Ne, Corollary 13].

Theorem 6.6. *Let (V, G) be a well-generated reflection group.*

(1) There exists a distinguished principal $\mathbb{Z}_k$-root system $\mathfrak{R}$ with $G(\mathfrak{R}) = G$.
(2) If S is a set of reflections of cardinality $\dim V$ which generates G, and if the corresponding roots are $\mathfrak{r}_s = (\mathbb{Z}_k\alpha_s, \mathbb{Z}_k\alpha_s^\vee, \zeta_s)_{s\in S}$, then $(\alpha_s)_{s\in S}$ and $(\alpha_s^\vee)_{s\in S}$ are $\mathbb{Z}_k$-bases of $Q_\mathfrak{R}$ and $Q_\mathfrak{R}^\vee$ respectively.
(3) By (1) and (2), $Q_\mathfrak{R}$ is a free $\mathbb{Z}_k$-module.

Proof. Theorem 5.16 ensures that all genera of root systems for the well-generated imprimitive groups $G(d, 1, r)$ and $G(e, e, r)$ contain a principal $\mathbb{Z}_k$-root system. This, together with item (4) of Proposition 3.42, imply (1) and (2) directly. □

The next result shows in particular that, for a well generated group, the connection index can be easily computed from any Cartan matrix.

Proposition 6.7. *Assume that G is generated by $r = \dim V$ reflections $(s_i)_{1\le i\le r}$. For each i $(1 \le i \le r)$ choose $v_i \in L_{s_i}$, $w_i \in M_{s_i}$ such that $\langle v_i, w_i\rangle = 1 - \zeta_{s_i}$.*

Then the connection index of G is equal to $\det\big(\langle v_i, w_j\rangle\big)_{1\le i,j\le r}$.

Proof. The value of $\det\big(\langle v_i, f_j\rangle\big)_{1\le i,j\le r}$ is independent of the choices of the systems $v_i \in L_{s_i}$, $w_i \in M_{s_i}$ such that $\langle v_i, w_i\rangle = 1 - \zeta_{s_i}$. By item (2) of Theorem 6.6, we may choose $v_i = \alpha_i$ and $w_i = \alpha_i^\vee$ so that $(\alpha_i)_{1\le i\le r}$ and $(\alpha_i^\vee)_{1\le i\le r}$ are $\mathbb{Z}_k$-bases of respectively $Q_\mathfrak{R}$ and $Q_\mathfrak{R}^\vee$ for some distinguished principal $\mathbb{Z}_k$-root system $\mathfrak{R}$ such that $G(\mathfrak{R}) = G$.

Then the statement is nothing but a translation of the proof of Theorem 3.48. □

Remark 6.8. Let (V, G) be an irreducible and **not** well-generated reflection group. Let $r = \dim V$ and assume that G is generated by the set of reflections $S = \{s_1, \dots, s_{r+1}\}$. Let $\mathfrak{R}$ be a distinguished principal root system for G. For the roots $\mathfrak{r}_{s_1}, \dots \mathfrak{r}_{s_{r+1}}$ in $\mathfrak{R}$ corresponding to S, we choose $\alpha_1, \dots, \alpha_{r+1}$ and $\beta_1, \dots, \beta_{r+1}$ as in Remark 3.7, and we denote by C the corresponding Cartan matrix for $\mathfrak{R}_S$.

By item (2) of Proposition 3.42,

$$Q_\mathfrak{R} = \sum_{i=1}^{r+1} \mathbb{Z}_k\alpha_i \quad\text{and}\quad Q_\mathfrak{R}^\vee = \sum_{i=1}^{r+1} \mathbb{Z}_k\beta_i\,.$$

We will show (see Proposition 6.18 and the tables of Appendix B) that for every irreducible, not well-generated reflection group (V, G) there always exists at least one distinguished principal root system for which (keeping the notation of the previous paragraph) there exists i_0 and j_0 such that:

$$Q_{\mathfrak{R}} = \bigoplus_{i \neq i_0} \mathbb{Z}_k \alpha_i \quad \text{and} \quad Q^\vee_{\mathfrak{R}} = \bigoplus_{i \neq j_0} \mathbb{Z}_k \beta_i \,.$$

Then the connection index of $\mathfrak{R}$ is equal to the determinant of the sub-matrix of C where the i_0-th row and the j_0-th column have been dropped.

6.3. **Principal root bases and Cartan matrices for imprimitive well generated reflection groups.**

Throughout this subsection, ζ is a root of unity, and k is a number field containing ζ and closed under conjugation.

Recall (see Section 5) that $\mathfrak{r}^{(j)}_{kl} = (\mathbb{Z}_k(e_k - \zeta^j e_l), \mathbb{Z}_k(e'_k - \zeta^j e'_l), -1)$. Write $\alpha^{(j)}_{kl}$ for the vector $e_k - \zeta^j e_l$ where $1 \leq k < l \leq n$.

The case of $G(e, e, r)$.

A. $G(e, e, 2)$ with e even. The Cartan matrix of the root basis $\{\mathfrak{r}_s, \mathfrak{r}_t\}$ of the principal $\mathbb{Z}_k$-root system $\mathfrak{R}$ of Proposition 5.12 is:

$$C = \begin{pmatrix} 2 & -(2 + \zeta_e + \zeta_e^{-1}) \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -4\cos^2(\frac{\pi}{e}) \\ -1 & 2 \end{pmatrix}.$$

Hence:

Lemma 6.9. *The connection index of $G(e, e, 2)$ with e even is*

$$c_{(e,e,2)} = (1 - \zeta_e)(1 - \zeta_e^{-1}) \,.$$

B. The general case $G(e, e, r)$ with $r > 2$ or e odd, $r = 2$.

By Corollary 5.8, there is a unique genus of $\mathbb{Z}_k$-root system (necessarily distinguished, being in $G(e, e, r)$), and which contains the representative $\mathbb{Z}_k$-root system

$$\mathfrak{R} = \mathfrak{R}(e, r) = \left\{ \mathfrak{r}^{(j)}_{kl} \mid 1 \leq k < l \leq r, 0 \leq j < r \right\} .$$

Using the notation of Section 5, the set:

$$S = \left\{ s^{(0)}_{12}, s^{(0)}_{23}, \ldots, s^{(0)}_{(r-1),r}, s^{(1)}_{(r-1),r} \right\} ,$$

consisting of r involutive reflections, generates $G(e, e, r)$, with corresponding set of roots:

$$\Pi = \left\{ \mathfrak{r}^{(0)}_{12}, \mathfrak{r}^{(0)}_{23}, \ldots, \mathfrak{r}^{(0)}_{(r-1),r}, \mathfrak{r}^{(1)}_{(r-1),r} \right\} ,$$

where $\mathfrak{r}^{(j)}_{k,l} := \big(\mathbb{Z}_k(e_k - \zeta_e^j e_l), \mathbb{Z}_k(e'_k - \zeta_e^j e'_l), -1\big)$.

Again, by 3.42 (4), Π is a root basis. The Cartan matrix for Π is:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & & & & \\ -1 & 2 & -1 & 0 & \cdots & & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & & \\ \vdots & & \ddots & \ddots & \ddots & & & \\ & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\ & & \cdots & 0 & -1 & 2 & -1 & -1 \\ & & & \cdots & 0 & -1 & 2 & (1+\zeta_e^{-1}) \\ & & & \cdots & 0 & -1 & (1+\zeta_e) & 2 \end{pmatrix}$$

In order to compute the determinant of the above matrix, we prove the following lemma, which will also be useful later on.

Lemma 6.10. *Let $r \geq 3$. Consider an $r \times r$ matrix of type*

$$C_r := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & * & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & * \end{pmatrix} .$$

Let C_{r-1} (resp. C_{r-2}) be the $(r-1)\times(r-1)$-matrix (respectively, the $(r-2)\times(r-2)$-matrix) obtained by suppressing the first row and the first column (resp. the first two rows and the first two columns). Then

$$\det C_r = 2 \det C_{r-1} - \det C_{r-2} .$$

Proof. It is immediate by expanding with respect to the first row. □

Proposition 6.11. *For all choices of $e \geq 1$ and $r \geq 2$, the connection index of $G(e,e,r)$ is*

$$c_{(e,e,r)} = (1-\zeta_e)(1-\zeta_e^{-1}) .$$

Proof. This results from Lemma 6.10 and from Lemma 6.9. □

The case of $G(d,1,r)$.

In this case, $k = \mathbb{Q}(\zeta_d)$ and $\mathbb{Z}_k = \mathbb{Z}[\zeta_d]$.

Let $\mathfrak{a}$ be an integral ideal which divides $(1-\zeta_d)\mathbb{Z}_k$. By Corollary 5.14 every genus of distinguished root system for $G(d,1,r)$ contains a $\mathbb{Z}_k$-root system of the form $\mathfrak{R}_\mathfrak{a} := \mathfrak{a} \cdot \mathfrak{R}_1^1(d,r) \cup \mathfrak{R}_2(d,r)$ where $\mathfrak{a} = \mathbb{Z}_k$ or $(1-\zeta_d)\mathbb{Z}_k$ (so is principal). In the notation of Section 5, the set:

$$S = \left\{ s_{12}^{(0)}, s_{23}^{(0)}, \dots, s_{(r-1),r}^{(0)}, s_r \right\}$$

consisting of r reflections, generates $G(d,1,r)$, and has corresponding set of roots:

$$\Pi_\mathfrak{a} = \left\{ \mathfrak{r}_{12}^{(0)}, \mathfrak{r}_{23}^{(0)}, \dots, \mathfrak{r}_{(r-1),r}^{(0)}, \mathfrak{a} \cdot \mathfrak{r}_r^{(1)} \right\} ,$$

where

$$\mathfrak{a}\cdot\mathfrak{r}_r^{(1)} = \left(\mathfrak{a}e_r, (1-\zeta^{-1})\mathfrak{a}^{-*}e'_r, \zeta_d\right), \text{ and}$$
$$\mathfrak{r}_{k,l}^{(0)} = \left((e_k - e_l)\mathbb{Z}_k, (e'_k - e'_l)\mathbb{Z}_k, -1\right).$$

In particular, $\Pi := \Pi_{\mathbb{Z}_k}$ provides a principal $\mathbb{Z}_k$-basis for $Q_{\mathfrak{R}_{\mathbb{Z}_k}}$.

The Cartan matrix for Π is:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & & & & \\ -1 & 2 & -1 & 0 & \cdots & & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & & \\ \vdots & & \ddots & \ddots & \ddots & & & \\ & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\ & & \cdots & 0 & -1 & 2 & -1 & 0 \\ & & & \cdots & 0 & -1 & 2 & -(1-\zeta_d) \\ & & & \cdots & 0 & 0 & -1 & 1-\zeta_d \end{pmatrix},$$

Since $\mathfrak{a}$ is principal, then $\Pi_{\mathfrak{a}}$ is principal and its Cartan basis is obtained by conjugating the above matrix by $\mathrm{diag}(1,\dots,1,(1-\zeta_d))$. Applying Lemma 6.10 then gives:

Proposition 6.12. *The connection index of $G(d,1,r)$ $(d\geq 2)$ is*

$$c_{(d,1,r)} = 1-\zeta_d\,.$$

6.4. **The case of not well-generated groups $G(de,e,r)$.**

We assume here that $G = G(de,e,r)$ is not well-generated, thus $d>1$ and $e>1$.

It follows from Theorem 5.13, that $\mathfrak{R} := \mathfrak{R}_1^1(d,r)\cup\mathfrak{R}_2(de,r)$ is a distinguished $\mathbb{Z}_k$-root system for G, which is principal since with the notation from Section 5, $\mathfrak{R}_1^i(d,r)$ consists of:

$$\mathfrak{r}_k^i = \mathbb{Z}_k\cdot\left(e_k, (1-\zeta_d^{-i})e'_k, \zeta_d\right)$$

with $1\leq k\leq r$ and $0<i<d$, and $\mathfrak{R}_2(de,r)$ consists of:

$$\mathfrak{r}_{k,l}^{(j)} = \mathbb{Z}_k\cdot\left((e_k-\zeta_{de}^j e_l), (e'_k-\zeta_{de}^j e'_l), -1\right)$$

with $1\leq k<l\leq r$ and $0\leq j<de$.

Recall (Definition 3.40) that a subset $\Pi = ((I_\mathfrak{r}, J_\mathfrak{r}, \zeta_\mathfrak{r}))_{\mathfrak{r}\in\Pi}\subset\mathfrak{R}$ is a set of *root generators* if $Q_\mathfrak{R} = \sum_{i\in\Pi} I_{\mathfrak{r}_i}$. and a *root lattice basis* if $Q_\mathfrak{R} = \bigoplus_{\mathfrak{r}\in R} I_\mathfrak{r}$.

Proposition 6.13. *Assume that $d>1$ and $e>1$.*

(1) The set:

$$\Pi = \left\{\mathfrak{r}_{12}^{(0)}, \mathfrak{r}_{23}^{(0)}, \dots, \mathfrak{r}_{(r-1),r}^{(0)}, \mathfrak{r}_{(r-1),r}^{(1)}, \mathfrak{r}_r^1\right\}$$

forms a set of root generators for $G = G(de,e,r)$.

The corresponding set of $r+1$ reflections:

$$S_\Pi := \left\{s_{12}^{(0)}, s_{23}^{(0)}, \dots, s_{(r-1),r}^{(0)}, s_{(r-1),r}^{(1)}, s_r^1\right\}$$

generates G and no set of r reflections will do so.

(2) *If $(r, de) \neq (2, 2p^l)$ ($l \geq 1$ any integer and p any prime), the genera of distinguished root systems for G are in bijection with the integral ideals dividing $(1-\zeta_d)\mathbb{Z}_k$.*

More precisely, if $\mathfrak{a}$ is such a divisor, the corresponding genus contains the root system with set of root generators

$$\Pi_{\mathfrak{a}} = \left\{ \mathfrak{r}_{12}^{(0)}, \mathfrak{r}_{23}^{(0)}, \ldots, \mathfrak{r}_{(r-1),r}^{(0)}, \mathfrak{r}_{(r-1),r}^{(1)}, \mathfrak{a} \cdot \mathfrak{r}_r^1 \right\} .$$

Proof.

(1) The fact that S_Π generates G has already been observed in Subsection 5.1. It follows by Proposition 3.42(2) that the set Π is a set of root generators.

(2) The $(r+1) \times (r+1)$ Cartan matrix C of Π is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & & & & & \\ -1 & 2 & -1 & 0 & \cdots & & & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & & & \\ \vdots & & \ddots & \ddots & \ddots & & & & \\ & \cdots & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ & & \cdots & 0 & -1 & 2 & -1 & -1 & 0 \\ & & & \cdots & 0 & -1 & 2 & 1+\zeta_{de}^{-1} & \zeta_d - 1 \\ & & & \cdots & 0 & -1 & 1+\zeta_{de} & 2 & -\zeta_{de}(1-\zeta_d) \\ & & & \cdots & 0 & 0 & -1 & -\zeta_{de}^{-1} & 1-\zeta_d \end{pmatrix},$$

Theorem 5.13 ensures that (except when $r = 2$ and $de = 2p^l$, p prime, $l \geq 1$), every other genus of root system for G is obtained as $\mathfrak{a} \cdot \mathfrak{R}_1^1(d, r) \cup \mathfrak{R}_2(de, r)$ where $\mathfrak{a}$ is an integral ideal dividing $(1-\zeta_d)\mathbb{Z}_k$. This gives rise to root systems with root generators

$$\Pi_{\mathfrak{a}} = \left\{ \mathfrak{r}_{12}^{(0)}, \mathfrak{r}_{23}^{(0)}, \ldots, \mathfrak{r}_{(r-1),r}^{(0)}, \mathfrak{r}_{(r-1),r}^{(1)}, \mathfrak{a} \cdot \mathfrak{r}_r^1 \right\} .$$

When $\mathfrak{a} = a\mathbb{Z}_k$ is principal, these root generators are principal with Cartan matrix C_a conjugate to C by $\mathrm{diag}(1, \ldots, 1, a)$. □

We will describe root lattice bases for most genera of distinguished $\mathbb{Z}_k$-root system for G (and all genera of principal distinguished $\mathbb{Z}_k$-root systems). By Theorem 5.13, each genus corresponds to the choice of ideals $\mathfrak{b}_0$, $\mathfrak{b}_1$ and $\mathfrak{a}$.

