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# Periodic solutions of $n$ -dof autonomous vibro-impact oscillators with one lasting contact phase

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**ABSTRACT** In the continuum structural mechanics framework, a unilateral contact condition between two flexible bodies does not generate impulsive contact forces. However, finite-dimensional systems, derived from a Finite Element semi-discretisation in space for instance, and undergoing a unilateral contact condition, require an additional impact law: unilateral contact occurrences then become impacts of zero duration unless (i) the impact law is purely inelastic, or (ii) the pre-impact velocity is zero. This contribution explores autonomous periodic solutions with one contact phase per period and zero pre-impact velocity (case (ii)), for any  $n$ -dof mechanical systems involving linear free-flight dynamics together with a linear unilateral contact constraint. A recent work has shown that such solutions seem to be limits of periodic trajectories with  $k$  impacts per period as  $k$  increases. Minimal analytic equations governing the existence of such solutions are proposed and it is proven that, generically, they occur only for discrete values of the period. It is also shown that the graphs of such periodic solutions have two axes of symmetry in time. Results are illustrated on a spring-mass system and on a 4-dof two-dimensional system made of 1D Finite Elements. Animations of SPPs with up to 30 dofs are provided.

**KEYWORDS** vibration analysis; impact dynamics; periodic solutions; vibro-impact oscillators; nonsmooth modal analysis

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**1. Introduction** When a continuous medium collides with a rigid obstacle, (i) travelling waves emerge and (ii) the gap remains closed for some time until the two bodies separate. In the spatially semi-discretised framework, (i) travelling waves are not properly captured and this is commonly overcome via an additional impact law [3]. The impact law makes the finite-dimensional dynamics deterministic, meaning that any initial conditions determines a unique time-evolution. In general, (ii) lasting contact phases are only possible for inelastic impact laws, resulting in energy dissipation. However, some very specific initial conditions generate solutions which have the property of being uniquely determined without requiring an impact law and featuring lasting contact phases. Such non-impulsive solutions “avoid” impacts by smoothly closing the gap with zero velocity and zero acceleration every time contact is activated. Among this class of trajectories, this work investigates periodic ones with one contact phase per period. They are referred to as 1 Sticking phase Per Period (1 SPP or simply SPP) solutions of  $n$ -dof autonomous systems with piecewise-linear dynamics and one unilateral contact constraint. While the investigation of such trajectories was initially motivated by their similarity with the continuous framework (no requirement for an impact law and dissipation-free lasting contact phases), they also seem to be limits of periodic solutions with  $k$  impacts per period as  $k \rightarrow \infty$  [15, 16].

The dynamics of finite-dimensional systems subject to unilateral contact constraints has been widely

investigated for several decades. Forced response of 1-dof vibro-impact oscillators were investigated [20] with particular interest in bifurcations [9, 12]. Multidimensional systems were later studied, see e.g. [11] and further references. The well-known phenomenon of chattering, consisting in an infinite number of gap closures in finite time, was also investigated for 1-dof systems [2] and  $n$ -dof systems [1, 4]. When chattering is said to be *complete*, it is followed by a phase of lasting closed contact, referred to as *sticking*<sup>1</sup>, where the external force maintains the gap closed for a finite amount of time. Such sticking phases have also been widely investigated [7, 8, 17–19], including stability analysis [10]. The major difference in this work lies in the fact that autonomous periodic solutions with sticking phases are targeted: there is no external force. Such trajectories were initially investigated by Le Thi [6] for a 2-dof-spring-mass system with a unilateral constraint on one mass, where the terminology SPP was first introduced. The present contribution extends the results to  $n$ -degree-of-freedom systems, general mass and stiffness matrices such as those encountered in the Finite Element Method, as well as any affine unilateral constraint.

More precisely, we consider a system with mass matrix  $M$  and stiffness matrix  $K$ , both symmetric positive-definite, undergoing a unilateral contact condition  $g(x) \leq 0$  where  $x$  is the vector of generalized coordinates. The gap function  $g$  is assumed affine, *i.e.* there exists a unit vector  $w$  such that  $g(x) = w^\top x + g_0$  where  $g_0$  is the signed gap from the unconstrained resting position. Contact is closed if  $g(x) = 0$ . We now assume that  $M$ ,  $K$  and  $w$  are fixed once for all and in the remaining, the term *generically* is to be understood in the sense of *for almost every*  $M$ ,  $K$  and  $w$ .

Without loss of generality, the initial time  $t = 0$  is fixed such that the free phase corresponds to time  $t \in [-2t_1, 0]$  while the contact phase corresponds to  $t \in [0, 2t_2]$  (the coefficient 2 is introduced to simplify further derivations). The unknown period is  $T = 2t_1 + 2t_2$ . The governing equations on  $[-2t_1, 0]$  read

$$M\ddot{x} + Kx = 0 \quad (1.1)$$

complemented on  $[0, 2t_2]$ , by the Signorini conditions [5]:

$$\begin{cases} M\ddot{x} + Kx = w\lambda & (1.2a) \\ g(x) \geq 0 & (1.2b) \\ \lambda \geq 0 & (1.2c) \\ g(x)\lambda = 0. & (1.2d) \end{cases}$$

Since periodic solutions are targeted, the unknowns of the problem are the initial conditions  $x(0), \dot{x}(0) \in \mathbb{R}^n$ , the durations  $t_1, t_2 \in \mathbb{R}$  and the normal contact force  $\lambda \in \mathbb{R}$ . Ineq. (1.2b) implies that a zero contact velocity  $w^\top \dot{x}(0) = 0$  may occur only if  $w^\top \ddot{x}(0) > 0$  or if  $w^\top \ddot{x}(0) = 0$ . The first case corresponds to *grazing contact* and cannot lead to sticking because of ineq. (1.2c). The second case means that if the obstacle were to be removed,  $t = 0$  would correspond to an inflection point of the normal acceleration  $w^\top \ddot{x}$ . A sticking phase can emerge with a zero contact velocity through this mechanism only. The free phase of the trajectory is connected to the contact phase via continuity conditions on  $x$  and  $\dot{x}$ , together with the periodicity condition, the contact condition at 0 and  $2t_2$  and the zero contact velocity and acceleration equations:

$$\begin{cases} x^-(0) = x^+(0), & \dot{x}^-(0) = \dot{x}^+(0) & \text{[continuity]} & (1.3a) \\ x(-2t_1) = x(2t_2), & \dot{x}(-2t_1) = \dot{x}(2t_2) & \text{[periodicity]} & (1.3b) \\ w^\top \dot{x}(0) = 0, & w^\top \ddot{x}(0) = 0 & \text{[zero normal velocity and acceleration]} & (1.3c) \\ w^\top x(t) + g_0 = 0, t \in [0, 2t_2] & & \text{[contact condition]} & (1.3d) \end{cases}$$

with  $x \in \mathcal{C}^2([-2t_1, 2t_2])$ . The schematic gap and normal contact force of a SPP are represented in fig. 1.

