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# An asymptotic preserving multi-dimensional ALE method for a system of two compressible flows coupled with friction

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## Abstract

We present a multi-dimensional asymptotic preserving scheme for the approximation of a mixture of compressible flows. Fluids are modeled by two Euler systems of equations coupled with a friction term.

The asymptotic preserving property is mandatory for this kind of model, to derive a scheme that behaves well in all regimes (*i.e.* whatever the friction parameter value is). The method we propose is defined in ALE coordinates, using a Lagrange plus remap approach. This imposes a multi-dimensional definition and analysis the scheme.

*Keywords:* Compressible gas dynamics, multi-fluid, finite-volumes, unstructured meshes, asymptotic preserving, arbitrary-Lagrangian-Eulerian (ALE)

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## 1. Introduction

2        A multifluid model is a model for a fluid mixture for which each fluid is described by is  
3        own full set of variables (for instance density, velocity and energy). The model is generally  
4        closed in a way that defines interactions between the constituents, depending on the envolved  
5        physic. These models are widely used in different communities. One very popular model of this  
6        kind is the Baer-Nunziato model [1] for deflagration-to-detonation transition of reactive flows.  
7        Many numerical methods to approximate this model have been designed, let us just cite a few  
8        of them [2, 3, 4, 5, 6]. Such kind of models is also used in plasma physics to account for plas-  
9        mas collision or Non-Local-Thermodynamic-Equilibrium (NLTE) Ion-Electron interactions [7].  
10       Scannapieco and Cheng [8] also derive similar kind of model for turbulent flows and apply it to  
11       describe a mixing zone driven by Rayleigh-Taylor or Richtmyer-Meshkov instabilities [9].

12       In this paper, we present a multi-dimensional scheme to approximate solutions of two com-  
13       pressible inviscid fluids coupled with friction, refer to equation (1). This model is a slightly  
14       simplified version of the Scannapieco-Cheng [8] model where friction is considered uniform in  
15       space. It can also be viewed as a simplification of the model proposed in [7] for which the elec-  
16       tron effect is neglected or of the Baer-Nunziato model [1], neglecting the interfacial terms and in  
17       the case where there is no phase transition.

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18 Our goal in this paper is to address two difficulties. First one is inherent to this kind of model  
 19 and rely to the asymptotic preserving (AP) property [10, 11, 12] in the high friction regime or  
 20 infinite friction regime. In the former regime, the fluids interpenetration follows a diffusion law.  
 21 In the latter one, the mixture evolves as a single fluid, see (4)–(5). If no attention is paid to  
 22 these regimes, the scheme will fail to capture it at a reasonable calculation cost. The second  
 23 difficulty comes from the numerical framework we consider. We want our scheme to be able to  
 24 deal with Arbitrary-Lagrange-Euler (ALE) frame and unstructured meshes in order to properly  
 25 handle highly deformed calculation domains.

26 While authors [13, 14, 15] propose an asymptotic discretization for the system (1) in 1D in  
 27 the Eulerian frame, no asymptotic preserving scheme has been yet published for 2D unstructured  
 28 meshes for this model. Even for simpler model, only few unstructured asymptotic preserving  
 29 schemes have been developed (refer for instance to Berthon and Turpault [16] and Franck *et*  
 30 *al.* [17, 18]). The scheme we propose in section 4 has connections with [19, 20], where an  
 31 Euler with friction system is studied. However, it is not a direct extension of [19] to the bi-  
 32 fluid case. The scheme presented in this work is split into two steps. In the first step we solve  
 33 two Euler systems of equations coupled by friction. Since each fluid has its own velocity, the  
 34 Lagrangian mesh of each fluid will evolve separately during this step. Then, in the second step,  
 35 the conservative variables vector of each of the fluid will be projected onto a common mesh (not  
 36 necessarily identical to the initial mesh).

37 In the section 2 of this paper, we recall the properties of the model we consider, that are  
 38 conservation, hyperbolicity, and asymptotic limit model. In section 3, we recall the basis of  
 39 the solver (Glace [21] or Eucclhyd [22]) used to compute the Lagrangian step. The section 4 de-  
 40 scribes the Lagrangian step of the proposed scheme. It is demonstrated that the scheme preserves  
 41 the properties of conservation, stability and consistency with respect to the continuous model for  
 42 all regimes (independantly of the value of the friction parameter). Then in section 5, our ALE  
 43 strategy is described. Finally, section 6 is devoted to numerical experiments on several problems  
 44 (Sod shock tube, triple point and Rayleigh-Taylor). Some comparisons with a non-AP scheme  
 45 are provided.

## 46 2. A two fluids model with friction

47 Let us consider a mixture of two fluids  $f_1$  and  $f_2$ . In the following, we will denote by “multi-  
 48 fluid model”, a model for which each fluid  $\alpha \in \{f_1, f_2\}$  is represented by its own set of variables:  
 49  $(\rho^\alpha, \mathbf{u}^\alpha, E^\alpha)$ . Conversely, we will refer as “mono-fluid model”, a model describing a mixture  
 50 where mean quantities are considered  $(\rho, \mathbf{u}, E)$ , each fluid position being precised by an addi-  
 51 tional equation on the concentration (e.g.  $c^\alpha = \frac{\rho^\alpha}{\rho^\alpha + \rho^\beta}$ ).

52 In this part, we present a simplified version of Scannapieco-Cheng’s model where the inter-  
 53 action between the two constituents reduces to a friction term. In semi-Lagrangian coordinates,  
 54 for each fluid  $\alpha \in \{f_1, f_2\}$  ( $\beta$  denoting the other fluid), the model writes

$$\begin{aligned}
 \rho^\alpha D_t^\alpha \tau^\alpha &= \nabla \cdot \mathbf{u}^\alpha, \\
 \rho^\alpha D_t^\alpha \mathbf{u}^\alpha &= -\nabla p^\alpha - \nu \rho \delta \mathbf{u}^\alpha, \\
 \rho^\alpha D_t^\alpha E^\alpha &= -\nabla \cdot (p^\alpha \mathbf{u}^\alpha) - \nu \rho \delta \mathbf{u}^\alpha \cdot \bar{\mathbf{u}},
 \end{aligned}
 \tag{1}$$

55 where  $\rho^\alpha, \mathbf{u}^\alpha$  and  $E^\alpha$  respectively denote the mass density, the velocity and the total energy  
 56 density of fluid  $\alpha$ . Also,  $\tau^\alpha = \frac{1}{\rho^\alpha}$  denotes the specific volume. The pressure  $p^\alpha$  satisfies the  
 57 equation of state  $p^\alpha := p^\alpha(\rho^\alpha, e^\alpha)$ , where  $e^\alpha$ , the internal energy density, is defined by  $e^\alpha :=$

58  $E^\alpha - \frac{1}{2}\|\mathbf{u}^\alpha\|^2$ . The total density  $\rho$  and the mean velocity  $\bar{\mathbf{u}}$  are defined as  $\rho := \rho^\alpha + \rho^\beta$  and  
 59  $\rho\bar{\mathbf{u}} := \rho^\alpha\mathbf{u}^\alpha + \rho^\beta\mathbf{u}^\beta$ . The term  $\delta\mathbf{u}^\alpha$  is the velocity difference, the  $\delta(\cdot)^\alpha$  operator being defined by  
 60  $\delta\phi^\alpha = -\delta\phi^\beta = \phi^\alpha - \phi^\beta$ . Finally,  $\nu$  is the friction parameter. Also, remark that the Lagrangian  
 61 derivative  $D_t^\alpha := \partial_t + \mathbf{u}^\alpha \cdot \nabla$ , is obviously not the same for each fluid.

62 The entropy  $\eta^\alpha$  defined by Gibbs formula  $T^\alpha d\eta^\alpha = de^\alpha + p^\alpha d\tau^\alpha$  satisfies the following entropy  
 63 inequality

$$T^\alpha D_t^\alpha \eta^\alpha \geq \nu \frac{\tau^\alpha}{\tau^\beta} \delta\mathbf{u}^\alpha \cdot \delta\mathbf{u}^\alpha \geq 0. \quad (2)$$

64 Prior to establishing a numerical scheme that discretizes this set of six equations, we recall  
 65 some properties of the model itself.

66 **Property 1** (Conservation). *The model (1) is conservative in volume and mass for each fluid.*  
 67 *Also, it is conservative in the sum of momenta and in the sum of the total energies of the two*  
 68 *fluids.*

69 *Proof.* Conservation of mass and volume is obvious since the first equation of (1) is the continu-  
 70 ity equation written for each fluid.

71 Conservation of momenta sum and total energies sum requires more cautiousness, since La-  
 72 grangian derivative are not the same for each fluid. To establish them one rewrites (1) in an  
 73 Eulerian framework.

74 Developing Lagrangian derivatives and using the identity  $\partial_t(\rho^\alpha\tau^\alpha) = 0$  elementary calcula-  
 75 tions allow to rewrite (1) as

$$\begin{aligned} \partial_t \rho^\alpha + \nabla \cdot (\rho^\alpha \mathbf{u}^\alpha) &= 0, \\ \partial_t (\rho^\alpha \mathbf{u}^\alpha) + \nabla \cdot (\rho^\alpha \mathbf{u}^\alpha \otimes \mathbf{u}^\alpha) + \nabla p^\alpha + \nu \rho \delta \mathbf{u}^\alpha &= \mathbf{0}, \\ \partial_t (\rho^\alpha E^\alpha) + \nabla \cdot (\rho^\alpha E^\alpha \mathbf{u}^\alpha) + \nabla \cdot (p^\alpha \mathbf{u}^\alpha) + \nu \rho \delta \mathbf{u}^\alpha \cdot \bar{\mathbf{u}} &= 0. \end{aligned} \quad (3)$$

Summing the two later equations over  $\alpha$  gives a system of the conservative form  $\partial_t \mathbf{U} + \nabla \cdot F(\mathbf{U}) = \mathbf{0}$ , where

$$\mathbf{U} = \begin{pmatrix} \rho^\alpha \mathbf{u}^\alpha + \rho^\beta \mathbf{u}^\beta \\ \rho^\alpha E^\alpha + \rho^\beta E^\beta \end{pmatrix},$$

and

$$F(\mathbf{U}) = \begin{pmatrix} \rho^\alpha \mathbf{u}^\alpha \otimes \mathbf{u}^\alpha + \rho^\beta \mathbf{u}^\beta \otimes \mathbf{u}^\beta + (p^\alpha + p^\beta) I \\ \rho^\alpha E^\alpha \mathbf{u}^\alpha + \rho^\beta E^\beta \mathbf{u}^\beta + p^\alpha \mathbf{u}^\alpha + p^\beta \mathbf{u}^\beta \end{pmatrix},$$

76 where  $I$  is the identity matrix of  $\mathbb{R}^{2 \times 2}$ . □

77 **Property 2** (Hyperbolicity). *The model (1) is hyperbolic.*

78 *Proof.* The proof is straightforward but calculatory, see [15] for details. □

**Asymptotic model.** *When  $\nu \rightarrow +\infty$ , (1) behaves has the following five equations model*

$$\rho D_t \mathbf{u} = -\nabla (p^\alpha + p^\beta), \quad (4)$$

while, for each fluid  $\alpha \in \{f_1, f_2\}$ ,  $\beta$  denoting the other one, one has

$$\begin{aligned} \rho^\alpha D_t \tau^\alpha &= \nabla \cdot \mathbf{u}, \\ \rho^\alpha D_t E^\alpha &= -\frac{\rho^\alpha}{\rho} \mathbf{u} \cdot \nabla (p^\alpha + p^\beta) - p^\alpha \nabla \cdot \mathbf{u}, \end{aligned} \quad (5)$$

79 where  $\mathbf{u}$  is the same velocity for both fluids, and thus the Lagrangian derivative is also the same.

80 *Formal derivation (established in [15]).* Let  $\epsilon = \nu^{-1}$  so that (1) rewrites

$$\begin{aligned}\rho^\alpha D_t^\alpha \tau^\alpha &= \nabla \cdot \mathbf{u}^\alpha, \\ \rho^\alpha D_t^\alpha \mathbf{u}^\alpha &= -\nabla p^\alpha - \frac{1}{\epsilon} \rho \delta \mathbf{u}^\alpha, \\ \rho^\alpha D_t^\alpha E^\alpha &= -\nabla \cdot (p^\alpha \mathbf{u}^\alpha) - \frac{1}{\epsilon} \rho \delta \mathbf{u}^\alpha \cdot \bar{\mathbf{u}}.\end{aligned}\tag{6}$$

81 We will now study its limit while  $\epsilon \rightarrow 0^+$  focusing first on the momentum equations since the  
82 friction term's goal is to impose that  $\delta \mathbf{u}^0 \xrightarrow{\epsilon \rightarrow 0} \mathbf{0}$ .

Developing the Lagrangian derivatives and dividing each momentum equation by  $\rho^\alpha > 0$ , one has

$$\partial_t \mathbf{u}^\alpha + (\nabla \mathbf{u}^\alpha) \mathbf{u}^\alpha = -\frac{\nabla p^\alpha}{\rho^\alpha} - \frac{1}{\epsilon} \frac{\rho}{\rho^\alpha} \delta \mathbf{u}^\alpha.$$

Since fluid  $\beta$  satisfies the same equation and recalling that  $\delta \phi^\alpha = -\delta \phi^\beta = \phi^\alpha - \phi^\beta$ , one gets

$$\partial_t (\delta \mathbf{u}^\alpha) + \delta ((\nabla \mathbf{u}) \mathbf{u})^\alpha = -\delta \left( \frac{\nabla p}{\rho} \right)^\alpha - \frac{1}{\epsilon} \lambda \delta \mathbf{u}^\alpha, \quad \text{where } \lambda = \frac{\rho^2}{\rho^\alpha \rho^\beta}.$$

83 We now perform an Hilbert expansion for all variables in the equation, that is  $\phi = \phi^0 + \epsilon \phi^1 +$   
84  $\mathcal{O}(\epsilon^2)$ . One has

$$\partial_t (\delta \mathbf{u}^{\alpha,0}) + \delta ((\nabla \mathbf{u}) \mathbf{u})^{\alpha,0} = -\delta \left( \frac{\nabla p}{\rho} \right)^{\alpha,0} - \lambda^0 \left( \frac{1}{\epsilon} \delta \mathbf{u}^{\alpha,0} + \delta \mathbf{u}^{\alpha,1} \right) - \lambda^1 \delta \mathbf{u}^{\alpha,0} + \mathcal{O}(\epsilon).\tag{7}$$

85 Multiplying this equation by  $\epsilon$  one has  $\lambda^0 \delta \mathbf{u}^{\alpha,0} = \mathcal{O}(\epsilon)$ , which gives  $\delta \mathbf{u}^{\alpha,0} = \mathbf{0}$  when  $\epsilon \rightarrow 0$  since  
86  $\lambda > 0$ .

87 So, when  $\epsilon \rightarrow 0$ , formula (7) recasts

$$\delta \mathbf{u}^{\alpha,1} = -\frac{1}{\lambda^0} \delta \left( \frac{\nabla p}{\rho} \right)^{\alpha,0}.\tag{8}$$

Now, we perform an Hilbert expansion for the whole system (6), neglecting the non negative powers of  $\epsilon$ . Choosing  $\alpha \in \{f_1, f_2\}$ ,  $\beta$  being the other one, it reads

$$\begin{aligned}\rho^{\alpha,0} D_t^\alpha \tau^{\alpha,0} &= \nabla \cdot \mathbf{u}^{\alpha,0}, \\ \rho^{\alpha,0} D_t^\alpha \mathbf{u}^{\alpha,0} &= -\nabla p^{\alpha,0} - \rho^0 \left( \frac{1}{\epsilon} \delta \mathbf{u}^{\alpha,0} + \delta \mathbf{u}^{\alpha,1} \right) - \rho^1 \delta \mathbf{u}^{\alpha,0}, \\ \rho^{\alpha,0} D_t^\alpha E^{\alpha,0} &= -\nabla \cdot (p^{\alpha,0} \mathbf{u}^{\alpha,0}) - \rho^0 \left( \frac{1}{\epsilon} \delta \mathbf{u}^{\alpha,0} \cdot \bar{\mathbf{u}}^{\alpha,0} + \delta \mathbf{u}^{\alpha,1} \cdot \bar{\mathbf{u}}^{\alpha,0} + \delta \mathbf{u}^{\alpha,0} \cdot \bar{\mathbf{u}}^{\alpha,1} \right) \\ &\quad - \rho^1 \delta \mathbf{u}^{\alpha,0} \cdot \bar{\mathbf{u}}^{\alpha,0},\end{aligned}$$

Since we just established  $\delta \mathbf{u}^{\alpha,0} = \mathbf{0}$ , one has  $\mathbf{u}^0 = \bar{\mathbf{u}}^0 = \mathbf{u}^{\alpha,0} = \mathbf{u}^{\beta,0}$ . Also, since  $D_t^\alpha \phi = \partial_t \phi + \mathbf{u}^{\alpha,0} \cdot \nabla \phi + \mathcal{O}(\epsilon)$ , Lagrangian derivatives are the same when  $\epsilon \rightarrow 0$ , so that using (8) the

system simplifies to

$$\begin{aligned}\rho^{\alpha,0} D_t \tau^{\alpha,0} &= \nabla \cdot \mathbf{u}^0, \\ \rho^{\alpha,0} D_t \mathbf{u}^0 &= -\nabla p^{\alpha,0} + \rho^0 \frac{1}{\lambda^0} \delta \left( \frac{\nabla p}{\rho} \right)^{\alpha,0}, \\ \rho^{\alpha,0} D_t E^{\alpha,0} &= -\nabla \cdot (p^{\alpha,0} \mathbf{u}^0) + \rho^0 \frac{1}{\lambda^0} \delta \left( \frac{\nabla p}{\rho} \right)^{\alpha,0} \cdot \mathbf{u}^0,\end{aligned}$$

Recalling  $\lambda = \frac{\rho^2}{\rho^\alpha \rho^\beta}$  and developing  $\delta \left( \frac{\nabla p}{\rho} \right)^{\alpha,0}$ , momentum equation rewrites

$$\begin{aligned}\rho^{\alpha,0} D_t \mathbf{u}^0 &= -\nabla p^{\alpha,0} + \frac{\rho^{\alpha,0} \rho^{\beta,0}}{\rho^0} \left( \frac{\nabla p^{\alpha,0}}{\rho^{\alpha,0}} - \frac{\nabla p^{\beta,0}}{\rho^{\beta,0}} \right), \\ &= -\frac{\rho^{\alpha,0}}{\rho^0} \nabla (p^{\alpha,0} + p^{\beta,0}).\end{aligned}$$

Proceeding the same way with total energy equation, one gets

$$\begin{aligned}\rho^{\alpha,0} D_t E^{\alpha,0} &= -\nabla \cdot (p^{\alpha,0} \mathbf{u}^0) + \frac{\rho^{\alpha,0} \rho^{\beta,0}}{\rho^0} \left( \frac{\nabla p^{\alpha,0}}{\rho^{\alpha,0}} - \frac{\nabla p^{\beta,0}}{\rho^{\beta,0}} \right) \cdot \mathbf{u}^0, \\ &= -\frac{\rho^{\alpha,0}}{\rho^0} (\nabla p^{\alpha,0} + \nabla p^{\beta,0}) \cdot \mathbf{u}^0 - p^{\alpha,0} \nabla \cdot \mathbf{u}^0,\end{aligned}$$

88

□

**Remark 1.** Defining  $E := \frac{\rho^\alpha E^\alpha + \rho^\beta E^\beta}{\rho}$  and  $\tau := \rho^{-1}$ , it is easy to check that if  $(\rho^\alpha, \rho^\beta, \mathbf{u}, E^\alpha, E^\beta)$  is a solution of the asymptotic model (4)–(5), one has

$$\begin{aligned}\rho D_t \tau &= \nabla \cdot \mathbf{u}, \\ \rho D_t \mathbf{u} &= -\nabla (p^\alpha + p^\beta), \\ \rho D_t E &= -\nabla \cdot ((p^\alpha + p^\beta) \mathbf{u}).\end{aligned}$$

89 One recognizes Euler equations for the mixture. The mixing pressure follows Dalton's law as  
90 one could have expected since we consider here non-reactive gases.

