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► **To cite this version:**

| Olivier Lablée. Trace formulas and Zeta functions in Spectral Geometry. 2017. hal-01504945

**HAL Id: hal-01504945**

**<https://hal.science/hal-01504945>**

Preprint submitted on 10 Apr 2017

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# Trace formulas and Zeta functions in Spectral Geometry

Olivier Lablée

10 April 2017

*Abstract.* Trace formulas have been one of the most powerful tools in spectral geometry. This article is an invitation to spectral geometry via trace formulas paradigm. The main goal of the present survey is to give some classical results concerning trace formulas in Riemannian geometry and to present some nice applications in conformal geometry and in dynamical systems. More precisely, we explain very recent results concerning the study of the geodesic flows on manifolds with negative variable curvature via the semi-classical zeta functions for contact Anosov flows.

**Key-words :** *Trace formulas, Spectral geometry, Length spectrum, Geodesic flow, Zeta functions, Hyperbolic surfaces.*

**AMS Classification :** 11F72, 11M36, 35P05, 35P20, 35K08, 37D20, 53D25, 58J50, 58J53, 58E10.

## 1 Introduction

The *Laplacian* in our three-dimensional Euclidean space  $\mathbb{R}^3$  is the usual linear differential operator defined by  $\Delta := \partial_x^2 + \partial_y^2 + \partial_z^2$ . This operator can be generalized to a compact Riemannian manifold  $(M, g)$  : this generalization is called the *Laplace-Beltrami operator* (or just Laplacian) and denoted by the symbol  $\Delta_g$ . The spectral theory of this operator is a central object in differential geometry. Indeed, the spectrum of  $\Delta_g$  contains vast information about the geometry and the topology of the associated manifold. The study of this operator and in particular the study of its spectrum is called *Spectral Geometry*. Spectral geometry have numerous connections with others fields of mathematical or physics, e.g. for understanding the relationships between the formalism of classical mechanics and quantum mechanics : this is semi-classical analysis. Spectral geometry is also very important in the study of dynamical systems, for example in classical and quantum chaos. To understand the fundamental relationships between the sequence of eigenvalues of  $-\Delta_g$  and the geometrical properties of the manifold  $(M, g)$ , one of the most powerful tools in this way is the principle of trace formulas. More generally, traces formulas are also useful techniques in analysis of PDE, in spectral theory, physics mathematics, in number theory, etc ...

In this article we focus on trace formulas applied to spectral geometry and to dynamical systems on a hyperbolic surface via some special zeta functions. First, we present a simple exact computation of trace formulas on a particular manifold : a flat torus. This example of trace formula have a fundamental application in spectral geometry : the famous Milnor Theorem. Next, we interest with in the powerfull notion of heat kernel, indeed, using trace formulas, we can establish the Minakshisundaram-Pleijel expansion and the asymptotic Weyl law for eigenvalues. Then, we finish with trace formulas on hyperbolic manifolds : in a first time we present the relationships between spectrum of the manifold and between the lenght spectrum (i.e. the set of lengths of closed geodesics). And, in a second time, we present Selberg formula and its applications in dynamical system, in particular for counting periodic geodesics on hyperbolic surface with an adapted zeta function.

## 2 Background on spectral geometry

Start here by some basics notions concerning the spectrum of the Laplacian on a compact Riemannian manifold, for more details classical references on this subject are the book of M. Berger, P. Gauduchon and E. Mazet [BGM], the book of P. Bérard [Bér4], of I. Chavel [Cha], the book of S. Rosenberg [Ros], see also the book [Lab].

### 2.1 Formal principle of trace formulas

The formal principle of trace formulas is the following : let us consider an unbounded self-adjoint linear operator  $H$  on a Hilbert space, and suppose that the spectrum of  $H$  is discrete :

$$\text{Spec}(H) = \{\lambda_k, k \geq 0\}.$$

This means that for all integer  $k \geq 0$ , there exists a vector  $\varphi_k \neq 0$  such that :  $H\varphi_k = \lambda_k\varphi_k$ . Let us also consider  $f$  a "nice" function. The fundamental principle of trace formulas is to compute the trace of the operator  $f(H)$  in two ways :

- the first way : with the eigenvalues of the linear operator  $f(H)$  :

$$\text{Trace}(f(H)) = \sum_{k=0}^{+\infty} f(\lambda_k).$$

- The second way : with the integral kernel of  $f(H)$  : if for all  $x \in M$

$$f(H)\varphi(x) = \int_M K_f(x, y)\varphi(y) dy$$

then

$$\text{Trace}(f(H)) = \int_M K_f(x, x) dx.$$

In consequence we get the following equality :

$$\sum_{k=0}^{+\infty} f(\lambda_k) = \int_M K_f(x, x) dx.$$

In practice, the main difficulty is to find a “good” choice for the function  $f$ . The usual choices are :

- $f(x) = e^{-xt}$  where  $t \geq 0$  (heat function),
- $f(x) = \frac{1}{x^s}$ , where  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 1$  (zeta Riemann function),
- $f(x) = e^{-\frac{ix}{\hbar}}$  where  $t \geq 0$  (Schrödinger function)
- etc ...

## 2.2 The Laplacian operator

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 1$ . The *Laplacian* on  $(M, g)$  is the linear operator defined by  $f \mapsto \Delta_g(f) := \text{div}(\nabla f)$ . In local coordinates we have : for any  $\mathcal{C}^2$  real valued function  $f$  on  $M$  and for any local chart  $\phi : U \subset M \rightarrow \mathbb{R}^n$  of  $M$ , the Laplacian  $\Delta_g$  is given by the local expression :

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk} \frac{\partial (f \circ \phi^{-1})}{\partial x_k} \right)$$

where  $g = \det(g_{ij})$  and  $g^{jk} = (g_{jk})^{-1}$ . This operator appears in many diffusion equations, for example :

- The heat equation : Consider here a domain  $\Omega \subset \mathbb{R}^3$  with a boundary  $\Gamma := \partial\Omega$ . The *heat equation* on  $\Omega$  describe the heat diffusion process on this domain. The heat equation is the following linear equation :

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t).$$

Here  $u(x, t)$  is the temperature at the time  $t$  of the point  $x \in \Omega$ .

- The Schrödinger equation : in Quantum Mechanics, a physical particle on a manifold  $M$  in a time interval  $I$  is describe by a *wave function*, that is a function  $(x, t) \in M \times I \mapsto \psi(x, t)$ , where the quantity  $\int_{\Omega} \psi(x, t) dx$  represents the probability to find the particle into the domain  $\Omega \subset M$  at the time  $t$ . In particular, this requires that  $\|\psi\|_{L^2(M)}^2 = \int_{\Omega} \psi(x, t) dx = 1$ . The quantum dynamics is governed by the famous *Schrödinger equation* :

$$i\hbar \frac{\partial \psi}{\partial t}(x, t) = -\Delta_g \psi(x, t) + V(x) \psi(x, t).$$

Here  $V$  is a function of  $L_{loc}^\infty(M)$  such that  $\lim_{|x| \rightarrow \infty} V(x) = +\infty$  : this is the potential.

### 2.3 The spectrum of a compact manifold

For simplify we suppose here that the manifold  $(M, g)$  is a closed manifold, i.e.  $M$  is compact and without boundary : for example the sphere, the torus...  
*The closed spectral* problem is : find all real numbers  $\lambda$  such that there exists a function  $u \in C^\infty(M)$  with  $u \neq 0$  such that :

$$-\Delta_g u = \lambda u.$$

From a spectral point of view we want to find the (ponctual) spectrum of the unbounded operator  $-\Delta_g$  on the domain  $D = C^\infty(M)$ . One of the pillars of spectral geometry is the following fact (see for example [Bér4], [Cha], [Lab]) : the spectrum and the ponctual spectrum of the operator  $-\Delta_g$  coincides and it consists of a real infinite sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

such that  $\lambda_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ . Moreover, each eigenvalue has a finite multiplicity, recall that the *multiplicity* of an eigenvalue  $\lambda_k$  is the dimension of the vector space  $\ker(\lambda_k I_d + \Delta_g)$ . In the notation  $(\lambda_k)_{k \geq 0}$  eigenvalues are counting with their multiplicities : in particular the notation  $0 = \lambda_0 < \lambda_1$  means that the first eigenvalue is simple and equal to zero. As it turns out, this result is also available for manifolds with boundary (but we have not always  $\lambda_0 = 0$ ). The incovenient of the notation  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$  is relative to concerning the multiplicity of each eigenvalues, we cant see explicitly the multiplicity of eigenvalues : so for some contexts we also use the following alternative notation : let us denote the distincts eigenvalues of  $(M, g)$  by :

$$(0 = \tilde{\lambda}_0, m_1 = 1) < (\tilde{\lambda}_1, m_1) < (\tilde{\lambda}_2, m_2) \dots < (\tilde{\lambda}_k, m_k) < \dots$$

where the integer  $m_i$  denote the multiplicity of the eigenvalue  $\tilde{\lambda}_i$ .

For summarize, for any closed (or more generally for any compact) Riemannian manifold there exists an unique sequence of reals numbers  $(\lambda_k)_{k \geq 0}$  such that

$$\text{Spec}(-\Delta_g) = \{\lambda_k, k \geq 0\}.$$

The previous sequence is called the *spectrum of the manifold*  $(M, g)$  and denoted by  $\text{Spec}(M, g)$ . We said that two (compact) Riemannian manifolds  $(M, g)$  and  $(M', g')$  are *isospectral* if and only if  $\text{Spec}(M, g) = \text{Spec}(M', g')$ . A *spectral invariant* on  $(M, g)$  is a quantity which is determined by  $\text{Spec}(M, g)$ .