Instead of solving these equations directly, the present approach consists in mimicking the contact phase by a free phase with a different stiffness matrix  $\tilde{K}$ . The new stiffness matrix should not affect the solutions of the original problem. That point is addressed in section 2, leading to the explicit characterisation of  $\tilde{K}$ . Necessary conditions on  $t_1, t_2$  and  $x(0), \dot{x}(0)$  for the existence of solutions with 1 SPP are given in section 3. The derivations are provided in section 4. Summary of the results and illustrations are given in section 5.

<sup>1</sup> The terminology might be confusing: there is no actual sticking in the sense that the sign of the normal force cannot change, *i.e.* there is no glue.

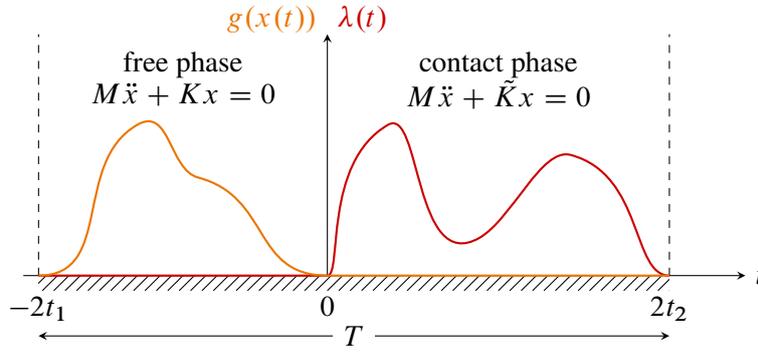


Figure 1: Example time-evolution of gap  $g(x)$  and normal contact force  $\lambda$ .

**2. Unilateral contact condition seen as a stiffness matrix switch** We consider a solution of the free system eq. (1.1) meeting the obstacle at  $t = 0$ , that is  $g(x(0)) = w^\top x(0) + g_0 = 0$ , with a zero contact velocity  $w^\top \dot{x} = 0$ . The aim is to describe the contact phase by a differential equation  $M\ddot{x} + \tilde{K}x = 0$  ( $\mathcal{S}$ ) such that:

- C1. **[gap closure]** all the solutions of  $\mathcal{S}$  satisfy the contact condition during the contact phase:  $w^\top \dot{x} = 0$  over  $[0, 2t_2]$ ;
- C2. **[energy conservation]** the solutions of  $\mathcal{S}$  which satisfy condition  $w^\top \dot{x} = 0$  preserve the energy.

**2.1. Consequences of energy conservation** The second condition means that  $w^\top \dot{x} = 0$  implies a constant energy. Equivalently, its time derivative must vanish:

$$0 = \dot{x}^\top M\ddot{x} + \dot{x}^\top Kx = \dot{x}^\top (M\ddot{x} + Kx) = \dot{x}^\top (-\tilde{K} + K)x. \quad (2.1)$$

This scalar number must be zero for all  $x$  so  $\dot{x}^\top (K - \tilde{K}) = 0$ , which must hold for any  $\dot{x}$  such that  $w^\top \dot{x} = 0$ . Hence, each column of  $K - \tilde{K}$  must be orthogonal to the hyperplane orthogonal to  $w$ , that is each column must be proportional to  $w$ . Accordingly, there exists a vector  $v$  such that  $K - \tilde{K} = wv^\top$ . We are now going to use the first condition to characterize the vector  $v$  such that:

$$\tilde{K} = K - wv^\top. \quad (2.2)$$

**2.2. Consequences of gap closure** Since  $w^\top \dot{x}(0) = 0$ ,  $w^\top \dot{x} = 0$  over  $[0, 2t_2]$  is equivalent to  $w^\top \ddot{x} = 0$  over  $[0, 2t_2]$ . All solutions of  $\mathcal{S}$  satisfy  $\ddot{x} = -M^{-1}\tilde{K}x$ , hence  $w^\top \ddot{x} = 0$  becomes  $w^\top M^{-1}\tilde{K} = 0$ . Invoking eq. (2.2) and  $w^\top M^{-1}w > 0$  because  $M$  is definite-positive yields:

$$w^\top M^{-1}(K - wv^\top) = 0 \quad \text{or} \quad w^\top M^{-1}K = (w^\top M^{-1}w)v \quad \text{or} \quad v = \frac{1}{w^\top M^{-1}w} KM^{-1}w. \quad (2.3)$$

We have just shown the following result.

**Theorem 2.1** *There exists a unique matrix  $\tilde{K}$  satisfying conditions C1 and C2, given by:*

$$\tilde{K} = \left( I_n - \frac{1}{w^\top M^{-1}w} ww^\top M^{-1} \right) K. \quad (2.4)$$

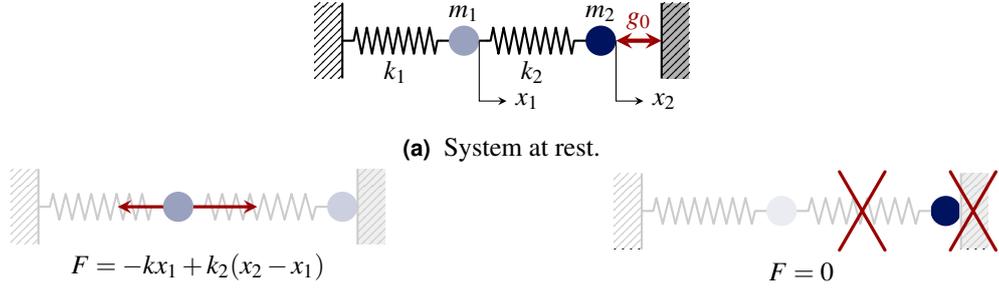
**2.3. Stiffness matrix switch illustrated on a simple example** This abstract result can be easily understood with a simple system. Consider the 2-dof spring-mass system illustrated in fig. 2 (left) as in [6]. The gap function is  $g(x) = g_0 - x_2 = w^\top x + g_0$  with  $w^\top = [0 \ -1]$ . The mass and stiffness matrices are expressed as:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}. \quad (2.5)$$

From theorem 2.1, the equivalent stiffness matrix  $\tilde{K}$  during the contact phase is

$$\tilde{K} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - m_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ 0 & 0 \end{bmatrix}. \quad (2.6)$$

This is interpreted as follows: the first mass does not see any change as its dynamics is still dictated by the action of both springs. The dynamics of the second mass is not restricted by a unilateral contact condition since  $M\ddot{x} + \tilde{K}x = 0$  but instead the connected spring induces no action: as a consequence, the second row of  $\tilde{K}$  vanishes. This can be understood as a ‘‘surgery’’ on the stiffness matrix, resulting in mimicking



**Figure 2:** Physical interpretation of  $\tilde{K}$  illustrated on a simple system.

the contact phase by a free phase and thus eliminating the contact force as an unknown—which can be recovered from:

$$\lambda w = M\ddot{x} + Kx = \underbrace{M\ddot{x} + \tilde{K}x}_{=0} - \tilde{K}x + Kx = (K - \tilde{K})x = \frac{1}{w^\top M^{-1}w} w w^\top M^{-1} Kx. \quad (2.7)$$

It is noteworthy that  $\tilde{K}$  is not symmetric, meaning it violates Maxwell–Betti reciprocal work theorem: it is not possible to build a linear (unconstrained!) mechanical system of stiffness matrix  $\tilde{K}$ .