91 However, notice that unless each fluid follows a barotropic equation of state ( $p^\alpha = p^\alpha(\rho^\alpha)$ ),  
92 equation (5) must be solved to determine  $e^\alpha$ .

### 93 3. Cell-centered schemes

94 We recall briefly the multi-dimensional finite volume schemes [23, 24, 22], since it is the  
95 basis of this work. For convenience, we use the notations defined in [21]. In the following, for  
96 all cell  $j$ , and for any quantity  $\phi$ , one defines its mean value  $\phi_j := \frac{1}{V_j} \int_j \phi$ , where  $V_j := \int_j 1$  is the  
97 cell volume. Also, let us denote the cell's mass as  $m_j := \int_j \rho = \rho_j V_j$ , which is constant in time in  
98 semi-Lagrangian coordinates ( $d_t m_j = 0$ ).

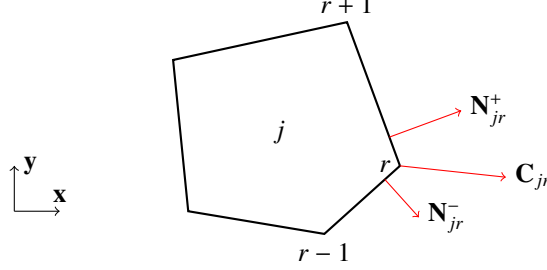


Figure 1: Illustration of  $\mathbf{C}_{jr}$  and  $\mathbf{N}_{jr}^i$  vectors at vertex  $r$  for a polygonal cell  $j$ .

We consider first-order schemes, so that one has the following relations

$$\begin{aligned} \frac{d}{dt} \int_j 1 &= m_j d_t \tau_j, & \frac{d}{dt} \int_j \rho_j &= 0, \\ \frac{d}{dt} \int_j \rho_j \mathbf{u}_j &= m_j d_t \mathbf{u}_j, & \frac{d}{dt} \int_j \rho_j E_j &= m_j d_t E_j. \end{aligned}$$

99 Let  $\mathcal{J}_r$  denote the set of cells connected to node  $r$  and let  $\mathcal{R}_j$  the set of nodes of cell  $j$ . Also,  
 100 let us introduce  $\mathbf{C}_{jr} := \nabla_{\mathbf{x}_r} V_j$ , the gradient of the volume of the polygonal cell  $j$ , according to the  
 101 position of one of its vertices  $r$ . Then, the cell-centered schemes we consider in this paper have  
 102 the following structure: for any cell  $j$  of the mesh one has

$$\begin{aligned} m_j d_t \tau_j &= \sum_{r \in \mathcal{R}_j} \mathbf{C}_{jr} \cdot \mathbf{u}_r, \\ d_t m_j &= 0, \\ m_j d_t \mathbf{u}_j &= - \sum_{r \in \mathcal{R}_j} \mathbf{F}_{jr}, \\ m_j d_t E_j &= - \sum_{r \in \mathcal{R}_j} \mathbf{F}_{jr} \cdot \mathbf{u}_r, \end{aligned} \tag{9}$$

where the fluxes  $\mathbf{u}_r$  and  $\mathbf{F}_{jr}$  are defined for any node  $r$

$$\forall j \in \mathcal{J}_r, \quad \mathbf{F}_{jr} = \mathbf{C}_{jr} p_j - A_{jr} (\mathbf{u}_r - \mathbf{u}_j), \tag{10}$$

$$\text{and } \sum_{j \in \mathcal{J}_r} \mathbf{F}_{jr} = \mathbf{0}. \tag{11}$$

103 In one hand, relation (10) is the matrix form of the acoustic Riemann solver (see for instance [25,  
 104 26]), while in the other hand (11) imposes conservation.

105 In the following to simplify notations, we omit sets  $\mathcal{R}_j$  and  $\mathcal{J}_r$  when there is no confusion.

- 106 • If  $A_{jr} := \rho_j c_j \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{\|\mathbf{C}_{jr}\|}$ , then (9)–(11) defines the Glace scheme [24, 21].
- 107 • Let  $\mathbf{N}_{jr}^+ = -\frac{1}{2}(\mathbf{x}_{r+1} - \mathbf{x}_r)^\perp$  and  $\mathbf{N}_{jr}^- = -\frac{1}{2}(\mathbf{x}_r - \mathbf{x}_{r-1})^\perp$ . If  $A_{jr} := \rho_j c_j \left( \frac{\mathbf{N}_{jr}^+ \otimes \mathbf{N}_{jr}^+}{\|\mathbf{N}_{jr}^+\|} + \frac{\mathbf{N}_{jr}^- \otimes \mathbf{N}_{jr}^-}{\|\mathbf{N}_{jr}^-\|} \right)$ , the  
 108 scheme (9)–(11) is Eucelhyd [22, 26]. One has  $\mathbf{N}_{jr}^+ + \mathbf{N}_{jr}^- = \mathbf{C}_{jr}$ , see figure 1.

109 These schemes are conservative in volume, mass, momentum and total energy. One easily  
 110 shows that they are entropy stable. These results can be found in [24, 22, 21, 26], for instance.  
 111 Also, a consistency result has been established in [27].

#### 112 4. Asymptotic Preserving scheme in semi-Lagrangian coordinates

113 We shall now present a multi-dimensional finite-volume scheme written in semi-Lagrangian  
 114 coordinates that preserves the asymptotic.

115 In this section, we present the Lagrangian step of our ALE method. In this step, each fluid is  
 116 associated to its own mesh. If the meshes may evolve differently, we assume that they coincide  
 117 at the beginning of the Lagrangian step. The rezoning/remapping procedure that is detailed in  
 118 section 5 is used to ensure that the meshes will coincide for the next Lagrangian step.

119 We first focus on the semi-discrete continuous in time scheme. Most of the properties of the  
 120 scheme are proved using this simpler formulation without any loss of generality. In paragraph 4.2,  
 121 we describe the fully discrete scheme. It is analysed in the remaining of this section.

##### 122 4.1. Continuous in time semi-discrete scheme

123 Let  $\omega \in [0, 2]$ , for each fluid  $\alpha \in \{f_1, f_2\}$ ,  $\beta$  denoting the other fluid, we define the scheme

$$\begin{aligned}
 m_j^\alpha d_t \tau_j^\alpha &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^\alpha, \\
 d_t m_j^\alpha &= 0, \\
 m_j^\alpha d_t \mathbf{u}_j^\alpha &= - \sum_r \mathbf{F}_{jr}^\alpha - \omega \sum_r \nu \rho_r B_{jr} \delta \mathbf{u}_j^\alpha - (1 - \omega) \sum_r \nu \rho_r B_{jr} \delta \mathbf{u}_r^\alpha, \\
 m_j^\alpha d_t E_j^\alpha &= - \sum_r \mathbf{F}_{jr}^\alpha \cdot \mathbf{u}_r^\alpha - \sum_r \nu \rho_r \bar{\mathbf{u}}_r^T B_{jr} \delta \mathbf{u}_r^\alpha + \omega \sum_r \nu \rho_r \bar{\mathbf{u}}_{jr}^T B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha),
 \end{aligned} \tag{12}$$

where the fluxes are given by

$$\mathbf{F}_{jr}^\alpha = \mathbf{C}_{jr} \rho_j^\alpha - A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) - \nu \rho_r B_{jr} \delta \mathbf{u}_r^\alpha, \quad \text{and} \tag{13}$$

$$\sum_j \mathbf{F}_{jr}^\alpha = \mathbf{0}. \tag{14}$$

124 In order to write (12), we introduced  $\rho_r^\alpha := \frac{1}{\#\mathcal{J}_r} \sum_{j \in \mathcal{J}_r} \rho_j^\alpha$  and  $\rho_r := \rho_r^\alpha + \rho_r^\beta$ . Also, we set

125  $\bar{\mathbf{u}}_r := \frac{\rho_r^\alpha \mathbf{u}_r^\alpha + \rho_r^\beta \mathbf{u}_r^\beta}{\rho_r^\alpha + \rho_r^\beta}$  and  $\bar{\mathbf{u}}_{jr} := \frac{\rho_j^\alpha \mathbf{u}_j^\alpha + \rho_j^\beta \mathbf{u}_j^\beta}{\rho_j^\alpha + \rho_j^\beta}$ .  $B_{jr}$  are symmetric and positive definite matrices such that  
 126  $\sum_{r \in \mathcal{R}_j} B_{jr} = V_j I$ . Matrices  $A_{jr}^\alpha$  are the standard ‘‘hydro-matrices’’ as defined in section 3.

127 **Remark 2.** One can choose  $B_{jr} := V_{jr} I$ , where  $V_{jr}$  is the volume of the subcell associated to  
 128 vertex  $r$  of cell  $j$ . Another obvious choice could be for instance  $B_{jr} := \frac{1}{\#\mathcal{R}_j} V_j I$ .

129 Observe that simple calculations allow to write

$$\rho_r \bar{\mathbf{u}}_r = \rho_r \mathbf{u}_r^\alpha - \rho_r^\beta \delta \mathbf{u}_r^\alpha \quad \text{and} \quad \rho_r \bar{\mathbf{u}}_{jr} = \rho_r \mathbf{u}_j^\alpha - \rho_r^\beta \delta \mathbf{u}_j^\alpha. \tag{15}$$



130 Injecting (13) in (12), and using (15), one gets the alternative form

$$\begin{aligned}
m_j^\alpha d_t \tau_j^\alpha &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^\alpha, \\
d_t m_j^\alpha &= 0, \\
m_j^\alpha d_t \mathbf{u}_j^\alpha &= \sum_r A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) + \omega \nu \sum_r \rho_r B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha), \\
m_j^\alpha d_t E_j^\alpha &= - \sum_r \mathbf{C}_{jr} p_j^\alpha \cdot \mathbf{u}_r^\alpha + \sum_r \mathbf{u}_r^{\alpha T} A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) + \nu \sum_r \rho_r^\beta {}^t \delta \mathbf{u}_r^\alpha B_{jr} \delta \mathbf{u}_r^\alpha \\
&\quad - \omega \nu \sum_r \rho_r^\beta {}^t \delta \mathbf{u}_j^\alpha B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha) + \omega \nu \sum_r \rho_r {}^t \mathbf{u}_j^\alpha B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha).
\end{aligned} \tag{16}$$

131 This form enlightens the fact that knowing the fluxes  $(\mathbf{u}_r^\alpha, \mathbf{u}_r^\beta)$  at any vertex  $r$  is enough to define  
132 the scheme. We shall now show that these nodal velocities are well defined.

Injecting (13) in (14) allows to calculate  $(\mathbf{u}_r^\alpha, \mathbf{u}_r^\beta)$ . Obviously, as soon as  $\nu \neq 0$ , both nodal velocities are coupled at vertex  $r$ . Omitting boundary conditions in the sake of simplicity, for each vertex of the mesh  $(\mathbf{u}_r^\alpha, \mathbf{u}_r^\beta)$  is the unique solution of the following linear system:

$$\underbrace{\sum_j \begin{pmatrix} A_{jr}^\alpha + \nu \rho_r B_{jr} & -\nu \rho_r B_{jr} \\ -\nu \rho_r B_{jr} & A_{jr}^\beta + \nu \rho_r B_{jr} \end{pmatrix}}_{\mathbb{A}_r^\nu :=} \begin{pmatrix} \mathbf{u}_r^\alpha \\ \mathbf{u}_r^\beta \end{pmatrix} = \underbrace{\sum_j \begin{pmatrix} A_{jr}^\alpha \mathbf{u}_j^\alpha + \mathbf{C}_{jr} p_j^\alpha \\ A_{jr}^\beta \mathbf{u}_j^\beta + \mathbf{C}_{jr} p_j^\beta \end{pmatrix}}_{\mathbf{b}_r :=}.$$

*Proof.* Since matrices  $A_{jr}^\alpha$  and  $B_{jr}$  are symmetric,  $\mathbb{A}_r^\nu$  is symmetric. To prove that  $(\mathbf{u}_r^\alpha, \mathbf{u}_r^\beta)$  is unique, it remains to show that it is positive definite. Elementary calculation gives,  $\forall (\mathbf{v}^\alpha, \mathbf{v}^\beta) \in \mathbb{R}^2 \times \mathbb{R}^2$ ,

$$\begin{aligned}
(\mathbf{v}^\alpha, \mathbf{v}^\beta)^T \mathbb{A}_r^\nu (\mathbf{v}^\alpha, \mathbf{v}^\beta) &= \mathbf{v}^{\alpha T} \left( \sum_j A_{jr}^\alpha \right) \mathbf{v}^\alpha + \mathbf{v}^{\beta T} \left( \sum_j A_{jr}^\beta \right) \mathbf{v}^\beta \\
&\quad + (\mathbf{v}^\alpha - \mathbf{v}^\beta)^T \left( \sum_j \nu \rho_r B_{jr} \right) (\mathbf{v}^\alpha - \mathbf{v}^\beta),
\end{aligned}$$

133 which is strictly positive if  $(\mathbf{v}^\alpha, \mathbf{v}^\beta) \neq (\mathbf{0}, \mathbf{0})$  since matrices  $\sum_j A_{jr}^\alpha$  and  $\sum_j \nu \rho_r B_{jr}$  are positive  
134 definite.  $\square$

135 The scheme being well-defined, we now establish its properties.

#### 136 4.1.1. Nodal velocities a priori estimates

137 Here, we establish estimates for the nodal velocities with regard to the frictionless case. These  
138 are actually some *instantaneous* stability results with regard to the mono-fluid schemes [21, 22],  
139 *i.e.* velocity fluxes are controlled by the frictionless ones.

**Property 3** (*A priori estimates*). For each fluid  $\alpha \in \{f_1, f_2\}$ , let  $\mathbf{u}_r^{\alpha, \nu}$  denote the nodal velocities at vertex  $r$ . Let  $A_r^\alpha := \sum_j A_{jr}^\alpha$  and  $B_r := \sum_j B_{jr}$ . Let  $\beta$  denote the other fluid, then one has the

following relations,  $\forall \nu \geq 0$

$$\mathbf{u}_r^{\alpha,\nu T} A_r^\alpha \mathbf{u}_r^{\alpha,\nu} + \mathbf{u}_r^{\beta,\nu T} A_r^\beta \mathbf{u}_r^{\beta,\nu} \leq \mathbf{u}_r^{\alpha,0 T} A_r^\alpha \mathbf{u}_r^{\alpha,0} + \mathbf{u}_r^{\beta,0 T} A_r^\beta \mathbf{u}_r^{\beta,0}, \quad (17)$$

$$\left( \mathbf{u}_r^{\alpha,\nu} - \mathbf{u}_r^{\beta,\nu} \right)^T B_r \left( \mathbf{u}_r^{\alpha,\nu} - \mathbf{u}_r^{\beta,\nu} \right) \leq \frac{1}{2\nu\rho_r} \left( \mathbf{u}_r^{\alpha,0 T} A_r^\alpha \mathbf{u}_r^{\alpha,0} + \mathbf{u}_r^{\beta,0 T} A_r^\beta \mathbf{u}_r^{\beta,0} \right), \quad (18)$$

$$\text{and } \left( \mathbf{u}_r^{\alpha,\nu} - \mathbf{u}_r^{\beta,\nu} \right)^T B_r \left( \mathbf{u}_r^{\alpha,\nu} - \mathbf{u}_r^{\beta,\nu} \right) \leq \left( \mathbf{u}_r^{\alpha,0} - \mathbf{u}_r^{\beta,0} \right)^T B_r \left( \mathbf{u}_r^{\alpha,0} - \mathbf{u}_r^{\beta,0} \right). \quad (19)$$

140 Let us first comment these estimates. The estimate (17) is a stability results. It shows that  
 141 the nodal velocity  $\|(\mathbf{u}_r^{\alpha,\nu}, \mathbf{u}_r^{\beta,\nu})\|_{\mathbb{A}_r^0}$  is bounded by  $\|(\mathbf{u}_r^{\alpha,0}, \mathbf{u}_r^{\beta,0})\|_{\mathbb{A}_r^0}$  independently of  $\nu$ . It shows that  
 142 friction nodal velocities are stable with regard to the classic frictionless case for a given state.