Remark 6.14. Assume $\mathfrak{b}_0, \mathfrak{b}_1$ and $\mathfrak{a}$ are principal, thus $\mathfrak{b}_0 = b_0\mathbb{Z}_k, \mathfrak{b}_1 = b_1\mathbb{Z}_k$ and $\mathfrak{a} = a\mathbb{Z}_k$, and let C' denote the matrix obtained by conjugating by $\mathrm{diag}(b_0, \ldots, b_0, b_1, a)$ the Cartan matrix C for Π. Let us explain how to determine if the root system with Cartan matrix C' has a root lattice basis or coroot lattice basis and how to compute the corresponding connection index.

Write l_1, l_2 and l_3 for the last three rows of C'. These rows satisfy the linear dependency relationship:

$$\frac{1}{b_0}l_1 - \frac{1}{b_1}l_2 + \frac{(1-\zeta_{de})}{a}l_3 = 0$$

Similarly, if c, c_2 and c_3 are the last three columns of C', then:

$$b_0c_1 - b_1c_2 + \frac{(1-\zeta_{de}^{-1})}{a'}c_3 = 0$$

where a' is the algebraic integer such that $aa' = (1-\zeta_d)$.

Denote P_{ij} the property that l_i is an integral linear combination of the other lines in $\{l_1, l_2, l_3\}$ and c_j is an integral linear combination of the other columns in $\{c_1, c_2, c_3\}$. If P_{ij} holds then the root system has a root lattice basis and a coroot lattice basis, and its connection index is $\det C'_{ij}$ where C'_{ij} is the matrix obtained from C' by removing the row l_i and the column c_j.

The bottom right corner of the matrix of cofactors of C' (the matrix of the $\det C'_{ij}$) has the following form:

$$\begin{pmatrix} 1-\zeta_d & \frac{b_1}{b_0}(1-\zeta_d) & \frac{a}{b_0}(1-\zeta_{de}^{-1}) \\ \frac{b_0}{b_1}(1-\zeta_d) & 1-\zeta_d & \frac{a}{b_1}(1-\zeta_{de}^{-1}) \\ b_0a'(1-\zeta_{de}) & b_1a'(1-\zeta_{de}) & (1-\zeta_{de})(1-\zeta_{de}^{-1}) \end{pmatrix}.$$

We describe case by case, according to values of d and e, the genera of $\mathbb{Z}_k$-root systems for G when $k = \mathbb{Q}[\zeta_{de}]$ (the field of definition of G), and compute connection indices.

By Theorem 5.13, genera of distinguished $\mathbb{Z}_k$-root systems are represented by $\mathfrak{a} \cdot \mathfrak{R}_1^1(d,r) \cup \mathfrak{b}_0 \cdot \mathfrak{R}_2^0(de,r) \cup \mathfrak{b}_1 \cdot \mathfrak{R}_2^1(de,r)$ where

- $\mathfrak{a}$ is an integral ideal dividing $(1-\zeta_d)\mathbb{Z}_k$,
- $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are relatively prime ideals dividing $\mathfrak{a}$ and $(1+\zeta_{de})\mathbb{Z}_k$.

We will first consider the cases where it is possible that $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are non-trivial, which (see Proposition 6.13) may happen when $r = 2$, e is even, and $de = 2p^l$. For $\mathfrak{a}$ to be non-trivial, we need that d be a prime power, which implies that $d = p^h$ or p odd, $d = 2$. In the second case $1-\zeta_d = 2$ is prime to $1+\zeta_{de} = 1-\zeta_{p^l}$, thus $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are trivial. Thus the only cases where $\mathfrak{b}_0$ or $\mathfrak{b}_1$ could be non-trivial is:

The case $r = 2$, $e = 2p^l$, $d = p^h$, with $l \geq 0, h \geq 1$.

Let $\mathfrak{p} := (1+\zeta_{de})\mathbb{Z}_k$. Then $1+\zeta_{de} = 1-(-\zeta_{de})$, and since $de = 2p^{h+l}$, $-\zeta_{de}$ has order p^{h+l} if p is odd, and has order $2p^{h+l}$ if $p = 2$. Thus by Proposition A.5, we have that $\mathfrak{p}$ is prime.

Furthermore, let $e' = e$ if $p = 2$, or $e' = e/2$ otherwise. Then it follows from Lemma A.4(2) that $(1-\zeta_d)\mathbb{Z}_k = \mathfrak{p}^{e'}$. Hence every ideal dividing $(1-\zeta_d)\mathbb{Z}_k$ is principal, equal to $\mathfrak{p}^n$ for $0 \leq n \leq e'$, thus the above root systems are principal.

Proposition 6.15. *Let $e = 2p^l$, $d = p^h$, with $l \geq 0$ and $h \geq 1$, and $\mathfrak{p} := (1+\zeta_{de})\mathbb{Z}_k$. Write $e' = e$ if $p = 2$, or $e' = e/2$ otherwise. Then every genus of distinguished root system for $G(2p^{h+l}, 2p^l, 2)$ contains one of the $3e'+1$ root systems:*

$$\mathfrak{a} \cdot \mathfrak{R}_1^1(d,r) \cup \mathfrak{b}_0 \cdot \mathfrak{R}_2^0(de,r) \cup \mathfrak{b}_1 \cdot \mathfrak{R}_2^1(de,r)\,,$$

where

$$(\mathfrak{b}_0, \mathfrak{b}_1, \mathfrak{a}) \in \{(\mathbb{Z}_k, \mathbb{Z}_k, \mathbb{Z}_k)\} \bigcup_{1 \leq n \leq e'} \{(\mathbb{Z}_k, \mathfrak{p}, \mathfrak{p}^n), (\mathfrak{p}, \mathbb{Z}_k, \mathfrak{p}^n), (\mathbb{Z}_k, \mathbb{Z}_k, \mathfrak{p}^n)\}\,.$$

These root systems are all principal, have a root lattice basis and a coroot lattice basis, and their connection index is one of $\mathbb{Z}_k$, $\mathfrak{p}$ or $\mathfrak{p}^2$.

Proof. We consider the cases $p = 2$ and $p > 2$ separately.

Suppose that $p = 2$. Thus $d = p^h$, $e = 2p^l$ and $e' = e$. In this case, $(1-\zeta_{de})\mathbb{Z}_k = (1+\zeta_{de})\mathbb{Z}_k = \mathfrak{p}$. The following table exhausts the possibilities for the root systems; we use the notation introduced in Remark 6.14, as well as "centre-dot" notation in the $\mathfrak{b}_0$ and $\mathfrak{b}_1$ columns, which indicates that $\mathfrak{b}_0$ (respectively $\mathfrak{b}_1$) may be either $\mathbb{Z}_k$ or $\mathfrak{p}$.

$\mathfrak{b}_0, \mathfrak{b}_1, \mathfrak{a}$	properties	connection index
$\mathbb{Z}_k, \mathbb{Z}_k, \mathbb{Z}_k$	P_{13}, P_{23}	$(1-\zeta_{de}^{-1})\mathbb{Z}_k = \mathfrak{p}$
$\cdot, \cdot, \mathfrak{p}$	P_{13} or P_{23}	$(1-\zeta_{de})\mathbb{Z}_k = \mathfrak{p}$
$\cdot, \cdot, \mathfrak{p}^j$ for $2 \leq j \leq e-1$	P_{33}	$\mathfrak{p}^2$
$\cdot, \cdot, \mathfrak{p}^e$	P_{32} or P_{31}	$\mathfrak{p}$

Both $\mathfrak{p}$ and $\mathfrak{p}^2$ occur as connection index, depending on the root system, which illustrates that for not well-generated groups, the connection index depends on the root system.

Suppose now that $p > 2$. Thus $d = p^h$, $e = 2p^l$ and $e' = p^l$. In this case, $(1-\zeta_{de})\mathbb{Z}_k = \mathbb{Z}_k$. Again, the table below exhausts the possibilities for the root systems; the notation is as above.

$\mathfrak{b}_0, \mathfrak{b}_1, \mathfrak{a}$	properties	connection index
$\mathbb{Z}_k, \mathbb{Z}_k, \mathbb{Z}_k$	P_{33}	$\mathbb{Z}_k$
$\mathbb{Z}_k, \mathfrak{p}, \mathbb{Z}_k$	P_{23}	$\mathbb{Z}_k$
$\mathfrak{p}, \mathbb{Z}_k, \mathbb{Z}_k$	P_{13}	$\mathbb{Z}_k$
$\cdot, \cdot, \mathfrak{p}^j$ for $1 \leq j \leq e'$	P_{33}	$\mathbb{Z}_k$

□

In the remaining cases we know that the genera of root systems are represented by the system $\mathfrak{a}\mathfrak{R}_1^1(d,r) \cup \mathfrak{R}_2^0(de,r) \cup \mathfrak{R}_2^1(de,r)$ where $\mathfrak{a}$ runs over integral ideals dividing $(1-\zeta_d)\mathbb{Z}_k$.

The case d composite.

Proposition 6.16. *Assume d composite. Then $G(de,e,r)$ has a single genus of root systems, represented by the principal root system:*

$$\mathfrak{R}_1^1(d,r) \cup \mathfrak{R}_2(de,r)\,,$$

which has a root lattice basis and a coroot lattice basis, and connection index $\mathbb{Z}_k$.

Proof. When d is composite, $1-\zeta_d$ is a unit thus $\mathfrak{a}$ is trivial. So there is a unique genus of root system, which contains the root system given in the statement of the proposition. This root system has root generators Π, for which the Cartan matrix is C. The property P_{13} holds and the connection index is $(1-\zeta_{de}^{-1})\mathbb{Z}_k = \mathbb{Z}_k$. □

The case d a prime power.

Assume $d = p^a$ with p prime, $a \geq 1$ and write $e = p^h n'$ with n' prime to p. Then by Proposition A.5, we have the decomposition in prime ideals $(1-\zeta_{p^a})\mathbb{Z}_k = (\mathfrak{p}_1 \dots \mathfrak{p}_\delta)^{p^h}$, where $\delta = \varphi(n')/s$ and s is the multiplicative order of $p \mod n'$.

We can say more when $\delta = 1$ than in the other cases.

Proposition 6.17. *Let $d = p^a$, $e = p^h n'$ with p prime, $a \geq 1, h \geq 0$ and n' prime to p and assume that p generates the multiplicative group $(\mathbb{Z}/n'\mathbb{Z})^\times$ (this includes the case $n' = 1$). Let $G := G(de,e,r) = G(p^{a+h}n', p^h n', r)$ and $\mathfrak{p} := (1-\zeta_{p^{a+h}})\mathbb{Z}_k$.*

Then (assuming we are not in the case 6.15) every genus of distinguished $\mathbb{Z}_k$-root system for G is represented by one of the p^h+1 principal root systems

$$\mathfrak{p}^n\mathfrak{R}_1^1(d,r) \cup \mathfrak{R}_2^0(de,r) \cup \mathfrak{R}_2^1(de,r)$$

where $0 \leq n \leq p^h$. These systems have a root lattice basis and a coroot lattice basis and connection index one of $\mathbb{Z}_k$, $\mathfrak{p}$ or $\mathfrak{p}^2$.

Proof. Since $\delta = 1$, the ideal $\mathfrak{p}$ is prime and $(1-\zeta_d)\mathbb{Z}_k = (1-\zeta_{p^a})\mathbb{Z}_k = \mathfrak{p}^{p^h}$, thus the divisors $\mathfrak{a}$ of $(1-\zeta_d)\mathbb{Z}_k$ are the principal ideals $\mathfrak{p}^n$ as described in the proposition.

The linear dependence relations noted in Remark 6.14 are still valid, that is if $n' \neq 1$ then P_{33} holds and the connection index is $\mathbb{Z}_k$. If $n' = 1$ then $(1-\zeta_{de})\mathbb{Z}_k = \mathfrak{p}$ and if $\mathfrak{a} = \mathbb{Z}_k$ then P_{13} holds and the connection index is $\mathfrak{p}$, if $a = \mathfrak{p}^{p^h}$ then P_{31} holds and the connection index is $\mathfrak{p}$, and if $a = \mathfrak{p}^n$ with $0 < n < p^h$ then P_{33} holds and the connection index is $\mathfrak{p}^2$. □

When $\delta > 1$ we can say less. There are $(p^h+1)^\delta$ distinct ideals $\mathfrak{p}_i$ dividing $(1-\zeta_d)\mathbb{Z}_k$, thus $(p^h+1)^\delta$ genera of distinguished $\mathbb{Z}_k$-root systems.

However, note that by Example A.6, for $G(39,3,3)$ for example the $\mathfrak{p}_i$ may be non-principal, which gives rise to genera containing no principal

root system, which illustrates the failure of Theorem 5.16 for non well-generated reflection groups.

There are still principal root systems for every group $G(de, e, r)$, such as:

$$\mathfrak{R}_1^1(d, r) \cup \mathfrak{R}_2(de, r),$$

which is distinguished, and:

$$\mathfrak{R}_1(d, r) \cup \mathfrak{R}_2(de, r),$$

which is complete and reduced. In particular, for all $d, e > 1$ we can fulfill the promise of Remark 6.8:

Proposition 6.18. *The principal distinguished $\mathbb{Z}_k$-root system*

$$\mathfrak{R}_1^1(d, r) \cup \mathfrak{R}_2(de, r)$$

for $G(de, e, r)$ ($d > 1$, $e > 1$) has a root lattice basis and a coroot lattice basis. Its connection index is $(1 - \zeta_{ed}^{-1})\mathbb{Z}_k$.

Proof. See proof of Proposition 6.16. □

7. Reflection groups and root systems over $\mathbb{R}$

7.1. Preliminary: positive Hermitian forms.

Let $\mathrm{Herm}(V; k)$ denote the k-vector space of Hermitian forms on V. Say that $\varphi : V \to W$ is *Hermitian* if for any $g \in \mathrm{GL}(V)$ we have $\varphi \circ g = g^\vee \circ \varphi$ (equivalently φ can be represented by an Hermitian matrix with respect to a basis $(e_i)_{i \in E}$ of V and its dual basis $(f_i)_{i \in E}$ of W). Let $\mathrm{Herm}(V, W)$ denote the k-vector space of Hermitian maps from V to W. Since the Hermitian pairing $V \times W \to k$ is non-degenerate, the linear map

$$\begin{cases} \mathrm{Herm}(V, W) \to \mathrm{Herm}(V; k)\,, \\ \varphi \mapsto (\,(v_1, v_2) \mapsto (v_1 \mid v_2)_\varphi := \langle v_1, \varphi(v_2)\rangle\,) \end{cases}$$

is an isomorphism. Moreover, the Hermitian form $(\cdot \mid \cdot)_\varphi$ is:

- non-degenerate if and only if φ is an isomorphism,
- positive (resp. positive definite) if and only if for all $v \in V - \{0\}$, $\langle v, \varphi(v)\rangle \geq 0$ (resp. $\langle v, \varphi(v)\rangle > 0$), in which case we say that φ is *positive* (resp. *positive definite*).