### 3. Solution method

The problem is now reduced to finding one solution satisfying

$$\forall t \in [-2t_1, 0], \quad \begin{cases} M\ddot{x}(t) + Kx(t) = 0 \\ g(x(t)) \geq 0 \end{cases} \quad (3.1a) \quad (3.1b)$$

during the free phase and

$$\forall t \in [0, 2t_2], \quad \begin{cases} M\ddot{x}(t) + \tilde{K}x(t) = 0 \\ w^\top M^{-1}Kx(t) \geq 0 \end{cases} \quad (3.2a) \quad (3.2b)$$

during the contact phase, together with the continuity conditions (1.3a), periodicity conditions (1.3b), zero contact velocity and acceleration (1.3c) and gap closure (1.3d) at  $t = 0$ . For the discussion to come, we introduce a terminology to distinguish actual solutions (admissible solutions) and solutions satisfying only the equalities, but not necessarily the two inequalities (potential solutions).

**Definition 3.1** A solution of eqs. (1.3), (3.1a) and (3.2a) is referred to as potential solution (or potential SPP). An admissible solution (or admissible SPP) is a solution of eqs. (1.3), (3.1) and (3.2).

By construction of the solution, eq. (1.2c) is always satisfied during the free phase since  $\lambda = 0$  and eq. (1.2b) is always satisfied during the contact phase, since  $g(x) = 0$ . This implies that eq. (1.2d) is also verified. It is worth mentioning that eq. (3.2b) follows from eq. (2.7), given that  $w^\top M^{-1}w > 0$ .

We are now showing that the dynamics of both phases can be written in the form  $\ddot{y} = -L^2y$  for some diagonalisable matrix  $L$  and  $y = M^{1/2}x$ .

#### 3.1. Free-flight dynamics

The matrix  $M$  has a unique symmetric and positive-definite root  $M^{1/2}$ ; let  $y = M^{1/2}x$ . Eq. (3.1a) becomes

$$\ddot{y} + M^{-1/2}KM^{-1/2}y = 0. \quad (3.3)$$

The matrix  $K_1 := M^{-1/2}KM^{-1/2}$  is symmetric hence orthogonally diagonalisable; let  $K_1 = Q\Omega^2Q^{-1}$  for some orthogonal matrix  $Q$  and  $\Omega = \text{diag}(\omega_1, \dots, \omega_n)$  a diagonal matrix of positive diagonal entries. Let  $L_1 = Q\Omega Q^{-1}$ , it follows that

$$\ddot{y} = -L_1^2y. \quad (3.4)$$

**3.2. Dynamics of the contacting phase** Expressing eq. (3.2) in terms of  $y$  gives:

$$\ddot{y} = -M^{-1/2} \tilde{K} M^{-1/2} y. \quad (3.5)$$

From theorem 2.1 and introducing  $w_1 = M^{-1/2} K$ , it comes that

$$\tilde{K} = K - \frac{1}{w_1^\top w_1} M^{1/2} w_1 w_1^\top M^{-1/2} K = M^{1/2} \left( I_n - \frac{1}{w_1^\top w_1} w_1 w_1^\top \right) K M^{1/2} \quad (3.6)$$

so that finally:

$$\ddot{y} = -J_1 K_1 y \quad \text{with} \quad J_1 = I_n - \frac{1}{w_1^\top w_1} w_1 w_1^\top. \quad (3.7)$$

$J_1$  is the matrix of the orthogonal projection on the hyperplane  $w_1^\perp$ , so there is an orthogonal change of basis of matrix  $P$  such that  $P^\top J_1 P$  is the orthogonal projection on the hyperplane  $\mathcal{H} = e_n^\perp$ , of diagonal matrix  $J = \text{diag}(1, \dots, 1, 0)$ . This change of basis transforms  $K_1$  into  $K_2 = P^\top K_1 P$  where  $K_2$  is symmetric positive-definite. The restriction of  $JK_2$  to  $\mathcal{H}$  is symmetric positive-definite, hence diagonalisable, invertible and has positive eigenvalues. The direct sum of  $\mathcal{H}$  and the kernel of  $JK_2$  is the whole vector space, hence  $JK_2$  is diagonalisable and so is  $J_1 K_1$ . Additionally, it follows that  $JK_2$  has non-negative eigenvalues (1 zero eigenvalue and  $n - 1$  strictly positive eigenvalues), implying that  $J_1 K_1$  has a square root  $L_2$ . In the end, we have shown that the dynamics during the contact phase can be described by the following ODE:

$$\ddot{y} = -L_2^2 y. \quad (3.8)$$

In practice,  $L_1$  and  $L_2$  are found numerically.

**3.3. Solutions of  $\ddot{y} = -L^2 y$  for some initial conditions  $y(0)$  and  $\dot{y}(0)$**  The dynamics of both the free and contact phases is captured by an ODE of the form  $\ddot{y} = -L^2 y$  (see eqs. (3.4) and (3.8)). This simple ODE can easily be solved as follows. Let

$$z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -L^2 & 0 \end{bmatrix}, \quad (3.9)$$

then the ODE becomes  $\dot{z} = Az$  so  $z(t) = \exp(tA)z(0)$ , where

$$\exp(tA) = \sum_{i=0}^{\infty} (-1)^i \frac{t^{2i}}{(2i)!} \begin{bmatrix} L^{2i} & 0 \\ 0 & L^{2i} \end{bmatrix} + \sum_{i=0}^{\infty} (-1)^i \frac{t^{2i+1}}{(2i+1)!} \begin{bmatrix} 0 & L^{2i} \\ -L^{2i+2} & 0 \end{bmatrix} \quad (3.10)$$

$$= \begin{bmatrix} \cos(tL) & t \text{sinc}(tL) \\ -L \sin(tL) & \cos(tL) \end{bmatrix}. \quad (3.11)$$

where  $\cos$ ,  $\sin$  and  $\text{sinc}$  are trigonometric functions of matrices. It follows that

$$\begin{cases} y(t) = \cos(tL)y(0) + t \text{sinc}(tL)\dot{y}(0) & (3.12a) \\ \dot{y}(t) = -L \sin(tL)y(0) + \cos(tL)\dot{y}(0). & (3.12b) \end{cases}$$

If  $L$  is invertible (in particular,  $L_1$  is invertible),  $t \text{sinc}(tL) = L^{-1} \sin(tL)$ . Since  $L$  can be made diagonal, the matrix operators  $\text{sinc}$  and  $\sin$  can be advantageously reduced to scalar operators acting on the eigenvalues  $(\lambda_1, \dots, \lambda_n)$  of  $L$ : if  $L = PDP^{-1}$ ,  $\text{sinc}(L) = P \text{diag}(\text{sinc}(\lambda_1), \dots, \text{sinc}(\lambda_n))P^{-1}$  and  $\sin(L) = P \text{diag}(\sin(\lambda_1), \dots, \sin(\lambda_n))P^{-1}$ .