### 2.4 The spectral partition function $Z_M$

**Definition 2.1.** Let  $(M, g)$  be a compact Riemannian manifold, we define the *spectral partition function* by the following serie of functions

$$Z_M(t) := \sum_{k=0}^{+\infty} e^{-t\lambda_k}.$$

We will see, using Weyl asymptotics (section 4.4) that the previous serie converge uniformly on  $\mathbb{R}_+^*$ . Using this uniform convergence it is obvious that

function  $t \mapsto Z_M(t)$  is continuous, non-increasing on  $\mathbb{R}_+^*$  and we have also  $\lim_{t \rightarrow +\infty} Z_M(t) = 0$  (because  $0 \in \text{Spec}(M, g)$  in the closed case). With the notation

$$(0 = \widetilde{\lambda}_0, m_1 = 1) < (\widetilde{\lambda}_1, m_1 = 1) < (\widetilde{\lambda}_2, m_2) < \cdots < (\widetilde{\lambda}_k, m_k) < \cdots$$

we have also the expression :

$$Z_M(t) = \sum_{k=1}^{+\infty} m_k e^{-t\widetilde{\lambda}_k}.$$

The main interest of the spectral partition function lies in the fact that this function  $Z_M$  determine the spectrum of the manifold  $(M, g)$ . Indeed, for any real number  $\mu > 0$  consider the function

$$e^{\mu t} Z_M(t) - e^{\mu t} = \sum_{k=0}^{+\infty} m_k e^{(\mu - \widetilde{\lambda}_k)t} - e^{\mu t} = \sum_{k=1}^{+\infty} m_k e^{(\mu - \widetilde{\lambda}_k)t}$$

therefore, if  $\mu < \widetilde{\lambda}_1$  then  $\lim_{t \rightarrow +\infty} e^{\mu t} Z_M(t) - e^{\mu t} = 0$ ; else if  $\mu = \widetilde{\lambda}_1$  then  $\lim_{t \rightarrow +\infty} e^{\mu t} Z_M(t) - e^{\mu t} = m_1$ ; else if  $\mu > \widetilde{\lambda}_1$  then

$$\lim_{t \rightarrow +\infty} e^{\mu t} Z_M(t) - e^{\mu t} = +\infty.$$

Thus  $\widetilde{\lambda}_1$  is the unique real number  $\mu > 0$  such that the function  $t \mapsto e^{\mu t} Z_M(t) - e^{\mu t}$  admits a finite limit as  $t$  tends to infinity. Consequently the function  $Z_M$  determine the first non-null eigenvalue  $\widetilde{\lambda}_1$ . By induction, with the function

$$e^{\mu t} Z_M(t) - \sum_{j=1}^{i-1} m_j e^{(\mu - \widetilde{\lambda}_j)t} = \sum_{j=i}^{+\infty} m_j e^{(\mu - \widetilde{\lambda}_j)t}$$

it is obvious to check that the function  $Z_M$  determine all eigenvalues  $\widetilde{\lambda}_k$   $k \geq 0$ . For finish, observe also that the function  $Z_M$  is associated to the *spectral density distribution* :

$$D_M(t) := \sum_{k=0}^{+\infty} \delta_{\lambda_k}$$

here  $\delta_{\lambda_k}$  denotes the Dirac mass distribution at the point  $\lambda_k$ . Indeed, using the Fourier Transform map (in the sense of distributions) we have :

$$\widehat{D}_M(t) := \sum_{k=0}^{+\infty} \widehat{\delta}_{\lambda_k} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{+\infty} e^{-it\lambda_k} = Z_M(it)$$

with the following Fourier Transform convention :

$$\widehat{f}(x) := \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} f(t) e^{-\frac{ixt}{\hbar}} dt.$$

### 3 A simple example of trace formula : the case of flat torus

#### 3.1 The Poisson summation formula for a lattice

The simplest example of exact trace formula is the *Poisson summation formula* : let  $f \in \mathcal{S}(\mathbb{R})$  and consider the function :

$$F(x) := \sum_{k \in \mathbb{Z}} f(x+k).$$

Since the function  $f$  belongs to the Schwartz's space  $\mathcal{S}(\mathbb{R})$  it is obvious that the serie  $F$  is absolutely convergent and is 1-periodic. Thanks to the Fourier's series theory we have the following equality

$$F(x) = \sum_{n \in \mathbb{Z}} c_n(F) e^{2i\pi n x}$$

with

$$c_n(F) = \int_0^1 F(t) e^{-2i\pi n t} dt.$$

Therefore we have

$$\begin{aligned} c_n(F) &= \int_0^1 \left( \sum_{k \in \mathbb{Z}} f(t+k) \right) e^{-2i\pi n t} dt = \sum_{k \in \mathbb{Z}} \left( \int_0^1 f(t+k) e^{-2i\pi n t} dt \right) \\ &= \sum_{k \in \mathbb{Z}} \int_k^{k+1} f(u) e^{-2i\pi n u} du = \widehat{f}(n) \end{aligned}$$

with the convention :

$$\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t) e^{-2i\pi x t} dt.$$

Consequently, for all  $x \in \mathbb{R}$ , we get the Poisson summation formula :

$$\sum_{k \in \mathbb{Z}} f(x+k) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2i\pi n x};$$

in particular with  $x = 0$  we obtain the usual classical form :

$$\sum_{k \in \mathbb{Z}} f(k) = \sum_{\ell \in \mathbb{Z}} \widehat{f}(\ell).$$

More generally, let  $a := (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$  such that for all  $i$ ,  $a_i \neq 0$ ; consider the associated lattice :

$$\Gamma := a_1 \mathbb{Z} + a_2 \mathbb{Z} + \dots + a_n \mathbb{Z}$$

(this is a sub-lattice in  $\mathbb{R}^n$  of rank  $n \geq 1$ ). The Poisson summation formula for this lattice reads :

$$\sum_{k \in \Gamma} f(k) = \frac{1}{\text{Vol}(\Gamma)} \sum_{\ell \in \Gamma^*} \widehat{f}(\ell)$$

with the notation :

$$\widehat{f}(y) := \int_{\mathbb{R}^n} f(t) e^{-2i\pi(y,t)} dt$$

and  $\Gamma^*$  is the dual lattice of  $\Gamma$  :

$$\Gamma^* := \{x \in \mathbb{R}^n; \forall y \in \Gamma, \langle x, y \rangle \in \mathbb{Z}\}.$$

Since  $(a_1, a_2, \dots, a_n)$  is a  $\mathbb{Z}$ -basis of  $\Gamma$ , the dual lattice  $\Gamma^*$  is given by the equality  $\Gamma^* = (A^{-1})^T \mathbb{Z}^n$  where

$$A := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix}.$$

Consequently  $(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n})$  is  $\mathbb{Z}$ -basis of the dual lattice  $\Gamma^*$  ; it follows that  $\text{Vol}(\Gamma^*) = \frac{1}{\text{Vol}(\Gamma)}$ . Moreover it is easy to show that  $\Gamma^{**} = \Gamma$  (we have not  $\Gamma^* = \Gamma$ ).

A classical calculus of Fourier transform show that the Fourier transform of the function

$$f(x) = e^{-\alpha\|x\|^2} \in \mathcal{S}(\mathbb{R}^n)$$

where  $\alpha > 0$  is given for all  $y \in \mathbb{R}^n$  by :

$$\widehat{f}(y) = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{\pi^2\|y\|^2}{\alpha}}.$$

Now, applying Poisson's formula, we get

$$\sum_{x \in \Gamma} e^{-\alpha\|x\|^2} = \frac{\pi^{\frac{n}{2}}}{\text{Vol}(\Gamma)\alpha^{\frac{n}{2}}} \sum_{y \in \Gamma^*} e^{-\frac{\pi^2\|y\|^2}{\alpha}}$$

replacing the lattice  $\Gamma$  by  $\Gamma^*$  we obtain the precious formula :

$$\sum_{x \in \Gamma^*} e^{-\alpha\|x\|^2} = \frac{\pi^{\frac{n}{2}} \text{Vol}(\Gamma)}{\alpha^{\frac{n}{2}}} \sum_{y \in \Gamma} e^{-\frac{\pi^2\|y\|^2}{\alpha}} \quad (3.1)$$

because  $\Gamma^{**} = \Gamma$  and  $\text{Vol}(\Gamma^*) = \frac{1}{\text{Vol}(\Gamma)}$ . In particular, with  $\alpha = 4\pi^2 t$ ,  $t > 0$ , we get the following formula :

$$\sum_{x \in \Gamma^*} e^{-4\pi^2 t\|x\|^2} = \frac{\text{Vol}(\Gamma)}{(4\pi t)^{\frac{n}{2}}} \sum_{y \in \Gamma} e^{-\frac{\|y\|^2}{4t}}. \quad (3.2)$$

### 3.2 Trace formula on a flat torus

Now, let us interpret this last formula in term of spectral geometry datas. For this we need to compute the spectrum of the torus associated to the lattice  $\Gamma$ . Start with the case of dimension one, i.e. the manifold the one dimensional torus  $\mathbb{T}_a$  of length  $a > 0$ , in others words the manifold id the circle of radius



a. We will show briefly that spectrum of the Laplacian  $-\frac{d^2}{dx^2}$  with closed conditions on this circle is given by :

$$\text{Spec}(\mathbb{T}_a) = \left\{ \frac{4\pi^2 n^2}{a^2}, n \in \mathbb{Z} \right\}$$

with associated eigenfunctions :  $x \mapsto e_n(x) = e^{\frac{2i\pi nx}{a}}, n \in \mathbb{Z}$ .

