



HAL
open science

Moral hazard in welfare economics: on the advantage of Planner's advices to manage employees' actions.

Thibaut Mastrolia

► **To cite this version:**

Thibaut Mastrolia. Moral hazard in welfare economics: on the advantage of Planner's advices to manage employees' actions.. 2017. hal-01504473

HAL Id: hal-01504473

<https://hal.science/hal-01504473>

Preprint submitted on 10 Apr 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Moral hazard in welfare economics: on the advantage of Planner's advices to manage employees' actions.*

By Thibaut Mastrolia[†]

April 10, 2017

Abstract: In this paper, we study moral hazard problems in contract theory by adding an exogenous Planner to manage the actions of Agents hired by a Principal. We provide conditions ensuring that Pareto optima exist for the Agents using the scalarization method associated with the multi-objective optimization problem and we solve the problem of the Principal by finding optimal remunerations given to the Agents. We illustrate our study with a linear-quadratic model by comparing the results obtained when we add a Planner in the Principal/multi-Agents problem with the results obtained in the classical second-best case. More particularly in this example, we give necessary and sufficient conditions ensuring that Pareto optima are Nash equilibria and we prove that the Principal takes the benefit of the action of the Planner in some cases.

Key words: Moral hazard, Nash equilibrium, Pareto efficiency, multi-objective optimization problems.

AMS 2000 subject classifications: Primary: 91A06, 91B40. Secondary: 91B15, 91B10, 91B69, 93E20.

JEL subject classifications: C61, C73, D60, D82, D86, J33, O21.

1 Introduction

In 1992, Maastricht treaty, considered as the key stage in the European Union construction, proposed the establishment of a single currency for its members to ensure a stability of prices inside the EU. This date marks the birth of the euro, as the common currency of the eurozone, with the creation of the European Institut Monetary to ensure the introduction of it, which has operated until the creation in 1998 of a central bank for Europe, the European Central Bank. The ECB is the decision-making center of the Eurosystem, which is one of the main component of the EU (see for instance [36, Chapter 5, Section 2.5] for a description of its general role inside the EU) to ensure the economical smooth functioning of the eurozone, and Maastricht Treaty contains

*This work is supported by the ANR project Pacman, ANR-16-CE05-0027 and the Chair Financial Risks (Risk Foundation, sponsored by Société Générale).

[†]CMAP-École Polytechnique, Route de Saclay 91128 Palaiseau, FRANCE, thibaut.mastrolia@polytechnique.edu

some criteria that member states are supposed to respect. For instance, the Article 104-C stipulates that "Member States shall avoid excessive government deficits" and some criteria to respect it concern the Government budget balance, which does not have to exceed 3% of the GDP and the debt-to-GDP ratio, which does not have to exceed 60%. If these benchmarks are exceeded, through the Stability and Growth Pact, members of eurozone have accepted to follow a precise procedure each year to reduce their debt. Even before the financial crisis of 2007-2008, these rules have been discussed (see [31]), and since the Greek government-debt crisis this controversy has been amplified.¹ The fact is that the eurozone is very heterogeneous in the respect of Maastricht criteria as showed in Table 1.

Table 1: *Gouvernement deficit/surplus in % of GDP and debt-to-GDP ratio in % for nine EU members in 2015. Data from Eurostat.*

EU members	gov. deficit/surplus in % of GDP	debt-to-GDP ratio in %
Estonia	+0.1	10.1
Finland	-2.8	63.6
France	-3.5	96.2
Germany	+0.7	71.2
Greece	-7.5	177.4
Ireland	-1.9	78.6
Italie	-2.6	132.3
Lithuania	-0.2	38.7
Spain	-5.1	99.8
Euro area (19 countries)	-2.1	90.4

The natural questions arising are the following:

- a. How the ECB can provide incentives to EU members to respect the rules induced by Maastricht treaty? Which kind of procedure is the more efficient?
- b. How EU members have to interact for the welfare of the European Union?

The Greek government crisis has impacted all the eurozone and showed that members are strongly correlated each others. Through this example we see that one difficulty of both the ECB and eurozone members is to reach an optimal decision and equilibria to succeed in the global european construction.

The example above, and more specially the problem a., describes the kind of investigations made in the incentives theory starting in the 70's (see among others [27]) and is an illustration of a Principal/Agent problem. More exactly, the classical framework considered is the following: a Principal (*she*) aims at proposing to an Agent (*he*) a contract. The Agent can accept or reject the contract proposed by the Principal and under acceptance conditions, he will provide a work to manage the wealth of the Principal. However, the Principal is potentially imperfectly informed about the actions of the Agent, by observing only the result of his work without any direct access on it. The Principal thus designs a contract which maximizes her own utility by considering

¹See for instance (see [13, 12, 20, 22]).

this asymmetry of information, given that the utility of the Agent is held to a given level (his reservation utility). From a game theory point of view, this class of problems can be identified with a Stackelberg game between the Principal and the Agents, *i.e.* the Principal anticipates the best-reaction effort of the Agent and takes it into account to maximize her utility. Moral hazard in contracting theory, *i.e.* the Principal has no access on the work of her Agent, has been developed during the 80's and was investigated in a particular continuous time framework by Holmström and Milgrom in [17]. We refer to the paper of Sung [38] and the monography of Cvitanic and Zhang [8] for nice reviews of the literature in this topic and different situations studied dealing with moral hazard in Principal/Agent problems. More recently, the noticeable work of Sannikov [35] investigates a stopping time problem in contract theory by emphasizing the fundamental impact of the value function of the Agent's problem to solve the problem of the Principal. This was then mathematically formalized with the works of Cvitanic, Possamaï and Touzi in [6, 7] by proposing a nice handleable procedure to solve a large panel of problems in moral hazard. More exactly, they have showed that the Stackelberg equilibrium between the Agent and the Principal may be reduced to two steps in the studying of general problems in contracts theory. First, given a fixed contract proposes by the Principal, the Agent aims at maximizing his utility by finding the best reaction effort associated with the proposed contract. It is well-known, since the works of Rouge and El Karoui [34] and Hu, Imkeller and Müller [18] that a utility maximization problem can be reduced to solve Backward Stochastic Differential Equations (BSDE for short, we refer to the works of Pardoux and Peng [29, 30] and El Karoui, Peng and Quenez [9] for general results related to this theory), and Cvitanic, Possamaï and Touzi have proved that more generally, when the Agent can control the volatility, the problem can be reduced to solve a second order BSDE (see for instance the seminal work [37] and the extension of it with more general conditions [32]). Then, it is proved in [6, 7] that the problem of the Principal can be reduced to solve a stochastic control problem with state variables of the problem the output and the value function of the Agent, by using the HJB equations associated with it and verification theorems.

An extension of moral hazard with a Principal and an Agent, which echoes the example of the ECB as the Principal and the EU members as the Agents presented above, consists in considering a Principal dealing with many Agents. Principal/Multi-Agents problems have been investigated in a one period framework by Hölmstrom [16] (among others) and then extended in the continuous time model by Koo, Shim and Sung in [21] and more recently by Elie and Possamaï in [11]. In the latter, Elie and Possamaï consider exponential utility functions for the Agents and the Principal and they assume that the Agents are rational so that the first step of the procedure to solve the Agents' problems remains to find Nash equilibria, which can be reduced to solve a multi-dimensional quadratic BSDE, as explained in [11]. Nevertheless, by recalling the example of the ECB and EU members above with question b., we can also consider an other type of interactions between the Agents.