143 The second estimate (18) shows that the nodal velocity difference  $\|\delta \mathbf{u}_r^{\alpha,\nu}\|_{B_r}$  is at most  $O(\nu^{-1/2})$   
 144 according to  $\|(\mathbf{u}_r^{\alpha,0}, \mathbf{u}_r^{\beta,0})\|_{\mathbb{A}_r^0}$ .

145 The last inequality (19) states that the nodal velocity difference is bounded by the frictionless  
 146 case independently of  $\nu$  in the  $\|\cdot\|_{B_r}$  norm, which is purely geometric.

*Proof of Property 3.*  $\forall \nu \geq 0$ ,  $(\mathbf{u}_r^{\alpha,\nu}, \mathbf{u}_r^{\beta,\nu})$  is the unique solution of

$$\begin{pmatrix} A_r^\alpha + \nu\rho_r B_r & -\nu\rho_r B_r \\ -\nu\rho_r B_r & A_r^\beta + \nu\rho_r B_r \end{pmatrix} \begin{pmatrix} \mathbf{u}_r^{\alpha,\nu} \\ \mathbf{u}_r^{\beta,\nu} \end{pmatrix} = \mathbf{b}_r, \quad \text{with } \mathbf{b}_r := \begin{pmatrix} \sum_j \mathbf{C}_{jr} p_j^\alpha \\ \sum_j \mathbf{C}_{jr} p_j^\beta \end{pmatrix}.$$

147 So, since  $\mathbf{b}_r$  is independent of  $\nu$ , one has

$$\forall \nu \geq 0, \quad (\mathbb{A}_r^0 + \nu\rho_r \Delta_r) \mathbf{u}_r^\nu = \mathbb{A}_r^0 \mathbf{u}_r^0, \quad (20)$$

where

$$\mathbb{A}_r^0 := \begin{pmatrix} A_r^\alpha & 0 \\ 0 & A_r^\beta \end{pmatrix}, \quad \Delta_r := \begin{pmatrix} B_r & -B_r \\ -B_r & B_r \end{pmatrix} \quad \text{and} \quad \mathbf{u}_r^\nu := \begin{pmatrix} \mathbf{u}_r^{\alpha,\nu} \\ \mathbf{u}_r^{\beta,\nu} \end{pmatrix}.$$

Multiplying on the left by  $\mathbf{u}_r^\nu$  yields  $\mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^\nu + \nu\rho_r \mathbf{u}_r^{\nu T} \Delta_r \mathbf{u}_r^\nu = \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^0$ . Since  $B_r$  is a positive matrix  $\Delta_r$  is also positive, and since  $\nu\rho_r \geq 0$ , one gets

$$\forall \nu \geq 0, \quad \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^\nu \leq \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^0.$$

Finally,  $\mathbb{A}_r^0$  being symmetric and positive definite, the simple following Youngs inequality,

$$\mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^0 \leq \frac{1}{2} \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^\nu + \frac{1}{2} \mathbf{u}_r^{0 T} \mathbb{A}_r^0 \mathbf{u}_r^0,$$

148 allows to prove (17).

The proof of (18) follows the same way. Multiplying (20) on the left by  $\mathbf{u}_r^\nu$ , one has

$$\forall \nu \geq 0, \quad \nu\rho_r \mathbf{u}_r^{\nu T} \Delta_r \mathbf{u}_r^\nu + \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^\nu = \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^0.$$

Then, using the same Youngs inequality, one gets after a few arrangements

$$\forall \nu \geq 0, \quad \nu\rho_r \mathbf{u}_r^{\nu T} \Delta_r \mathbf{u}_r^\nu + \frac{1}{2} \mathbf{u}_r^{\nu T} \mathbb{A}_r^0 \mathbf{u}_r^\nu \leq \frac{1}{2} \mathbf{u}_r^{0 T} \mathbb{A}_r^0 \mathbf{u}_r^0,$$

149 which yields to (18) since  $\mathbb{A}_r^0$  is positive.

The third inequality is a bit more difficult to establish. Let us introduce the quadratic form  $\mathbf{J}_v^v := \frac{1}{2} \mathbf{v}^T (\mathbb{A}_r^0 + \nu \rho_r \Delta_r) \mathbf{v} - \mathbf{b}_r \cdot \mathbf{v}$ . So, since  $\mathbf{u}_r^v$  is the unique solution of the linear system, one has

$$\forall v \geq 0, \forall \mathbf{v}, \quad \mathbf{J}_{\mathbf{u}_r^v}^v \leq \mathbf{J}_v^v.$$

In the particular case  $\mathbf{v} = \mathbf{u}_r^0$ , one gets  $\mathbf{J}_{\mathbf{u}_r^v}^v \leq \mathbf{J}_{\mathbf{u}_r^0}^0$ . It is then easy to check that

$$\mathbf{J}_{\mathbf{u}_r^v}^v = \frac{1}{2} \mathbf{u}_r^{0T} (\mathbb{A}_r^0 + \nu \rho_r \Delta_r) \mathbf{u}_r^0 - \mathbf{b}_r \cdot \mathbf{u}_r^0 = \mathbf{J}_{\mathbf{u}_r^0}^0 + \frac{\nu \rho_r}{2} \mathbf{u}_r^{0T} \Delta_r \mathbf{u}_r^0.$$

150 So, one has established a first inequality

$$\mathbf{J}_{\mathbf{u}_r^v}^v \leq \mathbf{J}_{\mathbf{u}_r^0}^0 + \frac{\nu \rho_r}{2} \mathbf{u}_r^{0T} \Delta_r \mathbf{u}_r^0. \quad (21)$$

Similarly, since  $\mathbf{u}_r^0$  is the unique solution of the linear system in the case  $\nu = 0$ , one has  $\mathbf{J}_{\mathbf{u}_r^0}^0 \leq \mathbf{J}_{\mathbf{u}_r^v}^v$ , which can be written as

$$\mathbf{J}_{\mathbf{u}_r^0}^0 \leq \mathbf{J}_{\mathbf{u}_r^v}^v - \frac{\nu \rho_r}{2} \mathbf{u}_r^{vT} \Delta_r \mathbf{u}_r^v.$$

This actually gives a lower bound to  $\mathbf{J}_{\mathbf{u}_r^v}^v$  which combined with its upper bound (21) yields

$$\mathbf{J}_{\mathbf{u}_r^0}^0 + \frac{\nu \rho_r}{2} \mathbf{u}_r^{vT} \Delta_r \mathbf{u}_r^v \leq \mathbf{J}_{\mathbf{u}_r^0}^0 + \frac{\nu \rho_r}{2} \mathbf{u}_r^{0T} \Delta_r \mathbf{u}_r^0.$$

151 Since  $\nu \rho_r$  is positive, elementary calculations allow to write (19). □

#### 152 4.1.2. Conservativity

153 **Property 4** (Conservation). *The scheme defined by (12)–(14) ensures conservation of mass and*  
 154 *volume for each fluid  $\alpha$  or  $\beta$ . It also ensures that the sum of the fluids' momenta and total energies*  
 155 *are conserved.*

156 *Proof.* Conservations of mass and volume for each fluid are obvious since the associated balance  
 157 equations are unchanged with regard to the mono-fluid schemes (see for instance [24, 21, 22,  
 158 26]).

Summing momenta equations in (12) for both fluids gives

$$\begin{aligned} m_j^\alpha d_t \mathbf{u}_j^\alpha + m_j^\beta d_t \mathbf{u}_j^\beta &= - \sum_r \mathbf{F}_{jr}^\alpha - \sum_r \mathbf{F}_{jr}^\beta \\ &\quad - \omega \sum_r \nu \rho_r B_{jr} (\delta \mathbf{u}_j^\alpha + \delta \mathbf{u}_j^\beta) - (1 - \omega) \sum_r \nu \rho_r B_{jr} (\delta \mathbf{u}_r^\alpha + \delta \mathbf{u}_r^\beta). \end{aligned}$$

Recalling that by definition,  $\delta \mathbf{u}_j^\alpha + \delta \mathbf{u}_j^\beta = \mathbf{0}$ , one has

$$m_j^\alpha d_t \mathbf{u}_j^\alpha + m_j^\beta d_t \mathbf{u}_j^\beta = - \sum_r \mathbf{F}_{jr}^\alpha - \sum_r \mathbf{F}_{jr}^\beta.$$

The conservativity proof is ended in a standard way. One now sums these equations over the cells which gives

$$\sum_j m_j^\alpha d_t \mathbf{u}_j^\alpha + \sum_j m_j^\beta d_t \mathbf{u}_j^\beta = - \sum_j \sum_{r \in \mathcal{R}_j} \mathbf{F}_{jr}^\alpha - \sum_j \sum_{r \in \mathcal{R}_j} \mathbf{F}_{jr}^\beta,$$

which rewrites,

$$\sum_j m_j^\alpha d_t \mathbf{u}_j^\alpha + \sum_j m_j^\beta d_t \mathbf{u}_j^\beta = - \sum_r \sum_{j \in \mathcal{J}_r} \mathbf{F}_{jr}^\alpha - \sum_r \sum_{j \in \mathcal{J}_r} \mathbf{F}_{jr}^\beta.$$

159 This proves that momenta sum is conserved using (14) and recalling that cell masses are La-  
160 grangian.

161 Conservation of total energies sum is obtained in the exact same way.  $\square$

#### 162 4.1.3. Stability

163 Before proving this result, let us recall that the fully discrete scheme's stability is presented  
164 bellow (see paragraph 4.2).

**Property 5** (Entropy). *The first-order continuous in time scheme defined by (12)–(14) satisfies,  $\forall \omega \in [0, 2]$ , the following entropy inequality  $\forall \alpha \in \{f_1, f_2\}$*

$$m_j^\alpha T_j^\alpha d_t \eta_j^\alpha \geq \left(1 - \frac{\omega}{2}\right) \sum_r \nu \rho_r^{\beta \text{ } t} \delta \mathbf{u}_r^\alpha B_{jr} \delta \mathbf{u}_r^\beta + \frac{\omega}{2} \sum_r \nu \rho_r^{\beta \text{ } t} \delta \mathbf{u}_j^\alpha B_{jr} \delta \mathbf{u}_j^\alpha \geq 0.$$

165 This inequality is consistent with (2).

166 Let us establish a simple technical Lemma that will be useful in the following and to demon-  
167 strate Property 5.

**Lemma 1.** *Let  $M$  denote a symmetric matrix of  $\mathbb{R}^{d \times d}$ . Let  $\omega \in \mathbb{R}$ , then*

$$\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \quad \mathbf{v}^T M \mathbf{v} - \omega \mathbf{w}^T M (\mathbf{v} - \mathbf{w}) = \left(1 - \frac{\omega}{2}\right) \mathbf{v}^T M \mathbf{v} + \frac{\omega}{2} \mathbf{w}^T M \mathbf{w} + \frac{\omega}{2} (\mathbf{w} - \mathbf{v})^T M (\mathbf{w} - \mathbf{v}).$$

*Proof.* Let  $\xi := \mathbf{v}^T M \mathbf{v} - \omega \mathbf{w}^T M (\mathbf{v} - \mathbf{w})$ . Obviously, one has

$$\xi = \mathbf{v}^T M \mathbf{v} + \omega \mathbf{w}^T M \mathbf{w} - \omega \mathbf{w}^T M \mathbf{v}.$$

168 Since  $M$  is symmetric, one has  $-\omega \mathbf{w}^T M \mathbf{v} = (\mathbf{v} - \mathbf{w})^T M (\mathbf{v} - \mathbf{w}) - \mathbf{v}^T M \mathbf{v} - \mathbf{w}^T M \mathbf{w}$ . Injecting this  
169 equality in the expression of  $\xi$  ends the demonstration.  $\square$

**Corollary 1.** *Let  $M$  denote a symmetric and positive matrix of  $\mathbb{R}^{d \times d}$ . Let  $\omega \geq 0$ , then*

$$\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \quad \mathbf{v}^T M \mathbf{v} - \omega \mathbf{w}^T M (\mathbf{v} - \mathbf{w}) \geq \left(1 - \frac{\omega}{2}\right) \mathbf{v}^T M \mathbf{v} + \frac{\omega}{2} \mathbf{w}^T M \mathbf{w}.$$

170 *Proof.* This is a direct consequence of Lemma 1, since  $\omega M$  is a positive matrix.  $\square$

171 We can now give the proof of Property 5.

*Proof of Property 5.* Gibbs formula reads  $T d\eta = de + pd\tau$ , so that one has

$$T_j^\alpha d_t \eta_j^\alpha = d_t e_j^\alpha + p_j^\alpha d_t \tau_j^\alpha,$$

which rewrites also

$$m_j^\alpha T_j^\alpha d_t \eta_j^\alpha = m_j^\alpha d_t E_j^\alpha - \mathbf{u}_j^\alpha \cdot m_j^\alpha d_t \mathbf{u}_j^\alpha + p_j^\alpha m_j^\alpha d_t \tau_j^\alpha.$$

Using (16), one gets

$$\begin{aligned} m_j^\alpha T_j^\alpha d_t \eta_j^\alpha &= - \sum_r \mathbf{C}_{jr} p_j^\alpha \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^{\alpha T} A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) + \nu \sum_r \rho_r^\beta \delta \mathbf{u}_r^\alpha B_{jr} \delta \mathbf{u}_r^\alpha \\ &\quad - \omega \nu \sum_r \rho_r^\beta \delta \mathbf{u}_j^\alpha B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha) + \omega \nu \sum_r \rho_r^\alpha \delta \mathbf{u}_j^\alpha B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha) \\ &\quad + \mathbf{u}_j^\alpha \cdot \left( \sum_r A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) + \omega \nu \sum_r \rho_r B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha) \right) + \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^\alpha p_j^\alpha, \end{aligned}$$

which simplifies as

$$m_j^\alpha T_j^\alpha d_t \eta_j^\alpha = \sum_r (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha)^T A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) + \nu \sum_r \rho_r^\beta \delta \mathbf{u}_r^{\alpha T} B_{jr} \delta \mathbf{u}_r^\alpha - \omega \nu \sum_r \rho_r^\beta \delta \mathbf{u}_j^{\alpha T} B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha).$$

Since  $B_{jr}$  matrices are symmetric and positive and since  $\omega \geq 0$ , one can apply Corollary 1 to obtain

$$\begin{aligned} m_j^\alpha T_j^\alpha d_t \eta_j^\alpha &\geq \sum_r (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha)^T A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) \\ &\quad + \left(1 - \frac{1}{2}\omega\right) \nu \sum_r \rho_r^\beta \delta \mathbf{u}_r^{\alpha T} B_{jr} \delta \mathbf{u}_r^\alpha + \frac{1}{2}\omega \nu \sum_r \rho_r^\beta \delta \mathbf{u}_j^{\alpha T} B_{jr} \delta \mathbf{u}_j^\alpha, \end{aligned}$$

Matrix  $A_{jr}^\alpha$  being positive, one finally has

$$m_j^\alpha T_j^\alpha d_t \eta_j^\alpha \geq \left(1 - \frac{1}{2}\omega\right) \nu \sum_r \rho_r^\beta \delta \mathbf{u}_r^{\alpha T} B_{jr} \delta \mathbf{u}_r^\alpha + \frac{1}{2}\omega \nu \sum_r \rho_r^\beta \delta \mathbf{u}_j^{\alpha T} B_{jr} \delta \mathbf{u}_j^\alpha,$$

172 which is positive as soon as  $\omega \in [0, 2]$ . □

#### 173 4.1.4. Asymptotic preserving

174 We now establish the main result of this paper. It consists in stating that when the friction  
175 parameter  $\nu$  tends to infinity, the scheme (12)–(14) behaves asymptotically as a scheme that is  
176 consistent with the asymptotic model (4)–(5).

177 To this end, we first compute the asymptotic scheme by means of Hilbert expansions, then we  
178 show its consistency with the asymptotic model. This later result relies strongly on B. Després's  
179 work [27].