Indeed, if the scalar number  $\lambda$  is in the spectrum of  $-\frac{d^2}{dx^2}$ , there exists a non trivial function  $u \in L^2(\mathbb{T}_a)$  such that

$$-u'' = \lambda u.$$

Thanks to the  $L^2$  Fourier's series theory the sequence of functions

$$\left( x \mapsto e_n(x) = e^{\frac{2i\pi nx}{a}} \right)_{n \in \mathbb{Z}}$$

is a Hilbert basis of  $L^2(\mathbb{T}_a)$ , hence there exists an unique sequence  $(c_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$  such that :

$$u = \sum_{n \in \mathbb{Z}} c_n e_n.$$

So we can rewrite the differential equation  $-u'' - \lambda u = 0$  as :

$$\sum_{n \in \mathbb{Z}} c_n \left( \frac{4\pi^2 n^2}{a^2} - \lambda \right) e_n = 0.$$

Since  $u \neq 0$ , there exists  $n_0 \in \mathbb{Z}$  such that  $c_{n_0} \neq 0$ , hence because  $(e_n)_{n \in \mathbb{Z}}$  is a basis we get :

$$-u'' - \lambda u = 0 \Rightarrow \lambda = \frac{4\pi^2 n_0^2}{a^2}.$$

Thus we have the inclusion  $\text{Spec}(\mathbb{T}_a) \subset \left\{ \frac{4\pi^2 n^2}{a^2}, n \in \mathbb{Z} \right\}$ . Conversely, it is clear that for any integer  $n \in \mathbb{Z}$  we have  $-\frac{d^2}{dx^2} e_n = \frac{4\pi^2 n^2}{a^2} e_n$ , so finally we get

$$\text{Spec}(\mathbb{T}_a) = \left\{ \frac{4\pi^2 n^2}{a^2}, n \in \mathbb{Z} \right\}.$$

Now, let us come back to the lattice  $\Gamma := a_1\mathbb{Z} + a_2\mathbb{Z} + \dots + a_n\mathbb{Z}$ . The associated flat torus is the following quotient space :

$$\mathbb{T}_a := \mathbb{R}^n / \Gamma.$$

Using multidimensional *Fourier's series theory*, the sequence of functions :

$$\left( x \mapsto e^{2i\pi \langle \omega, x \rangle} \right)_{\omega \in \Gamma^*} = \left( (x_1, x_2, \dots, x_n) \mapsto e^{\frac{2i\pi k_1 x_1}{a_1} + \frac{2i\pi k_2 x_2}{a_2} + \dots + \frac{2i\pi k_n x_n}{a_n}} \right)_{(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n}$$

is a Hilbert basis of  $L^2(\mathbb{T}_a)$ , so the same argument as in dimension one shows that :

$$\text{Spec}(\mathbb{T}_a) = \left\{ 4\pi^2 \left( \frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} + \dots + \frac{k_n^2}{a_n^2} \right), (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n \right\}.$$

In other words : the spectrum of the Laplacian (with closed conditions) on a flat torus  $\mathbb{R}^n/\Gamma$  is given by :

$$\text{Spec}(\mathbb{R}^n/\Gamma) = \left\{ 4\pi^2 \|\gamma\|_{\mathbb{R}^n}^2, \gamma^* \in \Gamma^* \right\}.$$

We will interpret this equality using the spectral function  $Z_M$  and the length spectrum of the flat torus  $\mathbb{R}^n/\Gamma$ . In the one-dimensional case, the length spectrum of the circle  $\mathbb{T}_a$  is obviously given by :

$$\Sigma = \{|n|a, n \in \mathbb{Z}\},$$

and in the multidimensional case we have

$$\Sigma = \left\{ \sqrt{\sum_{i=1}^n (k_i a_i)^2}, (k_1, k_2, \dots, k_n) \in \mathbb{Z} \right\}.$$

Since (see section 3.2) we have the equality :

$$\sum_{x \in \Gamma^*} e^{-4\pi^2 t \|x\|^2} = \frac{\text{Vol}(\Gamma)}{(4\pi t)^{\frac{n}{2}}} \sum_{y \in \Gamma} e^{-\frac{\|y\|^2}{4t}}$$

and because

$$Z_\Gamma(t) = \sum_{x \in \Gamma^*} e^{-4\pi^2 t \|x\|^2}$$

holds for all  $t \geq 0$ , we obtain the following trace formula :

$$Z_\Gamma(t) = \sum_{\lambda \in \text{Spec}(\Gamma)} e^{-\lambda t} = \frac{\text{Vol}(\Gamma)}{(4\pi t)^{\frac{n}{2}}} \sum_{\ell \in \Sigma} e^{-\frac{\ell^2}{4t}}. \quad (3.3)$$

In the previous equality on the right-hand side we have the geometrical informations of the manifold : dimension, volume,... and on the left-hand side we have spectral informations. This formula have a very important application : The Milnor counter-example of isospectrality.

### 3.3 A fundamental application : Milnor's counterexample of isospectrality

In 1966, M. Kac [Kac] in his famous article "Can one hear the shape of a drum?" investigate the following question : given a Riemannian manifold (the membrane of a drum), is the spectrum of  $\Delta_g$  (the harmonics of the drum) determine geometrically, up to an isometry, the manifold  $(M, g)$  ? The exact formulation of isospectrality question is : if two Riemannian manifolds  $(M, g)$  and  $(M', g')$  are isospectral, are they isometric ? This question may be traced back to H. Weyl in 1911-1912 and became popularized thanks to M. Kac's article. Before 1964, this problem was stay quite mysterious and J. Milnor [Mil] give the first counter-example : a pair of 16-dimensional flat torus which are isospectral and nonisometric. The proof of Milnor is based on the previous trace formula.

**Theorem 3.1. (Milnor, 1964).** *There exists two lattices  $\Gamma$  and  $\Gamma'$  of  $\mathbb{R}^{16}$  such that the associated tori  $\mathbb{T}^{16}(\Gamma)$  and  $\mathbb{T}^{16}(\Gamma')$  are isospectral but not isometric.*

Indeed, let us consider the lattice  $\Gamma(n)$  generated by the lattice  $\gamma_n \subset \mathbb{Z}^n$  defined by :

$$\gamma_n := \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n; \sum_{j=1}^n x_j \in 2\mathbb{Z} \right\}$$

with  $n = 8$  or  $n = 16$ . One checks immediately that :  $\Gamma^*(n) = \Gamma(n)$ ; in consequence the volume of  $\Gamma(n)$  is equal to 1.

The first step of the proof consist to show with basic arguments that the lattices  $\Gamma(16)$  and  $\Gamma(8) \oplus \Gamma(8)$  are non-isometric. Then, it follows immediately that their associated tori are also non-isometric.

The second step of this proof is based on trace formula. In this way, we introduce the lattice  $\Gamma_t(n) := \sqrt{t}\Gamma(n)$  with  $t > 0$  a parameter. We have :

$$\Gamma_t^*(n) = \frac{1}{\sqrt{t}}\Gamma^*(n) = \frac{1}{\sqrt{t}}\Gamma(n).$$

On one hand, we have :

$$\sum_{x \in \Gamma(n)} e^{-\pi t \|x\|^2} = \sum_{x \in \Gamma_t(n)} e^{-\pi \|x\|^2}$$

and, on the other hand, with the formula (3.1)

$$\begin{aligned} \sum_{x \in \Gamma_t(n)} e^{-\pi \|x\|^2} &= \frac{1}{\text{Vol}(\Gamma_t(n))} \sum_{\ell \in \Gamma_t^*(n)} e^{-\pi \|\ell\|^2} \\ &= \frac{1}{t^{\frac{n}{2}} \text{Vol}(\Gamma(n))} \sum_{\ell \in \Gamma_t^*(n)} e^{-\pi \|\ell\|^2} \\ &= \frac{1}{t^{\frac{n}{2}} \text{Vol}(\Gamma(n))} \sum_{\ell \in \Gamma(n)} e^{-\frac{\pi \|\ell\|^2}{t}}. \end{aligned}$$

In other words, if we introduce the following theta function :

$$\theta_\Gamma(t) := \sum_{x \in \Gamma} e^{-\pi t \|x\|^2};$$

then, we obtain for all  $t > 0$

$$\theta_{\Gamma(n)}(t) = \frac{1}{t^{\frac{n}{2}}} \theta_{\Gamma(n)}\left(\frac{1}{t}\right).$$

Moreover, observe that for all  $t > 0$

$$Z_{\Gamma(n)}\left(\frac{t}{4\pi}\right) = \sum_{\lambda \in \text{Spec}(\Gamma(n))} e^{-\lambda \frac{t}{4\pi}} = \frac{1}{t^{\frac{n}{2}}} \sum_{y \in \Gamma(n)} e^{-\frac{\pi \|y\|^2}{t}} = \frac{1}{t^{\frac{n}{2}}} \theta_{\Gamma(n)}\left(\frac{1}{t}\right);$$

therefore, finally, for all  $t > 0$ , we have

$$Z_{\Gamma(n)}\left(\frac{t}{4\pi}\right) = \frac{1}{t^{\frac{n}{2}}} \theta_{\Gamma(n)}\left(\frac{1}{t}\right) = \theta_{\Gamma(n)}(t).$$

The last part of the proof may be summarized in : first, to observe that the theta function

$$z \mapsto \theta_{\Gamma(n)}(z) - \frac{1}{z^{\frac{n}{2}}} \theta_{\Gamma(n)}\left(\frac{1}{z}\right)$$

admits a holomorphic expansion on  $\{z \in \mathbf{C}; \operatorname{Re}(z) > 0\}$ , consequently, for all  $z \in \mathbf{C}; \operatorname{Re}(z) > 0$  we have

$$\theta_{\Gamma(n)}(z) = \frac{1}{z^{\frac{n}{2}}} \theta_{\Gamma(n)}\left(\frac{1}{z}\right).$$

And the last part arguments of the proof consists to show that  $\theta_{\Gamma(16)}$  and  $\theta_{\Gamma(8) \oplus \Gamma(8)}$  are modular forms of weight equal to 4 such that

$$\theta_{\Gamma(16)}(\infty) = \theta_{\Gamma(8) \oplus \Gamma(8)}(\infty) = 1,$$

and it is possible to conclude that

$$\theta_{\Gamma(16)} = \theta_{\Gamma(8) \oplus \Gamma(8)}.$$

Finally, the tori  $\Gamma(16)$  and  $\Gamma(8) \oplus \Gamma(8)$  are isospectral (because theta function determine the spectrum).