**3.4. Junction at  $t = 0$  of the solutions for  $t < 0$  and  $t > 0$ , and symmetry** We consider one solution  $y \in \mathcal{C}^2(\mathbb{R})$  satisfying  $\ddot{y} = -L_1^2 y$  over  $\mathbb{R}^-$  and  $\ddot{y} = -L_2^2 y$  over  $\mathbb{R}^+$ , both sharing the same initial conditions  $y(0)$ ,  $\dot{y}(0)$ . Finding a periodic solution reduces to finding  $t_1, t_2 > 0$  such that there exists a nonzero solution  $y$  defined over  $[-2t_1, 2t_2]$ , with  $y(-2t_1) = y(2t_2)$  and  $\dot{y}(-2t_1) = \dot{y}(2t_2)$ . Introducing  $A_1, A_2$  corresponding to  $L_1, L_2$  in eq. (3.9), this condition can also be written as:

$$\exp(-2t_1 A_1)z(0) = \exp(2t_2 A_2)z(0) \quad \text{or} \quad \Delta z(0) = 0 \quad (3.13)$$

with  $\Delta = \exp(-2t_1 A_1) - \exp(2t_2 A_2)$ . Nonzero solutions may only exist for  $t_1, t_2$  such that  $\Delta$  is singular. We are going to show the following result.

**Theorem 3.1** *Instants  $t_1, t_2$  satisfy  $\det \Delta = 0$  iff there exists a nonzero solution  $y$  such that*

- *either  $y(-t_1) = y(t_2) = 0$ ; in that case,  $y$  has a **point of symmetry** at  $t_1$  over  $[-2t_1, 0]$ , and at  $t_2$  over  $[0, 2t_2]$ .*
- *or  $\dot{y}(-t_1) = \dot{y}(t_2) = 0$ ; in that case,  $y$  has one **axis of symmetry** at  $t_1$  on the interval  $[-2t_1, 0]$ , and another axis at  $t_2$  on the interval  $[0, 2t_2]$ .*

**Corollary 3.1** *All generic solutions have two axes of symmetry per period: one in the middle of the contact phase and one in the middle of the free flight phase.*

Corollary 3.1 directly results from theorem 3.1: if  $g_0 \neq 0$ , at the middle of the contact phase, that is  $g(x(t_2)) = 0$ , then  $w^\top x(t_2) = -g_0$  which is in contradiction with  $y$  having a punctual symmetry over  $[0, 2t_2]$  (we would have  $y(t_2) = 0$  so  $x(t_2) = 0$ ). It follows from the theorem that any solution has two axes of symmetry. If  $g_0 = 0$  and  $\det(T_2) = 0$ , by central symmetry  $w^\top x(-2t_1) = w^\top x(-t_1) = w^\top x(0) = 0$ . If for some  $t_0 \in (-2t_1, -t_1)$ ,  $w^\top x(t_0) > 0$ , then by punctual symmetry  $w^\top x(-2t_1 - t_0) < 0$  with  $-2t_1 - t_0 \in (-t_1, 0)$ , which is in contradiction with the unilateral contact condition  $w^\top x \geq 0$ .

To prove theorem 3.1, we start with the following simple observation.

**Symmetries of the solutions of  $\ddot{y} = -L^2 y$**  Eq. (3.12) implies that  $y$  is even iff  $\dot{y}(0) = 0$  and odd iff  $y(0) = 0$ . Similarly, via a  $t_0$  translation,  $y(t_0 + t) = y(t_0 - t)$  iff  $\dot{y}(t_0) = 0$  and  $y(t_0 - t) = -y(t_0 + t)$  iff  $y(t_0) = 0$ . The first case corresponds to an axial symmetry, the second one to a central symmetry.

**Proof of Theorem 3.1** Couples  $(t_1, t_2)$  satisfying periodicity conditions  $y(-2t_1) = y(2t_2)$  and  $\dot{y}(-2t_1) = \dot{y}(2t_2)$  are such that there exists a nonzero  $z(0)$  verifying  $\Delta z(0) = 0$ . Nonzero solutions may thus only exist if  $\Delta$  is singular. Using Eq. (3.11),  $\Delta$  reads:

$$\Delta = \begin{bmatrix} \cos(2t_1 L_1) - \cos(2t_2 L_2) & 2t_1 \operatorname{sinc}(2t_1 L_1) - 2t_2 \operatorname{sinc}(2t_2 L_2) \\ L_1 \sin(2t_1 L_1) + L_2 \sin(2t_2 L_2) & \cos(2t_1 L_1) - \cos(2t_2 L_2) \end{bmatrix}. \quad (3.14)$$

Then, using matrix trigonometric double-angle identities such as  $\operatorname{sinc}(P) \cos(P) = \operatorname{sinc}(2P)$  for any matrix  $P$ ,  $\Delta$  can be factorized in two different ways:

$$\Delta = 2 \begin{bmatrix} t_1 \operatorname{sinc}(t_1 L_1) & -t_2 \operatorname{sinc}(t_2 L_2) \\ \cos(t_1 L_1) & -\cos(t_2 L_2) \end{bmatrix} \begin{bmatrix} L_1 \sin(t_1 L_1) & \cos(t_1 L_1) \\ -L_2 \sin(t_2 L_2) & \cos(t_2 L_2) \end{bmatrix} := 2R_1 R_2 \quad (3.15)$$

and

$$\Delta = 2 \begin{bmatrix} \cos(t_1 L_1) & -\cos(t_2 L_2) \\ L_1 \sin(t_1 L_1) & L_2 \sin(t_2 L_2) \end{bmatrix} \begin{bmatrix} \cos(t_1 L_1) & t_1 \operatorname{sinc}(t_1 L_1) \\ \cos(t_2 L_2) & t_2 \operatorname{sinc}(t_2 L_2) \end{bmatrix} := 2T_1 T_2. \quad (3.16)$$

From the technical lemma A.1, it follows that  $\det(T_1) = \det(R_2)$  and  $\det(T_2) = \det(R_1)$ . In the end, the following equality holds:

$$\det(\Delta) = 2^n \det(R_2) \det(T_2). \quad (3.17)$$

For  $\Delta$  to be singular,

- either  $\det(R_2) = 0$ :

$$\begin{cases} L_1 \sin(t_1 L_1) y(0) + \cos(t_1 L_1) \dot{y}(0) = 0 & (3.18) \\ -L_2 \sin(t_2 L_2) y(0) + \cos(t_2 L_2) \dot{y}(0) = 0 & (3.19) \end{cases}$$

which, through Eq. (3.12), is equivalent to  $\dot{y}(t_1) = \dot{y}(t_2) = 0$ , that is  $y$  has two axial symmetries;

- or  $\det(T_2) = 0$ :

$$\begin{cases} \cos(t_1 L_1) y(0) + t_1 \operatorname{sinc}(t_1 L_1) \dot{y}(0) = 0 & (3.20) \\ \cos(t_2 L_2) y(0) + t_2 \operatorname{sinc}(t_2 L_2) \dot{y}(0) = 0 & (3.21) \end{cases}$$

which is equivalent to  $y(t_1) = y(t_2) = 0$ , that is  $y$  has two punctual symmetries.