In microeconomics, we can distinguish two type of interactions between connected agents. Agents can be considered as rational economical entities and aim at finding their best reaction functions to maximize their wealths/minimize their costs, in view of the actions of the others. This investigation consists in finding a Nash equilibrium and fits with a situation in which the (non-cooperative) agents cannot deviate from this equilibrium. However, as explained with the so-called prisoner's dilemma (see Table 2 for more explanation), the configuration obtained is not necessarily an optimal choice for the welfare of the system.

Table 2: This example was emphasized by Merrill Flood and Melvin Dresher in the 50's and then formalized by Albert W. Tucker. Considers two prisoners P1 and P2 waiting for their interrogation to determine if they are indeed guilty or not. The prisoners have two choices, they can keep silent or they can denounce the other prisoner. We sum up the payoffs-vector associated with this situation in the table below, the first (resp. second) component of the vector is the payoff of P1 (resp. P2). We see that Pareto optima are keep silent-keep silent, denounce-keep silent and keep silent-denounce and the Nash equilibrium is denounce-denounce, which is the dominant strategy. In particular, if the two prisoners are forced to play a Pareto optimum, their global payoff is 0 or -9 which is always better than the natural equilibrium with global payoff -10 .

Action of P2 \ Action of P1	keep silent	denounce
keep silent	(0, 0)	(1, -10)
denounce	(-10 , 1)	(-5 , -5)

Welfare economics is a part of microeconomics which aims at determining the degree of well-being of a situation and Pareto optima are considered as one criterion to measure it. Indeed, a Pareto optimum consists in finding a configuration in which all the considered entities cannot deviate without harming the state of an other entity. Unlike a Nash equilibrium, a Pareto optimum is not reached using dominant strategies but has to be imposed in a general situation, since rational entities will reasonably converge to a Nash equilibrium. Finding Pareto optima gives a lot of information in terms of general equilibrium inside a system of markets and supply/demand systems, as showed by (among others) Walras in [40] to explain price-setting mechanisms, and mathematically formulated by Arrow and Debreu in [3] with the celebrated two fundamental theorems of welfare economics. Roughly speaking, the first fundamental theorem of welfare economics states that any general equilibrium in a competitive market is (weakly) Pareto optimal. The second states that any Pareto optimum can lead to a general equilibrium by reallocating initial appropriations. We refer to [26, 2] and the monography [23] for more details on it. It is however² not clear that with information asymmetry this equivalence is always true (specially the second fundamental theorem does not hold), but studying the existence of Pareto optima seems to be interesting to have relevant information related to general equilibrium in view of the first theorem. Moral hazard problems dealing with Pareto optimality was studied in a one-period model in [33] and in a two-period model in [28]. As far as we know, it does not have been investigated in continuous time models and the present paper is the first who considers it. Before going further, let us explain how we have to understand the Principal/multi-Agents problem studied in this work. Since the actions of Agents lead naturally to a Nash equilibrium (which does not coincides *a priori* with a Pareto optimum), we have to assume that the Agents cannot manage their work themselves or are forced to follow a precise strategy. This induces to introduce a third player in the Principal/multi-Agents game, namely *the Planner*, who imposes an effort to the Agents for their well-being. More specifically, the Planner can be seen as a mediator inside a firm who managed the actions of agents or a regulator who forces the Agents to act for the global interest. The Planer can be for instance a Government who imposes some Labour laws, by thinking about the global interest of the employees or any other entity who manages actions of Agents hired by a Principal. In this paper, we will distinguish the case where no Planner impacts the Stackelberg equilibrium, which coincides with the second best case in contract the-

²We for instance refer to [1] for an investigation of this kind of issues.

ory and where rational Agents reach to Nash equilibria, to the case in which a Planner manages the work of the Agents. The structure of the paper is the following:

After having described the Economy studied in Section 2, we extend the result of [11] to general separable utilities for the Agents, and general utilities for the Principal in Section 3 for the classical second-best case, named the *no-Planner model*. We provide conditions on the data ensuring that the multi-dimensional BSDE associated with Nash equilibria for the Agents is well-posed as an application of the recent results in [15] (see Appendix B and Remark 3.1). Then, in Section 4 we turn to the model in which a Planner manages the action of Agents. Using the fact that finding Pareto optima can be reduced to solve a Multi-Objective Optimization Problem (MOOP for short), we give general forms of some Pareto optima through the solutions of BSDEs and we solve the problem of the Principal using the HJB equation associated with it. Finally, in Section 5, we apply our results to a linear-quadratic model similar to the applied model studied in [11] with two Agents having appetite coefficients and we give some interpretations on the relevant results. We compare the no-Planner model with models in which a Planner intervenes in the Principal/multi-Agents problem by providing sufficient and necessary conditions such that a Nash equilibrium is Pareto efficient, and by comparing the value functions of the Principal.

The notations and all the technical details are postponed to Appendix A and B and proofs of Section 4 are postponed to Appendix C to alleger the reading of the paper.

2 The Economy

We consider a finite number $N \geq 1$ of Agents hired by one Principal. Each Agent receives from the Principal a salary at a fixed horizon $T > 0$ to manage the project of the Principal described by an \mathbb{R}^N -valued process X with volatility an $\mathcal{M}_N(\mathbb{R})$ -valued map denoted by Σ depending on the output X , given by

$$X_t := \int_0^t \Sigma(s, X_s) dW_s, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad (2.1)$$

where W is an N -dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ filtered with the natural filtration of the Brownian motion $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. We assume in this paper that the volatility Σ is uniformly bounded, the map $(t, x) \mapsto \Sigma(t, X_t)$ is \mathbb{F} -progressively measurable, invertible and such that (2.1) has a unique strong solution with exponential moments.

Remark 2.1. *The boundedness of Σ is here for the sake of simplicity, while the invertibility is fundamental to define properly the weak formulation of the problem as explained in [11].*

2.1 Impact of the actions of N -interacting agent

Let A be a subset of \mathbb{R} . We assume that any agent can impact, with his effort, both the value of his assigned project and the values of the other projects. More explicitly, any i th Agent has an impact on the i th component on X and on the other components managed by the other Agents. We represent the general action of the N Agents by a matrix $a \in \mathcal{M}_N(A)$ where $a^{j,i}$ is the action of the i th Agent on the project j . Thus, $a^{:,i}$ is the A^N -valued column vector of the i th Agent's

actions. This matrix impacts³ the dynamic (2.1) of X by adding a drift b defined as a map from $[0, T] \times \Omega \times \mathcal{M}_N(A)$ such that for any time t , any space variable $x \in \mathbb{R}^N$ and any effort $a \in \mathcal{M}_N(A)$ of the Agents, we have

$$b(t, x, a) := \left(b^i \left(t, x, (a^{i,1}, \dots, a^{i,N})^\top \right) \right)_{1 \leq i \leq N}.$$

For technical reasons, we have to focus on *admissible* actions of the Agents as a subspace \mathcal{A} of $\mathcal{M}_N(A)$ -valued process for which the model is well-posed. We refer to Appendix A for a mathematical definition of it. Thus, under technical conditions given in the appendix, the dynamic of X , impacted by any admissible actions a of Agents, is given by

$$X_t = \int_0^t b(s, X_s, a_s) ds + \int_0^t \Sigma(s, X_s) dW_s^a, \quad (2.2)$$

where

$$W^a := W - \int_0^T \Sigma(s, X_s)^{-1} b(s, X_s, a_s) ds,$$

is a Brownian motion under some probability measure \mathbb{P}^a equivalent to \mathbb{P} . We thus work under the weak formulation of the problem (see [11, Sections 2.1 and 2.2] for more details on it).