As it turns out, in 1984 and 1985, respectively C. Gordon, E.N. Wilson [**Go-Wi**] and T. Sunada [**Sun**] gave a systematic construction for counter-example. In 1992, C. Gordon, D. Webb and S. Wolpert [**GWW1**], [**GWW2**] gave the first planar counter-example. See also the articles of P. Bérard [**Bér1**], [**Bér2**], [**Bér3**].

## 4 Heat kernel and trace formulas

In this section we present the fundamental notion of heat kernel and one of the most important result in geometry and topology : the Minakshisundaram-Pleijel expansion. Let  $(M, g)$  be a closed Riemannian manifold and let us denotes by  $(\lambda_k)_{k \geq 0}$  the spectrum of  $(M, g)$ .

### 4.1 The heat kernel

The Cauchy problem for the *heat equation* on a closed Riemannian manifold  $(M, g)$  is the following equation : find a function  $u : M \times \mathbb{R}_+^* \rightarrow \mathbb{R}$  such that :

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \Delta_g u(x, t) \\ u(x, 0) = u_0(x) \quad \forall x \in M. \end{cases}$$

where  $u_0$  is a given smooth function on  $M$ . Physically, for every point  $x \in M$  the number  $u_0(x)$  is an initial temperature data on the manifold  $M$  and the solution  $(x, t) \mapsto u(x, t)$  describe the evolution of the temperature for the point  $x \in M$  with the time  $t$ .

The first way to solve this equation is to use the spectral theory of  $\Delta_g$  and the bounded functional calculus of operator. Indeed, we can consider the bounded one-parameter semi-group  $\{U(t)\}_{t \geq 0}$  defined for all  $t \geq 0$  by

$$U(t) := e^{t\Delta_g}.$$

The generator of this semi-group is the operator  $\Delta_g$ . One sees immediately that the function  $u(x, t) = e^{t\Delta_g}u_0(x)$  is a solution of heat equation. By the bounded functional calculus, for any  $t \geq 0$  and for any integer  $k$  we also have

$$\left(e^{t\Delta_g}\right) e_k = \left(e^{-t\lambda_k}\right) e_k$$

where  $(e_k)$  is a Hilbert basis of eigenvectors. So, for any initial condition  $u_0 = \sum_{k=0}^{+\infty} a_k e_k \in L^2(M)$ , we have :

$$\begin{aligned} u(x, t) &= U(t)u_0(x) = \left(e^{t\Delta_g}\right) \left(\sum_{k=0}^{+\infty} a_k e_k(x)\right) \\ &= \sum_{k=0}^{+\infty} a_k e^{-t\lambda_k} e_k(x). \end{aligned}$$

In conclusion, for all  $x \in M$  and for all  $t \geq 0$ ,

$$u(x, t) = \sum_{k=0}^{+\infty} \langle u_0, e_k \rangle_{L^2} e^{-t\lambda_k} e_k(x).$$

Another way to solve heat equation is to use the *heat kernel* of the manifold. The theory of heat kernel on Riemannian manifolds is a deep and a fundamental theory providing many powerful tools for understanding the relationship between geometry and diffusion processes. The diffusion operator related to the Laplacian on a compact Riemannian manifold is the heat operator. This operator is an operator acts on smooth functions and admits a fundamental solution which is called the heat kernel. Indeed, the heat kernel is the fundamental solution of the heat process diffusion equation  $\frac{\partial}{\partial t}u = \Delta u$ . The heat kernel is a function

$$E : (x, y, t) \in M \times M \times \mathbb{R}_+^* \longmapsto E(x, y, t)$$

such that for all  $y \in M$  the function  $(x, t) \mapsto E(x, y, t)$  is a solution of the heat equation and for all  $x \in M$  and every test function  $\varphi \in \mathcal{D}(M)$  we have the initial data condition

$$\lim_{t \rightarrow 0^+} \int_M E(x, y, t) \varphi(y) d\mathcal{V}_g(y) = \varphi(x).$$

For a more complete introduction to heat kernel theory, see for example [Dav], [Ya-Sc], [BGV], [Gri], [Dod] and for some application see also the article [SaC].

**Definition 4.1.** The *heat kernel* (or *fundamental solution*) of the heat equation on a closed Riemannian manifold  $(M, g)$  is a function :

$$E : \begin{cases} M \times M \times \mathbb{R}_+^* \longrightarrow \mathbb{R} \\ (x, y, t) \longmapsto E(x, y, t) \end{cases}$$

such that :

(i)  $E$  is  $\mathcal{C}^0$  in the three variables  $(x, y, t)$ ;  $E$  is  $\mathcal{C}^2$  in the second variable  $y$  and  $\mathcal{C}^1$

in the third variable  $t$ .

(ii) For all  $(x, y, t) \in M^2 \times \mathbb{R}_+^*$  we have

$$\frac{\partial E}{\partial t}(x, y, t) = \Delta_{g,y}E(x, y, t),$$

where  $\Delta_{g,y}$  is the Laplacian for the second variable  $y$ ; in other words, for all fixed  $x \in M$ , the function  $(y, t) \mapsto E(x, y, t)$  is a solution of the heat equation.

(iii) For all  $x \in M$  and for all test function  $\varphi \in \mathcal{D}(M)$

$$\lim_{t \rightarrow 0^+} \int_M E(x, y, t) \varphi(y) d\mathcal{V}_g(y) = \varphi(x).$$

Since the manifold is compact, we have existence and uniqueness of a such function  $E$  (for the construction see [BGM], [Dod] or [Cha2]). We call this solution the heat kernel of  $M$ .

**Example 4.2.** In the Eucliden case  $M = \mathbb{R}^n$  with  $g = \text{can}$ , heat kernel is given by the expression

$$E(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|x-y\|^2}{4t}},$$

This exemple is very important because it can be used to it for construct the heat kernel in the general case.

Let us introduce the one parameter family integral operator :

$$P_t : \begin{cases} L^2(M) \rightarrow L^2(M) \\ \varphi \mapsto \int_M E(x, y, t) \varphi(y) d\mathcal{V}_g(y) \end{cases}$$

where  $t > 0$  is the time parameter, in other words for all  $x \in M$  we have

$$P_t(\varphi)(x) = \int_M E(x, y, t) \varphi(y) d\mathcal{V}_g(y).$$

For all  $t > 0$ , the associated integral kernel is  $(x, y) \mapsto E(x, y, t)$ . An important fact is that for all  $\varphi \in L^2(M)$ , the function  $P_t(\varphi)$  is a solution of the heat equation. As it turns out, the family  $\{P_t\}_{t \geq 0}$  is a non-negative semi-group of bounded operator and this semi-group coincide with  $U(t) := e^{t\Delta_g}$ .

Now, let us focus on the link between the heat kernel and the spectrum of the Laplacian. First, observe that we have the fundamental equality :

$$E(x, y, t) = \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) e_k(y),$$

in the sense that the serie converges uniformly and absolutely on  $\overline{M} \times \overline{M} \times [\varepsilon, +\infty[$  and its limit is equal to  $E(x, y, t)$ . Indeed, let us consider the function

$\mathcal{E} : M^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $\mathcal{E}(x, y, t) := \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) e_k(y)$ . We have

$$\Delta_{g,y} \mathcal{E}(x, y, t) = \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) \Delta_{g,y} e_k(y) = \sum_{k=0}^{+\infty} -\lambda_k e^{-t\lambda_k} e_k(x) e_k(y).$$

On the other hand,

$$\frac{\partial \mathcal{E}}{\partial t}(x, y, t) = \sum_{k=0}^{+\infty} -\lambda_k e^{-t\lambda_k} e_k(x) e_k(y)$$

hence for all  $(x, y, t) \in M^2 \times \mathbb{R}_+^*$  we obtain the equality

$$\frac{\partial \mathcal{E}}{\partial t}(x, y, t) = \Delta_{g,y} \mathcal{E}(x, y, t).$$

Moreover, for all  $\varphi \in \mathcal{D}(M)$  we have

$$\begin{aligned} \int_M \mathcal{E}(x, y, t) \varphi(y) d\mathcal{V}_g(y) &= \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) \int_M e_k(y) \varphi(y) d\mathcal{V}_g(y) \\ &= \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) \langle \varphi, e_k \rangle_{L^2} \end{aligned}$$

whence

$$\lim_{t \rightarrow 0^+} \int_M \mathcal{E}(x, y, t) \varphi(y) d\mathcal{V}_g(y) = \sum_{k=0}^{+\infty} e_k(x) \langle \varphi, e_k \rangle_{L^2} = \varphi(x).$$

Hence,  $E$  is a fundamental solution of heat equation which can be expressed as

$$E(x, y, t) := \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) e_k(y).$$

## 4.2 The spectral partition function $Z_M$ as a trace

Next, let us examine what happens with the heat kernel on the diagonal  $y = x$ . For a fixed  $\varepsilon > 0$ , for all  $t \geq \varepsilon$  and for all  $x \in M$  the serie

$$(x, t) \mapsto \sum_{k \geq 1} e^{-t\lambda_k} e_k^2(x)$$

is convergent and its limit is equal to  $E(x, x, t)$ . Consequently, for all  $t \in [\varepsilon, +\infty[$

$$\begin{aligned} \int_M E(x, x, t) d\mathcal{V}_g(x) &= \int_M \left( \sum_{k=1}^{+\infty} e^{-t\lambda_k} e_k^2(x) \right) d\mathcal{V}_g(x) \\ &= \sum_{k=1}^{+\infty} e^{-t\lambda_k} \int_M e_k^2(x) d\mathcal{V}_g(x) = \sum_{k=1}^{+\infty} e^{-t\lambda_k} \|e_k\|_{L^2}^2 = \sum_{k=1}^{+\infty} e^{-t\lambda_k}. \end{aligned}$$

So, we can define the trace of the operator  $e^{t\Delta_g}$  by

$$Z_M(t) := \text{trace} \left( e^{t\Delta_g} \right) = \int_M E(x, x, t) d\mathcal{V}_g(x) = \sum_{k=1}^{+\infty} e^{-t\lambda_k}. \quad (4.1)$$

We conclude that the serie of functions  $t \mapsto \sum_{k=1}^{+\infty} e^{-t\lambda_k}$  is uniformly and absolutely convergent on  $[\varepsilon, +\infty[$ .