This ends the proof of Theorem 3.1.

**Remark 3.1.** For some very specific values of  $M$  and  $K$ , it is possible to find  $t_1, t_2$  such that  $\det(T_2) = 0$ , leading to potential solutions with punctual symmetries instead of axial symmetries. Such an example is provided in section 6.2, however these cases are never admissible: corresponding  $\lambda$  has a point of symmetry, so cannot satisfy eq. (3.2b).

**4. Methodology to find SPPs** Put together, the previous derivations lead to a methodology to compute SPPs. We assume  $g_0 \neq 0$ .

**Periodicity as a first condition on  $(t_1, t_2)$**  The condition of periodicity is governed by  $\det(\Delta) = 0$ . However, from eq. (3.17) in the proof of Theorem 3.1,  $\det(\Delta) = 2^n \det(R_2) \det(T_2)$ . From Corollary 3.1,  $\det(T_2) = 0$  does not lead to admissible solutions, hence it suffices to find  $(t_1, t_2)$  solution to  $\det(R_2) = 0$ . This provides a first equation.

When the periodicity condition  $\det(R_2) = 0$  is satisfied, all the columns of  $\text{adj}(R_2)^\top$  are colinear and are in the kernel of  $R_2$ , hence in the kernel of  $\Delta$ , see eq. (3.15). Choosing  $z(0)$  as a nonzero column of  $\text{adj}(R_2)$  ensures that  $\det(\Delta)z(0) = 0$ , meaning that  $z(0)$  is proportional to a set of initial conditions leading to a potential SPP.

**Smooth normal contact velocity at  $t = 0$  as a second condition on  $(t_1, t_2)$**  The condition of zero pre-impact normal velocity is satisfied over  $[0, t_2]$  by construction of  $\tilde{K}$ : on  $[0, t_2]$ ,

$$w^\top \ddot{x} + \underbrace{w^\top M^{-1} \tilde{K} x}_{=0} = 0 \quad (4.1)$$

hence  $w^\top \dot{x}$  is constant and by symmetry, it vanishes over  $[0, t_2]$ . However eq. (1.3c) also includes  $w^\top \ddot{x}(0) = 0$ , which yields a condition on the free flight dynamics, since  $\ddot{x} = -M^{-1}Kx$  on  $[-2t_1, 0]$  so  $w^\top M^{-1}Kx(0) = 0$ . In terms of  $y = M^{1/2}x$ , the equality also writes

$$w^\top M^{-1}KM^{-1/2}y(0) = 0. \quad (4.2)$$

We insist that  $y(0)$  stores the  $n$  first rows of a non-zero column of  $\text{adj}(R_2)^\top$  and is known: it is completely determined by  $(t_1, t_2)$ . Equation (4.2) hence provides a second equation on  $(t_1, t_2)$ , which is generically independent of the first one. Unknowns  $(t_1, t_2)$  are thus solutions to two independent equations (except maybe for some peculiar  $M$  and  $K$ ), which yields the following important result.

**Theorem 4.1** *For generic  $M$  and  $K$ , if  $g_0 \neq 0$  then SPPs may only occur for isolated couples  $(t_1, t_2)$ , and thus isolated period  $T$ . If  $g_0 = 0$  and there is no SPP.*

In particular, there is no continuum of SPP, contrary to trajectories with  $k$  impacts per period [15].

The case  $g_0 = 0$  can be shown by observing that  $w^\top x(0) = 0$  yields an additional scalar condition on  $x(0)$ , *a priori* independent of the other ones, resulting in an over-determined system, generically with no solution. A subgeneric example where  $M$  and  $K$  have been specially designed to induce an SPP with  $g_0$  is provided in section 6.1. Until then, it is assumed that  $g_0 \neq 0$ .

**Gap closure at  $t = 0$**  We have just seen that an appropriate doublet  $(t_1, t_2)$  determines  $z(0)^\top = [y(0), \dot{y}(0)]$  chosen as a non-zero column of  $\text{adj}(R_2)^\top$ , up to a multiplicative constant. This constant is fixed by the condition  $g(x(0)) = 0$ :  $w^\top x(0) = -g_0$ , that is:

$$x(0) = -\frac{g_0}{w^\top M^{-1/2}y(0)} M^{-1/2}y(0). \quad (4.3)$$

This sets the initial velocities to

$$\dot{x}(0) = -\frac{g_0}{w^\top M^{-1/2}y(0)} M^{-1/2}\dot{y}(0). \quad (4.4)$$

The value  $g_0 \neq 0$  does not affect the potential existence of SPPs, only dictated by  $(t_1, t_2)$ . It shows that generically, there is no non-trivial SPP with  $g_0 = 0$ .

Recall that admissible SPPs must also satisfy the two following conditions, corresponding to eqs. (3.1b) and (3.2b).

- Gap non-negativeness on  $[-2t_1, 0]$ : The global solution on  $[-2t_1, 2t_2]$ , expended to  $\mathbb{R}$  via  $2(t_1 + t_2)$ -periodicity, can be recovered from  $x(0)$  and  $\dot{x}(0)$  through eq. (3.12). The inequality  $w^\top x + g_0 \geq 0$  can hence be verified, numerically.
- Non-sticking condition on  $[0, 2t_2]$ : The positiveness of the contact force also has to be verified numerically, *i.e.* using eq. (2.7):  $w^\top M^{-1}Kx \geq 0$  on  $[0, 2t_2]$ .

**Remark 4.1.** Whether the two above inequalities are satisfied depends on the sign of  $g_0$ , but not on its magnitude. In particular, if there is a SPP for some  $(t_1, t_2)$  and a  $g_0 \neq 0$ , there will be no SPP for the same  $(t_1, t_2)$  and the opposite gap  $-g_0$ .

## 5. Summary and examples

### 5.1. Algorithm for finding SPPs

To find an admissible SPP:

1. Compute  $\tilde{K}$  from (2.4), the positive definite square roots  $L_1$  of  $M^{-1}K$  and a square root  $L_2$  of  $M^{-1}\tilde{K}$ .

2. Find  $(t_1, t_2)$  such that:

$$\begin{cases} \det(R_2) = \begin{vmatrix} L_1 \sin(t_1 L_1) & \cos(t_1 L_1) \\ -L_2 \sin(t_2 L_2) & \cos(t_2 L_2) \end{vmatrix} = 0 \end{cases} \quad (5.1a)$$

$$\begin{cases} w^\top M^{-1} K M^{-1/2} y(0) = 0 \end{cases} \quad (5.1b)$$

where  $y(0)$  is a non-zero column of  $\text{adj}(R_2)^\top$ .