Interpretation of (2.2). We first focus on the dependance with respect to X in both the volatility Σ and the drift b of the project. This classical phenomenon expresses some friction effects between the different projects managed by the Agents so that each component of the general outcome X can depend on the values of the others. Turn now to the dependance with respect to the effort of the Agents. The dynamic of the project i depends on all the efforts provided by the N Agents and assuming that the i th Agent has preferences depending on the i th component of X , this model emphasizes exactly an interacting effect between Agents since each Agent can impact positively or negatively the project managed by an other.

2.2 Stackelberg equilibrium, Nash equilibrium and Pareto optimality

It is well known that a moral hazard problem can be identified with a Stackelberg equilibrium between the Principal and the Agents. In the considered system, the first step consists in finding a best respond effort provided by the Agents given a sequence of salaries proposed by the Principal. In the second step, taking into account the reactions of the Agents, the Principal aims at maximizing her own utility.

We assume that each Agent is penalized through a cost function depending on his effort. More precisely, we denote by $k^i : [0, T] \times \mathbb{R}^N \times A^N \rightarrow \mathbb{R}$ the cost function associated with the i th Agent.

For any $i \in \{1, \dots, N\}$, we set $U_0^i : \mathbb{R} \times \mathcal{M}_N(A) \rightarrow \mathbb{R}$ the utility of the i th Agent. We will consider in this paper separable utilities given a sequence $(\xi^i)_{1 \leq i \leq N}$ of salaries given by the Principal and an admissible action a chosen by the N -Agents, such that

$$U_0^i(\xi^i, a) := \mathbb{E}^{\mathbb{P}^a} \left[U_A^i(\xi^i) + \Gamma_i(X_T) - \int_0^T k^i(t, X_t, a^{i,i}) dt \right],$$

³See Appendix A for mathematical details.

where $U_A^i : \mathbb{R} \rightarrow \mathbb{R}$ is a concave and increasing map so that its inverse $U_A^{(-1)}$ is well-defined and increasing, $\Gamma_i : \mathbb{R}^N \rightarrow \mathbb{R}$ is the appetite function of Agent i to manage project i which satisfies a technical, but not too restrictive, assumption (see Assumption **(G)** in Appendix A). For instance, $\Gamma_i(x) = \gamma_i \left(x^i - \frac{1}{N-1} \sum_{j \neq i} x^j \right)$, where γ_i is the appetite of Agent i toward the project i compared to the other projects.

We recall that any Agent can accept or reject the contract proposed by the Principal through a reservation utility constrain. More exactly, we denote by \mathcal{C} a general set of *admissible*⁴ contracts proposed by the Principal to the Agents, which remains to put technical assumptions ensuring that all the mathematical objects used in this paper are well-defined and such that for any panel of contracts $\xi := (\xi^i)_{1 \leq i \leq N}$ the following reservation utility constrains are satisfied

$$\sup_{a \in \mathcal{A}} U_0^i(\xi^i, a) \geq R_0^i, \quad 1 \leq i \leq N, \quad (2.3)$$

with a reservation utility $R_0^i \in \mathbb{R}$ for the i th Agent. We denote $R_0 := (R_0^i)_{1 \leq i \leq N}$ the vector of reservation utilities.

An interesting problem concerns the intrinsic type of best reaction efforts provided by the Agents. Two different approaches can be investigated.

One can assume that each Agent provides an optimal effort in view of the actions of the others. In this case, each Agent aims at finding a best reaction effort given both a salary proposed by the Principal and other Agents efforts. This situation typically fits with a non-cooperative game between Agents which reach to find stable equilibria of type Nash equilibria. This problem was well investigated in [11] by proving that Agents play an equilibrium in view of performance of others players. Mathematically⁵, we define a Nash equilibrium for the Agents by the following

Definition 2.1 (Nash equilibrium). *Given an admissible contract $\xi \in \mathcal{C}$, a Nash equilibrium for the N Agents is an admissible effort $e^*(\xi) \in \mathcal{A}$ such that for any $i \in \{1, \dots, N\}$*

$$\sup_{e \in \mathcal{A}^i((e^*(\xi))^{:-i})} U_0^i(\xi^i, a \otimes_i (e^*(\xi))^{:-i}) = U_0^i(\xi^i, (e^*(\xi))^{:-i} \otimes_i (e^*(\xi))^{:-i}). \quad (2.4)$$

The main problem of Nash equilibrium is that it is not optimal in general, from a welfare economics point of view, of interacting entities (see the prisoner's dilemma in Table 2). It is why in this paper we will mainly focus on Pareto optima, mathematically defined by

Definition 2.2 (Pareto optimum). *An admissible action $a^* \in \mathcal{A}$ is Pareto optimal if it does not exist $a \in \mathcal{A}$ such that $U_0^i(\xi, a^*) \leq U_0^i(\xi, a)$ for any $i = 1, \dots, N$ and $U_0^i(\xi, a^*) < U_0^i(\xi, a)$ for at least one index i .*

The studied model coincides with a Principal/multi-Agents problem in which an exogenous entity, called the Planner, forces the Agents to act for the global interest. Given an optimal remuneration, the Planner aims at maximizing the global utility of the N -Agents by finding a Pareto equilibrium. Then, taking the best reaction effort of the Agents into account, the Principal chooses among the set of contracts those who maximize her utility.

⁴See Appendix A for mathematical details with the definition of \mathcal{C} .

⁵See paragraphs Notations and General model and definitions in Appendix A for the definitions of the operator \otimes and the space of admissible best reaction \mathcal{A}^i with (A.3).

3 Some reminders on the no-Planner model

We assume that no Planner intervenes in the Stackelberg games between Agents and the Principal and we extend merely the main results in [11] to the case of separable utilities and we omit the proof since they follow exactly the same lines that [11]. In that case, we consider rational Agents reaching a Nash equilibrium given an admissible sequence of contracts $(\xi^i)_{1 \leq i \leq N}$, which corresponds to the classical second-best case. Intuitively, we begin to set the paradigm of any i th Agent. Given a salary ξ^i given by the Principal and an effort $a^{i,-i}$ provided by the others, Agent i aims at solving

$$U_0^i(\xi, a^{i,-i}) := \sup_{a \in \mathcal{A}^i(a^{i,-i})} \mathbb{E}^{\mathbb{P}^{a \otimes_i a^{i,-i}}} \left[U_A^i(\xi^i) + \Gamma_i(X_T) - \int_0^T k^i(t, X_t, a^{i,i}) dt \right]. \quad (3.1)$$

Following the same computations than those in [11], by using martingale representation theorems, Problem (3.1) remains to solve

$$\mathcal{Y}_t^i = U_A^i(\xi^i) + \Gamma_i(X_T) + \int_t^T \sup_{a \in \mathcal{A}^i(a^{i,-i})} \{ b(s, X_s, a \otimes_i a^{i,-i}) \cdot \mathcal{Z}_s^i - k^i(s, X_s, a) \} ds - \int_t^T \mathcal{Z}_s^i \cdot \Sigma_s dW_s, \quad (3.2)$$

and by denoting a_{NA}^* the maximizer of the generator of this BSDE, we deduce that the A^N -valued process $a_{NA}^*(\cdot, X_\cdot, \mathcal{Z}^i, a^{i,-i})$ is the best reaction effort of Agent i given a salary ξ^i and effort of the other players $a^{i,-i}$, where $(\mathcal{Y}^i, \mathcal{Z}^i)$ is the unique solution to BSDE (3.2), under technical assumption stated in Appendix A.