Let G be a finite group of $\mathrm{GL}(V)$ (which may be viewed as a finite subgroup of the subgroup of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ which preserves the pairing). Then there exists a positive definite Hermitian kG-isomorphism $\varphi : V \xrightarrow{\sim} W$: given any basis $(e_1, \ldots, e_r)$ of V and its dual basis $(f_1, \ldots, f_r)$ of W (that is, $\langle e_i, f_j\rangle = \delta_{i,j}$), the isomorphism $\varphi : e_i \mapsto f_i$ is positive definite, and its average $\frac{1}{|G|}\sum_{g \in G} g^\vee \varphi g^{-1}$ is both G-stable and positive definite.

Remark 7.1. If V is an absolutely irreducible kG-module, the vector space of kG-morphisms $V \to W$ is one dimensional, hence the space of G-invariant Hermitian forms on V is also one dimensional.

In particular then, the trivial positive definite quadratic form on V need not be invariant under G: consider the case where $k = \mathbb{Q}(\sqrt{5})$ and G is the dihedral group of order 10. Then there is a G-invariant symmetric bilinear form on $V = k^2$ whose discriminant is $(5-\sqrt{5})/8$ – since it is the determinant (up to a square in $k^\times$) of the matrix:

$$\begin{pmatrix} 1 & \cos(2\pi/5) \\ \cos(2\pi/5) & 1 \end{pmatrix}.$$

The proof of the following lemma is immediate.

Lemma 7.2. *Assume that s is a reflection in G, whose reflecting line, reflecting hyperplane dual reflecting line, dual reflecting hyperplane are respectively L, H, M, K (see Definition 2.2). Let $\varphi : V \xrightarrow{\sim} W$ be any positive definite Hermitian G-stable isomorphism. Then*

(1) H is the orthogonal of L for the Hermitian form $(\cdot \mid \cdot)_\varphi$ on V,
(2) M is the orthogonal of H for the Hermitian pairing $\langle \cdot, \cdot \rangle$,
(3) $\varphi(L) = M$ and $\varphi(H) = K$, and
(4) for $x \in L$ and $v \in V$,

$$s(v) = v - \frac{(v \mid x)_\varphi}{(x \mid x)_\varphi}(1-\zeta)x\,.$$

7.2. **Families of simple reflections for real reflection groups.**

In this subsection, we assume that k is a subfield of the field $\mathbb{R}$ of real numbers. We denote by $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) the set of nonnegative (resp. nonpositive) real numbers, and we set $k^+ := \mathbb{R}^+ \cap k$ and $k^- := \mathbb{R}^- \cap k$. Thus $k = k^+ \cup k^-$ and $k^+ \cap k^- = \{0\}$.

Let V be a finite dimensional k-vector space, and let G be a finite subgroup of $GL(V)$. Let S be the set of reflections of G. Notice that the determinant of a reflection is always -1, and so the reflections $s \in S$ are in bijection with pairs (L_s, M_s) where L_s is the reflecting line of s (a line in V) while M_s is the dual reflecting line of s (a line in W).

We fix a positive definite Hermitian kG-isomorphism $\varphi : V \xrightarrow{\sim} W$ throughout.

Definition 7.3. *We call* admissible preorder *on V and W a preorder obtained as follows:*

We choose a nonzero element $v_0 \in V$ which belongs to no reflecting hyperplanes $H_s \subset V$ for $s \in S$, and such that $\varphi(v_0)$ belongs to no reflecting hyperplane $K_s \subset W$.

Such a choice induces a preorder on W and on V by:

- *defining $w \in W$ to be* positive *(which we denote $w > 0$) if $\langle v_0, w \rangle > 0$ and to be* negative *if $-w$ is positive, and*

- *for $v \in V$ defining $v > 0$, if $(v \mid v_0)_\varphi > 0$,*

The relation "$v_1 > v_2$ whenever $v_1 - v_2 > 0$" is preserved under vector addition and positive scalar multiplication, and so makes V (respectively W) a preordered vector space.

For $s \in S$ we set $L_s^+ := \{v \in L_s \mid v > 0\} \sqcup \{0\}$ and $L_s^- := -L_s^+$; we have $L_s = L_s^+ \sqcup \{0\} \sqcup L_s^-$. We define M_s^+ and M_s^- similarly.

Thus an admissible preorder determines a family of *positive half-lines* $(L_s^+)_{s\in S}$ in V and a family of *positive half-lines* $(M_s^+)_{s\in S}$ in W.

Definition 7.4. *Given a positive definite Hermitian kG-isomorphism $\varphi : V \xrightarrow{\sim} W$, and an admissible preorder on V and W, a family Σ of reflections of G satisfying:*

(A) $V = \bigoplus_{\sigma\in\Sigma} L_\sigma$ and $W = \bigoplus_{\sigma\in\Sigma} M_\sigma$, and

(B) for all $s \in S$, $L_s^+ \subset \sum_{\sigma\in\Sigma} L_\sigma^+$ and $M_s^+ \subset \sum_{\sigma\in\Sigma} M_\sigma^+$,

is called a family of simple reflections *for G.*

Proposition 7.5. *Let G be a finite subgroup of $\mathrm{GL}(V)$, let S be the set of reflections of G, let $\varphi : V \xrightarrow{\sim} W$ be a positive definite Hermitian kG-isomorphism, and suppose given an admissible preorder on V and W. Then there exists a unique family of simple reflections for G.*

Thanks to Proposition 7.5 it makes sense to say that a set of reflections is a "set of simple reflections" if it is the set of simple reflections determined by an admissible preorder.

Notice that all assertions concerning W are analogous to those concerning V, so from now on we only state (and prove) assertions concerning V.

We remark that there are subsets of S which satisfy the criterion (2) of the definition of a family of simple reflections – as indeed, S itself has that property. It turns out that a minimal such subset is precisely a family of simple reflections.

Lemma 7.6. *Suppose that $\Sigma \subseteq S$ is minimal subject to satisfying:*

(B) for all $s \in S$, $L_s^+ \subset \sum_{\sigma\in\Sigma} L_\sigma^+$.

Then

(1) For all distinct reflections $\sigma_1, \sigma_2 \in \Sigma$ we have $(L_{\sigma_1}^+ \mid L_{\sigma_2}^+)_\varphi \subset k^-$.

(2) For distinct reflections $\sigma_1, \sigma_2 \in \Sigma$, and $v_2 \in L_{\sigma_2}^+$, then $\sigma_1(v_2) = v_2 + v_1$ for some $v_1 \in L_{\sigma_1}^+$ — that is, $\sigma_1(L_{\sigma_2}^+) = L_{\sigma_1\sigma_2\sigma_1^{-1}}^+$.

Proof. We prove first (1), arguing by contradiction: assume there are $v_1 \in L_{\sigma_1}^+$ and $v_2 \in L_{\sigma_2}^+$ such that $(v_1 \mid v_2)_\varphi > 0$. Thus

$$\sigma_1(v_2) = v_2 - 2\frac{(v_2 \mid v_1)_\varphi}{(v_1 \mid v_1)_\varphi} v_1 = v_2 - \lambda v_1 \in L_s$$

where $\lambda > 0$ and $s = \sigma_1\sigma_2\sigma_1^{-1}$. Either $\sigma_1(v_2)$ is positive or negative.

Suppose first that $\sigma_1(v_2)$ is positive. Then $\sigma_1(v_2) = \sum_{\sigma\in\Sigma} v_\sigma$ where each $v_\sigma \in L_\sigma^+$. There is some $\lambda_{\sigma_2} > 0$ such that $v_{\sigma_2} = \lambda_{\sigma_2} v_2$, hence

$$\sum_{\sigma\in\Sigma-\{\sigma_2\}} v_\sigma + \lambda v_1 = (1-\lambda_{\sigma_2})v_2\,.$$

The expression on the left indicates this is a (strictly) positive vector; whence $1-\lambda_{\sigma_2} > 0$. So $v_2 \in \sum_{\sigma\in\Sigma-\{\sigma_2\}} L_\sigma^+$, a contradiction with the minimality of Σ.

If $\sigma_1(v_2)$ is negative, then $-\sigma_1(v_2) = \sum_{\sigma\in\Sigma} v_\sigma$ for $v_\sigma \in L_\sigma^+$, so

$$\sum_{\sigma\in\Sigma-\{\sigma_1\}} v_\sigma + v_2 = (\lambda-\lambda_{\sigma_1})v_1\,.$$

where $v_{\sigma_1} = \lambda_{\sigma_1} v_1$. Since this is an expression for a positive vector, $\lambda-\lambda_{\sigma_1} > 0$, so $v_1 \in \sum_{\sigma\in\Sigma-\{\sigma_1\}} L_\sigma^+$, again contradicting the minimality of Σ.

(2) is an immediate corollary of (1). □

Proof of Proposition 7.5. As remarked earlier, there are subsets of S which satisfy the criterion (B) for a family of simple reflections – including S itself.

Suppose that Σ is a minimal subset of S satisfying criterion (B). Take any partition $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and a vector v such that:

$$v \in \left(\sum_{\sigma_1\in\Sigma_1} L_{\sigma_1}^+\right) \bigcap \left(\sum_{\sigma_2\in\Sigma_2} L_{\sigma_2}^+\right)\,.$$

By Lemma 7.6(1), we get $(v \mid v)_\varphi \le 0$; and since the form is definite positive, $v = 0$. Thus $V = \bigoplus_{\sigma\in\Sigma} L_\sigma$ (criterion (A)) holds.

Finally, we prove that such a Σ is unique. Again we argue by contradiction, by assuming the existence of $\Sigma' \neq \Sigma$ which also satisfies criteria (A) and (B) of Definition 7.4. Choose $\sigma_0 \in \Sigma - \Sigma'$ and $v_{\sigma_0} \in L_{\sigma_0}^+$. There exists a family $(v_{\sigma'})_{\sigma'\in\Sigma'}$ with $v_{\sigma'} \in L_{\sigma'}^+$ such that $v_{\sigma_0} = \sum_{\sigma'\in\Sigma'} v_{\sigma'}$. Now for each σ' for which $v_{\sigma'} \neq 0$, there exists a family $(v_{\sigma,\sigma'})_{\sigma\in\Sigma}$ where $v_{\sigma,\sigma'} \in L_\sigma^+$ such that $v_{\sigma'} = \sum_{\sigma\in\Sigma} v_{\sigma,\sigma'}$. Since $\sigma' \neq \sigma_0$, there exists $\sigma \in \Sigma$, $\sigma \neq \sigma_0$, such that $v_{\sigma,\sigma'} \neq 0$. Such a vector $v_{\sigma,\sigma'}$ appears then in the decomposition of v_{σ_0} onto $\bigoplus_{\sigma\in\Sigma} L_\sigma$, which is a contradiction. □

Lemma 7.7. *Assume given an admissible preorder on V and W, that σ is a simple reflection and that s is any other reflection in S, distinct from σ. Then:*

(1) $\sigma(L_s^+) = L_{\sigma s\sigma^{-1}}^+$,
(2) $s(L_\sigma^+) = L_{s\sigma s^{-1}}^+$ if and only if $(L_s^+ \mid L_\sigma^+) \le 0$, and
(3) L_σ^+ is the only positive half-line made negative by σ.

Proof. For any reflections s_1 and s_2, $s_1(L_{s_2}) = L_{s_1 s_2 s_1^{-1}}$. We have to show in each case that the positive half-line remains positive.

(1) Choose $v_s \in L_s^+$. Take $\sigma_1 \neq s$; then $\sigma_1(v_s) \in L_{\sigma_1 s \sigma_1^{-1}}$. We want to show that it is positive. So there exist $v_\sigma \in L_\sigma^+$ ($\sigma \in \Sigma$) for which $v_s = \sum_{\sigma\in\Sigma} v_\sigma$. By Lemma 7.6(2),

$$\sigma_1(v_s) = \sum_{\sigma\in\Sigma-\{\sigma_1\}} (v_\sigma + v_{\sigma,1}) + \sigma_1(v_{\sigma_1}) \,,$$

where, for all σ simple reflections, $v_{\sigma,1} \in L_{\sigma_1}^+$. Thus

$$\sigma_1(v_s) = \sum_{\sigma\in\Sigma-\{\sigma_1\}} v_\sigma + u_1 \,,$$

where $u_1 \in L_{\sigma_1}$. Since $\sigma_1 \neq s$, there is a $\sigma \in \Sigma$ for which v_σ is non-zero, that is, strictly positive. So $\sigma_1(v_s)$ is in $L^+_{\sigma_1 s \sigma_1^{-1}}$, the positive part of $L_{\sigma_1 s \sigma_1^{-1}}$.

(2) For $v_\sigma \in L_\sigma^+$, we have $s(v_\sigma) = v_\sigma - 2\frac{(v_\sigma|v_s)}{(v_s|v_s)} v_s$ where without loss of generality we may assume that $v_s \in L_s^+$. Since $\sigma \neq s$, there is some strictly positive $v_{\sigma'}$ in the simple positive half-line decomposition of v_s, where $\sigma' \neq \sigma$. Thus $s(v_\sigma) \in L^+_{s\sigma s^{-1}}$ if and only if $(L_s^+ \mid L_\sigma^+) \leq 0$.

Item (3) follows from (1). □

Lemma 7.8. *Let G be a finite subgroup of $GL(V)$. Any two families of positive half-lines for G are conjugate under G.*

Proof. Suppose we have two families of positive half-lines: $\mathcal{L}_S^+ := (L_s^+)_{s\in S}$ and $\mathcal{L}_S^{++} := (L_s^{++})_{s\in S}$ which determine families of simple roots denoted by Σ and Σ' respectively. We shall prove by induction on $|\mathcal{L}_S^+ \cap (-\mathcal{L}_S^{++})|$ that Σ and Σ' are conjugate under G.

If $|\mathcal{L}_S^+ \cap (-\mathcal{L}_S^{++})| = 0$, then $\mathcal{L}_S^+ = \mathcal{L}_S^{++}$ and so $\mathcal{L}_S^+$ and $\mathcal{L}_S^{++}$ are conjugate by the identity element of G.

If $|\mathcal{L}_S^+ \cap (-\mathcal{L}_S^{++})| > 0$, then there exists $\sigma_0 \in \Sigma$ such that $L_{\sigma_0}^+ \in -(\mathcal{L}_S^{++})$. Indeed, if not, then $\mathcal{L}_S^+ \subset \mathcal{L}_S^{++}$, hence $\mathcal{L}_S^+ = \mathcal{L}_S^{++}$. By assertion (1),

$$\sigma_0(\mathcal{L}_S^+) = \left(\mathcal{L}_S^+ - \{L_{\sigma_0}^+\}\right) \sqcup \{-L_{\sigma_0}^+\} \,,$$

which shows that $\sigma_0(\mathcal{L}_S^+)$ is a family of positive half-lines (for the order conjugate under σ_0) such that

$$|\sigma_0(\mathcal{L}_S^+) \cap (-\mathcal{L}_S^{++})| = |\mathcal{L}_S^+ \cap (-\mathcal{L}_S^{++})| - 1 \,.$$

By the induction hypothesis, we get that $\sigma_0(\mathcal{L}_S^+)$ and $\mathcal{L}_S^{++}$ are conjugate under G, which shows that $\mathcal{L}_S^+$ and $\mathcal{L}_S^{++}$ are conjugate under G. □

Proposition 7.9. *Let V be a finite dimensional vector space on a real field k. Let G be a finite subgroup of $\mathrm{GL}(V)$ and let S the set of all reflections of G.*

(1) G acts transitively on the set of families of simple reflections.
(2) Let Σ be a family of simple reflections.
  (a) Every reflection of G is conjugate to an element of Σ.
  (b) Σ generates the (normal) subgroup of G generated by S.