**Asymptotic scheme.** *Let  $\omega \neq 0$ . If  $\forall \alpha \in \{f_1, f_2\}$ ,  $\forall j$ ,  $(\rho_j^\alpha, \mathbf{u}_j^\alpha, E_j^\alpha)$  are constant cell data, then the scheme (12)–(14), behaves asymptotically as*

$$(m_j^\alpha + m_j^\beta) d_t \mathbf{u}_j = - \sum_r \mathbf{F}_{jr}^\alpha - \sum_r \mathbf{F}_{jr}^\beta, \quad (22)$$

$$d_t V_j = m_j^\alpha d_t \tau_j^\alpha = \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r, \quad (23)$$

$$d_t m_j^\alpha = 0,$$

$$m_j^\alpha d_t E_j^\alpha = - \sum_r \mathbf{C}_{jr} p_j^\alpha \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^T A_{jr}^\alpha (\mathbf{u}_r - \mathbf{u}_j) - \frac{\rho_j^\alpha \rho_j^\beta}{\rho_j} \sum_r \mathbf{u}_j^T \delta \left( \frac{A_{jr}}{\rho_j} \right)^\alpha (\mathbf{u}_r - \mathbf{u}_j), \quad (24)$$

where  $\mathbf{u}_j = \mathbf{u}_j^\alpha = \mathbf{u}_j^\beta$ , and where nodal velocities  $\mathbf{u}_r = \mathbf{u}_r^\alpha = \mathbf{u}_r^\beta$  satisfy

$$\begin{aligned} \mathbf{F}_{jr}^\alpha + \mathbf{F}_{jr}^\beta &= \mathbf{C}_{jr} (p_j^\alpha + p_j^\beta) - (A_{jr}^\alpha + A_{jr}^\beta) (\mathbf{u}_r - \mathbf{u}_j), \\ \text{and } \sum_j \mathbf{F}_{jr}^\alpha &= \mathbf{0}. \end{aligned} \quad (25)$$

*Formal derivation.* Let  $\alpha \in \{f_1, f_2\}$ ,  $\beta$  denoting the other fluid. Let us introduce  $\epsilon := \nu^{-1}$ . One rewrites (16) as

$$m_j^\alpha d_t \tau_j^\alpha = \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^\alpha, \quad (26)$$

$$d_t m_j^\alpha = 0,$$

$$m_j^\alpha d_t \mathbf{u}_j^\alpha = \sum_r A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) - \frac{1}{\epsilon} \omega \sum_r \rho_r B_{jr} (\delta \mathbf{u}_j^\alpha - \delta \mathbf{u}_r^\alpha), \quad (27)$$

$$\begin{aligned} m_j^\alpha d_t E_j^\alpha &= - \sum_r \mathbf{C}_{jr} p_j^\alpha \cdot \mathbf{u}_r^\alpha + \sum_r {}^t \mathbf{u}_r^\alpha A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) + \frac{1}{\epsilon} \sum_r \rho_r^\beta (\delta \mathbf{u}_r^\beta)^T B_{jr} \delta \mathbf{u}_j^\alpha \\ &\quad + \frac{1}{\epsilon} \omega \sum_r (\rho_r \mathbf{u}_j^{\alpha T} - \rho_r^\beta \delta \mathbf{u}_j^{\alpha T}) B_{jr} (\delta \mathbf{u}_r^\alpha - \delta \mathbf{u}_j^\alpha), \end{aligned} \quad (28)$$

180 and

$$\sum_j A_{jr}^\alpha \mathbf{u}_r^\alpha + \sum_j \frac{1}{\epsilon} \rho_r B_{jr} \delta \mathbf{u}_r^\alpha = \sum_j A_{jr}^\alpha \mathbf{u}_j^\alpha + \sum_j \mathbf{C}_{jr} p_j^\alpha. \quad (29)$$

181 Following the analysis of the asymptotic model, we perform an Hilbert expansion.

The first information one gets is from equation (29) that writes

$$\begin{aligned} \sum_j A_{jr}^{\alpha,0} \mathbf{u}_r^{\alpha,0} + \sum_j \frac{1}{\epsilon} \rho_r^0 B_{jr} \delta \mathbf{u}_r^{\alpha,0} + \sum_j \rho_r^0 B_{jr} \delta \mathbf{u}_r^{\alpha,1} + \sum_j \rho_r^1 B_{jr} \delta \mathbf{u}_r^{\alpha,0} \\ = \sum_j A_{jr}^{\alpha,0} \mathbf{u}_j^{\alpha,0} + \sum_j \mathbf{C}_{jr} p_j^{\alpha,0} + \mathbf{O}(\epsilon), \end{aligned}$$

182 so that multiplying this equation by  $\epsilon$  leads to  $\rho_r^0 (\sum_j B_{jr}) \delta \mathbf{u}_r^{\alpha,0} = \mathbf{0}$  that is

$$\delta \mathbf{u}_r^{\alpha,0} = \mathbf{0}, \quad (30)$$

183 since  $\sum_j B_{jr}$  is symmetric positive definite and  $\rho_r = \rho_r^0 + \mathcal{O}(\epsilon) > 0$  so that  $\rho_r^0 > 0$  when  $\epsilon \rightarrow 0$ .

184 One gets volume conservation equation (23).

Now, the momentum equation (27) is considered, using (30), one has

$$\begin{aligned} m_j^\alpha d_t \mathbf{u}_j^{\alpha,0} &= \sum_r A_{jr}^{\alpha,0} (\mathbf{u}_r^{\alpha,0} - \mathbf{u}_j^{\alpha,0}) - \frac{1}{\epsilon} \omega \sum_r \rho_r^0 B_{jr} \delta \mathbf{u}_j^{\alpha,0} \\ &\quad - \omega \sum_r \rho_r^1 B_{jr} \delta \mathbf{u}_j^{\alpha,0} - \omega \sum_r \rho_r^0 B_{jr} (\delta \mathbf{u}_j^{\alpha,1} - \delta \mathbf{u}_r^{\alpha,1}) + \mathbf{O}(\epsilon), \end{aligned}$$

185 which gives

$$\delta \mathbf{u}_j^{\alpha,0} = \mathbf{0}. \quad (31)$$

186 Using, (31) and (30), one defines  $\mathbf{u}_j^0 := \mathbf{u}_j^{\alpha,0} = \mathbf{u}_j^{\beta,0}$  and  $\mathbf{u}_r^0 := \mathbf{u}_r^{\alpha,0} = \mathbf{u}_r^{\beta,0}$ .

So, Hilbert expansions of equations (26), (27) and (28) simplify as

$$\begin{aligned} m_j^\alpha d_t \tau_j^{\alpha,0} &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^0, \\ m_j^\alpha d_t \mathbf{u}_j^0 &= \sum_r A_{jr}^{\alpha,0} (\mathbf{u}_r^0 - \mathbf{u}_j^0) - \omega \sum_r \rho_r^0 B_{jr} (\delta \mathbf{u}_j^{\alpha,1} - \delta \mathbf{u}_r^{\alpha,1}), \end{aligned} \quad (32)$$

$$\begin{aligned} m_j^\alpha d_t E_j^{\alpha,0} &= - \sum_r (\mathbf{C}_{jr} p_j^{\alpha,0} - A_{jr}^{\alpha,0} (\mathbf{u}_r^0 - \mathbf{u}_j^0)) \cdot \mathbf{u}_r^0 \\ &\quad + \omega \sum_r \rho_r^0 \mathbf{u}_j^{\alpha,0T} B_{jr} (\delta \mathbf{u}_r^{\alpha,1} - \delta \mathbf{u}_j^{\alpha,1}), \end{aligned} \quad (33)$$

Our aim is now to evaluate the term  $\omega \sum_r \rho_r^0 \mathbf{u}_j^{\alpha,0T} B_{jr} (\delta \mathbf{u}_r^{\alpha,1} - \delta \mathbf{u}_j^{\alpha,1})$ . To do so, we divide momentum equation (32) by  $\rho_j^\alpha (> 0)$ , which gives

$$V_j d_t \mathbf{u}_j^0 = \frac{1}{\rho_j^\alpha} \sum_r A_{jr}^{\alpha,0} (\mathbf{u}_r^0 - \mathbf{u}_j^0) - \omega \sum_r \frac{\rho_r^0}{\rho_j^\alpha} B_{jr} (\delta \mathbf{u}_j^{\alpha,1} - \delta \mathbf{u}_r^{\alpha,1}).$$

The same relation can be written for fluid  $\beta$ . The difference of these two equations writes, recalling that  $\delta \phi^\alpha = -\delta \phi^\beta$ ,

$$\mathbf{0} = \sum_r \delta \left( \frac{A_{jr}^0}{\rho_j} \right)^\alpha (\mathbf{u}_r^0 - \mathbf{u}_j^0) - \frac{\rho_j}{\rho_j^\alpha \rho_j^\beta} \omega \sum_r \rho_r^0 B_{jr} (\delta \mathbf{u}_j^{\alpha,1} - \delta \mathbf{u}_r^{\alpha,1}).$$

187 Injecting this relation in (33) gives the limit scheme total energy balance equation (24). The  
188 momentum equation (22) is obtained the same way or by simply summing equations (32) for  
189 both fluids  $\alpha$  and  $\beta$ .  $\square$

190 In order to establish that the scheme is asymptotic preserving, it remains to show that the  
191 limit scheme (22)–(25) is consistent with the asymptotic model (4)–(5).

192 Before establishing this result, we recall the fundamental result by B. Després [27], that we  
193 adapt to the present context.

**Property 6** (B. Després). *Let  $m_j := m_j^\alpha + m_j^\beta$ ,  $\rho_j := \rho_j^\alpha + \rho_j^\beta$ ,  $\tau_j = \rho_j^{-1}$  and  $E_j := \frac{\rho_j^\alpha E_j^\alpha + \rho_j^\beta E_j^\beta}{\rho_j}$ . Then, the scheme*

$$\begin{aligned} d_t m_j &= 0, \\ m_j d_t \tau_j &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r, \\ m_j d_t \mathbf{u}_j &= - \sum_r \mathbf{F}_{jr}, \\ m_j d_t E_j &= - \sum_r \mathbf{F}_{jr} \cdot \mathbf{u}_r, \end{aligned}$$

$$\text{where } \mathbf{F}_{jr} = \mathbf{C}_{jr} (p_j^\alpha + p_j^\beta) - (A_{jr}^\alpha + A_{jr}^\beta) (\mathbf{u}_r - \mathbf{u}_j),$$

$$\text{and } \sum_j (A_{jr}^\alpha + A_{jr}^\beta) \mathbf{u}_r = \sum_j (A_{jr}^\alpha + A_{jr}^\beta) \mathbf{u}_j + \sum_j \mathbf{C}_{jr} (p_j^\alpha + p_j^\beta),$$

is weakly consistent with the following system of equations

$$\begin{aligned}\rho D_t \tau &= \nabla \cdot \mathbf{u}, \\ \rho D_t \mathbf{u} &= -\nabla(p^\alpha + p^\beta), \\ \rho D_t E &= -\nabla \cdot (p^\alpha + p^\beta) \mathbf{u}.\end{aligned}$$

194 *Proof.* The proof can be found in [27]. □

195 **Property 7.** The limit scheme (22)–(25) is weakly consistent with the asymptotic model (4)–(5).

196 *Proof.* Consistency for volume, mass and momentum is a direct consequence of Property 6, it  
197 remains to show the consistency for total energy.

We rewrite equation (5) using a more convenient form

$$\rho^\alpha D_t E^\alpha = -\nabla \cdot (p^\alpha + p^\beta) \mathbf{u} + p^\beta \nabla \cdot \mathbf{u} + \frac{\rho^\beta}{\rho} \nabla(p^\alpha + p^\beta) \cdot \mathbf{u}.$$

As a starting point we recall (25) for fluid  $\alpha$

$$m_j^\alpha d_t E_j^\alpha = -\sum_r \mathbf{C}_j p_j^\alpha \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^T A_{jr}^\alpha (\mathbf{u}_r - \mathbf{u}_j) - \frac{\rho_j^\alpha \rho_j^\beta}{\rho_j} \sum_r \mathbf{u}_j^T \delta \left( \frac{A_{jr}^\alpha}{\rho_j} \right) (\mathbf{u}_r - \mathbf{u}_j),$$

that we rewrite

$$\begin{aligned}m_j^\alpha d_t E_j^\alpha &= -\sum_r \mathbf{C}_j (p_j^\alpha + p_j^\beta) \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^T (A_{jr}^\alpha + A_{jr}^\beta) (\mathbf{u}_r - \mathbf{u}_j) \\ &\quad + \sum_r \mathbf{C}_j p_j^\beta \cdot \mathbf{u}_r - \sum_r \mathbf{u}_r^T A_{jr}^\beta (\mathbf{u}_r - \mathbf{u}_j) - \frac{\rho_j^\alpha \rho_j^\beta}{\rho_j} \sum_r \mathbf{u}_j^T \left( \frac{A_{jr}^\alpha}{\rho_j^\alpha} - \frac{A_{jr}^\beta}{\rho_j^\beta} \right) (\mathbf{u}_r - \mathbf{u}_j).\end{aligned}$$

Simple algebraic manipulations on the later term allow to write

$$\begin{aligned}m_j^\alpha d_t E_j^\alpha &= -\sum_r \mathbf{C}_j (p_j^\alpha + p_j^\beta) \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^T (A_{jr}^\alpha + A_{jr}^\beta) (\mathbf{u}_r - \mathbf{u}_j) \\ &\quad + \sum_r \mathbf{C}_j p_j^\beta \cdot \mathbf{u}_r - \sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta (\mathbf{u}_r - \mathbf{u}_j) - \frac{\rho_j^\beta}{\rho_j} \mathbf{u}_j^T \sum_r (A_{jr}^\alpha + A_{jr}^\beta) (\mathbf{u}_r - \mathbf{u}_j).\end{aligned}$$

- According to Property 6 the term

$$\frac{1}{V_j} \left( -\sum_r \mathbf{C}_j (p_j^\alpha + p_j^\beta) \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^T (A_{jr}^\alpha + A_{jr}^\beta) (\mathbf{u}_r - \mathbf{u}_j) \right),$$

198 is weakly consistent with  $(-\nabla \cdot (p^\alpha + p^\beta) \mathbf{u})|_{\mathbf{x}_j}$ .

- Also since  $\frac{1}{V_j} (\sum_r \mathbf{C}_j \cdot \mathbf{u}_r)$  is weakly consistent with  $\nabla \cdot \mathbf{u}$ ,

$$\frac{1}{V_j} \left( p_j^\beta \sum_r \mathbf{C}_j \cdot \mathbf{u}_r \right) \approx (p^\beta \nabla \cdot \mathbf{u})|_{\mathbf{x}_j}.$$



- Now, since  $\sum_r \mathbf{C}_{jr} = \mathbf{0}$ , one has

$$-\sum_r \mathbf{F}_{jr} = \sum_r (A_{jr}^\alpha + A_{jr}^\beta)(\mathbf{u}_r - \mathbf{u}_j),$$

so that Property 6 implies that

$$\frac{1}{V_j} \left( -\frac{\rho_j^\beta}{\rho_j} \mathbf{u}_j^T \sum_r (A_{jr}^\alpha + A_{jr}^\beta)(\mathbf{u}_r - \mathbf{u}_j) \right) \approx \left( \frac{\rho_j^\beta}{\rho} \nabla(p^\alpha + p^\beta) \cdot \mathbf{u} \right) \Big|_{\mathbf{x}_j}.$$

To conclude, it remains to prove for the remaining term

$$\frac{1}{V_j} \left( -\sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta (\mathbf{u}_r - \mathbf{u}_j) \right) \approx 0.$$

Let  $\zeta^\alpha$  denote its limit:

$$\frac{1}{V_j} \left( -\sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta (\mathbf{u}_r - \mathbf{u}_j) \right) \xrightarrow{V_j \rightarrow 0} \zeta^\alpha.$$

We have shown

$$\rho_j^\alpha d_t E_j^\alpha \approx \left( -\nabla \cdot (p^\alpha + p^\beta) \mathbf{u} + p^\beta \nabla \cdot \mathbf{u} + \frac{\rho_j^\beta}{\rho} \nabla(p^\alpha + p^\beta) \cdot \mathbf{u} \right) \Big|_{\mathbf{x}_j} + \zeta^\alpha.$$

Since the same result holds for fluid  $\beta$ , simple calculations lead to

$$\rho_j^\alpha d_t E_j^\alpha + \rho_j^\beta d_t E_j^\beta = \rho_j d_t E_j \approx \left( -\nabla \cdot (p^\alpha + p^\beta) \mathbf{u} \right) \Big|_{\mathbf{x}_j} + \zeta^\alpha + \zeta^\beta.$$

According to Property 6

$$\rho_j d_t E_j \approx \left( -\nabla \cdot (p^\alpha + p^\beta) \mathbf{u} \right) \Big|_{\mathbf{x}_j},$$

199 so that  $\zeta^\alpha + \zeta^\beta \approx 0$ .