3.1 Nash equilibrium and multidimensional BSDE

Now, each Agent computes his best reaction effort at the same time. We thus have to assume that for any $(t, z, x) \in [0, T] \times \mathcal{M}_N(\mathbb{R}) \times \mathbb{R}^N$ there exists a fixed point $a_{NA}^*(t, z, x) \in \mathcal{M}_N(A)$ inspiring by the best reaction effort of Agent i made before. We consider that the following assumption holds in the following

Assumption 3.1. *For any $(t, z, x) \in [0, T] \times \mathcal{M}_N(\mathbb{R}) \times \mathbb{R}^N$, there exists $a_{NA}^*(t, z, x) \in \mathcal{M}_N(A)$ such that*

$$(a_{NA}^*)^{i,i}(t, z, x) \in \arg \max_{a \in A^N} \left\{ \sum_{j=1}^N b^j(t, x, (a \otimes_i (a_{NA}^*)^{j,-i}(t, z, x))^{j,:}) z^{j,i} - k^i(t, x, a) \right\}.$$

We denote by $\mathcal{A}_{NA}^*(t, z, x)$ the set of fixed point $a_{NA}^*(t, z, x)$.

As emphasized by [11], finding a Nash equilibrium for any sequence of salaries $(\xi^i)_{1 \leq i \leq N}$ is strongly linked to find a solution to the following multidimensional BSDE

$$\mathcal{Y}_t = (U_A^i(\xi^i))_{1 \leq i \leq N} + (\Gamma_i(X_T))_{1 \leq i \leq N} + \int_t^T f_{NA}^*(s, X_s, \mathcal{Z}_s) ds - \int_t^T \mathcal{Z}_s^\top \Sigma_s dW_s, \quad (3.3)$$

with

$$f_{NA}^{*,i}(t, x, z) := \sum_{j=1}^N b^j(t, x, (a_{NA}^*)^{j,:}) z^{j,i} - k^i(t, x, (a_{NA}^*)^{i,i}), \quad 1 \leq i \leq N.$$

We recall the following definition of a solution to BSDE (3.3)

Definition 3.1. A solution of BSDE (3.3) is a pair $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^N) \times (\mathcal{S}^\infty(\mathbb{R}^N) \times \mathcal{H}_{\text{BMO}}^m(\mathbb{R}^N))$ for some $m > 1$ satisfying Relation (3.3).

Similarly to Theorem 4.1 in [11], we can now state that there is a one to one correspondance between a Nash equilibrium and a solution to BSDE (3.3). The proof of this result follows exactly the same lines that the proof of Theorem 4.1 in [11].

Theorem 3.1. There exists a Nash equilibrium $e^*(\xi)$ if and only if there is a unique solution of BSDE (3.3) in the sense of Definition 3.1. In this case, we have $e_t^*(\xi) \in \mathcal{A}_{NA}^*(t, X_t, Z_t)$ and conversely, any $a_{NA} \in \mathcal{A}_{NA}^*(t, X_t, Z_t)$ is a Nash equilibrium.

Remark 3.1. The previous theorem emphasizes that the existence of a Nash equilibrium is connected to the existence of a unique solution in the sense of Definition 3.1. As explained in [14], for instance, the problem is mathematically ill-posed in general and [11] circumvents this problem by imposing the existence of a solution to BSDE (3.1) in the definition of admissible contracts. However, as soon as the class of admissible contracts is sufficiently regular, recent results [41, 15] can be applied to ensure that the multidimensional BSDE (3.3) is well-posed. We refer to Appendix B for more details on this class of admissible contracts and Proposition B.1 which gives conditions ensuring that there exists a unique solution in the sense of Definition 3.1 of this BSDE.

3.2 The problem of the Principal and multidimensional HJB equation

As previously, this section is very informal to allege the reading since it is a mere extension of [11] to separable and general utilities. The results are again completely similar to those in [11] and can be proved by following exactly the same lines. For the sake of readability, we prefer to omit it to focus on the proofs of Section 4 below, which is the real contribution of this paper.

Assume that BSDE (3.3) admits a unique solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^N) \times (\mathcal{S}^\infty(\mathbb{R}^N) \times \mathcal{H}_{\text{BMO}}^m(\mathbb{R}^N))$ for some $m > 1$. Let a_{NA}^* be a Nash equilibrium selected by the Agents (see [11, Section 4.1.4] for some selection criterion) or the Principal⁶ if the Agents cannot select it. Recall that the Principal aims at solving

$$U_0^{P,NA} = \sup_{\xi=(\xi^i)_{1 \leq i \leq N} \in \mathcal{C}} \mathbb{E}^{\mathbb{P}^{a_{NA}^*(\cdot, X, Z)}} \left[\mathcal{U}_P \left(\ell(X_T) - \sum_{i=1}^N \xi^i \right) \right]. \quad (3.4)$$

As explained in [35, 6, 7], by mimicking [11, Section 4.2] and in view of the decomposition (3.3) of admissible contracts, one can show⁷ that solving this problem remains to solve a stochastic control problem with two state variables: the value of the firm X and the value function of

⁶In fact, if the Agents cannot select a Nash equilibrium, then the problem of the Principal is

$$U_0^{P,NA} = \sup_{\substack{\xi=(\xi^i)_{1 \leq i \leq N} \in \mathcal{C} \\ a_{NA}^* \in \mathfrak{A}_{NA}^*(X, Z)}} \mathbb{E}^{\mathbb{P}^{a_{NA}^*(\cdot, X, Z)}} \left[\mathcal{U}_P \left(\ell(X_T) - \sum_{i=1}^N \xi^i \right) \right],$$

where \mathfrak{A}_{NA}^* is the restriction of \mathcal{A}^* with selection criterion for the Agents, and the results below are completely similar. It is why for the sake of simplicity, we assume that only one Nash equilibrium is selected here, to allege the reading.

⁷See Section 4.2 which provides a path to do it or [11] for exponential utilities.

the i th Agent for any $i \in \{1, \dots, N\}$. This suggests to introduce the following Hamiltonian $H^{NA} : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\begin{aligned} H^{NA}(t, x, p, q, P, Q, R) := & \sup_{z \in \mathcal{M}_N(\mathbb{R})} \left\{ b(t, x, a_{NA}^*(t, x, z)) \cdot p + f_{NA}^*(t, x, z) \cdot q \right. \\ & + \frac{1}{2} \text{Tr} \left(\Sigma(t, x) \Sigma(t, x)^\top P \right) + \frac{1}{2} \text{Tr} \left(z^\top \Sigma(t, x) \Sigma(t, x)^\top z Q \right) \\ & \left. + \text{Tr} \left(\Sigma(t, x) \Sigma(t, x)^\top z R \right) \right\}. \end{aligned}$$

Thus, the HJB equation associated with (3.4) is

$$\text{(HJB-Na)} \begin{cases} -\partial_t v(t, x, y) - H^{NA}(t, x, \nabla_x v, \nabla_y v, \Delta_{xx} v, \Delta_{xy} v, \Delta_{yy} v) = 0, & (t, x, y) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \\ v(T, x, y) = \mathcal{U}_P \left(\ell(x) - \sum_{i=1}^N (U_A^i)^{(-1)}(y^i - \Gamma_i(x)) \right). \end{cases}$$

As usual, using a classical verification theorem (see for instance [39]) we get the following result