3. For such  $(t_1, t_2)$  and  $y(0)$ , define the potential SPP via its initial conditions:

$$\begin{cases} x(0) = -\frac{g_0}{w^\top M^{-1/2} y(0)} M^{-1/2} y(0) \end{cases} \quad (5.2a)$$

$$\begin{cases} \dot{x}(0) = -\frac{g_0}{w^\top M^{-1/2} y(0)} M^{-1/2} \dot{y}(0). \end{cases} \quad (5.2b)$$

4. Check the admissibility of the potential SPP determined by  $(x_0, \dot{x}_0)$  via eq. (3.12) and definition  $y = M^{1/2}x$ :

$$\begin{cases} w^\top x(t) + g_0 \geq 0, \quad \forall t \in [-2t_1, 0] \end{cases} \quad (5.3a)$$

$$\begin{cases} w^\top M^{-1} K x(t) \geq 0, \quad \forall t \in [0, 2t_2]. \end{cases} \quad (5.3b)$$

5. If not admissible, return to step 2.

*Remark 5.1.* A solution is actually solution to the initial problem formulated by eqs. (1.2) and (1.3). First, let's focus on  $[-2t_1, 0)$ . By construction  $M\ddot{x} + Kx = 0$  corresponding to eq. (1.2a) with  $\lambda = 0$ , so eqs. (1.2c) and (1.2d) are also satisfied. Inequality (1.2b) is satisfied because of step 3 above. Then, over  $(0, 2t_2]$ :  $M\ddot{x} + \tilde{K}x = 0$  or equivalently  $M\ddot{x} + Kx = w\lambda$  with  $\lambda = w^\top M^{-1} Kx(t)$ . By construction of  $\tilde{K}$ ,  $g(x) = 0$  so inequality (1.2b) and eq. (1.2d) are satisfied. Inequality (1.2c) is not violated because of step 3. Continuity of  $x$  and  $\dot{x}$  at  $t = 0$  (eq. (1.3a)) is ensured by construction, and periodicity (eq. (1.3b)) is ensured as  $\det(\Delta) = 0$ . Equation (1.3d) is verified by choice of the prefactor of  $M^{-1/2}y(0)$  in step 2. Finally, eq. (1.3c) is satisfied because of symmetry and periodicity.

Animated SPPs are provided in ref. [14], for all the following examples as well as larger systems with up to 30 dofs.

### 5.2. SPPs of 5-dof spring-mass system

The mathematical results are first illustrated using the spring-mass system depicted in fig. 3. For simplicity, all masses and stiffnesses are chosen equal to one, so that:

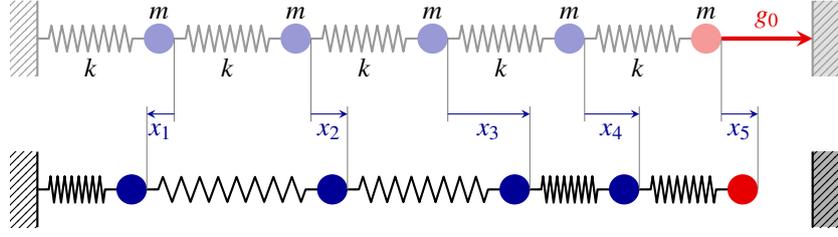
$$M = I_n, \quad K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}. \quad (5.4)$$

The resting positions of the masses are zero so that positions and displacements are equal. The gap writes  $g_0 \geq x_5$  or  $g(x) \geq 0$  with  $g(x) = w^\top x + g_0$  and  $w^\top = [0 \ 0 \ 0 \ 0 \ -1]$ , and  $g_0$  is arbitrarily chosen equal to one.

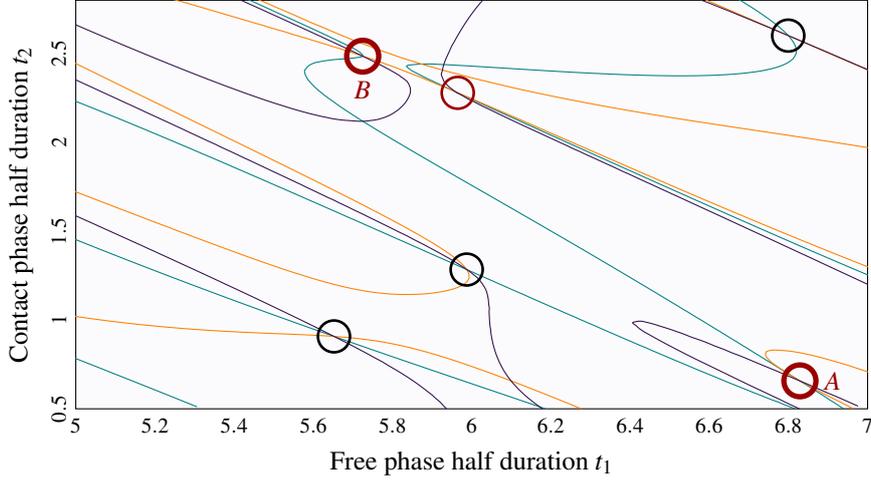
Equation (2.4) yields

$$\tilde{K} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.5)$$

which physically corresponds to disconnecting the last mass, only from the point of view of the last mass



**Figure 3:** First model used to illustrate SPPs.



**Figure 4:** Roots of the system (5.1). Each curve  $i$  corresponds to  $(t_1, t_2)$  such that  $w^\top M^{-1} K M^{-1/2}$  is orthogonal to column  $i$  of  $\text{adj}(R_2)^\top$ .  $\circ$  Potential SPP.  $\circ$  Admissible SPP.  $\circ$  Admissible SPP illustrated in 5.

(see 2.3). Square roots  $L_1$  and  $L_2$  can be easily computed and we can proceed with finding numerically  $(t_1, t_2)$  solutions of eq. (5.1), see fig. 4. Instead of plotting the roots of  $\det(R_2)$ , in this figure each curve corresponds to  $(t_1, t_2)$  such that a column  $c_i$  of  $\text{adj}(R_2)^\top$  is orthogonal to  $v = w^\top M^{-1} K M^{-1/2}$ . This “trick” reduces the computational cost and originates from the fact that when  $R_2$  is singular,  $\text{adj}(R_2)$  is generally of rank one; the conditions  $vc_i = 0$  for  $i \in \{1, \dots, n\}$  become identical, meaning that solution points  $(t_1, t_2)$  are located at the intersections of all the root curves of  $vc_i = 0$ . Figure 4 illustrates theorem 4.1: SPPs are admissible points.