### 4.3 From Minakshisundaram-Pleijel expansion ...

A very important corollary of the of heat kernel construction is the *Minakshisundaram-Pleijel expansion*:

**Theorem 4.3. (Minakshisundaram-Pleijel expansion).** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n$ , and let us denote by  $E$  the heat kernel of  $(M, g)$ . We have the expansion*

$$E(x, x, t) \underset{t \rightarrow 0^+}{\sim} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left( u_0(x, x) + u_1(x, x)t + \cdots + u_k(x, x)t^k + \cdots \right)$$

where the functions  $x \mapsto u_k(x, x)$  are smooths on  $M$  and depend only on the curvature tensor and its covariant derivatives.

If for any integer  $k$ , we denote

$$a_k := \int_M u_k(x, x) d\mathcal{V}_g(x),$$

then we obtain the nice equality

$$Z_M(t) = \sum_{k=1}^{+\infty} e^{-t\lambda_k} \underset{t \rightarrow 0^+}{\sim} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left( a_0 + a_1 t + \cdots + a_k t^k + \cdots \right) \quad (4.2)$$

i.e., for any integer  $N > 0$  and for all  $t > 0$ ,

$$Z_M(t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{k=0}^N a_k t^k + O\left(t^{N-\frac{n}{2}+1}\right).$$

The computation of the coefficients  $a_k$  is difficult. Nevertheless, we have :

$$a_0 = \text{Vol}(M, g)$$

and

$$a_1 = \frac{1}{6} \int_M \text{Scal}_g d\mathcal{V}_g.$$

The expression of the term  $a_2$  is also know:

$$a_2 = \frac{1}{360} \int_M \left( 2|R|^2 - 2|\text{Ric}|^2 + 5\text{Scal}_g^2 \right) d\mathcal{V}_g,$$

where  $R$  is the Riemann curvature tensor. The others coefficients  $a_k$  for  $k \geq 2$  are very complicated to compute and they are quite futile, more details can be found in the books [BGM], [Ber]. Note that, in the case of surfaces (i.e.  $\dim(M) = 2$ ) the Gauss-Bonnet formula yields

$$a_1 = \frac{\pi}{3} \chi(M)$$

where  $\chi(M)$  is the Euler-Poincaré characteristic of the surface  $M$ . In consequence for any closed surface we have :

$$Z_M(t) = \frac{\text{Vol}(M, g)}{4\pi t} + \frac{\chi(M)}{12} + \frac{\pi t}{60} \int_M K^2 d\mathcal{V}_g + P(t)t^2$$



where  $t \mapsto P(t)$  is a bounded function on  $M$ .

A fundamental corollary of the Minakshisundaram-Pleijel expansion is the following fact: if we know the spectrum of  $(M, g)$ , then in particular we know

- the dimension of the manifold;
- the volume of the manifold;
- the integral of the scalar curvature  $\text{Scal}_g$  over the manifold.

More precisely :

**Corollary 4.4.** *If two Riemannian manifolds  $(M, g)$  and  $(M', g')$  are isospectral then :*

- (i) *the dimensions of  $M$  and  $M'$  are equal.*
- (ii) *The volumes of  $(M, g)$  and  $(M', g')$  are equal.*
- (iii) *The integrals of the scalar curvature over  $(M, g)$  and  $(M', g')$  are equal. In particular, if  $M$  and  $M'$  are surfaces, then  $\chi(M) = \chi(M')$ .*

In the case of surfaces, the dimension and the Euler-Poincaré characteristic are topological invariants.

#### 4.4 ... to Weyl asymptotic formula

To finish, let us present a very famous corollary : that is the *Weyl asymptotic formula*. This formula is based on the *Karamata Tauberian Theorem*, this theorem reads that for any positive measure  $\mu$  on  $\mathbb{R}_+$  if

$$\int_0^{+\infty} e^{-tx} d\mu(x) \underset{t \rightarrow 0}{\sim} \frac{a}{t^\alpha}$$

holds for any  $\alpha, a > 0$ , then

$$\int_0^\lambda d\mu(x) = \mu([0, \lambda]) \underset{\lambda \rightarrow \infty}{\sim} \frac{a}{\Gamma(\alpha + 1)} \lambda^\alpha.$$

Let us apply this result to the following positive measure  $\mu := \sum_{k=1}^{+\infty} \delta_{\lambda_k}$ , since

$$Z_M(t) \underset{t \rightarrow 0}{\sim} \frac{\text{Vol}(M, g)}{(4\pi t)^{\frac{n}{2}}},$$

and using the Karamata Tauberian theorem, we obtain

$$\int_0^\lambda d\mu(x) \underset{\lambda \rightarrow \infty}{\sim} \frac{\text{Vol}(M, g)}{(4\pi)^{\frac{n}{2}}} \frac{1}{\Gamma(\frac{n}{2} + 1)} \lambda^{\frac{n}{2}} = \frac{B_n \text{Vol}(M, g)}{(2\pi)^n} \lambda^{\frac{n}{2}}$$

with  $B_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}$ . And, on the other hand, the counting eigenvalues function satisfy :

$$\#(\{k \in \mathbb{N}, \lambda_k \leq \lambda\}) = \int_0^\lambda d\mu(x);$$

consequently, we deduce the :

**Theorem 4.5. (Weyl asymptotic formula).** Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n$ , and let us denote by  $(\lambda_k)_{k \geq 0}$  the eigenvalues of  $-\Delta_g$  on  $(M, g)$ . Then

$$\#(\{k \in \mathbb{N}, \lambda_k \leq \lambda\}) \underset{\lambda \rightarrow +\infty}{\sim} \frac{B_n \text{Vol}(M, g)}{(2\pi)^n} \lambda^{\frac{n}{2}}$$

where  $B_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$  is the volume of the closed unit ball in  $(\mathbb{R}^n, \text{can})$ .

An obvious consequence of this asymptotics is the equivalent one

$$\lambda_k \underset{k \rightarrow +\infty}{\sim} \left( \frac{(2\pi)^n}{B_n \text{Vol}(M, g)} \right)^{\frac{2}{n}} k^{\frac{2}{n}}.$$

## 4.5 Spectral zeta function and uniformization theorem

In this subsection we present the relationship between the eigenvalues of a closed surface and its associated conformal geometry. For more details on this amazing subject see for example [Cha] or [Gur].

The *spectral zeta function* associated to a surface  $(M, g_0)$  is given by the formula :

$$\zeta_{g_0}(s) := \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}$$

where  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 1$ . For all  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 1$  we have :

$$\zeta_{g_0}(s) = \sum_{n=1}^{+\infty} \lambda_n^{-s} = \sum_{n=1}^{+\infty} e^{-s \ln(\lambda_n)}$$

using the Weyl's formula ie  $\lambda_n \underset{n \rightarrow +\infty}{\sim} Cn$  where  $C := \frac{4\pi^2}{B_2 \text{Vol}(M, g)} > 0$ ; the nature

of the series  $\sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}$  and  $\sum_{n=1}^{+\infty} \frac{1}{n^s}$  are the same, the series  $\sum_{n=1}^{+\infty} \frac{1}{n^s}$  is the usual Riemann function which it is convergent for all  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 1$ . So the series  $\zeta_{g_0}(s) := \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}$  is well defined and convergent for all  $s \in \mathbb{C}$  such that

$\text{Re}(s) > 1$ .

Now we shall prove that the function  $\zeta_{g_0}$  admit a meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  : first recall that  $\Gamma$  the Euler Gamma function

$$\Gamma(s) := \int_0^{+\infty} e^{-t} t^{s-1} dt$$

converge for all  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 0$ , moreover for all  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 0$  we have the relation :

$$\frac{1}{\Gamma(s)} = \lim_{n \rightarrow +\infty} \frac{n! n^s}{s(s+1)(s+2) \cdots (s+n)}$$

the function  $\Gamma$  admit a meromorphic continuation to  $\mathbb{C} - \{0, -1, -2, \dots\}$  with simple poles at  $s = 0, -1, -2, \dots$ . By changing variables we have for all  $x > 0$

$$\frac{1}{x^s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} t^{s-1} dt$$

therefore, for all  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 1$  we get

$$\begin{aligned} \zeta_{g_0}(s) &= \sum_{n=1}^{+\infty} \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-\lambda_n t} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \left( \sum_{n=1}^{+\infty} e^{-\lambda_n t} \right) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} (Z_M(t) - 1) t^{s-1} dt \end{aligned}$$

because  $\lambda_0 = 0$  thus

$$\zeta_{g_0}(s) = \frac{1}{\Gamma(s)} \int_0^1 (Z_M(t) - 1) t^{s-1} dt + \frac{1}{\Gamma(s)} \int_1^{+\infty} (Z_M(t) - 1) t^{s-1} dt.$$

Start by studying the first integral : using the remark 7.6.2, we have for all  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 1$

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^1 (Z_M(t) - 1) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 \left( \frac{\operatorname{Vol}(M, g)}{4\pi t} + \frac{\chi(M)}{12} + \frac{\pi t}{60} \int_M K^2 d\mathcal{V}_g + P(t)t^2 - 1 \right) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 \left( \frac{\operatorname{Vol}(M, g)}{4\pi t} + \frac{\chi(M)}{12} + \frac{\pi t}{60} \int_M K^2 d\mathcal{V}_g - 1 \right) t^{s-1} dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^1 P(t) t^{s+1} dt. \end{aligned}$$

The function  $s \mapsto \frac{1}{\Gamma(s)} \int_0^1 P(t) t^{s+1} dt$  is holomorphic for  $\operatorname{Re}(s) > -2$  and

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^1 \left( \frac{\operatorname{Vol}(M, g)}{4\pi t} + \frac{\chi(M)}{12} + \frac{\pi t}{60} \int_M K^2 d\mathcal{V}_g - 1 \right) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \left[ \frac{t^{s-1}}{s-1} \frac{\operatorname{Vol}(M, g)}{4\pi} + \frac{\chi(M)t^s}{12s} + \frac{\pi t^{s+1}}{60(s+1)} \int_M K^2 d\mathcal{V}_g - \frac{t^s}{s} \right]_{t=0}^{t=1} \end{aligned}$$

this quantity converges for all  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 1$  and has a meromorphic continuation to  $\mathbb{C}$  with a simple pole in  $s = 1$  (because  $\Gamma(s) = \lim_{n \rightarrow +\infty} \frac{s(s+1)(s+2)\dots(s+n)}{n!n^s}$ ).