Proof. Since the choice of a family of positive half-lines determines a single family of simple reflections, item (2) of the preceding lemma shows the transitivity of G on the families of positive half-lines.

We now turn to assertion (2)(a). Choose an admissible preorder on V and W, and denote by $\mathcal{L}_S^+ := (L_s^+)_{s\in S}$ the corresponding family of positive half-lines and by Σ the corresponding family of simple reflections. For each $\sigma \in \Sigma$, we choose $e_\sigma \in L_\sigma^+$ so that $(e_\sigma)_{\sigma\in\Sigma}$ is a basis of V, and for all $s \in S$ and $v \in L_s^+$, we have $v = \sum_{\sigma\in\Sigma} \lambda_\sigma(v)e_\sigma$ with $\lambda_\sigma(v) \geq 0$ for all $\sigma \in \Sigma$. We set

$$h(v) := \sum_{\sigma\in\Sigma} \lambda_\sigma(v)\,.$$

For each $s \in S$, we choose an element $v \in L_s$, $v \neq 0$, and we denote by Ω the union of the orbits of these vectors under G. Since S and G are finite, so is Ω.

Denote by Ω^+ the positive vectors of Ω, and define:

$$m_\Omega := \min\{h(v) \mid v \in \Omega^+\}\,.$$

Let $s \in S - \Sigma$. Let $v = \sum_{\sigma\in\Sigma} \lambda_\sigma(v)e_\sigma \in \Omega \cap L_s^+$. Since

$$(v \mid v)_\varphi = \sum_{\sigma\in\Sigma} \lambda_\sigma(v)(v \mid e_\sigma)_\varphi > 0\,,$$

there exists $\sigma_0 \in \Sigma$ such that $(v \mid e_{\sigma_0})_\varphi > 0$. By item (1) of Lemma 7.7, we know that $v' := \sigma_0(v) \in \Omega^+$. Since $v' = v - 2\frac{(v\,|\,e_{\sigma_0})_\varphi}{(e_{\sigma_0}\,|\,e_{\sigma_0})_\varphi}e_{\sigma_0}$, we see that $h(v') < h(v)$.

This proves in particular that if, for $v \in L_s$, $h(v) = m_\Omega$, then $s \in \Sigma$, and finally that there exists $g \in G$ such that $gsg^{-1} \in \Sigma$, which is (2)(a).

This also proves that every element of S is conjugate to an element of Σ by an element of the group generated by Σ, which proves that Σ generates the subgroup of G generated by S, *i.e.*, assertion (2)(b). □

The notion of *Coxeter system* is defined in [BouLie, Chap. IV, §1, 3, Déf. 1.3] .

Theorem 7.10. *Let V be a finite dimensional vector space on a real field k. Let G be a finite subgroup of* $\mathrm{GL}(V)$ *generated by reflections, and let S be the set of all reflections of G. Then the pair (G, Σ) is a Coxeter system for every is a family of simple reflections Σ.*

Proof. For all $\sigma \in \Sigma$, we set $P_\sigma := \{g \in G \mid g^{-1}(L_\sigma^+)$ is positive $\}$. We shall check that P_σ satisfies the hypotheses of [BouLie, Ch. IV, §1, 7, Proposition 6] and this will prove that

- (G, Σ) is a Coxeter system,
- P_σ comprises of all elements $g \in G$ such that $l_\Sigma(\sigma g) > l_\Sigma(g)$,

where $l_\Sigma(g)$ denotes the length of the shortest decomposition of g in terms of simple reflections.

It is clear that $P_\sigma \cap \sigma(P_\sigma) = \emptyset$. Now take $g \in P_\sigma$ and $\sigma' \in \Sigma$ such that $g\sigma' \notin P_\sigma$, that is $g^{-1}(L_\sigma^+)$ is positive and $\sigma' g^{-1}(L_\sigma^+)$ is negative. As σ' only changes the sign on $L_{\sigma'}$, we must have $g^{-1}(L_\sigma) = L_{\sigma'}$, which implies $g^{-1}\sigma g = \sigma'$. □

When the family of simple reflections Σ is clear from the context, we will just write $l(g)$ for the length of an element g of G with respect to the generating set Σ.

Corollary 7.11. *If $g \in G$ and σ a simple reflection, then $g(L_\sigma^+) > 0$ is equivalent to $l(g\sigma) = l(g) + 1$.*

7.3. **Highest half-lines.**

The definition of the highest root given in [BouLie, Ch. VI, §1.8] seems *a priori* not to make sense in our setting. For instance, for the group $G(5,5,2)$ we have $\mathbb{Z}_k = \mathbb{Z}[\phi]$ where $\phi = \frac{1+\sqrt{5}}{2} > 1$ is a unit of $\mathbb{Z}_k$. As a root is only defined up to multiplication by a unit, it will have no well-defined "length".

However, the following, a consequence of the definition in [BouLie], does make sense:

Definition 7.12. *Let $\mathcal{L}_S^+$ be a family of positive half-lines. We call L^+ in $\mathcal{L}_S^+$ a* highest half-line *if $(L^+ \mid L_s^+) \geq 0$ for all $s \in S$.*

Proposition 7.13. *If $l_\Sigma(s)$ is maximal over the conjugacy class of s in the Coxeter system (G, Σ), then L_s^+ is a highest half-line.*

Proof. Let Σ be a family of simple reflections and $\sigma \in \Sigma$. We want to show that for any positive half-line $L_s^+ \in \mathcal{L}_S^+$, if $(L_s^+ \mid L_\sigma^+) < 0$ then $l(\sigma s \sigma) > l(s)$.

Indeed, suppose otherwise: since $l(s\sigma) = l(s) + 1$ we must have $l(\sigma s \sigma) = l(s\sigma) - 1$. By Corollary 7.11 this implies $(s\sigma)^{-1}(L_\sigma^+) = \sigma s(L_\sigma^+) < 0$, thus σ changes the sign of $s(L_\sigma^+)$. By (3) of Lemma 7.7, L_σ^+ is the only half-line changed sign by σ so we have $s(L_\sigma^+) = L_\sigma^+$. Finally, this implies $(L_s^+ \mid L_\sigma^+) = 0$, a contradiction. □

The next lemma is a particular case of Theorem 7.17 below.

Lemma 7.14. *In a dihedral group, there is exactly one highest half-line in each conjugacy class.*

Proof. This follows from Proposition 7.13 and the fact that there is exactly one reflection of longest length in each conjugacy class of reflections. □

Proposition 7.15. *In a finite Coxeter system two non-commuting reflections are conjugate if and only if they are conjugate in the dihedral group they generate.*

Proof. Let s and t be two non-commuting reflections. Let P be the fixator of the intersection of the two reflecting hyperplanes. Then P

is a parabolic subgroup of G of rank at most two. It is of rank two since s and t are in P and do not commute. Up to conjugacy, we may suppose that P is a standard parabolic subgroup, so is defined by two vertices s_1 and s_2 of the Coxeter diagram (note that s_1 and s_2 may be different from s and t). Then s_1 and s_2 must be adjacent in the Coxeter diagram, otherwise P is of type $A_1 \times A_1$ and does not contain non-commuting reflections. It follows from the fact that s_1 and s_2 are adjacent and the description (see Theorem 2.35) of linear characters of Coxeter groups that any linear character of P extends to G. Furthermore, Theorem 2.35 ensures that linear characters separate conjugacy classes of reflections, so it follows that s and t are conjugate if and only if they are conjugate in P, which is a dihedral group.

Finally, if H is a dihedral subgroup of a dihedral group G, two reflections non-conjugate in H are not conjugate in G: for H to have two classes of reflections, it has to have an even bond, and then H has as many linear characters as G. □

Lemma 7.16. *Assume G irreducible, and that L_s^+ is a highest half-line. Then for any positive half-line $L_{s'}^+$, we have $(L_s^+ \mid L_{s'}^+) > 0$.*

Proof. Let $v \in L_s^+$ and write $v = \sum_{\sigma\in\Sigma} \lambda_\sigma(v) e_\sigma \in L_s^+$ where the e_σ and $\lambda_\sigma(v) \geq 0$ are as in the proof of Proposition 7.9. Let $I \subset \Sigma$ be the set of σ such that $\lambda_\sigma(v) > 0$, and let $J = \Sigma - I$, containing reflections σ' for which $\lambda_{\sigma'}(v) = 0$.

Now for $\sigma' \in J$, we have $(v \mid e_{\sigma'})_\varphi \geq 0$ by definition of a highest half-line, but also $(e_\sigma \mid e_{\sigma'}) \leq 0$ for any $\sigma \in I$ by Lemma 7.6(1). It follows that we must have $(e_\sigma \mid e_{\sigma'}) = 0$ for any $\sigma \in I, \sigma' \in J$, which contradicts the irreducibility of G unless $J = \emptyset$.

Thus $(L_s \mid L_\sigma) > 0$ for all simple σ; now any $v'_s \in L_{s'}^+$ is a non-negative linear combination of e_σ, with at least one non-zero coefficient, and so the result follows. □

Theorem 7.17. *If G is a finite real reflection group, there is exactly one highest half-line in each conjugacy class of reflections.*

Proof. Suppose L_s^+ and $L_{s'}^+$ are distinct highest half-lines in the same conjugacy class. We will derive a contradiction.

Two conjugate reflections belong to the same irreducible component of G, so we may assume G irreducible. Then, by Lemma 7.16, L_s^+ and $L_{s'}^+$ are not orthogonal, so s and s' do not commute. Thus, by Proposition 7.15, they are conjugate in the dihedral subgroup they generate; and L_s^+ and $L_{s'}^+$ are still highest half-lines in this subgroup. This contradicts Lemma 7.14. □

7.4. **Real root systems.**

Our approach allows us to extend the theory of root systems for Weyl groups to root systems for finite Coxeter groups. We spell out in this

subsection how our definitions translate in this case. As in Subsection 7.2, we assume $k \subset \mathbb{R}$, and we use the notation k^+ and k^-.

Let $\mathfrak{R}$ be a *reduced* $\mathbb{Z}_k$-root system, and let $G := G(\mathfrak{R})$. For s a reflection in G we denote by $\mathfrak{r}_s$ the element of $\mathfrak{R}$ associated with s, and for $\mathfrak{r} \in \mathfrak{R}$ we denote by $s_\mathfrak{r}$ the reflection of G determined by $\mathfrak{r}$.

We assume chosen a positive definite Hermitian kG-isomorphism $\varphi : V \xrightarrow{\sim} W$, and v_0 a nonzero element of V defining an order on V and W (as in Subsection 7.2). For $\mathfrak{r} = (I_\mathfrak{r}, J_\mathfrak{r}, \zeta_\mathfrak{r}) \in \mathfrak{R}$, we set

$$I_\mathfrak{r}^+ := I_\mathfrak{r} \cap V^+ \quad \text{and} \quad J_\mathfrak{r}^+ = J_\mathfrak{r} \cap W^+ .$$

If Σ is the family of simple reflections determined by the choice of the order on V (see Proposition 7.5), we set

$$\mathfrak{R}_\Sigma := \{\mathfrak{r} \in \mathfrak{R} \mid s_\mathfrak{r} \in \Sigma\} \quad \text{and} \quad \mathfrak{R}_\Sigma^\vee := \{\mathfrak{r}^\vee \in \mathfrak{R}^\vee \mid s_\mathfrak{r} \in \Sigma\} .$$

Theorem 7.18. *Under the above hypotheses and notation,*

(1) $\mathfrak{R}_\Sigma$ is a root basis and $\mathfrak{R}_\Sigma^\vee$ is a coroot basis,

(2) Whenever $\mathfrak{r} \in \mathfrak{R}$,

$$I_\mathfrak{r}^+ \subset \bigoplus_{\sigma \in \Sigma} I_{\mathfrak{r}_\sigma}^+ .$$

Proof. Assertion (1) results from the fact that Σ generates G (Proposition 7.9, (2)(b)) and from Proposition 3.42, (2). Assertion (2) is an immediate consequence of Proposition 7.5, (2). □

8. Bad numbers

Let W be an irreducible Weyl group. Then

(1) By [BouLie, Ch.vi, §2, Proposition 7],

$$|W| = r! c_W (n_1 \cdots n_r) .$$

where $n_1, \ldots, n_r$ are the coefficients of the longest root on the basis of simple roots, and c_W is he connection index of W.

(2) The set of *bad primes* for W is defined as the set of prime divisors of the product $n_1 \cdots n_r$.

The above definition for the set of bad primes is equivalent to several others, see [SpSt, 4.3(c)].

Theorem 3.48 ensures that the connection index for any irreducible, well-generated complex reflection group (V, G) is well-defined, independent of the choice of root system. Moreover, by Propositions 6.11 and 6.12, and the fact that the rings of integers for the fields of definition of the primitive reflection groups are principal, we know that the connection index of an irreducible, well-generated complex reflection group is always principal. We write c_G for a generator of the connection index, which is then well-defined up to a unit. This allows us to extend the definition of bad primes as follows:

Theorem–Definition 8.1. *For every irreducible well-generated complex reflection group G of rank r:*

(1) $c_G r!$ *divides* $|G|$.
(2) We define the bad prime ideals *for G as the set of prime ideals of $\mathbb{Z}_k$ which divide $|G|/(c_G r!)$.*

Proof. The proof of (1) is by inspection, using the values of c_G given in the tables of Appendix B for primitive well-generated groups, and in 6.12 and 6.11 for imprimitive groups. □

Now let W denote an irreducible *spetsial group* – that is, either a member of one of the imprimitive families $G(e,e,r)$, $G(d,1,r)$ or a primitive complex reflection group denoted G_n where

$$n \in \{4,6,8,14,23,24,25,26,27,28,29,30,32,33,34,35,36,37\}$$

according to the Shephard-Todd notation. (Note that spetsial groups are all well-generated.) For the definitions relative to the program *"Spetses"*, we refer the reader to [BMM2].

Definition 8.2. *We denote by* Bad_W *the largest integral ideal* $\mathfrak{a}$ *such that, whenever $S(x) \in \mathbb{Z}_k[x,x^{-1}]$ is a Schur element of an irreducible character of the spetsial Hecke algebra of W, then $S(x)\mathfrak{a}^{-1} \subset \mathbb{Z}_k[x,x^{-1}]$.*

An equivalent definition, which can be more convenient to use, is that whenever $D(x) \in k[x]$ is a unipotent degree of the principal series attached to W, then $\mathfrak{a}D(x) \subset \mathbb{Z}_k[x]$.

Conjecture 8.3. Bad_W *can also be characterized as the largest ideal* $\mathfrak{a}$ *such that whenever $D(x) \in k[x]$ is any unipotent degree attached to W, then $\mathfrak{a}D(x) \subset \mathbb{Z}_k[x]$.*

Remark 8.4. We were unable to answer the following question: is Bad_W always a principal ideal?

It is apparent from the values of unipotent degrees computed by Lusztig that the following theorem holds in the case of Weyl groups.