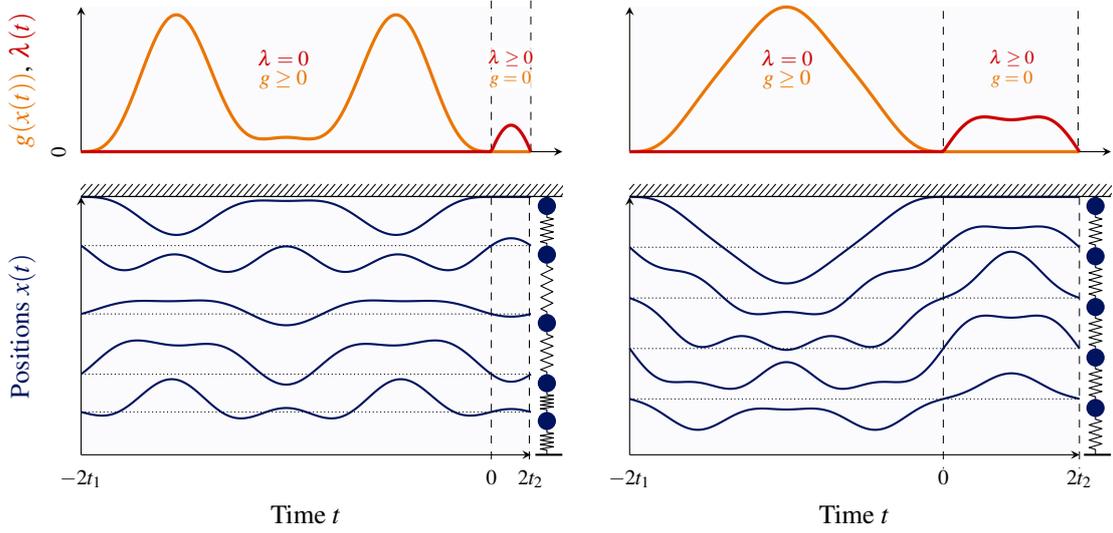
Then, the admissibility of the trajectories emanating from each solution point of fig. 4 has to be verified numerically (steps 3 and 4). Two admissible SPPs, corresponding to points  $(t_1, t_2)$  marked as  $A$  and  $B$  in fig. 4, are depicted in fig. 5. Conclusions of theorem 3.1 are observed: both solutions have an axis of symmetry at  $t = -t_1$  and  $t = t_2$ .

**5.3. SPPs with slip and non-diagonal mass matrix** The presented results also apply to more complex geometries and general mass or stiffness matrices. In this subsection, an application is presented on the 4-dof system depicted in fig. 6. Contrary to the previous system with concentrated masses, here the mass and stiffness matrices are chosen as

$$M = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1.5 \\ 0 & 0 & 1.5 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2.5 & -2.5 \\ 0 & 0 & -2.5 & 5 \end{bmatrix} \quad (5.6)$$

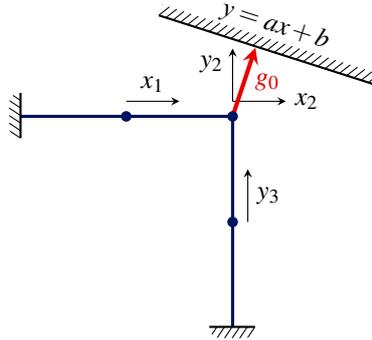
so their structure matches bar elements in the Finite Element Method. With a zero resting position and the equation of the obstacle given by  $y = ax + b$ , the gap writes

$$g(x) = \frac{y - ax - b}{\sqrt{1 + a^2}} = w^\top x + g_0 \quad \text{with} \quad w = \frac{1}{\sqrt{1 + a^2}} \begin{bmatrix} 0 \\ a \\ -1 \\ 0 \end{bmatrix}, \quad g_0 = \frac{b}{\sqrt{1 + a^2}} \quad (5.7)$$



**Figure 5:** Time-evolution of two SPP trajectories of the 5-dof spring-mass system. Left: point  $A$  of fig. 4. Right: point  $B$  of fig. 4. Top: gap and normal contact force. Bottom: positions. See ref. [14] for the animated versions.

with  $x^\top = [x_1 \ x_2 \ y_2 \ y_3]$  corresponding to fig. 6. We arbitrarily place the obstacle at position



**Figure 6:** FEM-model with exact slip condition to illustrate SPPs. Rod elements with linear mass.

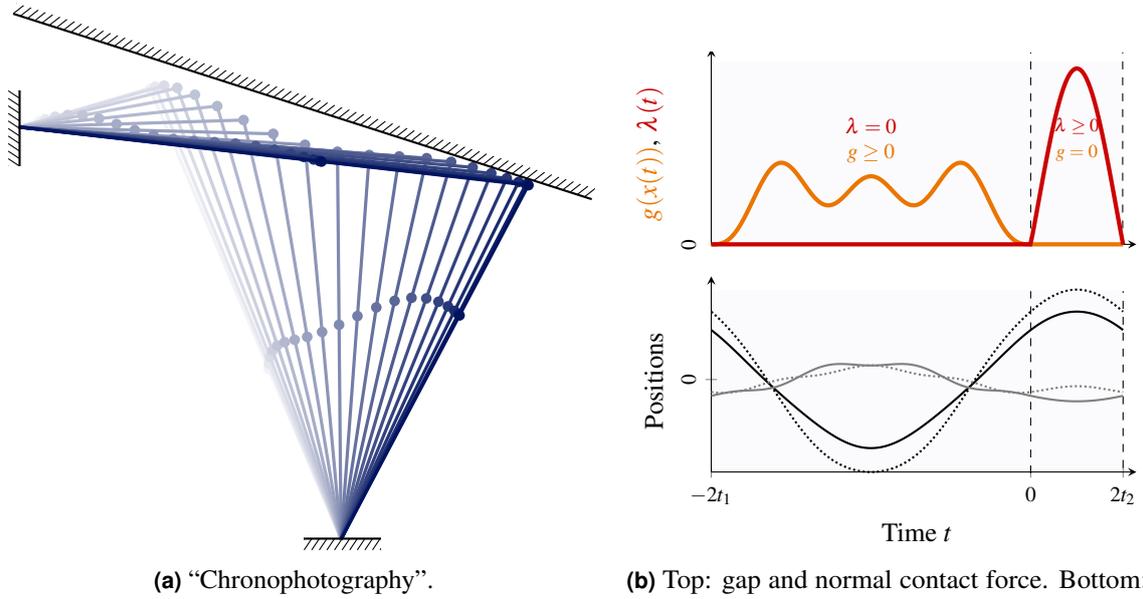
$4y = 1 - x$  so that  $w^\top = [0 \ -1 \ -4 \ 0]/\sqrt{17}$  and  $g_0 = 1/17$ . As previously, computing  $\tilde{K}$ ,  $L_1$  and  $L_2$  is straightforward. Solutions of system eq. (5.1) have to be found numerically, then ineqs. 5.3 have to be verified (steps 3 and 4). Solutions of an admissible SPP are represented in fig. 7 (left) together with the non-negative gap and the non-negative normal contact force (right). Again, the trajectories feature two axes of symmetry, at  $t = -t_1$  and  $t = t_2$ . (theorem 4.1).

## 6. Special behaviours for specific parameters

In this section, subgeneric cases are investigated, for completeness.