Actually, one has

$$\frac{1}{V_j} \left( \sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta (\mathbf{u}_r - \mathbf{u}_j) \right) + \frac{1}{V_j} \left( \sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\alpha (\mathbf{u}_r - \mathbf{u}_j) \right) \rightarrow 0,$$

since  $A_{jr}^\alpha$  and  $A_{jr}^\beta$  are positive matrices, one has finally

$$\frac{1}{V_j} \left( \sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\alpha (\mathbf{u}_r - \mathbf{u}_j) \right) \rightarrow 0 \quad \text{and} \quad \frac{1}{V_j} \left( \sum_r (\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta (\mathbf{u}_r - \mathbf{u}_j) \right) \rightarrow 0,$$

200 which ends the proof. □

201 *4.2. Discrete scheme*

We now describe the fully discrete scheme. According to the previously established results, let  $\omega \in ]0, 2]$ . One defines the following scheme for each fluid  $\alpha \in \{f_1, f_2\}$ ,  $\beta$  denoting the other one.

$$m_j^\alpha \frac{\tau_j^{\alpha n+1} - \tau_j^{\alpha n}}{\Delta t} = \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n}, \quad (34)$$

$$m_j^\alpha \frac{\mathbf{u}_j^{\alpha n+1} - \mathbf{u}_j^{\alpha n}}{\Delta t} = - \sum_r \mathbf{F}_{jr}^{\alpha, n} - \omega \sum_r \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_j^{\alpha n+1} - (1 - \omega) \sum_r \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha n}, \quad (35)$$

$$m_j^\alpha \frac{E_j^{\alpha n+1} - E_j^{\alpha n}}{\Delta t} = - \sum_r \mathbf{F}_{jr}^{\alpha, n} \cdot \mathbf{u}_r^{\alpha n} - \sum_r \nu \rho_r^n {}^t \bar{\mathbf{u}}_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \sum_r \nu \rho_r^n {}^t \bar{\mathbf{u}}_{jr}^{n+1} B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}), \quad (36)$$

where the fluxes are computed explicitly as

$$\mathbf{F}_{jr}^{\alpha, n} = \mathbf{C}_{jr}^n p_j^{\alpha n} - A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) - \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha n}, \quad (37)$$

$$\text{and} \quad \sum_j A_{jr}^{\alpha, n} \mathbf{u}_r^{\alpha n} + \sum_j \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha n} = \sum_j A_{jr}^{\alpha, n} \mathbf{u}_j^{\alpha n} + \sum_j \mathbf{C}_{jr}^n p_j^{\alpha n}. \quad (38)$$

202 To complete the scheme definition, observe that we introduced the following mean velocities

$$203 \quad \bar{\mathbf{u}}_{jr}^{n+1} = \frac{\rho_r^{\alpha n} \mathbf{u}_j^{\alpha n+1} + \rho_r^{\beta n} \mathbf{u}_j^{\beta n+1}}{\rho_r^{\alpha n} + \rho_r^{\beta n}} \quad \text{and} \quad \bar{\mathbf{u}}_r^n = \frac{\rho_r^{\alpha n} \mathbf{u}_r^{\alpha n} + \rho_r^{\beta n} \mathbf{u}_r^{\beta n}}{\rho_r^{\alpha n} + \rho_r^{\beta n}}, \quad \text{which rewrite}$$

$$\rho_r^n \bar{\mathbf{u}}_{jr}^{n+1} = \rho_r^n \mathbf{u}_j^{\alpha n+1} - \rho_r^{\beta n} \delta \mathbf{u}_j^{\alpha n+1} \quad \text{and} \quad \rho_r^n \bar{\mathbf{u}}_r^n = \rho_r^n \mathbf{u}_r^{\alpha n} - \rho_r^{\beta n} \delta \mathbf{u}_r^{\alpha n}. \quad (39)$$

204 Similarly to the semi-discrete case, for convinience, we inject the fluxes expression into momen-  
205 tum and total energy balance equation and use (39)

$$m_j^\alpha \frac{\mathbf{u}_j^{\alpha n+1} - \mathbf{u}_j^{\alpha n}}{\Delta t} = \sum_r A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) + \omega \nu \sum_r \rho_r^n B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}), \quad (40)$$

$$\begin{aligned} m_j^\alpha \frac{E_j^{\alpha n+1} - E_j^{\alpha n}}{\Delta t} &= - \sum_r \mathbf{C}_{jr}^n p_j^{\alpha n} \cdot \mathbf{u}_r^{\alpha n} + \sum_r {}^t \mathbf{u}_r^{\alpha n} A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) \\ &\quad + \nu \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \nu \sum_r \rho_r^n {}^t \mathbf{u}_j^{\alpha n+1} B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}) \\ &\quad - \omega \nu \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_j^{\alpha n+1} B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}). \end{aligned} \quad (41)$$

206 *4.3. Stability of the discrete scheme*

207 In this section we establish that the scheme is stable for arbitrary equation of state: there  
208 exists  $\Delta t > 0$  such that for each fluid  $\alpha \in \{f_1, f_2\}$ ,  $\tau_j^{\alpha n+1} > 0$ ,  $e_j^{\alpha n+1} > e(T = 0)$  and  $\eta_j^{\alpha n+1} \geq \eta_j^{\alpha n}$ .  
209 For the sake of simplicity, and without loss of generality, we will consider in the following the  
210 case  $e_j^{\alpha n+1} > 0$ .

211 Actually, we will provide explicit timesteps for the positivity of density and internal energy,  
 212 but we will only show that the increasing physical entropy timestep will be greater than  
 213 one of the mono-fluid case for given velocity fluxes, for which we established Property 3. The  
 214 main reason is that there only exists existence results for entropy stability for cell-centered semi-  
 215 Lagrangian schemes (even in 1D), see [28, 29].

#### 216 4.3.1. Positivity of density

217 Since  $p = p(\rho, e)$  one has to ensure that density cannot be made negative.

**Property 8** (Positivity of density). *Assuming that  $\forall \alpha \in \{f_1, f_2\}, \forall j \in \mathcal{M}, \rho_j^{\alpha n} > 0$ . Denoting  $\mathcal{C}^{\alpha n}$  the set of compressive cells for each fluid  $\alpha$ ,  $\mathcal{C}^{\alpha n} := \{j \in \mathcal{M} / \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} < 0\}$ , there exists  $\Delta t^\rho > 0$  such that,*

$$\forall \alpha \in \{f_1, f_2\}, \forall j \in \mathcal{C}^{\alpha n}, \quad \Delta t^\rho < \frac{V_j^{\alpha n}}{-\sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n}}.$$

Then, the scheme (34)–(38) defined by  $\Delta t \in ]0, \Delta t^\rho]$  ensures that

$$\forall \omega \in [0, 2], \forall \alpha \in \{f_1, f_2\}, \forall j \in \mathcal{M}, \quad \rho_j^{\alpha n+1} > 0.$$

218 Observe that, as expected, only compressive cells ( $j \in \mathcal{C}^{\alpha n}$ ) can lead to negative densities, so  
 219 in the case of non-compressive flows,  $\Delta t^\rho$  may be arbitrarily large. Also, in the case of triangular  
 220 meshes, this constrain implies that no cell will tangle during the timestep.

*Proof.* Obviously, this is equivalent to show that  $\tau_j^{\alpha n+1} = \frac{1}{\rho_j^{\alpha n+1}} > 0$ . According to (34), one has

$$\tau_j^{\alpha n+1} = \tau_j^{\alpha n} + \frac{\Delta t}{m_j^\alpha} \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n}.$$

221 So, one has the following alternative:

- 222 • if  $j \notin \mathcal{C}^{\alpha n}$  that is  $\sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} \geq 0$ , then  $\forall \Delta t > 0$  one has  $\tau_j^{\alpha n+1} > 0$ ,
  - 223 • else if  $j \in \mathcal{C}^{\alpha n}$ , one has  $\sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} < 0$ , then  $\forall \Delta t < \tau_j^{\alpha n} \frac{m_j^\alpha}{-\sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n}}$ , one has  $\tau_j^{\alpha n+1} > 0$ .
- 224 Since  $\frac{m_j^\alpha}{-\sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n}} > 0$ , the existence of such a  $\Delta t > 0$  is obvious.

225

□

#### 226 4.3.2. Positivity of internal energy

First, as a primary result, we give internal energy variation for fluid  $\alpha \in \{f_1, f_2\}$ ,  $\beta$  denoting the other one. Internal energy is updated as

$$\begin{aligned} e_j^{\alpha n+1} = e_j^{\alpha n} + \frac{\Delta t}{m_j^\alpha} & \left[ \sum_r {}^t(\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) A_{jr}^{\alpha n} (\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) - \sum_r \rho_j^{\alpha n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} \right] \\ & + \nu \frac{\Delta t}{m_j^\alpha} \left[ \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha n} \mathbf{B}_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_j^{\alpha n+1} \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha n+1} - \delta \mathbf{u}_r^{\alpha n}) \right] \\ & - \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \sum_r A_{jr}^{\alpha n} (\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) \right)^2 + \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \omega \nu \sum_r \rho_r^{\beta n} \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha n+1} - \delta \mathbf{u}_r^{\alpha n}) \right)^2. \quad (42) \end{aligned}$$

*Proof.* Rewriting  $e_j^{\alpha^{n+1}} = -\frac{1}{2}\|\mathbf{u}_j^{\alpha^{n+1}}\|^2 + E_j^{\alpha^{n+1}}$  and using (41), one gets after a few arrangements

$$\begin{aligned}
e_j^{\alpha^{n+1}} &= \frac{1}{2}\|\mathbf{u}_j^{\alpha^n}\|^2 - \frac{1}{2}\|\mathbf{u}_j^{\alpha^{n+1}}\|^2 \\
&\quad + e_j^{\alpha^n} - \frac{\Delta t}{m_j^\alpha} \left( \sum_r p_j^{\alpha^n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} + \sum_r {}^t \mathbf{u}_r^{\alpha^n} A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) \right) \\
&\quad + \nu \frac{\Delta t}{m_j^\alpha} \left\{ \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_r^{\alpha^n} \mathbf{B}_{jr}^n \delta \mathbf{u}_r^{\alpha^n} - \omega \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_r^{\alpha^{n+1}} \mathbf{B}_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_r^{\alpha^{n+1}}) \right. \\
&\quad \left. + \omega \sum_r \rho_r^{\beta^n} {}^t \mathbf{u}_j^{\alpha^{n+1}} \mathbf{B}_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_r^{\alpha^{n+1}}) \right\}. \quad (43)
\end{aligned}$$

As a first step one estimates kinetic energy variation

$$-\Delta \mathcal{K}_j^\alpha = \frac{1}{2}\|\mathbf{u}_j^{\alpha^n}\|^2 - \frac{1}{2}\|\mathbf{u}_j^{\alpha^{n+1}}\|^2 = \frac{\mathbf{u}_j^{\alpha^n} + \mathbf{u}_j^{\alpha^{n+1}}}{2} \cdot (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_j^{\alpha^{n+1}}),$$

which rewrites using (40)

$$\begin{aligned}
-\Delta \mathcal{K}_j^\alpha &= \left( \mathbf{u}_j^{\alpha^n} - \frac{\Delta t}{2m_j^\alpha} \left[ \sum_r A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) + \omega \nu \sum_r \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right] \right) \\
&\quad \cdot \frac{\Delta t}{m_j^\alpha} \left[ \sum_r A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) + \omega \nu \sum_r \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right],
\end{aligned}$$

that is

$$\begin{aligned}
-\Delta \mathcal{K}_j^\alpha &= \frac{\Delta t}{m_j^\alpha} \left( \sum_r {}^t \mathbf{u}_j^{\alpha^n} A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) + \omega \nu \sum_r {}^t \mathbf{u}_j^{\alpha^n} \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right) \\
&\quad - \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \sum_r A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) + \omega \nu \sum_r \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right)^2.
\end{aligned}$$

So, one has

$$\begin{aligned}
e_j^{\alpha^{n+1}} &= e_j^{\alpha^n} + \frac{\Delta t}{m_j^\alpha} \left[ \sum_r {}^t (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) - \sum_r p_j^{\alpha^n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} \right. \\
&\quad \left. - \frac{\Delta t}{2m_j^\alpha} \left( \sum_r A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) + \omega \nu \sum_r \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right)^2 \right] \\
&\quad + \nu \frac{\Delta t}{m_j^\alpha} \left[ \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_r^{\alpha^n} \mathbf{B}_{jr}^n \delta \mathbf{u}_r^{\alpha^n} + \omega \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_j^{\alpha^{n+1}} \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right] \\
&\quad + \omega \nu \frac{\Delta t}{m_j^\alpha} \sum_r \rho_r^{\beta^n} {}^t (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_j^{\alpha^{n+1}}) \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}),
\end{aligned}$$

227 which using (38) is nothing but (42). □

Actually, (42) can be rewritten as

$$e_j^{\alpha n+1} = e_{h_j}^{\alpha n+1} + \nu \frac{\Delta t}{m_j^\alpha} \left[ \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha n} \mathbf{B}_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_j^{\alpha n+1} \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha n+1} - \delta \mathbf{u}_r^{\alpha n}) \right] + \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \omega \nu \sum_r \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha n+1} - \delta \mathbf{u}_r^{\alpha n}) \right)^2, \quad (44)$$

228 where  $e_{h_j}^{\alpha n+1}$  denotes the obtained internal energy without friction: *i.e.* injecting nodal velocities  
229  $\mathbf{u}_r^{\alpha n}$  into the classic mono-fluid scheme. The remaining terms can be viewed as the heating do to  
230 friction.

Since  $\omega \in ]0, 2]$ , using Corollary 1 allows to minorate  $e_j^{\alpha n+1}$

$$e_j^{\alpha n+1} \geq e_{h_j}^{\alpha n+1} + \nu \frac{\Delta t}{m_j^\alpha} \left[ \left(1 - \frac{\omega}{2}\right) \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha n} \mathbf{B}_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \frac{\omega}{2} \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_j^{\alpha n+1} \mathbf{B}_{jr}^n \delta \mathbf{u}_j^{\alpha n+1} \right] + \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \omega \nu \sum_r \rho_r^n \mathbf{B}_{jr}^n (\delta \mathbf{u}_j^{\alpha n+1} - \delta \mathbf{u}_r^{\alpha n}) \right)^2, \quad (45)$$

231 which implies  $e_j^{\alpha n+1} \geq e_{h_j}^{\alpha n+1}$ , since friction terms are positive.

**Property 9** (Positivity of internal energy). *Assuming that  $\forall \alpha \in \{f_1, f_2\}$ ,  $\forall j \in \mathcal{M}$ ,  $e_j^{\alpha n} > 0$ , there exists  $\Delta t^\epsilon > 0$  such that the scheme (34)–(38) ensures that*

$$\forall \omega \in [0, 2], \forall \Delta t \in ]0, \Delta t^\epsilon[, \forall \alpha \in \{f_1, f_2\}, \forall j \in \mathcal{M}, \quad e_j^{\alpha n+1} > 0.$$

232 *Proof.* The proof is obvious since  $e_j^{\alpha n+1} \geq e_{h_j}^{\alpha n+1}$  and since  $e_{h_j}^{\alpha n+1}(\Delta t)$  is a polynomial of degree 2  
233 satisfying  $e_{h_j}^{\alpha n+1}(0) = e_j^{\alpha n} > 0$ .  $\Delta t^\epsilon$  is nothing but the smallest root of these polynomial for each  
234 cells of each fluid.  $\square$

#### 235 4.3.3. Entropy stability for general equations of state

236 In the previous paragraph, we provided explicitly a choice of  $\Delta t > 0$  that ensures positivity  
237 of internal energy and density for the proposed scheme, but this is not sufficient for stability. In  
238 this section, we give an existence result of a strictly positive timestep  $\Delta t$  that ensures production  
239 of physical entropy for arbitrary physical equation of state.

Let  $U = (\tau, \mathbf{u}^T, E)^T$  and let  $\eta$  be the entropy of the fluid. Gibbs formula reads  $T d\eta = de + p d\tau$ . Following [23, 30], we estimate the entropy change, by means of a third-order Taylor expansion, due to the proposed scheme:

$$\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n}) = (U_j^{\alpha n+1} - U_j^{\alpha n})^T \frac{\partial \eta}{\partial U} \Big|_{U_j^{\alpha n}} + \frac{1}{2} (U_j^{\alpha n+1} - U_j^{\alpha n})^T \frac{\partial^2 \eta}{\partial U^2} \Big|_{U_j^{\alpha n}} (U_j^{\alpha n+1} - U_j^{\alpha n}) + \mathcal{O}((U_j^{\alpha n+1} - U_j^{\alpha n})^3).$$

One has  $\frac{\partial \eta}{\partial U} \Big|_{U_j^{\alpha^n}} = \frac{1}{T_j^{\alpha^n}} (p_j^{\alpha^n}, -\mathbf{u}_j^{\alpha^n}, 1)^T$  and the variable change  $V = (p, -\mathbf{u}, \eta)^T$  reads

$$(U_j^{\alpha^{n+1}} - U_j^{\alpha^n})^T \frac{\partial^2 \eta}{\partial U^2} \Big|_{U_j^{\alpha^n}} (U_j^{\alpha^{n+1}} - U_j^{\alpha^n}) = (V_j^{\alpha^{n+1}} - V_j^{\alpha^n})^T \frac{\partial^2 \eta}{\partial V^2} \Big|_{V_j^{\alpha^n}} (V_j^{\alpha^{n+1}} - V_j^{\alpha^n}) + \mathcal{O}((U_j^{\alpha^{n+1}} - U_j^{\alpha^n})^3),$$

where, see [28, 23] for instance,

$$\frac{\partial^2 \eta}{\partial V^2} \Big|_{V_j^{\alpha^n}} = -\frac{1}{T_j^{\alpha^n}} \begin{pmatrix} ((\rho c)_j^{\alpha^n})^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let  $O_1 := (U_j^{\alpha^{n+1}} - U_j^{\alpha^n})^T \frac{\partial \eta}{\partial U} \Big|_{U_j^{\alpha^n}}$ , using (34), (40) and (41), one gets

$$\begin{aligned} O_1 = \frac{1}{T_j^{\alpha^n}} \frac{\Delta t}{m_j^\alpha} & \left\{ p_j^{\alpha^n} \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} \right. \\ & - {}^t \mathbf{u}_j^{\alpha^n} \left( \sum_r A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) + \omega \nu \sum_r \rho_r^n B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right) \\ & - \sum_r \mathbf{C}_{jr}^n p_j^{\alpha^n} \cdot \mathbf{u}_r^{\alpha^n} + \sum_r {}^t \mathbf{u}_r^{\alpha^n} A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) \\ & + \nu \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha^n} B_{jr}^n \delta \mathbf{u}_r^{\alpha^n} - \omega \nu \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha^{n+1}} B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \\ & \left. + \omega \nu \sum_r \rho_r^n {}^t \mathbf{u}_j^{\alpha^{n+1}} B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right\}. \end{aligned}$$

which simplifies as

$$\begin{aligned} O_1 = \frac{1}{T_j^{\alpha^n}} \frac{\Delta t}{m_j^\alpha} & \left\{ \sum_r {}^t (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) \right. \\ & \left. + \nu \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha^n} B_{jr}^n \delta \mathbf{u}_r^{\alpha^n} - \omega \nu \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha^{n+1}} B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right\} \\ & + \frac{1}{T_j^{\alpha^n}} \frac{\Delta t}{m_j^\alpha} \left\{ \omega \nu \sum_r \rho_r^n {}^t (\mathbf{u}_j^{\alpha^{n+1}} - \mathbf{u}_j^{\alpha^n}) B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right\}. \end{aligned}$$

Now using Lemma 1, one gets

$$\begin{aligned} O_1 = \frac{1}{T_j^{\alpha^n}} \frac{\Delta t}{m_j^\alpha} & \left\{ \left(1 - \frac{1}{2} \omega\right) \nu \sum_r \rho_r^\beta {}^t \delta \mathbf{u}_r^{\alpha^n} B_{jr}^n \delta \mathbf{u}_r^{\alpha^n} + \frac{1}{2} \omega \nu \sum_r \rho_r^\beta {}^t \delta \mathbf{u}_j^{\alpha^{n+1}} B_{jr}^n \delta \mathbf{u}_j^{\alpha^{n+1}} \right\} \\ & + \frac{1}{T_j^{\alpha^n}} \frac{\Delta t}{m_j^\alpha} \left\{ \sum_r {}^t (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) + \omega \nu \sum_r \rho_r^{\beta n} {}^t (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right\} \\ & + \frac{1}{T_j^{\alpha^n}} \frac{\Delta t}{m_j^\alpha} \left\{ \omega \nu \sum_r \rho_r^n {}^t (\mathbf{u}_j^{\alpha^{n+1}} - \mathbf{u}_j^{\alpha^n}) B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right\}. \end{aligned}$$

240

Observe that later term is second-order in time, so that one retrieves as expected the entropy production of the continuous in time scheme established in Property 5 page 11.