Theorem 8.5. *For any spetsial complex reflection group W,*

(1) Bad_W *divides* $|W|/(r!c_W)$.
(2) All bad prime ideals divide Bad_W.

or, equivalently,

Bad_W *divides* $|W|/(c_W r!)$ *and* $|W|/(c_W r!)$ *divides a power of* Bad_W.

It follows that the "bad primes" defined in [BK, above Proposition 1.31] are the same as ours.

Proof. For the primitive spetsial reflection groups, the proof is by inspection of:

- the tables of unipotent degrees of the principal series given in [BMM2], and

- the values of c_W given in the tables of Appendix B.

The values of Bad_W are reported in the tables of Appendix B. In every case, Bad_W is also equal to the largest integral ideal $\mathfrak{a}$ such that for *all* unipotent degrees $\mathfrak{a}D(x) \subset \mathbb{Z}[x]$; that is, Conjecture 8.3 holds.

We now prove the theorem for the imprimitive groups $G(d,1,r)$. Set $\zeta := \exp(2\pi i/d)$. By Proposition 6.12, $c_W = 1-\zeta$, thus $|W|/(c_W r!) = d^r/(1-\zeta)$. To compute Bad_W, we use the formula for the unipotent degrees given in [Ma, 3.8]. Let us recall the setup.

- The unipotent degrees of the principal series of $G(d,1,r)$ are parameterized by the d-symbols $S = (S_0, \dots, S_{d-1})$ of rank r and of shape $(|S_0|, \dots, |S_{d-1}|) = (m+1, m, \dots, m)$ for $m \in \mathbb{N}$ large enough.
- The unipotent degree attached to S is of the form $P_S(x)/f_S$ where
  - ▷ $P_S(x) \in \mathbb{Z}_k[x]$ is a monic polynomial and
  - ▷ $f_S = \tau(d)^m/(\prod_{0\le i<j\le d-1}(\zeta^i-\zeta^j)^{|S_i\cap S_j|})$ up to a unit, where $\tau(d) = \prod_{0\le i<j\le d-1}(\zeta^i-\zeta^j)$, so that

$$f_S = \prod_{0\le i<j\le d-1} (\zeta^i-\zeta^j)^{m-|S_i\cap S_j|}\,. \tag{8.6}$$

Due to the shape of the symbols, $m \ge |S_i \cap S_j|$, and so $f_S \in \mathbb{Z}_k$. Notice also that f_S depends only on the equivalence class of S, since shifting all the S_i increases by 1 all the $|S_i \cap S_j|$, and also increases m by 1, thus leaving invariant f_S.

To prove item (1) of the theorem, we have to show that f_S divides $|W|/(r!c_W) = d^r/(1-\zeta)$. We will show this by induction on r. By [Ma, §3.C], a symbol of the principal series is not 1-cuspidal, thus admits a $(1,1)$-hook; that is, any of the considered symbols of rank r can be obtained from a symbol of rank $r-1$ by increasing by 1 one of the entries $\lambda \in S_i, \lambda+1 \notin S_i$ for some i.

The effect of this is to reduce by at most 1 the $|S_i \cap S_j|$ for $j \ne i$, that is to multiply f_S by at most $\prod_{j\in[0\dots d-1], j\ne i}(\zeta^i-\zeta^j)$, which is equal to d up to a unit. But increasing r by 1 multiplies $d^r/(1-\zeta)$ by d, so the divisibility of $d^r/(1-\zeta)$ by f_S is preserved. It remains to show the starting point of the induction, which is that when $r=1$, f_S divides $(d/(\zeta-1))$. This results from [BMM2, §5.2], which finishes the proof of item (1) for $G(d,1,r)$. Note that by [BMM2, §5.3] it can be seen that Conjecture 8.3 holds in this case.

Let us prove item (2) for $G(d,1,r)$. First, note that for a d-symbol S of rank r, if S' is the symbol of rank $r+1$ obtained by increasing by 1 the highest entry in S, then f_S divides $f_{S'}$. Indeed, the numbers $m - |S_i \cap S_j|$ can only increase when going from S to S'. Thus it is sufficient to show that for $r=1$ any prime ideal dividing d (which is the same as dividing $d/(1-\zeta)$) divides Bad_W. For instance (see [BMM2,

§5.2]) for the character denoted ρ_1 in *loc.cit.* we have $f_S = d/(\zeta - 1)$, which proves the result (since $(1-\zeta)\mathbb{Z}_k = (1-\zeta^{-1})\mathbb{Z}_k$ divides $d/(\zeta-1)$).

To complete the proof, we consider the case of the groups $G(d,d,r)$ ($r \geq 2$) following [Ma, §6]. The setup is as follows:

- The unipotent characters of the principal series are parameterized by d-symbols of rank r and shape $(m,\dots,m)$.
- By [Ma, 6.4], the unipotent degree attached to S is of the form $P_S(x)/f'_S$ where $P_S(x) \in \mathbb{Z}_k[x]$ is a monic polynomial and, up to unit, we have $f'_S = f_S\gamma(S)/d$ where f_S is as in equation 8.6 and $\gamma(S)$ is the cardinality of the subgroup of $\mathbb{Z}/d$ leaving S invariant, where $i \in \mathbb{Z}/d$ acts on S by mapping it to the symbol S' such that $S'_j = S_{(j+i) \pmod d}$.

This time we have $|W|/(r!c_W) = d^{r-1}/((1-\zeta)(1-\zeta^{-1}))$ (see Proposition 6.11). To show that $f_S\gamma(S)/d$ divides that number is equivalent to showing that

$$f_S\gamma(S) \text{ divides } \frac{d^r}{(1-\zeta)(1-\zeta^{-1})} \tag{8.7}$$

We will show this by a double induction. When $\gamma(S) = 1$ we proceed by induction on r. Just as in the case of $G(d,1,r)$, the symbol admits a $(1,1)$-hook and we reduce the problem to the case of rank $r-1$. The starting case is $r = 2$.

In this case we may look at [Lu, 4.1] where the unipotent degrees of the dihedral groups $G(d,d,2)$ are given in the form $P_S(x)/f_S$ where P is an integral polynomial and $f \in \mathbb{Z}_k$ divides $d/((1-\zeta)(1-\zeta^{-1}))$. The divisibility is obvious except when d is even and $f_S = d/2$, where one needs Corollary A.8 below. It can be seen also that for $G(d,d,2)$ Conjecture 8.3 holds.

The other case of the induction is when $\gamma := \gamma(S) > 1$. In this case if we set $d' = d/\gamma$, then S is the concatenation of γ copies of a d'-symbol S' of rank $r' := r/\gamma$.

Lemma 8.8. *Up to a unit, we have $f_S = f_{S'}^{\gamma}$.*

Proof. Given $0 \leq i,j \leq d-1$, there are unique expressions $i = i' + i''d'$ and $j = j' + j''d'$ for $i',j' \in [0,\dots,d'-1]$. Using that $\zeta^{d'} = \zeta_\gamma$ where $\zeta_\gamma := \exp 2i\pi/\gamma$, and that $S_i = S_{i'}, S_j = S_{j'}$, we can write

$$f_S = \prod_{0\leq i''\leq j''\leq \gamma-1} \prod_{\substack{0\leq i',j'\leq d'-1 \\ i'<j' \text{ if } i''=j''}} \left(\zeta^{i'}\zeta_\gamma^{i''} - \zeta^{j'}\zeta_\gamma^{j''}\right)^{m-|S_{i'}\cap S_{j'}|}$$

We make the following observations on the above formula:

- We can assume $i' \neq j'$ since the terms where $i' = j'$ have zero exponent.

- The term indexed by i', j' is the negative of the term indexed by j', i'. Thus we can decide to retain only the terms where $i' < j'$, up to doubling the exponent when $i'' \neq j''$.

Doubling the exponent is compensated by making the product over all i'', j'', giving:

$$f_S = \prod_{0 \leq i'', j'' \leq \gamma - 1} \prod_{0 \leq i' < j' \leq d' - 1} \left(\zeta^{i'} \zeta_\gamma^{i''} - \zeta^{j'} \zeta_\gamma^{j''} \right)^{m - |S_{i'} \cap S_{j'}|}.$$

Apply the formula $\prod_{0 \leq j'' \leq \gamma - 1} (a - b\zeta_\gamma^{j''}) = a^\gamma - b^\gamma$ to get:

$$f_S = \prod_{0 \leq i' < j' \leq d' - 1} \left(\zeta^{\gamma i'} - \zeta^{\gamma j'} \right)^{\gamma (m - |S_{i'} \cap S_{j'}|)},$$

which is what we want since $\zeta^\gamma = \exp 2\pi i / d'$. □

The lemma can be used to deal with the symbols for which $\gamma > 1$. We distinguish two cases.

The first case is $r' = 1$. Let us recall that f_S is invariant (up to sign) by the action of $\mathbb{Z}/d'$ — this can be seen directly from its formula or from the fact that according to [Ma] the whole unipotent degree is invariant (up to sign) by that action. Then, up to $(\mathbb{Z}/d')$-action there is only one d'-symbol of rank 1, given by $S' = (\{1\}, \{0\}, \ldots, \{0\})$. A direct computation shows that $f_{S'} = d'$ up to unit. We thus have to show that $d'^\gamma \gamma$ divides $d^r / ((1 - \zeta)(1 - \zeta^{-1}))$. Using that $d^r = (\gamma d')^\gamma$ it remains to see that $(1 - \zeta)(1 - \zeta^{-1})$ divides $\gamma^{\gamma - 1}$. This follows from item (1) of Corollary A.8 (see Appendix A).

The other case is $r' > 1$. The case $\gamma = 1$, already treated above, ensures that S' satisfies the condition 8.7, that is:

$$f_{S'} \gamma(S') \text{ divides } \frac{d'^{r'}}{(1 - \zeta')(1 - \zeta'^{-1})}$$

where $\zeta' = \exp(2\pi i / d')$. Using the fact that $\gamma(S') = 1$, raising both sides to the power of γ gives:

$$(f_{S'}^\gamma \gamma) \text{ divides } \frac{d'^{r'\gamma} \gamma}{(1 - \zeta')^\gamma (1 - \zeta'^{-1})^\gamma} = \frac{d'^r \gamma}{(1 - \zeta')^\gamma (1 - \zeta'^{-1})^\gamma}$$

It suffices now to show that $d'^r \gamma / ((1 - \zeta')^\gamma (1 - \zeta'^{-1})^\gamma)$ divides $d^r / ((1 - \zeta)(1 - \zeta^{-1}))$. Using that $d^r = d'^r \gamma^r$ and simplifying, it suffices to show that $(1 - \zeta)(1 - \zeta^{-1})$ divides $\gamma^{r-1} (1 - \zeta')^\gamma (1 - \zeta'^{-1})^\gamma$, which is an immediate consequence of Corollary A.8 since $r > 1$.

It remains to prove item (2) for $G(d, d, r)$. By the same argument as for $G(d, 1, r)$ given a symbol S of rank r we may find a symbol S' of rank $r + 1$ such that f_S divides $f_{S'}$. We proceed by induction on the rank, starting from the base case $r = 2$.

For $d \notin \{2,3\}$, according to [Lu, 4.1], for $G(d,d,2)$ there is a symbol S such that $f_S = d/((1-\zeta)(1-\zeta^{-1}))$. Then item (2) of Corollary A.8 completes the proof in this case.

For $d = 2$, the group $G(2,2,2)$ is not irreducible and we do not have to consider it. The group $G(2,2,3)$ is the Weyl group of type A_3, and $|W|/(r!c_W) = 1$ and there is nothing to prove. We start the induction at $G(2,2,4)$, the Weyl group of type D_4 and for the symbol $S = (\{1,2\},\{0,3\})$, for instance, we find $f_S = 2$.

For $d = 3$, the group $G(3,3,2)$ is the Weyl group of type A_2, and $|W|/(r!c_W) = 1$ and there is nothing to prove. We start the induction at $G(3,3,3)$, and for the symbol $S = (\{0,1\},\{1,2\},\{0,2\})$, for instance, we find $f_S = (1-\zeta_3)$ up to a unit. □

9. Classification of distinguished root systems for irreducible primitive reflection groups

As noticed previously (see for example [Ne]), it can be checked that whenever G is a primitive irreducible reflection group, its field of definition $k = \mathbb{Q}_W$ (see 2.30) has class number 1, *i.e.*, the ring $\mathbb{Z}_k$ is a principal ideal domain.

In this case, every root system is principal in the sense of Definition 3.9, and hence gives rise to a Cartan matrix as in Definition 6.2.

We present here a classification of the *distinguished $\mathbb{Z}_k$-root systems for primitive groups* (up to genus), based on the data in the `CHEVIE` package of `GAP3`. The classification may be summarised by looking at Appendix B, which exhibits, for each primitive irreducible reflection group G:

- a diagram describing its presentation;
- a Cartan matrix C which corresponds to the data in `CHEVIE` as well as diagonal matrices giving Cartan matrices for all other genera of root system by conjugation of C;
- the ring of integers of the field of definition, a generator of the connection index and a generator of the ideal Bad_G.

A complete legend for the table is on page 78.

In `GAP3`, all vectors are row vectors, matrices operate from the right, and in `CHEVIE` the pairing used is not Hermitian. Consequently, to change from `CHEVIE` conventions to our conventions, one has to transpose and conjugate the list of coroots. Thus the Cartan matrices given in the tables of Appendix B are the transpose of one obtained by applying the conventions of the preceding sections.

For each primitive irreducible reflection group G, `CHEVIE` contains a Cartan matrix C which satisfies the assumptions of Proposition 6.5. Thus there exists a (principal) root system for G since C satisfies the following set of properties:

(1) C is the Cartan matrix of a ordered set of distinguished roots $\mathfrak{R}_0$ for G, whose root lines generate V and such that the corresponding reflections generate G. The ordered set of reflections corresponding to $\mathfrak{R}_0$ are called the "standard" generators of G.
(2) The entries of C are elements of $\mathbb{Z}_k$. [3]
(3) For each triple $(\alpha_i, \beta_i, \zeta_i)$ defining an element of $\mathfrak{R}_0$, its orbit under G is finite, and whenever two such triples define the same reflection they differ by the action of an element of $\mathbb{Z}_k^\times$.

It follows from the above properties that the root system $\mathfrak{R}$ determined by C is actually distinguished. Indeed:

- all roots in $\mathfrak{R}$ are distinguished,
- there is only one root in $\mathfrak{R}$ attached to each distinguished reflection in G.

It follows from Lemma 6.3 that any other distinguished root system $\mathfrak{R}'$ for the standard generators taken in the same order corresponds to conjugating C by a diagonal matrix. So the first step to classify genera of root systems is to determine whether C can be modified by a diagonal matrix so that the entries remain integral.

9.1. **Cases with only one genus of distinguished root systems.**

By Proposition 3.27, if G has a single orbit of distinguished reflections, it has a single genus of distinguished root systems. This proves uniformly the uniqueness (provided they exist) of distinguished root systems for 19 of the 34 primitive irreducible reflection groups, that is the groups G_n where $n \in \{4, 8, 12, 16, 20, 22, 23, 24, 25, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37\}$. The existence of a distinguished root system for these groups is shown by the `CHEVIE` root data and gives the Cartan matrices (with integral entries) in Appendix B.