**6.1. Closed gap at rest ( $g_0 = 0$ )** Theorem 4.1 assumes  $g_0 \neq 0$ . We now investigate the case when  $g_0 = 0$ . The gap closure at  $t = 0$  implies  $w^\top x(0) = -g_0 = 0$ : non-trivial solutions can be found only if  $x(0)$  is orthogonal to  $w$ . This adds one independent scalar equation to the system (5.1), requiring an additional unknown. To illustrate SPPs with zero initial gap, we consider a 2-dof spring-mass system, with an unknown stiffness  $k_2$ :

$$M = I_2, \quad K = \begin{bmatrix} 1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad (6.1)$$

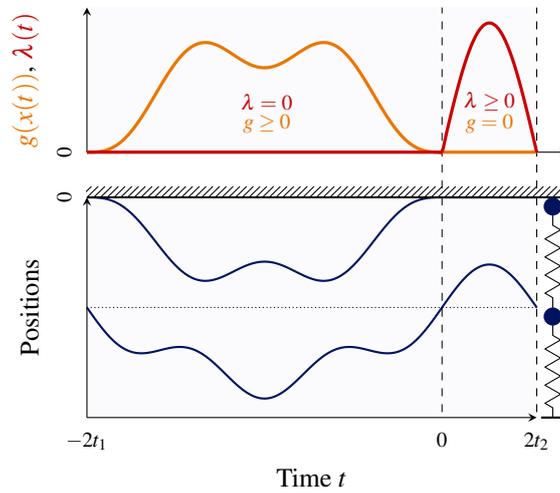


**Figure 7:** Time-evolution of a SPP of the FEM-model with slip. See [14] for the animated version.

Introducing an additional unknown  $k_2$  (compared to 5.2, for example) compensate the additional equation  $w^\top x(0)$  and solutions can be found numerically by solving:

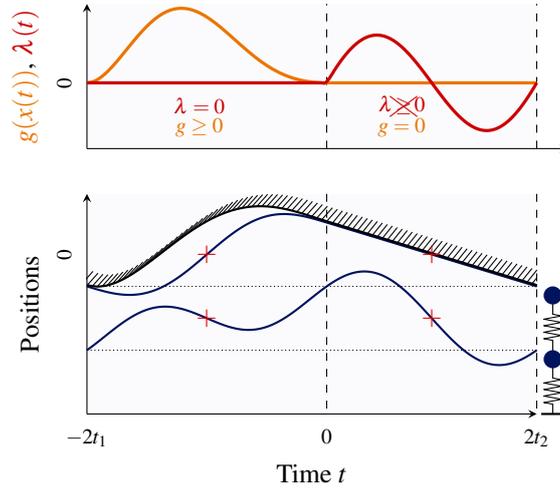
$$\begin{cases} \det R_2(t_1, t_2) = 0 & (6.2) \\ w^\top M^{-1} K x(0) = 0 & (6.3) \\ w^\top x(0) = 0, & (6.4) \end{cases}$$

as illustrated in Fig. 8.



**Figure 8:** 2-dof SPP with  $g_0 = 0$ . Such motions occur only for specific values of  $M$  and  $K$ : here,  $k_2 \approx 5.716$ . See [14] for the animated version.

**6.2. Non-symmetric potential SPPs with moving obstacle** The system of the previous subsection is considered. Solving for  $\det(T_2) = 0$  instead of  $\det(R_2) = 0$  provides  $(t_1, t_2)$  leading to SPPs with *points* of symmetry in position, see fig. 9. However, since on  $[0, 2t_2]$ ,  $\lambda w$  and  $w^\top M^{-1} K x$  have the same sign (see eq. (2.7)), and because  $x$  has a point of symmetry at  $t = t_2$ ,  $w^\top M^{-1} K x$  is either zero on  $[0, 2t_2]$  or its sign changes, breaking condition 1.2c. In other words, there is no admissible SPP with axial symmetry. Moreover, because of the punctual symmetry of the positions, the obstacle cannot be still (except if  $g_0 = 0$ ) and may only have a linear velocity during the contact phase. In fig. 9, the motion of the obstacle has been smoothly extrapolated on  $[-2t_1, 0]$  to satisfy periodicity.



**Figure 9:** 2-dof potential SPP with punctual symmetry. Such motions may happen only if  $\det(T_2) = 0$  (here, with  $m \approx 4.875$ ). Signs + indicates points of symmetry over  $[-2t_1, 0]$  and  $[0, 2t_2]$ . Like all SPPs with punctual symmetry, it is not admissible because  $\lambda$  is not non-negative over  $[-2t_1, 2t_2]$ . See [14] for the animated version.

**7. Conclusion** Previous works have shown that autonomous periodic motions with one sticking phase per period (SPP) play a particular role in the frequency–energy plot of piecewise-linear dynamical systems with one unilateral constraint. Indeed, they seem to be the limit of autonomous periodic trajectories with  $k$  impacts per period (IPP) as  $k$  goes to infinity. Additionally, SPPs are uniquely determined by a set of initial conditions even though no impact law is specified.

The minimal set of equations governing 1 SPPs was derived in the general  $n$ -dof framework with a single unilateral constraint. The first step is to solve a system of two scalar equations (one nonlinear and one linear) for the duration  $2t_1$  of the free flight phase and the duration  $2t_2$  of the contact phase. As with  $k$  IPP solutions for which the sequence of free flight durations determine appropriate initial conditions, these two durations then determine 1 SPPs. It was shown that if  $g_0 \neq 0$ , 1 SPPs may only be isolated, irrespective of the sign of  $g_0$ . The displacements of SPPs trajectories were shown to exhibit two axis of symmetry in time: one in the middle of the free flight ( $t = -t_1$ ) and one in the middle of the contact phase ( $t = t_2$ ). It was proven that 1 SPP occur only for discrete values of  $(t_1, t_2)$ . Results were illustrated using various geometries, unilateral constraints and mass matrices. Using the proposed methodology, computation of SPPs with 30 dofs was straightforward. No attempt with a larger number of dofs was made, but the equations are indeed expected to be challenging to solve numerically.

While it is clear that *potential* SPPs exist, there is no result on the existence of admissible SPPs.

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## A. Technical lemma

**Lemma A.1** *Let  $A, B, C, D$  be four matrices of order  $n$ . If  $AB = BA$  and  $CD = DC$ , then*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} B & -D \\ A & -C \end{vmatrix} = \det(DA - CB). \quad (\text{A.1})$$

This result is known [13], however it can be briefly proven.

**Proof.** Let’s first assume  $A$  is invertible. Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ CA^{-1} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} A & B \\ 0 & I_n \end{bmatrix} \quad (\text{A.2})$$

so that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(D - CA^{-1}B) \det(A) = \det(D - CBA^{-1}) \det(A) = \det(DA - CB). \quad (\text{A.3})$$

If  $A$  is singular, for all  $x$ ,  $A - xI$  and  $B$  commute and the equality

$$\begin{vmatrix} A - xI & B \\ C & D \end{vmatrix} = \det(D(A - xI) - CB) \quad (\text{A.4})$$

holds at least for any  $x$  which is not eigenvalue of  $A$ , hence for all  $x$  (it’s a polynomial equality), in particular for  $x = 0$ . Similarly, if  $C$  is invertible,

$$\begin{bmatrix} B & -D \\ A & -C \end{bmatrix} = \begin{bmatrix} I_n & D \\ 0 & C \end{bmatrix} \begin{bmatrix} B - DC^{-1}A & 0 \\ C^{-1}A & -I_n \end{bmatrix} \quad (\text{A.5})$$

and it comes that

$$\begin{vmatrix} B & -D \\ A & -C \end{vmatrix} = \det(DA - CB), \quad (\text{A.6})$$

which also holds for singular  $C$ . ■