241

One now focuses on the second-order term of the entropy variation

$$O_2 := \frac{1}{2}(V_j^{\alpha^{n+1}} - V_j^{\alpha^n})^T \frac{\partial^2 \eta}{\partial V^2} \Big|_{V_j^{\alpha^n}} (V_j^{\alpha^{n+1}} - V_j^{\alpha^n}),$$

which rewrites

$$O_2 = \frac{1}{2}(\Delta \Psi)^T \begin{pmatrix} ((\rho c)_j^{\alpha^n})^{-2} & 0 \\ 0 & 1 \end{pmatrix} \Delta \Psi, \quad \text{with } \Delta \Psi = \begin{pmatrix} p_j^{\alpha^{n+1}} - p_j^{\alpha^n} \\ -\mathbf{u}_j^{\alpha^{n+1}} + \mathbf{u}_j^{\alpha^n} \end{pmatrix}.$$

One has to estimate  $p_j^{\alpha^{n+1}} - p_j^{\alpha^n}$ . Assuming that the equation of state  $p : (\tau, e) \rightarrow p(\tau, e)$  is regular enough, one has

$$p_j^{\alpha^{n+1}} - p_j^{\alpha^n} = (\tau_j^{\alpha^{n+1}} - \tau_j^{\alpha^n}) \frac{\partial p}{\partial \tau} \Big|_{jn} + (e_j^{\alpha^{n+1}} - e_j^{\alpha^n}) \frac{\partial p}{\partial e} \Big|_{jn} + O(\Delta t^2).$$

Using (34) and (42) and keeping only first-order terms, one has

$$\begin{aligned} p_j^{\alpha^{n+1}} - p_j^{\alpha^n} &= \frac{\Delta t}{m_j^\alpha} \left( \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} \right) \frac{\partial p}{\partial \tau} \Big|_{jn} \\ &\quad + \frac{\Delta t}{m_j^\alpha} \left\{ \left( \sum_r {}^t(\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) - \sum_r p_j^{\alpha^n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} \right) \right. \\ &\quad \left. + \nu \left( \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_r^{\alpha^n} B_{jr}^n \delta \mathbf{u}_r^{\alpha^n} + \omega \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_j^{\alpha^{n+1}} B_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right) \right. \\ &\quad \left. + \omega \nu \left( \sum_r \rho_r^n {}^t (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_j^{\alpha^{n+1}}) B_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right) \right\} \frac{\partial p}{\partial e} \Big|_{jn} + O(\Delta t^2). \end{aligned}$$

Then, using (40), one gets

$$\begin{aligned} O_2 &= -\frac{1}{T_j^{\alpha^n}} \frac{\Delta t^2}{2m_j^{\alpha^2}} \left[ \left\{ \left( \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} \right) \frac{\partial p}{\partial \tau} \Big|_{jn} \right. \right. \\ &\quad \left. \left. + \left( \sum_r {}^t(\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) A_{jr}^{\alpha^n} (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_r^{\alpha^n}) - \sum_r p_j^{\alpha^n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha^n} \right) \right. \right. \\ &\quad \left. \left. + \nu \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_r^{\alpha^n} B_{jr}^n \delta \mathbf{u}_r^{\alpha^n} + \omega \nu \sum_r \rho_r^{\beta^n} {}^t \delta \mathbf{u}_j^{\alpha^{n+1}} B_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right. \right. \\ &\quad \left. \left. + \omega \nu \sum_r \rho_r^n {}^t (\mathbf{u}_j^{\alpha^n} - \mathbf{u}_j^{\alpha^{n+1}}) B_{jr}^n (\delta \mathbf{u}_j^{\alpha^{n+1}} - \delta \mathbf{u}_r^{\alpha^n}) \right\} \frac{\partial p}{\partial e} \Big|_{jn} \right]^2 ((\rho c)_j^{\alpha^n})^{-2} \\ &\quad + \left\{ \sum_r A_{jr}^{\alpha^n} (\mathbf{u}_r^{\alpha^n} - \mathbf{u}_j^{\alpha^n}) + \omega \nu \sum_r \rho_r^n B_{jr}^n (\delta \mathbf{u}_r^{\alpha^n} - \delta \mathbf{u}_j^{\alpha^{n+1}}) \right\}^2 \Big] + O(\Delta t^3). \end{aligned}$$

Finally, putting all the pieces together, one has

$$\begin{aligned} & \eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n}) \\ &= \frac{1}{T_j^{\alpha n}} \frac{\Delta t}{m_j^\alpha} \left\{ \left(1 - \frac{1}{2}\omega\right) \nu \sum_r \rho_r^{\beta t} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \frac{1}{2} \omega \nu \sum_r \rho_r^{\beta t} \delta \mathbf{u}_j^{\alpha n+1} B_{jr}^n \delta \mathbf{u}_j^{\alpha n+1} \right\} \\ & \quad + \frac{1}{T_j^{\alpha n}} \frac{\Delta t}{m_j^\alpha} \left( a - \frac{\Delta t}{m_j^\alpha} (b+c) + O(\Delta t^2) \right), \end{aligned}$$

242 with  $a \geq 0$  and  $b \geq 0$ .

243 Thus it remains to study the positiveness of  $a - \frac{\Delta t}{m_j^\alpha} (b+c) + O(\Delta t^2)$ . There are two possibilities.

*Case  $a > 0$ .* In that case, there obviously exists  $\Delta t > 0$  such that

$$\begin{aligned} T_j^{\alpha n} m_j^\alpha \frac{\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n})}{\Delta t} &\geq \left(1 - \frac{1}{2}\omega\right) \nu \sum_r \rho_r^{\beta t} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} \\ & \quad + \frac{1}{2} \omega \nu \sum_r \rho_r^{\beta t} \delta \mathbf{u}_j^{\alpha n+1} B_{jr}^n \delta \mathbf{u}_j^{\alpha n+1}. \end{aligned}$$

*Case  $a = 0$ .* If  $a = 0$ , one has

$$\sum_r {}^t(\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) + \omega \nu \sum_r \rho_r^{\beta n t} (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}) B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}) = 0.$$

Since  $\omega \geq 0$  and since  $A_{jr}^{\alpha, n}$  and  $B_{jr}^n$  are positive matrices, all the terms in the sum are zeros. Let us first focus on  ${}^t(\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) = 0$  terms. Two cases occur. In case of Euclhyd scheme,  $A_{jr}^{\alpha, n}$  is positive definite so that one has  $\mathbf{u}_r^{\alpha n} = \mathbf{u}_j^{\alpha n}$ . For Glace scheme

$${}^t(\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) = \frac{(\rho c)_j^{\alpha n}}{\|\mathbf{C}_{jr}^n\|} \|\mathbf{C}_{jr}^n \cdot (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n})\|^2 = 0.$$

244 So, for both scheme, one has  $\mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} = \mathbf{C}_{jr}^n \cdot \mathbf{u}_j^{\alpha n}$  and  $A_{jr}^{\alpha, n} (\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) = \mathbf{0}$ . Recalling that  
245  $\sum_r \mathbf{C}_{jr}^n = \mathbf{0}$ , one also has  $\sum_r \rho_j^{\alpha n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} = 0$ .

246 One now analyzes  $\omega (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1})^T B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1}) = 0$ . Here again two cases occur  
247  $\omega = 0$  or  $\omega > 0$ . In that second case, since  $B_{jr}^n$  are positive definite, this implies  $\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1} = \mathbf{0}$ .  
Finally, if  $a = 0$ , one has

$$\begin{aligned} T_j^{\alpha n} m_j^\alpha \frac{\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n})}{\Delta t} &= \nu \sum_r \rho_r^{\beta t} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} \\ & \quad - \frac{\Delta t}{2m_j^\alpha} \left( \nu \sum_r \rho_r^{\beta n t} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} \right)^2 \frac{\partial p}{\partial e} \Big|_{jn} (\rho c)_j^{\alpha n}{}^{-2} + O(\Delta t^2). \end{aligned}$$

Before enunciating the result, one should remark that in the general case, one has

$$\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n}) = \frac{1}{T_j^{\alpha n}} \frac{\Delta t}{m_j^\alpha} \left( (a + a_\nu) - \frac{\Delta t}{m_j^\alpha} (b+c) + O(\Delta t^2) \right),$$



248 with  $a \geq 0$ ,  $a_\nu \geq 0$  and  $b \geq 0$ . Again, one has two alternatives  $a + a_\nu > 0$  or  $a + a_\nu = 0$ . In the  
 249 first case, there exists  $\Delta t$  such that  $\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n}) > 0$ . In the second case, one has  $a = a_\nu = 0$   
 250 so as previously,  $a = 0 \implies \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} = \mathbf{C}_{jr}^n \cdot \mathbf{u}_j^{\alpha n}$ ,  $A_{jr}^{\alpha, n}(\mathbf{u}_r^{\alpha n} - \mathbf{u}_j^{\alpha n}) = \mathbf{0}$  and  $\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1} = \mathbf{0}$ .  
 251 Also, since  $a_\nu = 0$  and since  $B_{jr}^n$  is positive definite one has  $\delta \mathbf{u}_r^{\alpha n} = \mathbf{0}$ , so the scheme (34)–(36)  
 252 gives  $U_j^{\alpha n+1} = U_j^{\alpha n}$  and finally one has  $\forall \Delta t > 0$ ,  $\eta(U_j^{\alpha n+1}) = \eta(U_j^{\alpha n})$ .

253 The obtained results are summarized in the following Property.

254 **Property 10** (Entropy). Let  $U := (\tau, \mathbf{u}^T, E)^T$  and let  $\eta$  the entropy. There exists  $\Delta t^n > 0$ , such  
 255 that  $\forall \alpha, \beta \in \{f_1, f_2\}$ , such that  $\alpha \neq \beta$ , if the pressure law  $p^\alpha : (\rho, e) \rightarrow p^\alpha(\rho, e)$  is a differentiable  
 256 function, then the scheme (34)–(38) defined by  $\Delta t = \Delta t^n$  and  $\forall \omega \in [0, 2]$ , ensures that,

1. the scheme is entropy stable:

$$\forall j \in \mathcal{M}, \quad \eta(U_j^{\alpha n+1}) \geq \eta(U_j^{\alpha n}),$$

2. and  $\forall j \in \mathcal{M}$ , one has the following alternative. If  $\forall r \in \mathcal{R}_j$ ,  $\mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} = \mathbf{C}_{jr}^n \cdot \mathbf{u}_j^{\alpha n}$  and  
 $\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1} = \mathbf{0}$ , then

$$T_j^{\alpha n} m_j^\alpha \frac{\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n})}{\Delta t} \geq \nu \sum_r \rho_r^{\beta n} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \mathcal{O}(\Delta t),$$

else

$$T_j^{\alpha n} m_j^\alpha \frac{\eta(U_j^{\alpha n+1}) - \eta(U_j^{\alpha n})}{\Delta t} \geq \nu \sum_r \rho_r^{\beta n} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n}.$$

257

258 **Remark 3.** Let us comment point 2 of property 10. Actually, this is a consistency result with  
 259 regard to (2). In the first case (if  $\forall r \in \mathcal{R}_j$ ,  $\mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} = \mathbf{C}_{jr}^n \cdot \mathbf{u}_j^{\alpha n}$  and  $\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n+1} = \mathbf{0}$ ), the scheme  
 260 gives following values  $\rho_j^{\alpha n+1} = \rho_j^{\alpha n}$ ,  $\mathbf{u}_j^{\alpha n+1} = \mathbf{u}_j^{\alpha n}$  and  $e_j^{\alpha n+1} = e_j^{\alpha n} + \frac{\Delta t}{m_j^\alpha} \nu \sum_r \rho_r^{\beta n} \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n}$ . In  
 261 this case, the scheme acts simply as a first-order ODE solver. Since then  $d\eta = de$  and since  $\eta$  is  
 262 strictly convex, a time integration error is to be expected.

263 To sum up, we proved that the proposed scheme is stable, meaning that there exists  $0 < \Delta t \leq$   
 264  $\min(\Delta t^\rho, \Delta t^e, \Delta t^n)$  such that the scheme is entropy stable and preserves positivity of density and  
 265 internal energy. Moreover, it is consistent with (2).

#### 266 4.3.4. Minoration of $\Delta t^n$

267 As stated before, to prove that the scheme is asymptotic preserving, it remains to show that  
 268  $\lim_{\nu \rightarrow +\infty} \Delta t^n \neq 0$ . Even if we will not provide here an explicit value, we will give a lower bound  
 269 independent of  $\nu$ .

**Property 11.** Let  $\omega \in ]0, 2]$ .  $\forall j \in \mathcal{M}$ , let  $(\tau_j^n, \mathbf{u}_j^{nT}, E_j^n)^T$  denote the initial state of fluid  $\alpha \in$   
 $\{f_1, f_2\}$ . Let  $\{\mathbf{u}_r\}_{r \in \mathcal{R}_j}$  be an arbitrary set of nodal velocities (or velocity fluxes). Then, if  $\forall \nu \geq 0$ ,  
 $(\tau_j^{\nu, n+1}, e_j^{\nu, n+1})$  denotes the thermodynamic state obtained by scheme (34)–(36), one has

$$\eta(\tau_j^{\nu, n+1}, e_j^{\nu, n+1}) \geq \eta(\tau_j^{0, n+1}, e_j^{0, n+1}),$$

270 where  $\eta := \eta(\tau, e)$  is the physical entropy expressed according to the independent variables  $\tau$   
 271 and  $e$ .

272 *Proof.* Gibbs formula reads  $\nabla_{\tau,e}\eta = \frac{1}{T}\left(\frac{p}{1}\right)$ , where  $T := T(\tau, e)$  is a positive function. So, for any  
 273  $\tau, \eta(\tau, \cdot)$  is an increasing function.

Since (34) is independent of  $\nu$  and according to (45), one has

$$\forall \{\mathbf{u}_r\}_{r \in \mathcal{R}_j}, \forall \nu \geq 0, \forall \Delta t \quad \tau_j^{v,n+1} = \tau_j^{0,n+1} \quad \text{and} \quad e_j^{v,n+1} \geq e_j^{0,n+1},$$

so

$$\forall \{\mathbf{u}_r\}_{r \in \mathcal{R}_j}, \forall \nu \geq 0, \forall \Delta t \quad \eta\left(e_j^{v,n+1}, \tau_j^{v,n+1}\right) \geq \eta\left(e_j^{0,n+1}, \tau_j^{0,n+1}\right).$$

274

□

275 **Remark 4.** Property 11 establishes that, for a given set of velocities  $\{\mathbf{u}_r\}_{r \in \mathcal{R}_j}$ , the maximum  
 276 timestep required for the scheme to be entropy stable is greater than the mono-fluid timestep  
 277 independently of  $\nu$ .

278 However, one emphasises that velocities  $\{\mathbf{u}_r\}_{r \in \mathcal{R}_j}$  are actually functions of  $\nu$  as expressed  
 279 in (38). It turns out that entropy stability timestep depends on  $\nu$  though the velocity fluxes and  
 280 can be either bigger or smaller than the mono-fluid timestep.

281 **Example 1** ( $\Delta t^{n0} > \Delta t^{nv}$ ). Let us consider two fluides in the monodimensional  $]0, 1[$  domain. Let  
 282 the first fluid  $\alpha$  be a very light fluid at rest  $\rho^\alpha = \epsilon$  with  $0 < \epsilon \ll 1$ ,  $\mathbf{u}^\alpha = \mathbf{0}$  and  $e^\alpha$  being set such  
 283 that the sound speed  $c^\alpha = 1$ . Let the second fluid  $\beta$  be the initial state of a Sod shock tube.

284 If  $\nu = 0$ , one has obviously  $\forall r, \mathbf{u}_r^\alpha = \mathbf{0}$ . So, (34)–(36) implies  $U_j^{\alpha n+1} = U_j^{\alpha n}$ , which is  
 285 unconditionally stable.