Proposition 9.1. *For each of the groups $G_9, G_{10}, G_{11}, G_{14}, G_{17}, G_{18}, G_{19}, G_{21}$ there is a unique genus of distinguished root systems; indeed, the matrix corresponding to the standard generators as given in* `CHEVIE` *is the unique one in its class modulo conjugation by diagonal matrices with integral entries.*

Proof. We first show how the proof goes for G_{11}. The Cartan matrix for the standard generators given in `CHEVIE` for G_{11} is:

$$\begin{pmatrix} 2 & \zeta_3^2(\sqrt{-3}-\sqrt{-2}) & 1-\zeta_{24} \\ 1 & 1-\zeta_3 & \zeta_{24}^{13} \\ 1+\zeta_{24}^5 & \zeta_{24}^7+\zeta_{24}^5 & 1-i \end{pmatrix}.$$

[3] Actually, the data in `CHEVIE` did not always verify (2) when there were several G-orbits of distinguished reflections, but we have found it possible to adjust the data by multiplying by a global scalar all roots in one of the orbits in order to satisfy (2). These modifications have now been incorporated in the `CHEVIE` database.

When we conjugate by the matrix $\mathrm{diag}(1,a,b)$ with $a,b\in k$ we get

$$\begin{pmatrix} 2 & a\zeta_3^2(\sqrt{-3}-\sqrt{-2}) & b(1-\zeta_{24}) \\ a^{-1} & 1-\zeta_3 & a^{-1}b\zeta_{24}^{13} \\ b^{-1}(1+\zeta_{24}^5) & ab^{-1}(\zeta_{24}^7+\zeta_{24}^5) & 1-i \end{pmatrix}.$$

Knowing that $\sqrt{-3}-\sqrt{-2}$, $1-\zeta_{24}$ and $1+\zeta_{24}^5$ are units (which is easily checked in `CHEVIE` if not obvious), we see that for the above matrix to have integral entries a,a^{-1},b,b^{-1} must be integral, which forces a and b to be units and $\mathrm{diag}(1,a,b)$ to be a matrix of units, thus to preserve the genus.

A similar reasoning, using the fact that particular entries of the Cartan matrix are units, also applies to the other cases of the proposition, whose Cartan matrices are given in Appendix B. To help the reader apply the above argument, the entries of the Cartan matrices which are units are in bold in Appendix B, and the same convention is applied for the matrices given in the next subsection. □

9.2. **The cases with more than one genus.**

The approach here follows the same lines as the proof of Proposition 9.1. The Cartan matrix cited in each case is the one given in `CHEVIE` satisfying the conditions (1), (2) and (3) of page 66, and is also the one given in Appendix B.

The case of G_5.

The Cartan matrix C is:

$$\begin{pmatrix} 1-\zeta_3 & \mathbf{1} \\ -2\zeta_3 & 1-\zeta_3 \end{pmatrix}.$$

A matrix $\mathrm{diag}(1,a)$ conjugates C to an integral matrix if and only if a is an integral divisor of 2. Since 2 is prime in $\mathbb{Z}[\zeta_3]$, this gives rise to two distinct genera of distinguished $\mathbb{Z}_k$-root systems.

The case of G_6.

The Cartan matrix C is:

$$\begin{pmatrix} 2 & \frac{(3+\sqrt{3})(\zeta_3-1)}{3} \\ -\mathbf{1} & 1-\zeta_3 \end{pmatrix}.$$

A matrix $\mathrm{diag}(1,a)$ conjugates C to an integral matrix if and only if a is an integral divisor of $\frac{(3+\sqrt{3})(\zeta_3-1)}{3}$, which up to a unit is equal to $i+1$. In $\mathbb{Z}_k=\mathbb{Z}[\zeta_{12}]$, the ideal $(i+1)\mathbb{Z}_k$ is prime of square $2\mathbb{Z}_k$. This gives rise to 2 distinct genera of distinguished $\mathbb{Z}_k$-root systems.

The case of G_7.

The Cartan matrix C is:

$$\begin{pmatrix} 2 & \zeta_3^2(1-i) & -\zeta_3^2(i+1) \\ \boldsymbol{\zeta_3(1-\mathrm{i}\zeta_3)} & 1-\zeta_3 & -\zeta_3(1-i) \\ \boldsymbol{\mathrm{i}\zeta_3(1-\mathrm{i}\zeta_3)} & i+1 & 1-\zeta_3 \end{pmatrix}.$$

A matrix $\mathrm{diag}(1,a,b)$ conjugates C to an integral matrix if and only if both a and b are integral divisors of $i+1$, which is prime in $\mathbb{Z}[\zeta_{12}]$. This gives rise to 4 distinct genera of distinguished $\mathbb{Z}_k$-root systems.

The group G_7 has an outer automorphism induced by an element of $\mathrm{GL}(V)$, induced by the embedding of reflection groups $G_7 \subset G_{15}$, where $[G_{15} : G_7] = 2$. This automorphism exchanges the conjugacy classes of the reflections corresponding to rows 2 and 3 of C, thus exchanges two of the genera of the root systems and leaves the others fixed. Thus we get one more root system than [Ne], which counts the systems up to isomorphism.

The case of G_{13}.

The Cartan matrix C is:

$$\begin{pmatrix} 2 & \sqrt{2} & i-1 \\ \boldsymbol{1+\sqrt{2}} & 2 & -1+\sqrt{-2} \\ \boldsymbol{-\zeta_8(1+\sqrt{2})} & -1-\sqrt{-2} & 2 \end{pmatrix}.$$

Now $i-1 = \zeta_8^3\sqrt{2}$, and $\sqrt{2} = \zeta_8^3(1+\sqrt{2})(1-\zeta_8)^2$ where $1+\sqrt{2}$ is a unit. Hence in terms of ideals, in $\mathbb{Z}_k = \mathbb{Z}[\zeta_8]$ we have $\sqrt{2}\mathbb{Z}_k = (1-i)\mathbb{Z}_k = ((1-\zeta_8)\mathbb{Z}_k)^2$ (see Corollary A.7 of Appendix A).

A matrix $\mathrm{diag}(1,a,b)$ conjugates C to an integral matrix if and only if a and b are equal up to a unit, and both are integral divisors of $\sqrt{2}$. This gives rise to 3 distinct genera of distinguished $\mathbb{Z}_k$-root systems, corresponding respectively to the values $\mathbb{Z}_k$, $(1-\zeta_8)\mathbb{Z}_k$ and $\sqrt{2}\mathbb{Z}_k = (1-i)\mathbb{Z}_k$ for $a\mathbb{Z}_k$.

The case of G_{15}.

The Cartan matrix C is:

$$\begin{pmatrix} 2 & -\zeta_{24}(\mathbf{1}-\zeta_{24}^{\mathbf{19}}) & \mathbf{1} \\ \mathbf{1}-\zeta_{24}^{-\mathbf{1}} & 1-\zeta_3 & \mathbf{1} \\ \zeta_8^{\mathbf{3}}(1-\zeta_8)^2 & \mathbf{u}(1-\zeta_8)^2 & 2 \end{pmatrix}$$

where $\mathbf{u} = (1+\sqrt{2})(\zeta_3+i)$ is a unit.

A matrix $\mathrm{diag}(1,a,b)$ conjugates C to an integral matrix if and only if a is a unit and b is an integral divisor of $(1-\zeta_8)^2$, which is a square in $\mathbb{Z}_k = \mathbb{Z}[\zeta_{24}]$. Hence there are 3 distinct genera of distinguished $\mathbb{Z}_k$-root systems.

The case of G_{26}.

The Cartan matrix C is:

$$\begin{pmatrix} 2 & \mathbf{-1} & 0 \\ \zeta_3 - 1 & 1 - \zeta_3 & \boldsymbol{\zeta_3^2} \\ 0 & -\boldsymbol{\zeta_3^2} & 1 - \zeta_3 \end{pmatrix}.$$

A matrix $\mathrm{diag}(1, a, b)$ conjugates C to an integral matrix if and only if a and b are equal up to a unit and both are integral divisors of $1 - \zeta_3$, which is prime in $\mathbb{Z}_k = \mathbb{Z}[\zeta_3]$. So there are 2 distinct genera of distinguished $\mathbb{Z}_k$-root systems.

The case of $G_{28} = F_4$.

The Cartan matrix C is:

$$\begin{pmatrix} 2 & \mathbf{-1} & 0 & 0 \\ \mathbf{-1} & 2 & \mathbf{-1} & 0 \\ 0 & -2 & 2 & \mathbf{-1} \\ 0 & 0 & \mathbf{-1} & 2 \end{pmatrix}.$$

A matrix $\mathrm{diag}(1, a, b, c)$ conjugates C to an integral matrix if and only if a is a unit and b and c are equal up to a unit and are integral divisors of 2. Since 2 is prime in $\mathbb{Z}$ this leaves 2 distinct genera of distinguished $\mathbb{Z}_k$-root systems.

Appendix A. On roots of unity

A.1. Notation and summary of known properties.

For any natural integer n, we denote by

- $\varphi(n)$ the order of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$,
- $\Phi_n(X)$ the n-th cyclotomic polynomial, monic element of $\mathbb{Z}[X]$ inductively defined by the equality
$$X^n - 1 = \prod_{d|n} \Phi_d(X)\,,$$
- $\mathbb{Z}_n$ (resp. $\mathbb{Q}_n$) the ring (resp. the field) generated by the group $\boldsymbol{\mu}_n$ of all n-th roots of unity.

The following omnibus proposition states properties which are either well known, or easy to establish.

Proposition A.1.

(1) $\Phi_n(X) = \prod_{\zeta \text{ of order } n}(X-\zeta)$, *and* $\deg \Phi_n(X) = \varphi(n)$.

(2) If $n = \prod_{p\in\mathcal{P}(n)} p^{v_p(n)}$ *(where* $\mathcal{P}(n)$ *denotes the set of prime numbers dividing* n*), then*
$$\varphi(n) = \prod_{p\in\mathcal{P}(n)} p^{v_p(n)-1}(p-1)\,.$$

(3) $n = \prod_{(d|n)(d\neq 1)} \Phi_d(1)$.

(4) We say that an integer n *is* composite *if it is divisible by at least two different prime numbers. Then*
$$\Phi_n(1) = \prod_{\zeta \text{ of order } n}(1-\zeta) = \begin{cases} 1 & \text{if } n \text{ is composite,} \\ p & \text{if } n = p^m \text{ (} p \text{ a prime number).} \end{cases}$$

(5) (Changing X *to* $-X$*)*
$$\Phi_d(-X) = \Phi_{2d}(X) \text{ for } d \text{ odd and } d > 1\,,$$
$$\Phi_d(-X) = \Phi_d(X) \text{ for } d \text{ divisible by } 4.$$

(6) The polynomial $\Phi_n(X)$ *is irreducible in* $\mathbb{Z}[X]$*, hence*
$$\mathbb{Q}_n \simeq \mathbb{Q}[X]/\Phi_n(X) \quad \text{and} \quad \mathbb{Z}_n \simeq \mathbb{Z}[X]/\Phi_n(X)\,.$$

(7) Let a *and* b *be relatively prime. Then* $\varphi(ab) = \varphi(a)\varphi(b)$*, so:*
  (a) $[\mathbb{Q}_{ab} : \mathbb{Q}_a] = \varphi(b)$,
  (b) $\Phi_b(X)$ *is irreducible in* $\mathbb{Q}_a[X]$*, hence in* $\mathbb{Z}_a[X]$,
  (c) $\mathbb{Z}_{ab} \simeq \mathbb{Z}_a[X]/\Phi_b(X)$.

(8) Whenever p *is a prime number,*
$$\begin{cases} \Phi_n(X^p) = \Phi_n(X)\Phi_{pn}(X) \text{ if } p \nmid n\,, \\ \Phi_n(X^p) = \Phi_{pn}(X) \text{ if } p \mid n\,. \end{cases}$$
In particular
$$\Phi_{p^a}(X) = \Phi_p(X^{p^{a-1}})\,.$$

Corollary A.2. *If m divides n, then $(1+\zeta_m)\mathbb{Z}_n \neq \mathbb{Z}_n$ precisely when $m = 2p^k$, p prime, $k \geq 0$.*

Proof. By A.1(4), $1+\zeta_m$ is not a unit if $-\zeta_m$ is a prime power. □

A.2. **Decomposition of the ideal $\mathfrak{I}_{m,n}$ in $\mathbb{Z}_n$.**

Let $m \in \mathbb{N}$ divide n. If ζ and ξ are two roots of unity, both of order m, then the elements ζ and ξ generate the same multiplicative group, and hence each of them is a power of the other. Thus the elements $(1-\zeta)$ and $(1-\xi)$ are multiples of one another in the ring $\mathbb{Z}_n$, and so generate the same principal ideal of $\mathbb{Z}_n$:

$$\mathfrak{I}_{m,n} := (1-\zeta)\mathbb{Z}_n = (1-\xi)\mathbb{Z}_n.$$

Lemma A.3. *Let m and n be natural integers such that m divides n. Then*

$$\begin{cases} \mathfrak{I}_{m,n} = \mathbb{Z}_n & \text{if } m \text{ is composite,} \\ \mathfrak{I}_{m,n}^{\varphi(m)} = p\mathbb{Z}_n & \text{if } m = p^a \ (p \text{ a prime number}). \end{cases}$$

Proof. This is an immediate consequence of the definition of $\mathfrak{I}_{m,n}$ and of item (4) of A.1. □

We now investigate the decomposition of $\mathfrak{I}_{m,n}$ into a product of prime ideals of $\mathbb{Z}_n$. By the preceding lemma, $\mathfrak{I}_{m,n}$ is invertible if m is composite. So from now on we assume that $m = p^a$ where p is a prime number and $a \geq 1$. The general result is provided by Proposition A.5 below; the next lemma gives the result in the particular case where n/m is prime to p.

Lemma A.4. *Let $m = p^a$ and $n = p^a n'$ be natural integers, where n' is prime to p.*

(1) If $n' = 1$ (i.e., $m = n$), the ideal $\mathfrak{I}_{m,m}$ is maximal in $\mathbb{Z}_m$. More precisely,

$$\mathbb{Z}_{p^a}/\mathfrak{I}_{p^a,p^a} \simeq \mathbb{F}_p\,.$$

(2) Let r be the order of p in the multiplicative group $(\mathbb{Z}/n'\mathbb{Z})^\times$ and let $d := \varphi(n')/r$. Then

$$\mathfrak{I}_{m,n} = \mathfrak{p}_1 \cdots \mathfrak{p}_d\,,$$

where $\mathfrak{p}_1, \cdots, \mathfrak{p}_d$ are the maximal ideals in $\mathbb{Z}_n$ such that $\mathbb{Z}_n/\mathfrak{p}_i$ is a finite field with p^r elements (for $i = 1, 2, \ldots, d$).

Proof. (1) The ideal $\mathfrak{I}_{m,m}$ is maximal, since (by item (4) of Proposition A.1) we have

$$\mathbb{Z}_m/\mathfrak{I}_{m,m} = \mathbb{Z}[X]/\langle \Phi_m(X), 1-X\rangle = \mathbb{Z}/\Phi_m(1) = \mathbb{Z}/p\mathbb{Z}\,.$$

(2) In $\mathbb{F}_p[X]$, $\Phi_{n'}(X)$ splits into irreducible polynomials of degree r, thus there are d of them. Thus the ring $\mathbb{F}_p[X]/\Phi_{n'}[X]$ is isomorphic to $\mathbb{F}_{p^r} \times \cdots \times \mathbb{F}_{p^r}$ (d factors).