For finish : let us study the second integral : for all  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 1$

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_1^{+\infty} (Z_M(t) - 1) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_1^{+\infty} \left( \sum_{n=1}^{+\infty} e^{-\lambda_n t} \right) t^{s-1} dt \end{aligned}$$

and for all  $t \geq 1$ ,  $e^{-\lambda_n t} = e^{-\lambda_1 t} e^{(\lambda_1 - \lambda_n)t}$ , so since the eigenvalues  $\lambda_n$  grows like  $n$  as  $n$  tends to infinity (Weyl asymptotic formula) and since  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  for  $t$  large enough we have

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} \leq C e^{-\lambda_1 t}$$

consequently using the holomorphic theorem under integral sign we get easily that the function  $s \mapsto \frac{1}{\Gamma(s)} \int_1^{+\infty} \left( \sum_{n=1}^{+\infty} e^{-\lambda_n t} \right) t^{s-1} dt$  is holomorphic on  $\mathbb{C}$ . Finally the function  $\zeta_{g_0}$  has a meromorphic expansion to  $\mathbb{C}$  with a simple pole in  $s = 1$ , in particular  $\zeta_{g_0}$  is analytic in 0, hence the quantity  $\zeta'_{g_0}(0)$  exist and

$$\zeta'_{g_0}(0) = \lim_{s \rightarrow 0} \frac{\zeta_{g_0}(s) - \zeta_{g_0}(0)}{s - 0}.$$

Next, the derivative of  $\zeta_{g_0}$  is

$$\zeta'_{g_0}(s) = \sum_{i=1}^{+\infty} -\ln(\lambda_i) \frac{1}{\lambda_i^s}$$

hence for  $s = 0$  we get

$$\zeta'_{g_0}(0) = - \sum_{i=1}^{+\infty} \ln(\lambda_i) = - \ln \left( \prod_{i=1}^n \lambda_i \right)$$

therefore we obtain :

$$e^{-\zeta'_{g_0}(0)} = \prod_{i=1}^{+\infty} \lambda_i = \det(-\Delta_{g_0}).$$

Thus we can define the determinant of  $\Delta_{g_0}$  by the formula :

$$\det(-\Delta_{g_0}) = e^{-\zeta'_{g_0}(0)}.$$

Now, let  $g \in [g_0]$ , so there exists  $u \in \mathcal{C}^\infty(M)$  such that  $g = e^{2u} g_0$ , the *Polyakov formula* (see for example [Cha]) is :

$$\ln \left( \frac{\det(-\Delta_g)}{\det(-\Delta_{g_0})} \right) = - \frac{1}{12\pi} \int_M (\|\nabla u\|^2 + K_{g_0} u) d\mathcal{V}_{g_0}$$

we can consider the following functional :

$$F : \begin{cases} \mathcal{C}^\infty(M) \rightarrow \mathbb{R} \\ u \mapsto \ln \left( \frac{\det(-\Delta_g)}{\det(-\Delta_{g_0})} \right). \end{cases}$$

Since the determinant is not a scale invariant we consider the *normalized functional determinant*

$$S : \begin{cases} \mathcal{C}^\infty(M) \rightarrow \mathbb{R} \\ u \mapsto -12\pi \ln \left( \frac{\det(-\Delta_g)}{\det(-\Delta_{g_0})} \right) + 2\pi\chi(M) \ln(\text{Vol}(M, g)) \end{cases}$$

where  $\chi(M)$  is the Euler's characteristic of  $M$ . The critical functions of  $S$  are functions  $u$  such that

$$-\Delta_{g_0} u + K_{g_0} = ce^{2u}$$

where  $c$  is a real constant. Using Gaussian curvature equation we have :  $u$  is a critical function of  $S$  if and only if the surface  $M$  with the conformal metric  $g := e^{2u}g_0$  admits a constant Gaussian curvature  $K_g$ . In 1998, B. Osgood, R. Phillips and P. Sarnak [OPS1], [OPS2] proved the following theorem :

**Theorem 4.6. (Osgood, Phillips, Sarnak, 1998).** *The supremum  $\sup_{u \in C^\infty(M)} S(u)$  of the previous functional exists and it is a maximum : there exists  $u_0 \in C^\infty(M)$  such that Gaussian curvature  $K_g$  (where  $g := e^{2u}g_0$ ) is constant on  $M$ .*

It is turn out that each surface can be endowed with a metric conformally equivalent to a metric with constant Gaussian curvature. The main consequence is the famous *Poincaré-Klein-Koebe uniformization theorem* :

**Theorem 4.7. (Uniformization theorem).** *Let  $(M, g_0)$  be a closed surface with a Riemannian metric  $g_0$ . Then there exists an unique metric<sup>1</sup>  $g$  conformal to  $g_0$  on  $M$  with a constant Gaussian curvature  $K_g \in \{+1, 0, -1\}$ .*

The values of the constant  $K_g \in \{+1, 0, -1\}$  depends on the topology of the surface  $M$ . Recall that the Gauss-Bonnet formula for a closed surface is :

$$\int_M K d\mathcal{V}_g = 2\pi\chi(M)$$

and for a surface with boundary

$$\int_M K d\mathcal{V}_g + \int_M K_g dS_g = 2\pi\chi(M)$$

where  $\chi(M)$  is the Euler-Poincaré characteristic of  $M$  and  $K_g$  is the geodesic curvature on the boundary. According to the Gauss-Bonnet formula the sign of the curvature is determined by the Euler-Poincaré characteristic<sup>2</sup>  $\chi(M)$  of  $M$ . Therefore the universal covering of every closed surface can be isometrically embedded onto :

1. the round sphere  $S^2$  for genus zero ;  $\chi(M) = 2 > 0$ , and in this case  $K_g \equiv +1$ .
2. The Euclidian plane  $\mathbb{R}^2$  for genus one :  $\chi(M) = 0$ , in this case  $K_g \equiv 0$ .
3. The hyperbolic space  $\mathbb{H}^2$  for genus  $g \geq 2$ :  $\chi(M) = -2 < 0$ , in this case  $K_g \equiv -1$ .

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<sup>1</sup>called a uniformization metric.

<sup>2</sup>For  $M$  a closed orientable surface  $\chi(M) = 2 - 2g$  where  $g$  is the genus of the surface.

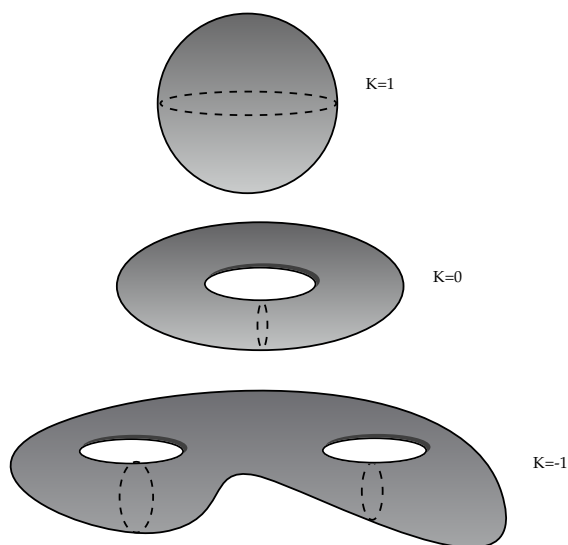


Fig. 1. Three geometries on closed surfaces.

## 5 Trace formulas on hyperbolic manifolds

In this section we investigate trace formulas on closed hyperbolic manifolds, in particular the case of surfaces.

### 5.1 Length spectrum versus spectrum

The length spectrum of a Riemannian manifold  $(M, g)$  is the set of lengths of closed geodesics on  $(M, g)$  counted with multiplicities (the multiplicity of a length is the number of free homotopy classes of closed curves containing a geodesic of the given length). In 1973, Y. Colin de Verdière [Col1], [Col2] showed that, in the compact case, up to a generic hypothesis (which it is always satisfied in the case of negative sectional curvature), the spectrum of the Laplacian determines the length spectrum. Colin de Verdière's proof is based on trace formulas. More precisely, in the first article [Col1], the author focus on closed hyperbolic surface and he use the heat kernel of the surface via the following trace formula (see formula 4.2) :

$$\sum_{k=0}^{+\infty} e^{-\frac{\lambda_k(M)}{z}} = \int_M E\left(x, x, \frac{1}{z}\right) d\mathcal{V}_g(x)$$

with a complex time  $t = \frac{1}{z} \in \mathbb{C}$ , and he used an approximation of the heat kernel  $E\left(x, x, \frac{1}{z}\right)$ . The main result of [Col1] is :

**Theorem 5.1. (Colin de Verdière, 1973).** *Let  $M$  be a compact hyperbolic surface with a constant sectional curvature :  $K \equiv -1$ . Let us denote by  $\mathcal{L}$  the set of free*

homotopy classes of  $M$ , and if  $a$  belongs to  $\mathcal{L}$  then  $|a|$  denote the length of the unique periodic geodesic of  $a$ . Then, for any  $z \in \mathbb{C}$  such that  $\text{Re}(z) \geq \xi_0 > 0$  we have :

$$\sum_{k=0}^{+\infty} e^{-\frac{\lambda_k}{z}} = \left(\frac{z}{4\pi}\right) \text{vol}(M) + \sqrt{\frac{z}{4\pi}} \sum_{a \in \mathcal{L}, \gamma \neq e}^{+\infty} u_\gamma e^{-\frac{z|a|^2}{4}} + O^+(1)$$

with  $u_\gamma$  are real numbers such that  $u_\gamma > 0$  and  $O^+(1)$  is a bounded function on the set of complex numbers  $z \in \mathbb{C}$  such that  $\text{Re}(z) \geq \xi_0 > 0$ .