286 Choosing  $\nu \gg 1$  and solving (38) implies that  $\delta \mathbf{u}_r^\alpha$  is arbitrary small, and we write  $\mathbf{u}_r = \mathbf{u}_r^\alpha =$   
 287  $\mathbf{u}_r^\beta$ , and the sum equations (38) can be rewritten has

$$\sum_j (A_{jr}^\alpha + A_{jr}^\beta) \mathbf{u}_r = \sum_j \mathbf{C}_{jr} p_j^\beta, \quad (46)$$

288 since  $\mathbf{u}^\alpha = \mathbf{u}^\beta = \mathbf{0}$  and  $p^\alpha$  is constant (recalling that  $\sum_j \mathbf{C}_{jr} = \mathbf{0}$ ). Since  $\rho^\alpha c^\alpha = \epsilon$ ,  $A_{jr}^\alpha$  is  
 289 neglectable with regard to  $A_{jr}^\beta$ . So, one has  $\sum_j A_{jr}^\beta \mathbf{u}_r = \sum_j \mathbf{C}_{jr} p_j^\beta$ , that is the timestep for fluid  $\alpha$   
 290 is the same as one imposed by the fluid  $\beta$  which is much smaller than the arbitrary one obtained  
 291 in the mono-fluid case  $\nu = 0$ .

292 **Example 2** ( $\Delta t^{n0} < \Delta t^{nv}$ ). Let us now consider a similar example. The two fluids in  $]0, 1[$  are  
 293 now described as follows. Let fluid  $\alpha$  be in the initial state of a Sod shock tube. Fluid  $\beta$  is this  
 294 time a very heavy fluid at rest:  $\rho^\beta = \frac{1}{\epsilon}$  with  $0 < \epsilon \ll 1$ ,  $\mathbf{u}^\beta = \mathbf{0}$  and  $e^\beta$  being set such that the  
 295 sound speed  $c^\beta = 1$ .

296 If  $\nu = 0$ , stability for the fluid  $\alpha$  is the one of the mono-fluid Sod shock tube.

297 Choosing  $\nu \gg 1$ , (46) holds again. Since,  $\rho^\beta = \frac{1}{\epsilon}$  is very large, one gets in the limit  $\mathbf{u}_r = \mathbf{0}$ ,  
 298 which provides unconditionally stability for both fluids.

#### 299 4.3.5. On the importance of the implicit velocities in (34)–(36)

Using the notations defined in section 4.2, let us consider the fully explicit scheme that consists in replacing momentum and total energy updates in (34)–(38) by their explicit counterparts

$$m_j^\alpha \frac{\mathbf{u}_j^{\alpha n+1} - \mathbf{u}_j^{\alpha n}}{\Delta t} = - \sum_r \mathbf{F}_{jr}^{\alpha, n} - \omega \sum_r \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_j^{\alpha n} - (1 - \omega) \sum_r \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha n},$$

$$m_j^\alpha \frac{E_j^{\alpha n+1} - E_j^{\alpha n}}{\Delta t} = - \sum_r \mathbf{F}_{jr}^{\alpha, n} \cdot \mathbf{u}_r^{\alpha n} - \sum_r \nu \rho_r^n {}^t \bar{\mathbf{u}}_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \sum_r \nu \rho_r^n {}^t \bar{\mathbf{u}}_r^n B_{jr}^n (\delta \mathbf{u}_r^{\alpha n} - \delta \mathbf{u}_j^{\alpha n}).$$

Using this scheme, one easily checks that internal energy variation reads

$$\begin{aligned}
e_j^{\alpha n+1} = e_j^{\alpha n} + \frac{\Delta t}{m_j^\alpha} & \left[ \sum_r {}^t(\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) A_{jr}^{\alpha n} (\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) - \sum_r \rho_j^{\alpha n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha n} \right] \\
& + \nu \frac{\Delta t}{m_j^\alpha} \left[ \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_j^{\alpha n} B_{jr}^n (\delta \mathbf{u}_j^{\alpha n} - \delta \mathbf{u}_r^{\alpha n}) \right] \\
& - \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \sum_r A_{jr}^{\alpha n} (\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) + \omega \nu \sum_r \rho_r^{\beta n} B_{jr}^n (\delta \mathbf{u}_j^{\alpha n} - \delta \mathbf{u}_r^{\alpha n}) \right)^2.
\end{aligned}$$

that is

$$\begin{aligned}
e_j^{\alpha n+1} = e_{h_j}^{\alpha n+1} + \nu \frac{\Delta t}{m_j^\alpha} & \left[ \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_r^{\alpha n} B_{jr}^n \delta \mathbf{u}_r^{\alpha n} + \omega \sum_r \rho_r^{\beta n} {}^t \delta \mathbf{u}_j^{\alpha n} B_{jr}^n (\delta \mathbf{u}_j^{\alpha n} - \delta \mathbf{u}_r^{\alpha n}) \right] \\
& - \frac{\Delta t^2}{m_j^{\alpha 2}} \left( \omega \nu \sum_r \rho_r^{\beta n} B_{jr}^n (\delta \mathbf{u}_j^{\alpha n} - \delta \mathbf{u}_r^{\alpha n}) \right) \cdot \left( \sum_r A_{jr}^{\alpha n} (\mathbf{u}_j^{\alpha n} - \mathbf{u}_r^{\alpha n}) \right) \\
& - \frac{\Delta t^2}{2m_j^{\alpha 2}} \left( \omega \nu \sum_r \rho_r^{\beta n} B_{jr}^n (\delta \mathbf{u}_j^{\alpha n} - \delta \mathbf{u}_r^{\alpha n}) \right)^2,
\end{aligned}$$

300 where  $e_{h_j}^{\alpha n+1}$  still denotes the obtained internal energy without friction. The later term being a  
301 negative factor of  $\nu^2$ , in the explicit case,  $\forall \Delta t > 0$  for large values of  $\nu$ , one can have  $e_j^{\alpha n+1} <$   
302  $e_{h_j}^{\alpha n+1}$ . So even if a similar result to Property 10 can be established (existence of an entropy  
303 stable timestep), one cannot prove an equivalent of Property 11. If cell velocities are explicit,  
304 one eventually gets  $\lim_{\nu \rightarrow +\infty} \Delta t^\epsilon = \lim_{\nu \rightarrow +\infty} \Delta t^\eta = 0$  for a given set of nodal velocities  $\{\mathbf{u}_r\}_{r \in \mathcal{R}_j}$ .

## 305 5. ALE scheme

306 The semi-Lagrangian scheme presented in this paper is defined assuming that both fluid  
307 meshes are identical at the beginning of the timestep. One understands easily that this is of huge  
308 help in the construction of an asymptotic preserving scheme. One could imagine a purely La-  
309 grangian approach, but even dealing with a non-AP approach seems very difficult since one  
310 would have to consider meshes intersections and complex geometrical calculations.

311 Thus, the algorithm, we propose in this paper, consists in ensuring that for each timestep both  
312 fluids meshes coincide. To do so an ALE formulation is mandatory.

313 Figure 2 depicts the general ALE case. Our ALE method is a Lagrange-rezoning-advection  
314 procedure which ensures that the solution is defined at time  $t^{n+1}$  on a unique mesh.

- 315 • At time  $t^n$  solutions are discretized on the meshes  $\mathcal{M}_\alpha^n = \mathcal{M}_\beta^n$
- 316 • In a first step (Lagrangian phase), each mesh evolves in a different way  $\tilde{\mathcal{M}}_\alpha^{n+1} \neq \tilde{\mathcal{M}}_\beta^{n+1}$ .  
317 Each mesh being defined by  $\tilde{\mathbf{x}}_r^{\alpha, n+1} = \mathbf{x}_r^n + \Delta t \mathbf{u}_r^{\alpha, n}$ .
- 318 • Then the meshes are smoothed in a way to obtain new meshes such that  $\mathcal{M}_\alpha^{n+1} \equiv \mathcal{M}_\beta^{n+1}$ . For  
319 each fluid  $\alpha$ , it allows to define an arbitrary velocity  $\mathbf{v}_r^{\alpha, n+1}$  such that  $\mathbf{x}_r^{n+1} = \tilde{\mathbf{x}}_r^{\alpha, n+1} + \Delta t \mathbf{v}_r^{\alpha, n+1}$ .

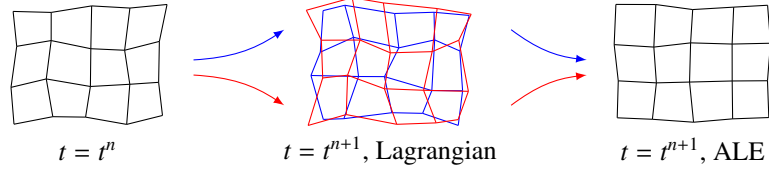


Figure 2: **Left**: at time  $t = t^n$ , both fluid share the same mesh. **Middle**: at the end of the Lagrangian phase, one gets two different meshes, one for each fluid. **Right**: meshes are displaced so that they coincide. Solution is remapped and a new timestep can be performed.

- Finally, for both fluids, the numerical solution is computed on the common mesh by remapping the conservative variables  $(\rho^\alpha, \rho^\alpha \mathbf{u}^\alpha, \rho^\alpha E^\alpha)^T$  at velocity  $-\mathbf{v}_r^{\alpha, n+1}$ , with a second-order accurate scheme. One can then compute another timestep.

## 6. Numerical tests

### 6.1. Reference scheme

We introduce the following scheme that will be used as a “non-AP” scheme to illustrate the advantages of scheme (22)–(25). For each fluid  $\alpha \in \{f_1, f_2\}$ , one writes

$$\begin{aligned}
 d_t m_j^\alpha &= 0, \\
 m_j^\alpha d_t \tau_j^\alpha &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^\alpha, \\
 m_j^\alpha d_t \mathbf{u}_j^\alpha &= - \sum_r \mathbf{F}_{jr}^\alpha - \sum_r \nu \rho_r B_{jr} \delta \mathbf{u}_j^\alpha, \\
 m_j^\alpha d_t e_j^\alpha &= - \sum_r \mathbf{F}_{jr}^\alpha \cdot \mathbf{u}_r^\alpha - \sum_r \nu \rho_r \bar{\mathbf{u}}_{jr}^T B_{jr} \delta \mathbf{u}_j^\alpha,
 \end{aligned} \tag{47}$$

where  $\bar{\mathbf{u}}_{jr}$  and  $\rho_r$  are defined as in section 4.1 and the fluxes are given by

$$\begin{aligned}
 \mathbf{F}_{jr}^\alpha &= \mathbf{C}_{jr} p_j^\alpha - A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) \\
 \sum_j A_{jr}^\alpha \mathbf{u}_r^\alpha &= \sum_j A_{jr}^\alpha \mathbf{u}_j^\alpha + \sum_j \mathbf{C}_{jr} D_j^\alpha.
 \end{aligned} \tag{48}$$

It can be showed that this scheme is entropic, conservative in volume and mass for each fluid and in the sum of momentums and total energies. Also, the scheme is weakly consistent with (1). One can moreover show that its associated discrete in time scheme, where only  $\mathbf{u}_j$  terms are implicit, is stable in the same way as scheme (34)–(38).

However, this scheme does not *a priori* preserve the asymptotic. For these reasons this scheme is a very good candidate for the comparisons that we perform in this section.

### 6.2. Tests conditions

In all the following tests, we choose  $A_{jr}^\alpha = \rho_j^\alpha c_j^\alpha \sum_i \frac{N_{jr}^i \otimes N_{jr}^i}{\|N_{jr}^i\|}$  (Euclhyd scheme) and  $B_{jr} = V_{jr} I_2$ , with  $V_{jr} = \frac{1}{\#\mathcal{R}_j} V_j$ . Also, for each test one chooses  $\gamma^\alpha = \gamma^\beta = 1.4$ .

337 Results are compared with the non-AP scheme (47)–(48). Also for the 2D tests, we compare  
 338 our results ( $\nu \gg 1$ ) to the mono-fluid case, where mass fraction  $\frac{\rho^\alpha}{\rho^\alpha + \rho^\beta}$  is treated as a passive  
 339 scalar.

340 As it is often the case for multi-velocity models [2], the scheme is only defined in regions  
 341 where both fluids are present. Thus in regions where a fluid should be absent, one keeps a  
 342 neglectable amount of it.

### 343 6.3. Sod shock tube

The computational domain we consider is  $\Omega := ]0, 1[ \times ]0, 0.1[$ . Initial data is given as  $U :=$   
 $(\rho, \mathbf{u}, p)^T$ , so that one defines  $U^L := (1, \mathbf{0}, 1)^T$ ,  $U^R := (0.125, \mathbf{0}, 0.1)^T$  and  $U^\epsilon := (\epsilon, \mathbf{0}, \epsilon)^T$ . For  
 both fluids initial states are then

$$U^\alpha = \mathbf{1}_{]0, 0.5[}(U^L - U^\epsilon) + \mathbf{1}_{]0.5, 1[}U^\epsilon \quad \text{and} \quad U^\beta = \mathbf{1}_{]0, 0.5[}U^\epsilon + \mathbf{1}_{]0.5, 1[}(U^R - U^\epsilon),$$

344 where  $\mathbf{1}_O$  denotes the characteristic function of the set  $O$  and where we take  $\epsilon = 10^{-3}$ .

345 Choosing  $\omega = 1$ , one figure 3, we compare the solution at time  $t = 0.14$  obtained by the  
 346 proposed scheme (34)–(38) to the reference scheme (47)–(48). One plots the density sum:  $\rho^\alpha + \rho^\beta$ .  
 347 The grid is  $200 \times 3$  cells and the solution is compared to a reference solution obtained using  
 348 a  $10^5 \times 3$  grid. The simulation is Eulerian: the smoothing strategy consists in a return to the  
 initial mesh.

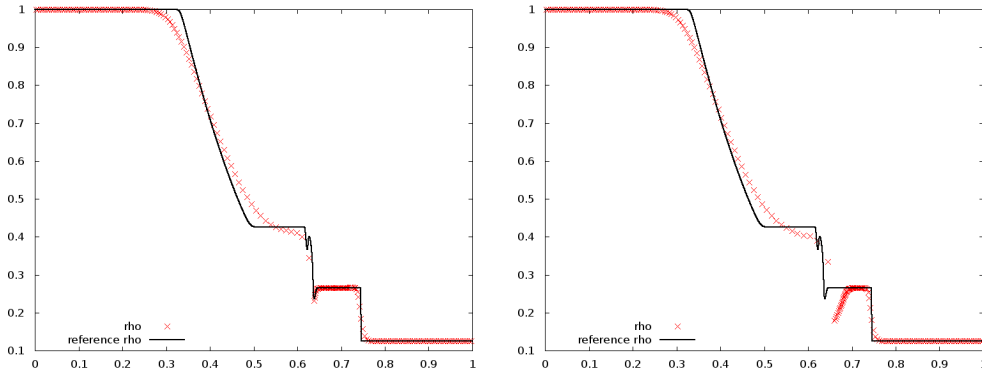


Figure 3:  $\nu = 100$ .  $\rho^\alpha + \rho^\beta$  profile. AP-scheme (left) gives a much better solution than the non-AP scheme (right).

349 The same test is performed for a friction parameter  $\nu = 10^6$ . The density sum is presented in  
 350 figure 4 at time  $t = 0.14$ .

351 One retrieves the results presented in [15], even if the scheme does not degenerate in 1D to  
 352 the scheme proposed in [15].

353 Finally we show the importance of choosing  $\omega \in ]0, 2]$  by comparing the velocity differences  
 354 in the cases  $\omega = 0$  and  $\omega = 1$  for  $\nu = 10^6$  using the scheme (34)–(38). One will note the oscil-  
 355 lations obtained for the choice  $\omega = 0$ , for which we could not show the asymptotic preserving  
 356 property.  
 357

### 358 6.4. Triple-point problem

359 The triple-point problem is a standard benchmark [31]. It is a multi-dimensional Riemann  
 360 problem whose data are close to the Sod shock tube. The self-similarity of the problem yields an

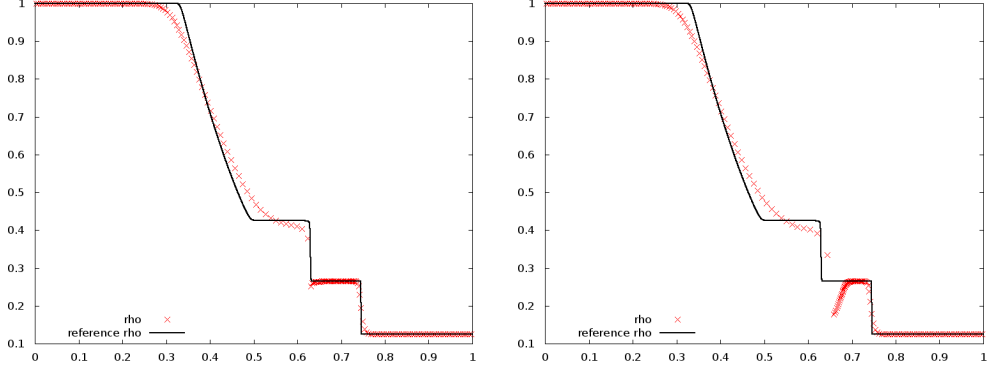


Figure 4:  $\nu = 10^6$ .  $\rho^\alpha + \rho^\beta$  profile. AP-scheme (left) gives a much better solution than the non-AP scheme (right). The expected solution is close to the classical mono-fluid case.