Theorem 3.2. *Assume that PDE (HJB-Na) admits a unique solution v continuously differentiable with respect to t and twice continuously differentiable with respect to its spaces variables and that for any $t, x, p, q, P, Q, R \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R})$, the supremum in $H^{NA}(t, x, p, q, P, Q, R)$ is attained for (at least) one $z_{NA}^*(t, x, p, q, P, Q, R)$. Moreover, for any $Y_0^i \geq R_0^i$, $1 \leq i \leq N$, we assume that the following coupled SDE*

$$\begin{cases} X_t^* & = x + \int_0^t \Sigma(s, X_s^*) dW_s, \\ Y_t^{*, Y_0} & = Y_0 - \int_0^t f_{NA}^*(s, X_s^*, z_s^{*, NA}) ds + \int_0^t (z_s^{*, NA})^\top \Sigma(s, X_s^*) dW_s, \end{cases}$$

admits a unique solution (X^*, Y^{*, Y_0}) where

$$z_t^{*, NA} := z_{NA}^*(t, X_t^*, \nabla_x v, \nabla_y v, \Delta_{xx} v, \Delta_{yy} v, \Delta_{xy} v).$$

If moreover $z^{*, NA} \in \mathcal{H}_{BMO}^m(\mathcal{M}_N(\mathbb{R}))$ and $(U_A^i)^{(-1)}((Y_T^{*, R_0})^i - \Gamma_i(X_T^*)) \in \mathcal{C}$ for any $1 \leq i \leq N$, then

$$\xi := \left((U_A^i)^{(-1)}((Y_T^{*, R_0})^i - \Gamma_i(X_T^*)) \right)_{1 \leq i \leq N}$$

is an optimal contract which solves the Principal problem (3.4) with

$$U_0^P = v(0, x, R_0).$$

4 Multi Objective Optimization Problem with separable preferences

Assume now that the Agent does not manage their efforts which are chosen by an exogenous Planner who acts for the welfare of the Agents. We fix a sequence of admissible contracts $\xi := (\xi^i)_{1 \leq i \leq N}$. As before, in all this section, we work under technical assumptions on the coefficients b, k given in Appendix A (see Assumption A.1).

4.1 Solving the multi objective problem using scalarization

Let $(\lambda_i)_{1 \leq i \leq N}$ be a sequence of nonnegative reals such that $\sum_{i=1}^N \lambda_i = 1$ and consider the multi-objective optimization problem

$$\sup_{a \in \mathcal{A}} \sum_{i=1}^N \lambda_i \mathbb{E}^{\mathbb{P}^a} \left[U_A^i(\xi^i) + \Gamma_i(X_T) - \int_0^T k^i(t, X_t, a_t^{:,i}) dt \right]. \quad (4.1)$$

The following proposition gives sufficient conditions to find Pareto optima through solutions to the MOOP.

Proposition 4.1 (Theorem 3.1.2 in [25]). *If there exists $\lambda_i > 0$ for any $i \in \{1, \dots, N\}$ with $\sum_{i=1}^N \lambda_i = 1$ such that the multi-objective optimization problem (4.1) has a solution denoted by $a^*(\xi, \lambda_1, \dots, \lambda_N)$, then $a^*(\xi, \lambda_1, \dots, \lambda_N)$ is Pareto optimal.*

The coefficient λ_i can be seen as the part chosen by the Planner of the utility of Agent i to maximize the general weighted utility of the Agents.⁸ Fix a sequence $\lambda := (\lambda_i)_{1 \leq i \leq N}$ with $\lambda_i > 0$, $i = 1, \dots, N$. We set for any $(t, x, a) \in [0, T] \times \mathbb{R}^N \times \mathcal{M}_N(A)$

$$U_A^\lambda(\xi) := \sum_{i=1}^N \lambda_i U_A^i(\xi^i), \quad \Gamma^\lambda(x) := \sum_{i=1}^N \lambda_i \Gamma_i(x), \quad k^\lambda(t, x, a) := \sum_{i=1}^N \lambda_i k^i(t, x, a^{:,i}),$$

Problem (4.1) thus becomes

$$u_0^\lambda(\xi) := \sup_{a \in \mathcal{A}} u_0^\lambda(\xi, a), \quad (4.2)$$

with

$$u_t^\lambda(\xi, a) := \mathbb{E}^{\mathbb{P}^a} \left[U_A^\lambda(\xi) + \Gamma^\lambda(X_T) - \int_t^T k^\lambda(s, X_s, a_s) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

We consider the following BSDE for any $a \in \mathcal{A}$

$$Y_t^{\lambda,a} = U_A^\lambda(\xi) + \Gamma^\lambda(X_T) + \int_t^T \left(b(s, X_s, a_s) \cdot Z_s^{\lambda,a} - k^\lambda(s, X_s, a_s) \right) ds - \int_t^T Z_s^{\lambda,a} \cdot dX_s. \quad (4.3)$$

We thus have the following Lemma, whose the proof is postponed to Appendix C

Lemma 4.1. *BSDE (4.3) admits a unique solution $(Y^{\lambda,a}, Z^{\lambda,a}) \in \mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ such that*

$$Y_t^{\lambda,a} = u_t^\lambda(\xi, a).$$

Let $\mathcal{A}^*(x, z, \lambda)$ be defined for any $(x, z, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, 1)^N$ by

$$\mathcal{A}^*(x, z, \lambda) := \left\{ \alpha \in \mathcal{A}, \alpha_s \in \operatorname{argmax}_{a \in \mathcal{M}_N(A)} \left\{ b(s, x, a) \cdot z - k^\lambda(s, x, a) \right\}, \text{ for a.e. } s \in [0, T] \right\}.$$

We define for any $a^*(x, z, \lambda) \in \mathcal{A}^*(x, z, \lambda)$

$$f^*(s, x, z, \lambda) := b(s, x, a^*(x, z, \lambda)) \cdot z - k^\lambda(s, x, a^*(x, z, \lambda)).$$

⁸See [23, Proposition 16.F.1], λ^i can also be seen as the inverse of the marginal utility of Agent i .

We consider the following BSDE

$$Y_t^\lambda = U_A^\lambda(\xi) + \Gamma^\lambda(X_T) + \int_t^T f^*(s, X_s, Z_s^\lambda, \lambda) ds - \int_t^T Z_s^\lambda \cdot dX_s \quad (4.4)$$

The following result solves the problem of the Planner and we refer to Appendix C for its proof.

Theorem 4.1. *BSDE (4.4) admits a (unique) solution denoted by $(Y^\lambda, Z^\lambda) \in \mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ such that*

$$Y_0^\lambda = u_0^\lambda(\xi),$$

and any a^* in $\mathcal{A}^*(X, Z^\lambda, \lambda)$ is Pareto optimal for any $\lambda \in (0, 1)^N$ such that $\lambda \cdot \mathbf{1}_N = 1$.