One consequence of this result is :

**Corollary 5.2.** For any compact hyperbolic surface  $M$  with a constant sectional curvature :  $K \equiv -1$ , the spectrum of the surface  $M$  determine the length spectrum of  $M$ .

In the second article [Col2], Y. Colin de Verdière gave a generalisation in the case of hyperbolic manifolds of dimension  $n \geq 2$  with a constant sectional curvature :  $K \equiv -1$ , for more recent approach and details see also [Col3].

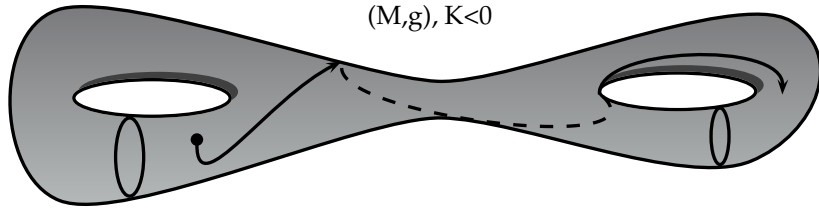


Fig. 2. A closed hyperbolic surface with some periodic geodesics.

## 5.2 The Selberg formula for hyperbolic surface

Now let us present some points from the fascinating story of Selberg formula on hyperbolic surfaces. Main application of this formula concerning dynamical system, in particular for understand trajectories and flows on negatively curved surfaces. The proof of this formula is based on many technical computations (see [Sel], [Mar] or [Vor]). The statement of this formula is the following : let  $(M, g)$  be a compact hyperbolic surface, and consider a test function  $h : \mathbb{C} \rightarrow \mathbb{C}$  such that :

- $h$  is analytic for  $z$  such that  $|\text{Im}(z)| \leq \sigma$  with  $\sigma > \frac{1}{2}$ ;
- $h$  is even;
- for all  $z$  in the strip  $|\text{Im}(z)| \leq \sigma$ , we have  $h(z) = O\left(\frac{1}{(1+|\text{Re}(z)|)^N}\right)$ .

Under this previous hypothesis, the Selberg formula read :

**Theorem 5.3. (Selberg Formula).** Using the previous hypothesis, we have :

$$\sum_{k=0}^{+\infty} h\left(\sqrt{\lambda_k - \frac{1}{4}}\right) = \frac{\text{Vol}(\Gamma)}{4\pi} \int_{-\infty}^{+\infty} h(r) \tanh(\pi r) r dr + \sum_{\gamma \in \Gamma} \sum_{n=1}^{+\infty} \frac{|\gamma| \widehat{h}\left(\frac{n|\gamma|}{2\pi}\right)}{4\pi \sinh\left(\frac{n|\gamma|}{2}\right)}$$

here  $\Gamma$  denotes the set of prime closed oriented geodesics of the surface  $M$ . In the formula we use the convention :

$$\widehat{f}(y) := \int_{-\infty}^{+\infty} f(t) e^{-2i\pi y t} dt.$$

The first application of this formula is to a Weyl law for surface :

$$\#(\{k \in \mathbb{N}, \lambda_k \leq \lambda\}) \underset{\lambda \rightarrow +\infty}{\sim} \frac{\text{Vol}(\Gamma)}{4\pi} \lambda.$$

### 5.3 The zeta Selberg function

One other important application of Selberg formula concerning the density of closed geodesics on a compact hyperbolic surface : let us introduce the following counting function :

$$\Pi(L) := \#(\{\gamma \in \Gamma, |\gamma| \leq L\}).$$

One's important hope is to understand this function, and more precisely to give a precise estimate of  $\Pi(L)$  for  $L \rightarrow +\infty$ . In the goal to estimate the function  $\Pi(L)$  for  $L \rightarrow +\infty$  in the case of a compact hyperbolic surface  $M$  with a constant sectional curvature :  $K \equiv -1$ , A. Selberg, in a famous paper of 1956 [Sel] introduce a special zeta function :  $s \in \mathbb{C} \mapsto \zeta_{\text{Selberg}}(s)$  for understand the dynamic of geodesics on this surface. This function zeta is defined for all  $s \in \mathbb{C}$  by the double product formula :

$$\zeta_{\text{Selberg}}(s) := \prod_{\gamma \in \Gamma} \prod_{k=0}^{+\infty} (1 - e^{-(s+k)|\gamma|}).$$

An another formulation of  $\zeta_{\text{Selberg}}$  is done by :

$$\zeta_{\text{Selberg}}(s) = \exp \left( - \sum_{\gamma \in \Gamma} \sum_{k=0}^{+\infty} \sum_{m=1}^{+\infty} \left( \frac{e^{-(s+k)m|\gamma|}}{m} \right) \right).$$

We have an important analogy with the zeta Riemann function. Before to study the function  $\zeta_{\text{Selberg}}$ , we briefly recall some classical elements about the classical Riemann zeta function. The Riemann zeta function is defined for all  $s \in \mathbb{C}, \text{Re}(s) > 1$  by the serie :

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

This function admits a holomorphic expansion to  $\mathbb{C} - \{1\}$ . Moreover, for all  $s \in \mathbb{C}$  such that  $\text{Re}(s) > 1$  we have we have also the Euler product :

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - e^{-s \ln p}}$$

here  $\mathbb{P}$  is the set of prime numbers. For all  $x > 0$ , we define the counting prime number functions



$$\pi(x) := \#(\{p \in \mathbb{P}, p \leq x\}).$$

The statement of the prime number theorem is : for  $x \rightarrow +\infty$  we have

$$\pi(x) = \ell_i(x) (1 + O(1))$$

where  $\ell_i$  is the log-integral function :  $\ell_i(x) := \int_2^x \frac{du}{\ln u}$ . The actual description of zeros for  $\zeta$  is the following :

- we have trivial zeros :  $\{-2k, k \geq 1\}$ ;
- there exists no zeros in the half plane  $\text{Re}(z) \geq 1$ ;
- and there exists an infinity of non-trivial zeros on the vertical line  $\text{Re}(z) = \frac{1}{2}$  ... and we have the famous Riemann hypothesis : all non-trivial zeros are on the vertical line  $\text{Re}(z) = \frac{1}{2}$  ?

Now, we come back to the function  $\zeta_{\text{Selberg}}$ , main results concerning this function are summarized in :

**Theorem 5.4. (Selberg, 1956).** *Let  $M$  be a compact hyperbolic surface  $M$  with a constant sectional curvature :  $K \equiv -1$  and let us denote by  $\Gamma$  the set of prime closed oriented geodesics of  $M$ . Then :*

- (i) *The previous function  $\zeta_{\text{Selberg}}$  is absolutely convergent for  $s$  such that  $\text{Re}(s) > 1$ .*
- (ii) *This function admits an analytic extension to  $\mathbb{C}$ .*
- (iii) *The zeros of the previous extension on  $\mathbb{C}$  are :*

$$\{-n, n \geq 0\} \cup \left\{ \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_k}, k \geq 0 \right\}$$

where  $(\lambda_k)_{k \geq 0}$  are the eigenvalues of the closed surface  $M$ .

- (iv) *Moreover, for all  $s \in \mathbb{C}$ , we have the relation of duplication :*

$$\zeta_{\text{Selberg}}(1-s) = \zeta_{\text{Selberg}}(s) \exp \left( 2(g-1) \int_0^{s-\frac{1}{2}} \pi x \tan(\pi x) dx \right)$$

here  $g \geq 2$  is the genus of the surface  $M$ .

Let us examine the item (iii) : we have two types of zeros :

- trivial zeros :  $-n, n \geq 0$ .
- and non-trivial zeros :  $s_k := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_k}, k \geq 0$ .

Since  $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$  with  $\lambda_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ , it is obvious that there exists an integer  $N > 0$  such that

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \leq \frac{1}{4} < \lambda_{N+1}$$

thus, for all integer  $k \in \{0, \dots, N\}$

$$s_k = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_k} \in [0, 1];$$

and for all integer  $k \geq N + 1$

$$s_k = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_k} \in \left\{ z \in \mathbb{C}, \operatorname{Re}(z) = \frac{1}{2} \right\}.$$

Observe that since  $\lambda_0 = 0$  then 0 and 1 are zeros of  $\zeta_{\text{Selberg}}$ . Moreover, the repartition of the real zeros  $\{s_k, 0 \leq k \leq 1\}$  are symmetric with respect to the vertical line  $\operatorname{Re}(z) = \frac{1}{2}$ ; and the complex zeros  $\{s_k, k \geq N + 1\}$  are symmetric with respect to the horizontal axe  $\operatorname{Re}(z)$ . For summarize : without a finite number, all non trivial zeros of the zeta function  $\zeta_{\text{Selberg}}$  are localised on the line  $\operatorname{Re}(z) = \frac{1}{2}$ . In some sense the Riemann Hypothesis is true on every compact hyperbolic surface with a constant sectional curvature.