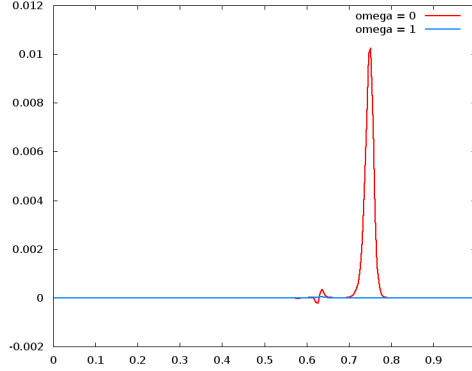


Figure 5:  $\nu = 10^6$ . Comparison of the  $\delta \mathbf{u}_j$  obtained for  $\omega = 0$  (red) and  $\omega = 1$  (blue) at time  $t = 1.4$  for a  $200 \times 3$  grid.

361 infinitely rolling vortex, the quantity of the details generated by the secondary Kelvin-Helmholtz  
 362 instabilities depends only on the numerical dissipation of the scheme. Figure 6 depicts the initial  
 363 geometry and the initial three states.

Let us define  $\rho^L = 1$ ,  $\rho^l = 0.125$ ,  $p^L = 1$  and  $p^l = 0.1$ . Also,  $\Omega_1 = ]0, 1[ \times ]0, 3[$ ,  $\Omega_2 = ]1, 7[ \times ]0, 1.5[$  and  $\Omega_3 = ]1, 7[ \times ]1.5, 3[$ . This allows to define the initial states of both fluids:

$$U^\alpha = \mathbf{1}_{\Omega_2} \begin{pmatrix} \rho^L - \varepsilon \\ \mathbf{0} \\ p^L - \varepsilon \end{pmatrix} + \mathbf{1}_{\Omega_1 \cup \Omega_3} \begin{pmatrix} \varepsilon \\ \mathbf{0} \\ \varepsilon \end{pmatrix} \quad \text{and} \quad U^\beta = \mathbf{1}_{\Omega_1} \begin{pmatrix} \rho^L - \varepsilon \\ \mathbf{0} \\ p^L - \varepsilon \end{pmatrix} + \mathbf{1}_{\Omega_3} \begin{pmatrix} \rho^l - \varepsilon \\ \mathbf{0} \\ p^l - \varepsilon \end{pmatrix} + \mathbf{1}_{\Omega_2} \begin{pmatrix} \varepsilon \\ \mathbf{0} \\ \varepsilon \end{pmatrix}.$$

364 Symmetry boundary conditions are set at each straight boundary of the computational domain.

365 The ALE strategy we use for this test consists in a barycentric smoother for the grid of the  
 366 fluid  $\alpha$  and then to impose  $\mathbf{x}_r^\beta = \mathbf{x}_r^\alpha$ .

367 We run the test on a  $91 \times 40$  grid and setting  $\omega = 1$ . Choosing the friction parameter  $\nu =$   
 368  $10^6$ , we compare the obtained result to the solution of the mono-fluid solver and to the non-AP  
 369 scheme, see figure 7. For the comparison, we plot the mass fraction in each case:  $\frac{\rho^\alpha}{\rho^\alpha + \rho^\beta}$ . One

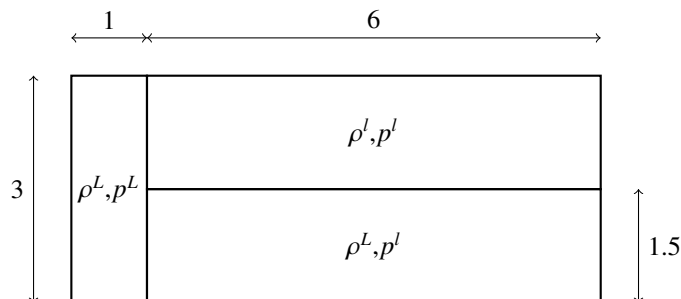


Figure 6: Geometry, pressures and densities for the triple-point problem at time  $t = 0$ .

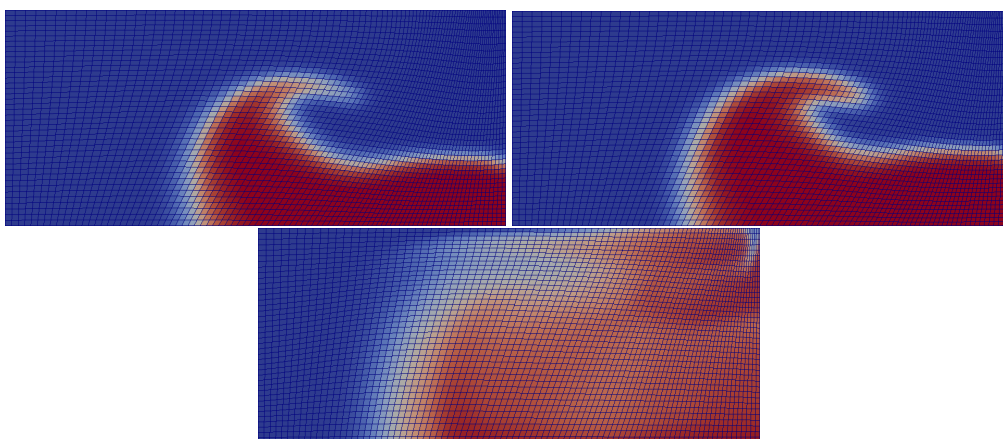


Figure 7:  $91 \times 40$  mesh. Mass fraction of fluid  $\alpha$  at time  $t = 5$ . **Left:** mono-fluid solution. **Right:** bi-fluid solution with  $\nu = 10^6$ . **Bottom:** bi-fluid solution with non-AP scheme with  $\nu = 10^6$ .

370 notices the nice agreement of the solution for the proposed scheme with regard to the mono-fluid  
 371 case, even for this small amount of cells, whereas the non-AP scheme is not even able to compute  
 372 the large structures of the flow at this grid resolution.

373 Then we study the effect of the friction parameter. Figure 8 presents the obtained solutions,  
 374 on a finer  $210 \times 90$  grid, for  $\nu \in \{10, 100, 10^6\}$ .

### 375 6.5. A Rayleigh Taylor instability

376 For this test, we modify the scheme in order to incorporate the gravity treatment. Obviously,  
 377 we use a well-balanced approach [32] to take this term into account. For this modified scheme,  
 378 the properties we established for (34)–(39) remain true. We did not take the gravity term into  
 379 account in section 4 to avoid a more complex presentation, since there is no additional difficulties  
 380 to overcome.

381 The interface perturbation is defined by the function  $f(y) = 0.05 \cos(8\pi y)$  and centered at  
 382  $x = 0.35$  in the computational domain  $\Omega = ]0, 0.7[ \times ]0, 0.25[$ . Thus, two regions are defined:  
 383  $\Omega_\alpha = \{(x, y) \in \Omega / x < 0.35 + f(y)\}$  and  $\Omega_\beta = \Omega \setminus \Omega_\alpha$ .

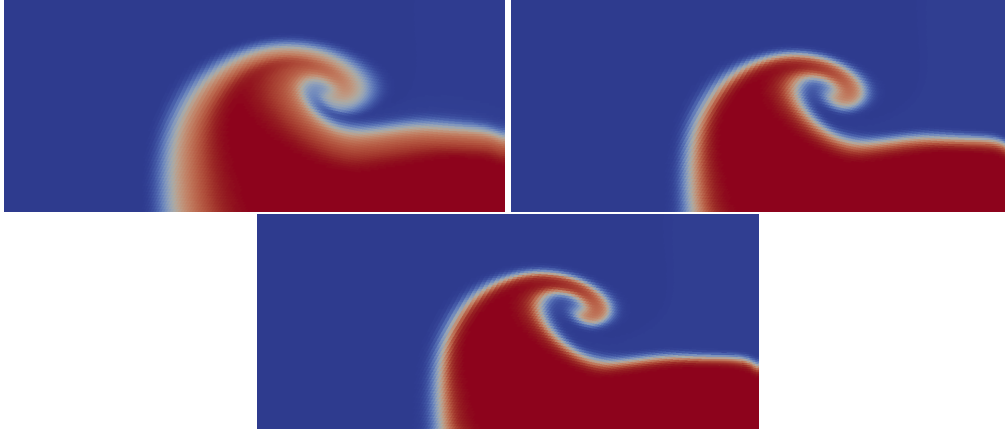


Figure 8:  $210 \times 90$  mesh. Time  $t = 5$ . Mass fraction of fluid  $\alpha$ . Effect of the friction parameter  $\nu$ . **Left:**  $\nu = 10$ . **Right:**  $\nu = 100$ . **Bottom:**  $\nu = 10^6$ .

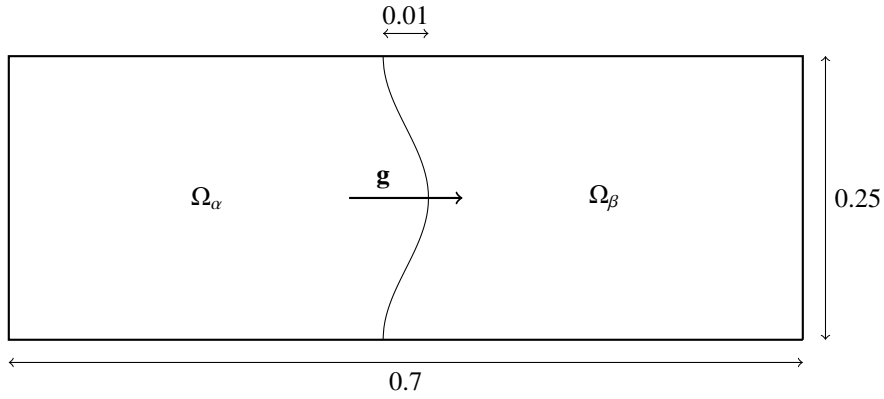


Figure 9: Rayleigh-Taylor test initial geometry. Fluid  $\alpha$  being heavier than fluid  $\beta$ , instability will grow.

Initially, velocities are set to  $\mathbf{0}$  in  $\Omega$ , and densities are defined as

$$\rho^\alpha = \mathbf{1}_{\Omega_\alpha}(0.8 - \varepsilon) + \mathbf{1}_{\Omega_\beta}\varepsilon, \quad \text{and} \quad \rho^\beta = \mathbf{1}_{\Omega_\alpha}\varepsilon + \mathbf{1}_{\Omega_\beta}(0.25 - \varepsilon).$$

Choosing the gravity acceleration as  $\mathbf{g} = 9.8 \mathbf{e}_x$ , we define the pressure in the whole domain at a quasi-equilibrium state (omitting the  $y$  dependency), that is

$$p(x) = \int_0^x (\rho^\alpha + \rho^\beta) \mathbf{g} \cdot \mathbf{e}_x.$$

384 Again, symmetry boundary conditions are imposed all over  $\partial\Omega$ . We represent the mass fraction  
 385 of fluid  $\alpha$  that is  $\frac{\rho^\alpha}{\rho^\alpha + \rho^\beta}$ . One takes  $\omega = 1$ . We use the same ALE strategy as in the previous test:  
 386 a barycentric remapping is performed on the mesh of fluid  $\alpha$  and we set  $\mathcal{M}_\beta^{n+1} = \mathcal{M}_\alpha^{n+1}$  to allow  
 387 the calculation of timestep  $n + 1$ .



388 At first, we validate the approach by comparing the obtained result to the mono-fluid scheme.  
 389 The results are presented on figure 6.5, one observes again a very good agreement even on a  $112 \times 40$  coarse grid. As expected, the non-AP scheme clearly shows lack of convergence.

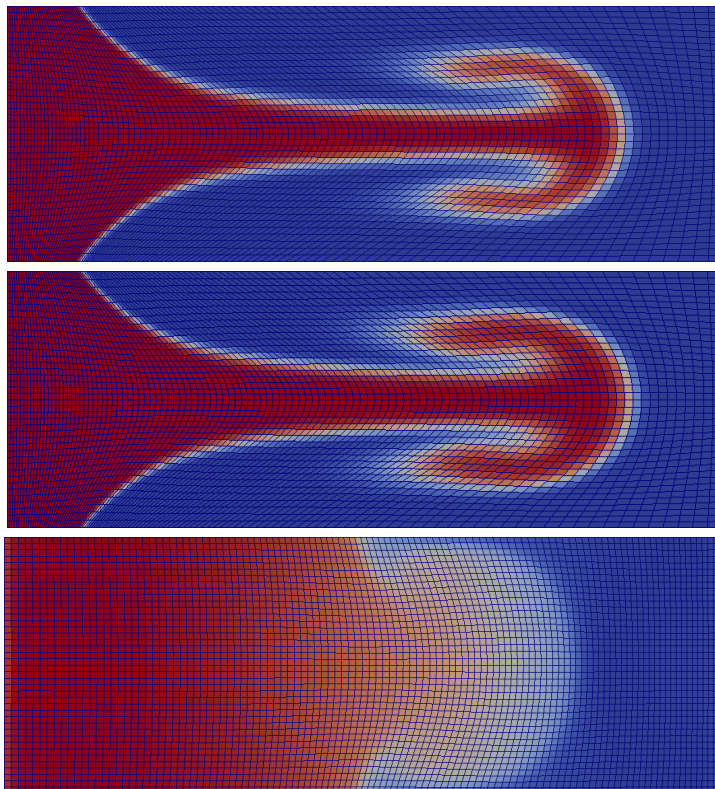


Figure 10:  $112 \times 40$  mesh. Mass fraction of fluid  $\alpha$ . Time  $t = 0.7s$ . **Top**: mono-fluid solution. **Middle**: bi-fluid solution with  $\nu = 10^6$ . **Bottom**: bi-fluid solution with non-AP scheme with  $\nu = 10^6$ .

390 Finally, we study the influence of the friction parameter  $\nu$  for successive values of 100, 1000  
 391 and  $10^6$ . A slightly finer grid ( $224 \times 80$ ) is used for it.  
 392

## 393 7. Conclusion

394 In this paper, we presented a multi-dimensionnal asymptotic preserving scheme to solve a  
 395 bi-fluid model defined as a set of two Euler systems coupled with a friction term. The originality  
 396 of the approach is that the scheme is ALE: the only constrain being that meshes must coincide at  
 397 the beginning of each timestep.

398 The scheme is conservative and weakly-consistent by construction. Moreover, we showed  
 399 that it is at least as stable as the underlying hydro-scheme in the sense that the timestep required  
 400 to increase entropy does not tend to zero when friction increases. We showed consistency of  
 401 the limit scheme ( $\nu \rightarrow +\infty$ ) to the limit model. So, we proved that the scheme is asymptotic  
 402 preserving. On the way we proved some stability results with regard to the fluxes  $\mathbf{u}_r$ , which

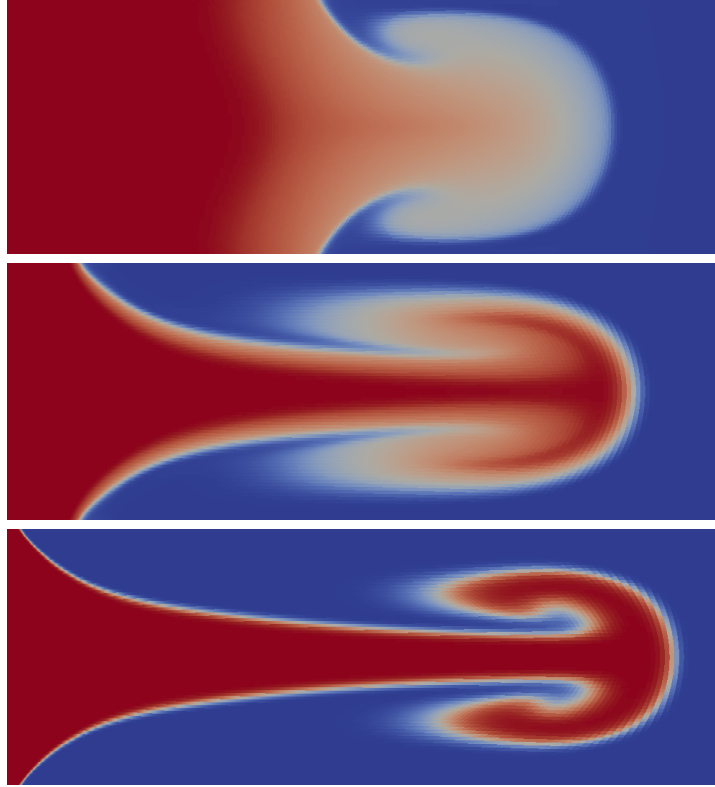


Figure 11:  $224 \times 80$  mesh. Time  $t = 0.7$ . Mass fraction of fluid  $\alpha$ . Influence of the friction parameter. **Top:**  $\nu = 100$ . **Middle:**  $\nu = 1000$ . **Bottom:**  $\nu = 10^6$ .

403 give some bounds independently of  $\nu$  (Property 3), which complete the numerical analysis of the  
 404 scheme.

405 The numerical results show that the scheme behaves as expected and appears to be a good  
 406 candidate to study interpenetration mixing [8], which is the goal of this study. Actually, all  
 407 the results<sup>1</sup> can be established with a varying positive friction  $\nu$ . In the paper we kept  $\nu$  constant  
 408 for the sake of simplicity. The numerical analysis and tests are performed in 2D, however the  
 409 analysis in 3D is completely unchanged.

410 On the numerical point of view, a second-order accurate version of the scheme would be of  
 411 interest. However, this is not an easy task for two main reasons. First, on the theoretical point of  
 412 view, establishing properly the asymptotic preserving property would be challenging. Second,  
 413 using a Runge-Kutta-like approach to get second-order accuracy in time would probably impose  
 414 to incorporate the remeshing into the time integration or to consider a one-step approach.

415 Another extension is to introduce more Physics in the model. The friction coupling is a  
 416 very simple approach, one could use more appropriate closures based on the presented work, for  
 417 instance [8] or [33] where Lorentz forces are taken into account in a ion-electron mixture.

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<sup>1</sup>If friction parameter depends on the cell data ( $\nu = \nu_j$ ), Property 3 takes a slightly different form.

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