Remark 4.1. *Concerning the choice of the parameter λ by the Planner, one can assume that the Planner is penalized by a bad choice of Pareto optima given some external criteria. For instance, the Planner could be forced to choose a class of λ depending on the performance of any Agents relatively to the others, e.g. $\lambda_i \in [\frac{k^{ii}}{\sum_{j=1}^N k^{ji}} \pm \varepsilon_i]$ for some $\varepsilon_i > 0$ with $k^{ji} > 0$ the underlying cost of the Agent i to manage project j . Mathematically, the Planner aims at solving for instance*

$$\min_{\lambda \in (0,1)^N} \mathbb{E}^{\mathbb{P}^{a^*(X, Z^\lambda, \lambda)}} \left[\int_0^T c^{Pe}(s, X_s, \lambda) ds \right],$$

for some cost function c^{Pe} depending on the characteristics of the Agents. An other section criterion could be to take any parameter λ and the corresponding Pareto optima which are Nash equilibria. Thus, we deal in the following with a parameter λ^* chosen by the Planner due to some selection criterions. If this λ^* is not unique, we assume that the Principal have the final say on this choice and maximizes her utility over all the selected Pareto optima. We also assume that in this case that $\mathcal{A}^*(X, Z^{\lambda^*}, \lambda^*)$ is reduced to a singleton for the sake of simplicity (if not and as usual, the Principal also maximizes her utility on any Pareto optima with parameter λ^*).

4.2 Characterization of the set of admissible contracts

Let (Y^λ, Z^λ) be the (unique) solution to BSDE (4.4). We fix λ and the Pareto optimum a^* in $\mathcal{A}^*(X, Z^\lambda, \lambda)$ (reduced to a singleton for the sake of simplicity as explained in Remark 4.1). We consider the following BSDE

$$Y_t^i = U_A^i(\xi^i) + \Gamma_i(X_T) + \int_t^T \left(b(s, X_s, a^*(s, X_s, Z_s^\lambda, \lambda)) \cdot Z_s^i - k^i(s, X_s, (a^*(s, X_s, Z_s^\lambda, \lambda))^{:,i}) \right) ds - \int_t^T Z_s^i \cdot dX_s \quad (4.5)$$

We thus have the fundamental following Proposition and we refer to Appendix C for its proof

Proposition 4.2. *For any $i = 1, \dots, N$, BSDE (4.5) admits a unique solution $(Y^i, Z^i) \in \mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ such that*

$$Y_0^i = U_0^i(\xi, a^*(s, X_s, Z_s^\lambda, \lambda)).$$

Besides, let (Y^λ, Z^λ) be the (unique) solution of BSDE (4.4), we have

$$Y^\lambda = \sum_{i=1}^N \lambda_i Y^i, \quad Z^\lambda = \sum_{i=1}^N \lambda_i Z^i$$

and the following decomposition for any admissible contracts holds for any $1 \leq i \leq N$

$$U_A^i(\xi^i) = Y_0^i - \Gamma_i(X_T) - \int_0^T \left(b(s, X_s, a^*(s, X_s, Z_s^\lambda, \lambda)) \cdot Z_s^i - k^i(s, X_s, (a^*(s, X_s, Z_s^\lambda, \lambda))^{:,i}) \right) ds + \int_0^T Z_s^i \cdot dX_s. \quad (4.6)$$

In view of Proposition 4.2 above, we have

$$Y_t^i = Y_0^i + \int_0^t k^i(s, X_s, (a^*(s, X_s, Z_s^\lambda, \lambda))^{:,i}) ds + \int_0^t Z_s^i \cdot \Sigma(s, X_s) dW_s^*, \quad Y_T^i = U_A^i(\xi^i) + \Gamma_i(X_T).$$

This suggests to introduce a set of contracts Ξ as the set of random variables $(U_A^i)^{(-1)}(Y_T^{i,Z^i,Y_0} - \Gamma_i(X_T))$ with $i \in \{1, \dots, N\}$ defined by

$$Y_t^{i,Z^i,Y_0} = Y_0^i + \int_0^t k^i(s, X_s, (a^*(s, X_s, Z_s^\lambda, \lambda))^{:,i}) ds + \int_0^t Z_s^i \cdot \Sigma(s, X_s) dW_s^*, \quad Z^\lambda := \sum_{i=1}^N \lambda_i Z^i$$

for any controls $((Z^i)_{1 \leq i \leq N}, Y_0) \in \mathbb{D}(\mathbb{R}^N) \times ([R_0^i, +\infty))_{1 \leq i \leq N}$, where $\mathbb{D}(\mathbb{R}^N)$ denotes the set of controls \mathbb{F} -predictable processes $(Z^i)_{1 \leq i \leq N} \in \mathcal{H}^2(\mathbb{R}^N)$ such that $(U_A^i)^{(-1)}(Y_T^{i,Z^i,Y_0} - \Gamma_i(X_T)) \in \mathcal{C}$.

We set an \mathbb{F} -predictable process Z with values in $\mathcal{M}_N(\mathbb{R})$ such that its coefficient $Z^{i,j}$ is the i th element of the vector process Z^j . In this case we define the \mathbb{R}^N -valued process

$$Y_t^{Z,Y_0} := Y_0 + \int_0^t K(s, X_s, (a^*(s, X_s, Z_s, \lambda))) ds + \int_0^t Z_s^\top \Sigma(s, X_s) dW_s^*,$$

by setting (with an abuse of notation but justified by Proposition 4.2)

$$a^*(s, X_s, Z_s, \lambda) := a^*(s, X_s, Z_s^\lambda, \lambda) \quad (4.7)$$

with $Z^\lambda = \left(\sum_{i=1}^N \lambda_i Z^{j,i} \right)_{1 \leq j \leq N}$ and

$$K(s, X_s, (a^*(s, X_s, Z_s, \lambda))) := (k^i(s, X_s, (a^*(s, X_s, Z_s, \lambda))^{:,i}))_{1 \leq i \leq N}^\top.$$

In this case, from Proposition 4.2, we get⁹

$$\mathcal{C} = \Xi. \quad (4.8)$$

In the following we will denote $\mathbb{M}_N(\mathbb{R}^N)$ the set of \mathbb{F} -predictable process Z with values in $\mathcal{M}_N(\mathbb{R})$ such that $Z^{:,i} \in \mathbb{D}(\mathbb{R}^N)$ for any $1 \leq i \leq N$.

4.3 The general problem of the Principal with an exogenous Planner

Using the notation (4.7), we recall that the Principal solves

$$U_0^P = \sup_{\xi = (\xi^i)_{1 \leq i \leq N} \in \mathcal{C}} \mathbb{E}^{\mathbb{P}^{a^*(X, Z^\lambda, \lambda)}} \left[\mathcal{U}_P \left(\ell(X_T) - \sum_{i=1}^N \xi^i \right) \right], \quad (4.9)$$

⁹The inclusion $\Xi \subset \mathcal{C}$ is in the definition of Ξ . The inclusion $\mathcal{C} \subset \Xi$ comes from (4.6) by using martingale representation.

From Proposition 4.2 and Characterization (4.8), Problem (4.9) becomes

$$U_0^P = \sup_{\substack{Z \in \mathbb{M}_N(\mathbb{R}^N) \\ Y_0^i \geq R_0^i}} \mathbb{E}^{\mathbb{P}^{a^*(X,Z,\lambda)}} \left[\mathcal{U}_P \left(\ell(X_T) - \sum_{i=1}^N (U_A^i)^{-1} (Y_T^{i,Z^i,i,Y_0^i} - \Gamma_i(X_T)) \right) \right]. \quad (4.10)$$

Since \mathcal{U}_P and $(U_A^i)^{-1}$ are increasing, one can explicitly get the optimal Y_0^i for any i by saturating the reservation utility constrains of the Agents, *i.e.*, $Y_0^{i,*} = R_0^i$. Thus, Problem (4.10) becomes

$$U_0^P = \sup_{Z \in \mathbb{M}_N(\mathbb{R}^N)} \mathbb{E}^{\mathbb{P}^{a^*(X,Z,\lambda)}} \left[\mathcal{U}_P \left(\ell(X_T) - \sum_{i=1}^N (U_A^i)^{-1} (Y_T^{i,Z^i,i,R_0^i} - \Gamma_i(X_T)) \right) \right]. \quad (4.11)$$

As emphasized above and as an extension of [35, 6, 7], solving (4.11) remains to solve a stochastic control problem with

- control variable: $Z \in \mathbb{M}_N(\mathbb{R})$,
- two state variables: the value of the firm X and the value function of the i th Agent Y^{i,Z^i,i,R_0^i} for any $i \in \{1, \dots, N\}$.