By A.1 7(c) and the proof of (1), we have

$$\mathbb{Z}_{p^a n'}/\mathfrak{I}_{p^a,p^a n'} = \mathbb{Z}_{p^a}[X]/\langle \Phi_{n'}[X], \mathfrak{I}_{p^a,p^a}\rangle \simeq \mathbb{F}_p[X]/\Phi_{n'}[X]\,,$$

hence

$$\mathfrak{I}_{p^a,p^a n'} = \mathfrak{p}_1 \dots \mathfrak{p}_d$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_d$ are the maximal ideals of $\mathbb{Z}_n$ such that $\mathbb{Z}_n/\mathfrak{p}_i$ is a field with p^r elements. □

Proposition A.5. *Assume $m = p^a$ for some prime number p, and $n = p^{a+h}n'$ where n' is an integer not divisible by p. We denote by r the multiplicative order of p modulo n' and we set $d := \varphi(n')/r$. Then*

$$\mathfrak{I}_{m,n} = (\mathfrak{p}_1 \cdots \mathfrak{p}_d)^{p^h}\,,$$

where $\mathfrak{p}_1, \cdots, \mathfrak{p}_d$ are d maximal ideals in $\mathbb{Z}_n$ such that $\mathbb{Z}_n/\mathfrak{p}_i$ is a finite field with p^r elements (for $i = 1, 2, \dots, d$).

Proof. Proposition A.1(2) implies $\varphi(p^{a+h}) = p^h\varphi(m)$. By Lemma A.3,

$$p\mathbb{Z}_{p^{a+h}} = \mathfrak{I}_{p^{a+h},p^{a+h}}^{p^h\varphi(m)} = \mathfrak{I}_{m,p^{a+h}}^{\varphi(m)}\,.$$

Since $\mathfrak{I}_{p^{a+h},p^{a+h}}$ is a maximal ideal (Lemma A.4(1)), it follows from the uniqueness of the decomposition of an ideal into a product of prime ideals in the Dedekind domain $\mathbb{Z}_n$ that

$$\mathfrak{I}_{m,p^{a+h}} = \mathfrak{I}_{p^{a+h},p^{a+h}}^{p^h}\,,$$

which implies

$$\mathfrak{I}_{m,n} = \mathfrak{I}_{p^{a+h},n}^{p^h}\,.$$

Applying item (2) of Lemma A.4, we get

$$\mathfrak{I}_{m,n} = (\mathfrak{p}_1 \cdots \mathfrak{p}_d)^{p^h}\,.$$

□

Example A.6. Take $p = 13$, $a = 1$, $h = 0$, $n' = 3$, hence $n = 39$, $r = 1$, $d = 2$. Then (see proof of item (2) of A.4):

$$\mathbb{Z}_{39}/\mathfrak{I}_{13,39} = \mathbb{F}_{13}[X]/\Phi_3(X)\,.$$

The decomposition of $\Phi_3(X)$ in $\mathbb{F}_{13}[X]$ is $\Phi_3(X) = (X-3)(X-9)$. Define two ideals of $\mathbb{Z}_{39}$ as follows:

$$\mathfrak{p}_1 := \mathfrak{I}_{13,39} + (\zeta_3 - 3)\mathbb{Z}_{39}\,,$$
$$\mathfrak{p}_2 := \mathfrak{I}_{13,39} + (\zeta_3 - 9)\mathbb{Z}_{39}\,.$$

Then $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct maximal ideals of $\mathbb{Z}_{39}$ such that $\mathbb{Z}_{39}/\mathfrak{p}_1 \simeq \mathbb{Z}_{39}/\mathfrak{p}_2 \simeq \mathbb{F}_{13}$, and we have

$$\mathfrak{I}_{13,39} = \mathfrak{p}_1\mathfrak{p}_2\,.$$

It can be checked, for example with the PARI-GP command:

```
f=bnfinit(polcyclo(39));
bnfisprincipal(f,idealprimedec(f,13)[1])
```

that the ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are not principal ideals.

Corollary A.7. *Assume that $m' \mid m \mid n$ and that m is a power of a prime p. Then*

$$\mathfrak{I}_{m',n} = \mathfrak{I}_{m,n}^{m/m'} .$$

Proof. This is an immediate consequence of the above Proposition A.5, since the integer r and the maximal ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_d$ depend only on the pair (n, p). □

Corollary A.8. *Let m be a natural integer and let ζ be a root of unity of order m. Let n be a multiple of m.*

(1) For any natural integer $m' > 1$ dividing m, $(1-\zeta)(1-\zeta^{-1})$ divides m' in $\mathbb{Z}_n$.

(2) If $m \notin \{2, 3\}$, every prime factor of m divides $\dfrac{m}{(1-\zeta)(1-\zeta^{-1})}$ in $\mathbb{Z}_n$.

Proof.

(1) We first notice that if $m = m'$ the result follows from the equality $\prod_{i=1}^{m-1}(1-\zeta^i) = m$. We may thus assume $m > m'$, and we do so. If m is composite, by Lemma A.3, $\mathfrak{I}_{m,n} = \mathbb{Z}_n$ and there is nothing to prove. Otherwise, by Corollary A.7, we have $\mathfrak{I}_{m',n} = \mathfrak{I}_{m,n}^{m/m'}$; so in particular $\mathfrak{I}_{m',n}$, which divides $m'\mathbb{Z}_n$, is divisible by $\mathfrak{I}_{m,n}^2 = [(1-\zeta)\mathbb{Z}_n]\,[(1-\zeta^{-1})\mathbb{Z}_n]$.

(2) The statement is equivalent to saying that any prime factor of m divides $m\mathbb{Z}_n/\mathfrak{I}_{m,n}^2$. By Lemma A.3, $\mathfrak{I}_{m,n} = \mathbb{Z}_n$ if m is composite. Thus we need only consider the case where m is a prime power. Since

$$m\mathbb{Z}_n = \prod_{i=1}^{m-1}(1-\zeta^i)\mathbb{Z}_n = \prod_{i=1}^{m-1} \mathfrak{I}_{m/\gcd(i,m)} ,$$

then $m\mathbb{Z}_n$ is divisible by $\mathfrak{I}_m$ to the power at least $\varphi(m)$. For $m \notin \{2, 3, 4, 6\}$, $\varphi(m) > 2$ and the assertion holds. The cases $m = 2$ and 3 are excluded by the hypothesis, and $m = 6$ is composite. By the formula above for $m = 4$ we have $4\mathbb{Z}_n = \mathfrak{I}_{4,n}^2\mathfrak{I}_{2,n}$, and by Corollary A.7, we have $\mathfrak{I}_{2,n} = \mathfrak{I}_{4,n}^2$, completing the proof. □

A.3. **On cyclotomic fields.**

On real subfields of cyclotomic fields.

Proposition A.9. *Let $e > 1$ be an integer. We set $\zeta_e := \exp(2\pi i/e)$.*

(1) The field $\mathbb{Q}[\cos(2\pi/e)]$ is the largest real subfield of the cyclotomic field $\mathbb{Q}(\zeta_e)$, i.e., $\mathbb{Q}[\cos(2\pi/e)] = \mathbb{Q}(\zeta_e) \cap \mathbb{R}$.

(2) If e is odd, $\mathbb{Q}(\zeta_e) = \mathbb{Q}(\zeta_{2e})$ and $\mathbb{Q}[\cos(2\pi/e)] = \mathbb{Q}[\cos(\pi/e)]$.
If e is even, $\mathbb{Q}(\zeta_e) \subsetneq \mathbb{Q}(\zeta_{2e})$ and $\mathbb{Q}[\cos(2\pi/e)] \subsetneq \mathbb{Q}[\cos(\pi/e)]$.

Proof.

(1) Consider the sequence of polynomials $P_n(T) \in \mathbb{Z}[T]$ inductively defined by $P_0(T) = 2$, $P_1(T) = T$, and $P_{n+1}(T) = TP_n(T) - P_{n-1}(T)$. Then, for all $n \geq 1$,

$$X^n + X^{-n} = P_n(X + X^{-1}).$$

Since $\zeta_e + \zeta_e^{-1} = 2\cos(2\pi/e)$, it is clear that $\mathbb{Q}[\cos(2\pi/e)] \subset \mathbb{R} \cap \mathbb{Q}(\zeta_e)$. Conversely, any real element of $\mathbb{Q}(\zeta_e)$ is a $\mathbb{Q}$-linear combination of elements of the form $\zeta^j + \zeta^{-j}$, hence is a $\mathbb{Q}$-linear combination of $P_j(\zeta + \zeta^{-1})$, proving that it belongs to $\mathbb{Q}[\cos(2\pi/e)]$.

(2) If e is odd, $\zeta_{2e} = -\zeta_e^{(e+1/2)}$, hence $\mathbb{Q}(\zeta_e) = \mathbb{Q}(\zeta_{2e})$, and by part (1) this implies $\mathbb{Q}[\cos(2\pi/e)] = \mathbb{Q}[\cos(\pi/e)]$.

If e is even, $\varphi(2e) = 2\varphi(e)$, and so $\mathbb{Q}(\zeta_e) \subsetneq \mathbb{Q}(\zeta_{2e})$. Moreover, $i \in \mathbb{Q}(\zeta_{2e})$, hence an element $x + iy$ (with $x, y \in \mathbb{R}$) belongs to $\mathbb{Q}(\zeta_{2e})$ if and only if x and y belong to $\mathbb{Q}(\zeta_{2e})$. This shows that $\mathbb{Q}(\zeta_{2e}) = \mathbb{Q}[\cos(\pi/e)] \oplus i\mathbb{Q}[\cos(\pi/e)]$, hence that $[\mathbb{Q}[\cos(\pi/e)] : \mathbb{Q}] = \varphi(2e)/2 = \varphi(e)$. Since $\mathbb{Q}[\cos(2\pi/e)] \subsetneq \mathbb{Q}(\zeta_e)$, this shows that $\mathbb{Q}[\cos(2\pi/e)] \subsetneq \mathbb{Q}[\cos(\pi/e)]$. □

By [Was, Theorem 2.6. and Proposition 2.16],

Proposition A.10.

(1) The ring of integers of $\mathbb{Q}(\zeta_e)$ is $\mathbb{Z}[\zeta_e]$.
(2) The ring of integers of $\mathbb{Q}(\zeta_e + \zeta_e^{-1})$ is $\mathbb{Z}[\zeta_e + \zeta_e^{-1}]$.

The ring $\mathbb{Z}[\zeta_e + \zeta_e^{-1}]$ is not necessarily a P.I.D. For example (see [Mi, Theorem 1.1]), $\mathbb{Z}[\zeta_{163} + \zeta_{163}^{-1}]$ has class number 4. Nevertheless (*ibidem*),

Theorem A.11. *Let p be a prime number. Then the ring $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ is a P.I.D. if $p \leq 151$.*

The next lemma is used in section 7 above.

Lemma A.12. *Let ζ be a root of unity of order p^a, where p is a prime and $a \geq 1$ is a integer. Then the principal ideal of $\mathbb{Z}[\zeta + \zeta^{-1}]$ generated by $(1 - \zeta)(1 - \zeta^{-1}) = (2 - \zeta - \zeta^{-1})$ is prime.*

Proof. It suffices to prove that the norm $N_{\mathbb{Q}[\zeta+\zeta^{-1}]/\mathbb{Q}}(2 - \zeta - \zeta^{-1})$ of $(2 - \zeta - \zeta^{-1})$ is equal to p. By item (4) of Proposition A.1, we see that

$$N_{\mathbb{Q}[\zeta]/\mathbb{Q}}(1 - \zeta) = N_{\mathbb{Q}[\zeta]/\mathbb{Q}}(1 - \zeta^{-1}) = p.$$

Since

$$N_{\mathbb{Q}[\zeta]/\mathbb{Q}[\zeta+\zeta^{-1}]}(1 - \zeta) = N_{\mathbb{Q}[\zeta]/\mathbb{Q}[\zeta+\zeta^{-1}]}(1 - \zeta^{-1}) = (1 - \zeta)(1 - \zeta^{-1}),$$

the assertion follows from the fact that

$$N_{\mathbb{Q}[\zeta]/\mathbb{Q}} = N_{\mathbb{Q}[\zeta+\zeta^{-1}]/\mathbb{Q}} \cdot N_{\mathbb{Q}[\zeta]/\mathbb{Q}[\zeta+\zeta^{-1}]}.$$

□

Further properties of cyclotomic fields.

The following property may be found, for example, in [Sa, §6.5].

Proposition A.13. *Let p be an odd prime. Then the cyclotomic field $\mathbb{Q}(\zeta_p)$ contains a single quadratic extension, namely $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$.*

Corollary A.14. *Let p be an odd prime. Then*

- $\mathbb{Q}[\sqrt{p}] \subset \mathbb{Q}(\zeta_p)$ *if* $p \equiv 1 \pmod 4$.
- $\mathbb{Q}[\sqrt{-p}] \subset \mathbb{Q}(\zeta_p)$ *if* $p \equiv 3 \pmod 4$.

In the second case we have $\mathbb{Q}[\sqrt{-p}] \subset \mathbb{Q}(\zeta_{4p})$.

Let us recall that the fields $\mathbb{Q}[\sqrt{-p}]$ for $p \equiv 3 \pmod 4$ are principal ideal domains if and only if $p \in \{3, 7, 11, 19, 43, 67, 163\}$.

The next result shows that a cyclotomic extension cannot contain the roots of some rational numbers.

Proposition A.15. *A cyclotomic field $\mathbb{Q}(\zeta_n)$ cannot contain an element α whose minimal polynomial is $X^m - a$ for $a \in \mathbb{Q}$ and $m \geq 3$.*

Sketch of proof. If $\mathbb{Q}(\zeta_n)$ contains such an element α, it also contains the Galois closure of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$, namely $\mathbb{Q}(\alpha, \zeta_m)$. This is impossible since the Galois group of $\mathbb{Q}(\alpha, \zeta_m)/\mathbb{Q}$ is not abelian. □

The next theorem maybe found in [Was, Chap. 11, Theorem 11.1] .

Theorem A.16. *A cyclotomic field $\mathbb{Q}(\zeta_m)$ is a principal ideal domain if and only if $m \leq 22$ or $m \in \{24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 40, 42, 44, 45, 48, 50, 54, 60, 66, 70, 84, 90\}$.*

Appendix B. A table of Cartan matrices

The following table gives, for each exceptional irreducible finite complex reflection group $G_4, \ldots, G_{34}$:

- The diagram describing its presentation, using the same conventions as in [Bro2, Appendix A]; a subset of nodes representing each conjugacy class of hyperplanes have been given labels in the set $\{s, t, u\}$.
- The Cartan matrix C of a principal distinguished $\mathbb{Z}_k$-root system for the standard generators corresponding to the diagram. If there are several genera of $\mathbb{Z}_k$-root systems, below C we list the diagonal matrices which conjugate C to the Cartan matrix for other genera are listed below C.

  For badly generated groups, the rows (resp. columns) of the Cartan matrix which are integral linear combinations of the others are indicated by $*$. That there is always at least one such row and column fulfills the promise of Remark 6.8.
- The value of $\mathbb{Z}_k$. The values of $\mathbb{Z}_k$ for $G_{14}, G_{20}, G_{21}, G_{22}$ and G_{27} can be readily seen using Exercise 4.5.13 in [MuEs].
- A generator of the connection index for each root system.
- A generator of the (principal) ideal Bad_G.