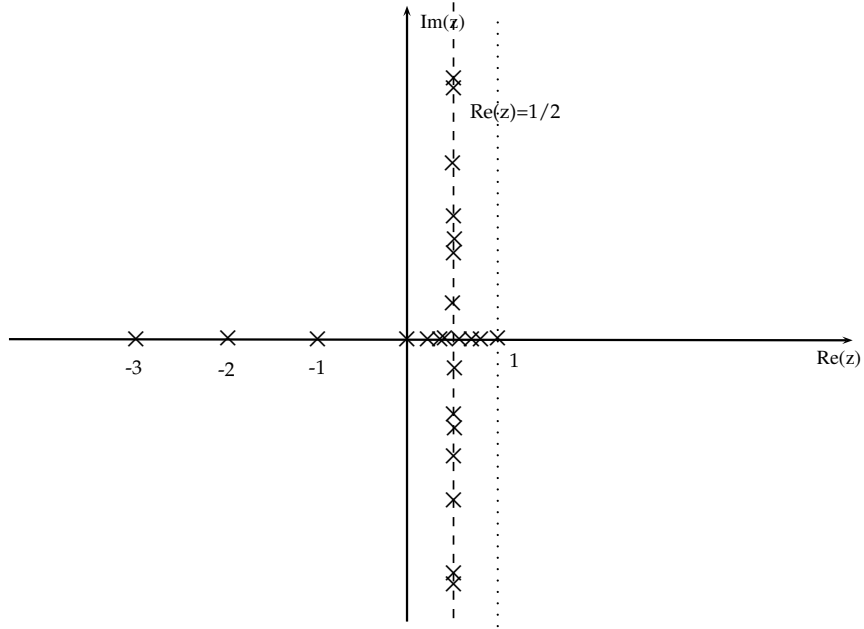


Fig. 3. The zeros of zeta function  $\zeta_{\text{Selberg}}$  in the complex plane.

A consequence of the previous theorem 5.4 is concerning the counting function  $\Pi(L)$  :

**Theorem 5.5.** *There exists  $\varepsilon > 0$  such that :*

$$\Pi(L) = \#(\{\gamma \in \Gamma, |\gamma| \leq L\}) = \ell_i(e^L) + O_{L \rightarrow +\infty}(1^{(1-\varepsilon)L})$$

where  $L_i$  is the log-integral function  $L_i(x) := \int_2^x \frac{du}{\ln u}$ .

This theorem is a called prime geodesics theorem. In particular, we deduce the equivalent :

$$\#(\{\gamma \in \Gamma, |\gamma| \leq L\}) \underset{L \rightarrow +\infty}{\sim} \frac{e^L}{L}.$$

Note that again the analogy with the Riemann zeta function  $\zeta$  and the prime number theorem. The definition of the function  $\zeta_{\text{Selberg}}$  can be define to any compact Riemannian manifold and the problem of counting closed geodesics can be generalized to counting orbit associated to a flow, indeed the case of geodesics appears as trajectories associated to the dynamics of classical Hamiltonian flow of  $H = -\Delta_g$  (geodesic flow) in the phase plane, identified with the cotangent bundle  $T^*M$ .

In 1967, in a fundamenal paper, S. Smale [**Sma**] has the idea to use the function  $\zeta_{\text{Selberg}}$  in a more general context : the case of of Anosov flows. Note that a geodesic flow on a compact manifold with a negative curvature is a particular case of Anosov flow. Unfortunately, since the crux of the proof for theorem 5.4 concerning the zeta function  $\zeta_{\text{Selberg}}$  is based on a trace formula for the heat kernel of a surface with a constant sectional curvature, then it was not clear that this function was very usefull for understand general Anosov flows on manifold with no constant curavture. More precisely, S. Smale ask himself the question “Does the item (i) of the theorem 5.4 is always true for Anosov Flow ? Or on a compact hyperbolic surface with a non constant sectional curvature  $K < 0$  ? : The author wrote : “Does  $\zeta_{\text{Selberg}}$  have a meromorphic continuation to all  $\mathbb{C}$  ? An affirmative answer would be roughly necessary and sufficient condition for  $\zeta_{\text{Selberg}}$  to be useful. I must admit a positive answer would be a little shocking!”

## 5.4 More recent results

Later, in 2013, P. Giulietti, C. Liverani, M. Pollicott [**GLP**], and in 2016, S. Dyatlov and M. Zworski [**Dy-Zw**] shows that the items (i) and (ii) on the function  $\zeta_{\text{Selberg}}$  holds for  $C^\infty$  Anosov flows (note that a geodesic flow on a compact manifold with a negative curvature is a Anosov flow). But what happens for item (iii) ? Let us present a very recent result in this way. We have seen that the fundamental hypothesis of the works of Selberg is that the sectional curvature of the surface is constant equal to  $-1$ . In the case of hyperbolic surface with a variable sectional curvature  $K < 0$ , the situation is quite delicate ... As it turns out the “good” zeta function in the case of variable curvature is the following *Semi-Classical zeta function* (for a details on this fact see [**Fa-Ts2**] :

$$\zeta_{\text{SC}}(s) := \exp \left( - \sum_{\gamma \in \Gamma} \sum_{m=1}^{+\infty} \left( \frac{e^{-sm|\gamma|}}{m \sqrt{|\det(I_d - D_\gamma^m)|}} \right) \right)$$

where  $D_\gamma$  is the Jacobian matrix of the Poincaré map for the orbit  $\gamma \in \Gamma$ . From a physical point of view, in quantum chaos, it is natural to study this semi-classical zeta function  $\zeta_{\text{SC}}$  rather than the zeta function  $\zeta_{\text{Selberg}}$ . Before 2016, the mathematical study of this semi-classical zeta function seems to be limited to the special case of constant curvature, where Selberg trace formula is available. Note that if  $K \equiv -1$  then for all  $s \in \mathbb{C}$  we have

$$\zeta_{\text{SC}}(s) = \zeta_{\text{Selberg}} \left( s + \frac{1}{2} \right).$$

In 2016, F. Faure and M. Tsujii [**Fa-Ts1**], [**Fa-Ts2**] shows that item (iii) holds for this new function  $\zeta_{\text{SC}}(s)$  in the case of geodesic flows on a manifold with

negative variable curvature  $K < 0$ . Indeed the main theorem (see Theorem 1.2) of the article [Fa-Ts2] is the first result for the study of the semi-classical zeta function for the geodesic flows on manifolds with negative variable curvature. The statement of this theorem is :

**Theorem 5.6. (Faure, Tsujii, 2016).** *Let  $M$  be a compact hyperbolic surface  $M$  and let us denotes by  $\Gamma$  the set of prime closed oriented geodesics of  $M$ . Then we have :*

- (i) *The function  $\zeta_{SC}$  admits an meromorphic extension to  $\mathbf{C}$ .*
- (ii) *For all  $\varepsilon > 0$ , without a finite number, all zeros and poles of  $\zeta_{SC}$  are localised in*

$$\{z \in \mathbf{C}, |Re(z)| \leq \varepsilon\} \cup \{z \in \mathbf{C}, Re(z) \leq -\lambda + \varepsilon\}$$

here  $\lambda$  is a constant.

- (iii) *Moreover, the set  $\{z \in \mathbf{C}, |Re(z)| \leq \varepsilon\}$  admits finite number of poles and infinite number of zeros of  $\zeta_{SC}$ .*

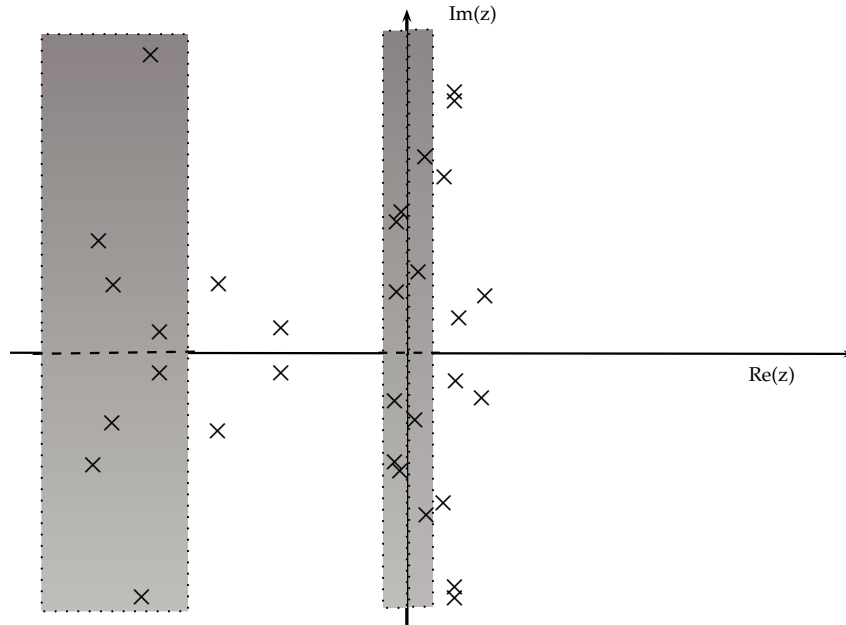


Fig. 4. Poles and zeros of semi-classical zeta function  $\zeta_{SC}$  in the complex plane.

In the article [Fa-Ts2], the authors gave also the following exciting open question :

**Conjecture 5.7.** *For all  $\varepsilon > 0$ , without a finite number, all zeros of the function  $\zeta_{SC}$  are in*

$$\{z \in \mathbf{C}, |Re(z)| \leq \varepsilon\} \cup \{z \in \mathbf{C}, |Im(z)| \leq C\}$$

here  $C$  is a constant.

The huge paper [Fa-Ts2] is a very deep and technical article; it is no reasonable to explain here the mains arguments of the main theorem. However, let us just mention that the authors use several complex techniques of semi-classical analysis. In particular they consider and study spectral properties and trace formulas applied to some transfer operators, theses operators are associated to a “prequantum” Anosov flow.

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