We define $H^{Par} : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times (0, 1)^N \rightarrow \mathbb{R}$ by

$$\begin{aligned} H^{Par}(t, x, p, q, P, Q, R, \lambda) := & \sup_{z \in \mathcal{M}_N(\mathbb{R})} \left\{ b(t, x, a^*(t, x, z, \lambda)) \cdot p + K(t, x, (a^*(t, x, z, \lambda))) \cdot q \right. \\ & + \frac{1}{2} \text{Tr} \left(\Sigma(t, x) \Sigma(t, x)^\top P \right) + \frac{1}{2} \text{Tr} \left(z^\top \Sigma(t, x) \Sigma(t, x)^\top z Q \right) \\ & \left. + \text{Tr} \left(\Sigma(t, x) \Sigma(t, x)^\top z R \right) \right\}. \end{aligned}$$

We thus consider the following HJB equation associated with (4.11)

$$\text{(HJB-Par)} \quad \begin{cases} -\partial_t v(t, x, y) - H^{Par}(t, x, \nabla_x v, \nabla_y v, \Delta_{xx} v, \Delta_{xy} v, \Delta_{yy} v, \lambda) = 0, & (t, x, y) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \\ v(T, x, y) = \mathcal{U}_P \left(\ell(x) - \sum_{i=1}^N (U_A^i)^{-1} (y^i - \Gamma_i(x)) \right). \end{cases}$$

As usual, using a classical verification theorem (see for instance [39]) we get the following result

Theorem 4.2. *Assume that PDE (HJB-Par) admits a unique solution v continuously differentiable with respect to t and twice continuously differentiable with respect to its spaces variables and that for any $t, x, p, q, P, Q, R, \lambda \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times (0, 1)^N$, the supremum in $H^{Par}(t, x, p, q, P, Q, R, \lambda)$ is attained for (at least) one $z^*(t, x, p, q, P, Q, R, \lambda)$. Moreover, for any $Y_0^i \geq R_0^i$, $1 \leq i \leq N$, we assume that the following coupled SDE*

$$\begin{cases} X_t^* &= x + \int_0^t \Sigma(s, X_s^*) dW_s, \\ Y_t^* &= R_0 + \int_0^t \left(K(s, X_s^*, (a^*(s, X_s^*, z_s^*, \lambda))) - (z_s^*)^\top b(s, X_s^*, (a^*(s, X_s^*, z_s^*, \lambda))) \right) ds \\ &+ \int_0^t (z_s^*)^\top \Sigma(s, X_s^*) dW_s, \end{cases}$$

admits a unique solution (X^*, Y^*) where

$$z_t^* := z^*(t, X_t^*, \nabla_x v, \nabla_y v, \Delta_{xx} v, \Delta_{yy} v, \Delta_{xy} v, \lambda).$$

If moreover $(z^*)^{\cdot i} \in \mathbb{D}(\mathbb{R}^N)$ then

$$\xi := \left((U_A^i)^{-1} ((Y_T^*)^i - \Gamma_i(X_T^*)) \right)_{1 \leq i \leq N} \in \mathcal{C}$$

is an optimal contract which solves the Principal problem (4.9), with

$$U_0^P = v(0, x, R_0).$$

5 Application to a bidimensional linear-quadratic model

We compare the case where a Planner intervenes in the Stackelberg equilibrium between the Principal and the Agent with the case in which no-Planner acts in the linear-quadratic model developed in [11].

5.1 The model and characterization of Pareto optima

Let $a \in \mathcal{M}_2(A)$, $b : \mathcal{M}_2(\mathbb{R}) \rightarrow \mathbb{R}^2$ and $k : \mathcal{M}_2 \rightarrow \mathbb{R}^2$.

$$dX_t = b(a_t)dt + dW_t^a$$

$a^{i,j}$ action of Agent j on the project i with

$$b(a) = \begin{pmatrix} a^{11} - a^{12} \\ a^{22} - a^{21} \end{pmatrix}, \quad k(a) = \begin{pmatrix} \frac{k^{11}}{2}|a^{11}|^2 + \frac{k^{21}}{2}|a^{21}|^2 \\ \frac{k^{22}}{2}|a^{22}|^2 + \frac{k^{12}}{2}|a^{12}|^2 \end{pmatrix}.$$

We also assume that there exists $\gamma_1, \gamma_2 > 0$ such that for any $i \in \{1, 2\}$

$$\Gamma_i(x) = \Gamma_i \cdot x, \quad x \in \mathbb{R}^2, \quad j \neq i,$$

with

$$\Gamma_1 = \gamma(1, -1)^\top, \quad \Gamma_2 = \gamma(-1, 1)^\top, \quad \gamma \geq 0.$$

Then, for any parameter $\lambda = (\lambda_1, \lambda_2) \in (0, 1)^2$ with $\lambda_1 + \lambda_2 = 1$, we have for any $z := (z^{i,j})_{1 \leq i, j \leq 2} \in \mathcal{M}_2(\mathbb{R})$

$$a^*(z, \lambda) = \begin{pmatrix} \frac{\lambda_1 z^{11} + \lambda_2 z^{12}}{\lambda_1 k^{11}} & -\frac{\lambda_1 z^{11} + \lambda_2 z^{12}}{\lambda_2 k^{12}} \\ -\frac{\lambda_1 z^{21} + \lambda_2 z^{22}}{\lambda_1 k^{21}} & \frac{\lambda_1 z^{21} + \lambda_2 z^{22}}{\lambda_2 k^{22}} \end{pmatrix}$$

5.2 Problem of the Principal under the action of a general Planner

$$U_0^P(\lambda) := \sup_{Z \in \mathcal{M}_2(\mathbb{R})} \mathbb{E}^{\mathbb{P}^{a^*(Z, \lambda)}} \left[-e^{-R_P(X_T \cdot \mathbf{1}_N - \xi^1 - \xi^2)} \right].$$

We get

$$\begin{aligned} U_0^P(\lambda) &= \sup_{Z \in \mathcal{M}_2(\mathbb{R})} \mathbb{E}^* \left[-e^{-R_P \left(\int_0^T (b^1 + b^2)(a^*(Z, \lambda)) ds + \int_0^T \mathbf{1}_2 \cdot dW_s^* - \int_0^T (k^1 + k^2)(a^*(Z, \lambda)) ds - \int_0^T (Z^{:,1} + Z^{:,2}) \cdot dW_s^* \right)} \right] \\ &= \sup_{Z \in \mathcal{M}_2(\mathbb{R})} \mathbb{E}^* \left[-\mathcal{E} \left(-R_P \int_0^T (\mathbf{1}_2 - (Z_s^{:,1} + Z_s^{:,2})) \cdot dW_s^* \right) e^{-R_P \int_0^T g(\lambda, Z_t) dt} \right] \end{aligned}$$

where $\mathbb{E}^* := \mathbb{E}^{\mathbb{P}^{a^*(z, \lambda)}}$ and $W^* := W^{a^*(z, \lambda)}$ and with, by setting $\bar{\lambda} := 1 - \lambda$

$$\begin{aligned}
g(\lambda, z) &= -\frac{R_P}{2} (z^{11} + z^{12} - 1)^2 - \frac{R_P}{2} (z^{22} + z^{21} - 1)^2 \\
&+ \left(\frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\lambda k^{11}} + \frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\bar{\lambda} k^{12}} \right) + \left(\frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\bar{\lambda} k^{22}} + \frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\lambda k^{21}} \right) \\
&- \frac{k^{11}}{2} \left| \frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\lambda k^{11}} \right|^2 - \frac{k^{21}}{2} \left| \frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\lambda k^{21}} \right|^2 - \frac{k^{22}}{2} \left| \frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\bar{\lambda} k^{22}} \right|^2 - \frac{k^{12}}{2} \left| \frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\bar{\lambda} k^{12}} \right|^2.
\end{aligned}$$