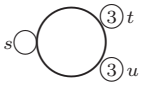
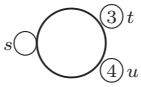
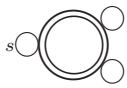
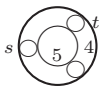
For the omitted exceptional groups $G_{35} = E_6$, $G_{36} = E_7$ and $G_{37} = E_8$, the diagram and the Cartan matrix are well known. We have $\mathbb{Z}_k = \mathbb{Z}$ in each case, connection indices are

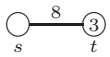
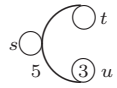
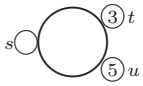
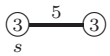
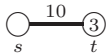
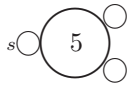
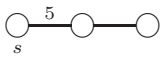
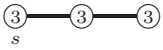
$$c_{E_6} = 3, c_{E_7} = 2 \text{ and } c_{E_8} = 1,$$

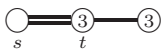
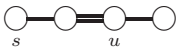
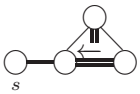
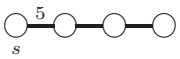
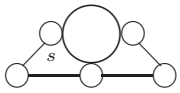
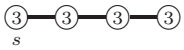
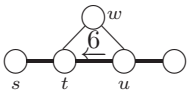
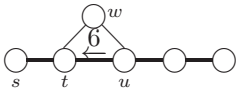
and the numbers Bad_G are:

$$\mathrm{Bad}_{E_6} = 6, \mathrm{Bad}_{E_7} = 6 \text{ and } \mathrm{Bad}_{E_8} = 120.$$

Finally, in the tables, we use the notation $\phi = \frac{1+\sqrt{5}}{2}$.

Name	Diagram	Cartan matrices	$\mathbb{Z}_k$	c_G	Bad_G
G_4		$\begin{pmatrix} 1-\zeta_3 & \boldsymbol{\zeta_3^2} \\ -\boldsymbol{\zeta_3^2} & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_3]$	2	$2\sqrt{-3}$
G_5		$\begin{pmatrix} 1-\zeta_3 & \mathbf{1} \\ -2\zeta_3 & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_3]$	1	
		$\mathrm{diag}(1,2)$			
G_6		$\begin{pmatrix} 2 & (\mathbf{i}+\boldsymbol{\zeta_3})(1+i) \\ -\mathbf{1} & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_{12}]$	$1+i$	$4\sqrt{3}$
		$\mathrm{diag}(i+1,1)$			
G_7		$\begin{array}{c} \\ * \\ * \\ * \end{array}\begin{array}{c} * \\ \begin{pmatrix} 2 & \zeta_3^2(1-i) & -\zeta_3^2(1+i) \\ \boldsymbol{\zeta_3}(\mathbf{1}+\mathbf{i}\boldsymbol{\zeta_3}) & 1-\zeta_3 & -\zeta_3(1-i) \\ \mathbf{i}\boldsymbol{\zeta_3}(\mathbf{1}+\mathbf{i}\boldsymbol{\zeta_3}) & 1+i & 1-\zeta_3 \end{pmatrix} \end{array}$	$\mathbb{Z}[\zeta_{12}]$	1	
		$\mathrm{diag}(i+1,1,i+1)$		1	
		$\mathrm{diag}(i+1,i+1,1)$		1	
		$\mathrm{diag}(i+1,1,1)$		1	
G_8		$\begin{pmatrix} 1-i & -i \\ 1 & 1-i \end{pmatrix}$	$\mathbb{Z}[i]$	1	12
G_9		$\begin{pmatrix} 2 & -\frac{\mathbf{2}+\sqrt{\mathbf{2}}}{\mathbf{i}+\mathbf{1}} \\ -\mathbf{1} & 1-i \end{pmatrix}$	$\mathbb{Z}[\zeta_8]$	1	
G_{10}		$\begin{pmatrix} 1-\zeta_3 & \mathbf{1} \\ -\mathbf{i}-\boldsymbol{\zeta_3} & 1-i \end{pmatrix}$	$\mathbb{Z}[\zeta_{12}]$	1	
G_{11}		$\begin{array}{c} \\ * \\ * \\ * \end{array}\begin{array}{c} \begin{array}{ccc} * & * & * \end{array} \\ \begin{pmatrix} 2 & \boldsymbol{\zeta_3^2}(\sqrt{-\mathbf{3}}-\sqrt{-\mathbf{2}}) & \mathbf{1}-\boldsymbol{\zeta_{24}} \\ \mathbf{1} & 1-\zeta_3 & \boldsymbol{\zeta_{24}^{13}} \\ \mathbf{1}+\boldsymbol{\zeta_{24}^5} & \boldsymbol{\zeta_{24}^7}+\boldsymbol{\zeta_{24}^5} & 1-i \end{pmatrix} \end{array}$	$\mathbb{Z}[\zeta_{24}]$	1	
G_{12}		$\begin{array}{c} \\ * \\ * \\ * \end{array}\begin{array}{c} \begin{array}{ccc} * & * & * \end{array} \\ \begin{pmatrix} 2 & -1-\sqrt{-2} & -1+\sqrt{-2} \\ -1+\sqrt{-2} & 2 & -1-\sqrt{-2} \\ -1-\sqrt{-2} & -1+\sqrt{-2} & 2 \end{pmatrix} \end{array}$	$\mathbb{Z}[\sqrt{-2}]$	1	
G_{13}		$\begin{array}{c} \\ * \end{array}\begin{array}{c} \begin{array}{ccc} * & * & * \end{array} \\ \begin{pmatrix} 2 & \sqrt{2} & i-1 \\ (\mathbf{1}+\sqrt{\mathbf{2}}) & 2 & -1+\sqrt{-2} \\ -\boldsymbol{\zeta_8}(\mathbf{1}+\sqrt{\mathbf{2}}) & -1-\sqrt{-2} & 2 \end{pmatrix} \end{array}$	$\mathbb{Z}[\zeta_8]$	1	
		$\mathrm{diag}(1-\zeta_8,1,1)$		1	
		$\mathrm{diag}(i+1,1,1)$		1	

Name	Diagram	Cartan matrices	$\mathbb{Z}_k$	c_G	Bad_G
G_{14}		$\begin{pmatrix} 2 & -\mathbf{1} \\ -\zeta_{\mathbf{3}}^{\mathbf{2}}(\sqrt{-\mathbf{3}}+\sqrt{-\mathbf{2}}) & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_3, \sqrt{-2}]$	1	$12\sqrt{-2}$
G_{15}		$\begin{array}{cc} & \begin{array}{ccc} * & * & * \end{array} \\ * & \begin{pmatrix} 2 & -\zeta_{\mathbf{24}}-\zeta_{\mathbf{24}}^{\mathbf{8}} & \mathbf{1} \\ \mathbf{1}-\zeta_{\mathbf{24}}^{-\mathbf{1}} & 1-\zeta_3 & \mathbf{1} \\ \zeta_{\mathbf{8}}^{\mathbf{3}}(1-\zeta_8)^2 & -\zeta_{\mathbf{8}}(\zeta_{\mathbf{3}}+\mathbf{i})\sqrt{2} & 2 \end{pmatrix} \end{array}$	$\mathbb{Z}[\zeta_{24}]$	1	
		$\mathrm{diag}(1,1,1-\zeta_8)$		1	
		$\mathrm{diag}(1,1,1+i)$		1	
G_{16}		$\begin{pmatrix} 1-\zeta_5 & \mathbf{1} \\ -\zeta_{\mathbf{5}} & 1-\zeta_5 \end{pmatrix}$	$\mathbb{Z}[\zeta_5]$	1	
G_{17}		$\begin{pmatrix} 2 & \mathbf{1} \\ \mathbf{1}-\zeta_{\mathbf{5}}-\zeta_{\mathbf{20}}^{\mathbf{7}} & 1-\zeta_5 \end{pmatrix}$	$\mathbb{Z}[\zeta_{20}]$	1	
G_{18}		$\begin{pmatrix} 1-\zeta_3 & -\zeta_{\mathbf{15}}^{\mathbf{4}} \\ \zeta_{\mathbf{15}}^{-\mathbf{1}}+\zeta_{\mathbf{15}} & 1-\zeta_5 \end{pmatrix}$	$\mathbb{Z}[\zeta_{15}]$	1	
G_{19}		$\begin{array}{cc} & \begin{array}{ccc} * & * & * \end{array} \\ \begin{array}{c} * \\ * \\ * \end{array} & \begin{pmatrix} 2 & \mathbf{1}-\zeta_{\mathbf{3}}\zeta_{\mathbf{60}}^{-\mathbf{1}} & -\mathbf{1} \\ \mathbf{1}+\zeta_{\mathbf{60}} & 1-\zeta_3 & -\zeta_{\mathbf{60}} \\ \zeta_{\mathbf{5}}-\mathbf{1}+\zeta_{\mathbf{20}}^{\mathbf{7}} & \zeta_{\mathbf{60}}^{\mathbf{19}}+\zeta_{\mathbf{60}}^{\mathbf{11}} & 1-\zeta_5 \end{pmatrix} \end{array}$	$\mathbb{Z}[\zeta_{60}]$	1	
G_{20}		$\begin{pmatrix} 1-\zeta_3 & -\zeta_{\mathbf{3}}\phi \\ \phi & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_3, \phi]$	1	
G_{21}		$\begin{pmatrix} 2 & \mathbf{1} \\ \mathbf{1}-\zeta_{\mathbf{3}}+\frac{\phi}{\zeta_{\mathbf{12}}} & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_{12}, \phi]$	1	
G_{22}		$\begin{array}{cc} & \begin{array}{ccc} * & * & * \end{array} \\ \begin{array}{c} * \\ * \\ * \end{array} & \begin{pmatrix} 2 & -i-\phi & -i+\phi \\ i-\phi & 2 & -1+i\phi \\ i+\phi & -1-i\phi & 2 \end{pmatrix} \end{array}$	$\mathbb{Z}[i, \phi]$	1	
$G_{23} = H_3$		$\begin{pmatrix} 2 & -\phi & 0 \\ -\phi & 2 & -\mathbf{1} \\ 0 & -\mathbf{1} & 2 \end{pmatrix}$	$\mathbb{Z}[\phi]$	2	$2\sqrt{5}$
G_{24}		$\begin{pmatrix} 2 & -\mathbf{1} & -\mathbf{1} \\ -\mathbf{1} & 2 & (1-\sqrt{-7})/2 \\ -\mathbf{1} & (1+\sqrt{-7})/2 & 2 \end{pmatrix}$	$\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$	1	$2\sqrt{-7}$
G_{25}		$\begin{pmatrix} 1-\zeta_3 & \zeta_{\mathbf{3}}^{\mathbf{2}} & 0 \\ -\zeta_{\mathbf{3}}^{\mathbf{2}} & 1-\zeta_3 & \zeta_{\mathbf{3}}^{\mathbf{2}} \\ 0 & -\zeta_{\mathbf{3}}^{\mathbf{2}} & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_3]$	$\sqrt{-3}$	6

Name	Diagram	Cartan matrices	$\mathbb{Z}_k$	c_G	Bad_G
G_{26}		$\begin{pmatrix} 2 & -\mathbf{1} & 0 \\ \zeta_3-1 & 1-\zeta_3 & \boldsymbol{\zeta_3^2} \\ 0 & -\boldsymbol{\zeta_3^2} & 1-\zeta_3 \end{pmatrix}$ $\mathrm{diag}(1, 1-\zeta_3, 1-\zeta_3)$	$\mathbb{Z}[\zeta_3]$	1	$18\sqrt{-3}$
G_{27}		$\begin{pmatrix} 2 & -\mathbf{1} & -\mathbf{1} \\ -\mathbf{1} & 2 & \frac{\zeta_3^2(\sqrt{-3}+\sqrt{5})}{2} \\ -\mathbf{1} & \frac{(\sqrt{5}-\sqrt{-3})\zeta_3}{2} & 2 \end{pmatrix}$	$\mathbb{Z}[\zeta_3, \phi]$	1	$6\sqrt{5}$
G_{28} $= F_4$		$\begin{pmatrix} 2 & -\mathbf{1} & 0 & 0 \\ -\mathbf{1} & 2 & -\mathbf{1} & 0 \\ 0 & -2 & 2 & -\mathbf{1} \\ 0 & 0 & -\mathbf{1} & 2 \end{pmatrix}$ $\mathrm{diag}(1, 1, 2, 2)$	$\mathbb{Z}$	1	24
G_{29}		$\begin{pmatrix} 2 & -\mathbf{1} & 0 & 0 \\ -\mathbf{1} & 2 & i+1 & -\mathbf{1} \\ 0 & 1-i & 2 & -\mathbf{1} \\ 0 & -\mathbf{1} & -\mathbf{1} & 2 \end{pmatrix}$	$\mathbb{Z}[i]$	1	20
G_{30} $= H_4$		$\begin{pmatrix} 2 & -\phi & 0 & 0 \\ -\phi & 2 & -\mathbf{1} & 0 \\ 0 & -\mathbf{1} & 2 & -\mathbf{1} \\ 0 & 0 & -\mathbf{1} & 2 \end{pmatrix}$	$\mathbb{Z}[\phi]$	1	120
G_{31}		$\begin{array}{c} \\ * \\ * \\ * \end{array} \begin{array}{c} \begin{array}{ccccc} * & * & * & & \end{array} \\ \begin{pmatrix} 2 & i+1 & 1-i & -\mathbf{i} & 0 \\ 1-i & 2 & 1-i & -\mathbf{1} & -\mathbf{1} \\ i+1 & i+1 & 2 & 0 & -\mathbf{1} \\ \mathbf{i} & -\mathbf{1} & 0 & 2 & 0 \\ 0 & -\mathbf{1} & -\mathbf{1} & 0 & 2 \end{pmatrix} \end{array}$	$\mathbb{Z}[i]$	1	
G_{32}		$\begin{pmatrix} 1-\zeta_3 & \boldsymbol{\zeta_3^2} & 0 & 0 \\ -\boldsymbol{\zeta_3^2} & 1-\zeta_3 & \boldsymbol{\zeta_3^2} & 0 \\ 0 & -\boldsymbol{\zeta_3^2} & 1-\zeta_3 & \boldsymbol{\zeta_3^2} \\ 0 & 0 & -\boldsymbol{\zeta_3^2} & 1-\zeta_3 \end{pmatrix}$	$\mathbb{Z}[\zeta_3]$	1	$120\sqrt{-3}$
G_{33}		$\begin{pmatrix} 2 & -\mathbf{1} & 0 & 0 & 0 \\ -\mathbf{1} & 2 & -\mathbf{1} & -\boldsymbol{\zeta_3^2} & 0 \\ 0 & -\mathbf{1} & 2 & -\mathbf{1} & 0 \\ 0 & -\boldsymbol{\zeta_3} & -\mathbf{1} & 2 & -\mathbf{1} \\ 0 & 0 & 0 & -\mathbf{1} & 2 \end{pmatrix}$	$\mathbb{Z}[\zeta_3]$	2	6
G_{34}		$\begin{pmatrix} 2 & -\mathbf{1} & 0 & 0 & 0 & 0 \\ -\mathbf{1} & 2 & -\mathbf{1} & -\boldsymbol{\zeta_3^2} & 0 & 0 \\ 0 & -\mathbf{1} & 2 & -\mathbf{1} & 0 & 0 \\ 0 & -\boldsymbol{\zeta_3} & -\mathbf{1} & 2 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} & 2 & -\mathbf{1} \\ 0 & 0 & 0 & 0 & -\mathbf{1} & 2 \end{pmatrix}$	$\mathbb{Z}[\zeta_3]$	1	42

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