One can show after tedious but easy computations that g is coercive in z^1 and z^2 and convex. The first order conditions are then given by

$$(\text{FOC}) \begin{cases} R_P(z^{11} + z^{12} - 1) + \left(\frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\lambda k^{11}} \right) + \frac{\lambda}{\bar{\lambda}} \left(\frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\bar{\lambda} k^{12}} \right) = \frac{1}{k^{11}} + \frac{\lambda}{\bar{\lambda} k^{12}} \\ \left(\frac{\lambda}{\bar{\lambda}} - 1 \right) R_P(z^{11} + z^{12} - 1) = 0 \\ R_P(z^{22} + z^{21} - 1) + \left(\frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\bar{\lambda} k^{22}} \right) + \frac{\bar{\lambda}}{\lambda} \left(\frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\lambda k^{21}} \right) = \frac{1}{k^{22}} + \frac{\bar{\lambda}}{\lambda k^{21}} \\ \left(\frac{\bar{\lambda}}{\lambda} - 1 \right) R_P(z^{22} + z^{21} - 1) = 0. \end{cases}$$

This implies to distinguish $\lambda = \frac{1}{2}$ or not.

5.2.1 Pareto optima and Optimal contracts with an exogenous Planner

For any $\lambda \neq \frac{1}{2}$ fixed by an exogenous Planner, (FOC) gives

$$\begin{aligned}
g(\lambda, z) &= \left(\frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\lambda k^{11}} + \frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\bar{\lambda} k^{12}} \right) + \left(\frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\bar{\lambda} k^{22}} + \frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\lambda k^{21}} \right) \\
&- \frac{k^{11}}{2} \left| \frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\lambda k^{11}} \right|^2 - \frac{k^{21}}{2} \left| \frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\lambda k^{21}} \right|^2 - \frac{k^{22}}{2} \left| \frac{\lambda z^{21} + \bar{\lambda} z^{22}}{\bar{\lambda} k^{22}} \right|^2 - \frac{k^{12}}{2} \left| \frac{\lambda z^{11} + \bar{\lambda} z^{12}}{\bar{\lambda} k^{12}} \right|^2.
\end{aligned}$$

By maximizing the function g , we get the following general Pareto optima conditions

$$\begin{cases} z_{Pe}^{11} &= \frac{|1-\lambda|^2 k^{12}}{|\lambda|^2 k^{11} + |1-\lambda|^2 k^{12}} \\ z_{Pe}^{21} &= \frac{|1-\lambda|^2 k^{22}}{|\lambda|^2 k^{21} + |1-\lambda|^2 k^{22}} \\ z_{Pe}^{22} &= \frac{|\lambda|^2 k^{21}}{|\lambda|^2 k^{21} + |1-\lambda|^2 k^{22}} \\ z_{Pe}^{12} &= \frac{|\lambda|^2 k^{11}}{|\lambda|^2 k^{11} + |1-\lambda|^2 k^{12}} \\ a^*(z_{Pe}, \lambda) &= \begin{pmatrix} \frac{\lambda z_{Pe}^{11} + (1-\lambda) z_{Pe}^{12}}{\lambda k^{11}} & -\frac{\lambda z_{Pe}^{11} + (1-\lambda) z_{Pe}^{12}}{(1-\lambda) k^{12}} \\ -\frac{\lambda z_{Pe}^{21} + (1-\lambda) z_{Pe}^{22}}{\lambda k^{21}} & \frac{\lambda z_{Pe}^{21} + (1-\lambda) z_{Pe}^{22}}{(1-\lambda) k^{22}} \end{pmatrix} \\ U_0^P(\lambda) &= -e^{-R_P T g(\lambda, z_{Pe})} \end{cases}$$

Thus, given a Pareto optimum of parameter $\lambda \in (0, 1) \setminus \{\frac{1}{2}\}$ chosen by the Planner, the optimal

contracts given to Agent 1 and Agent 2 are respectively

$$\begin{aligned}\xi^{1,\star} &= R_0^1 + \frac{T}{2} \left(\frac{|\lambda z_{P_e}^{11} + (1-\lambda)z_{P_e}^{12}|^2}{|\lambda|^2 k^{11}} + \frac{|\lambda z_{P_e}^{21} + (1-\lambda)z_{P_e}^{22}|^2}{|\lambda|^2 k^{21}} \right) \\ &\quad - T z_{P_e}^{11} \left(\frac{\lambda z_{P_e}^{11} + (1-\lambda)z_{P_e}^{12}}{\lambda k^{11}} + \frac{\lambda z_{P_e}^{11} + (1-\lambda)z_{P_e}^{12}}{(1-\lambda)k^{12}} \right) \\ &\quad - T z_{P_e}^{21} \left(\frac{\lambda z_{P_e}^{21} + (1-\lambda)z_{P_e}^{22}}{(1-\lambda)k^{22}} + \frac{\lambda z_{P_e}^{21} + (1-\lambda)z_{P_e}^{22}}{\lambda k^{21}} \right) \\ &\quad + (z_{P_e}^{11} - \gamma)X_T^1 + (z_{P_e}^{21} + \gamma)X_T^2.\end{aligned}$$

and

$$\begin{aligned}\xi^{2,\star} &= R_0^2 + \frac{T}{2} \left(\frac{|\lambda z_{P_e}^{11} + (1-\lambda)z_{P_e}^{12}|^2}{|1-\lambda|^2 k^{12}} + \frac{|\lambda z_{P_e}^{21} + (1-\lambda)z_{P_e}^{22}|^2}{|1-\lambda|^2 k^{22}} \right) \\ &\quad - T z_{P_e}^{12} \left(\frac{\lambda z_{P_e}^{11} + (1-\lambda)z_{P_e}^{12}}{\lambda k^{11}} + \frac{\lambda z_{P_e}^{11} + (1-\lambda)z_{P_e}^{12}}{(1-\lambda)k^{12}} \right) \\ &\quad - T z_{P_e}^{22} \left(\frac{\lambda z_{P_e}^{21} + (1-\lambda)z_{P_e}^{22}}{(1-\lambda)k^{22}} + \frac{\lambda z_{P_e}^{21} + (1-\lambda)z_{P_e}^{22}}{\lambda k^{21}} \right) \\ &\quad + (z_{P_e}^{12} + \gamma)X_T^1 + (z_{P_e}^{22} - \gamma)X_T^2.\end{aligned}$$

Notice that g admits a maximum in $\lambda = \frac{1}{2}$. We can deduce that the case $\lambda = \frac{1}{2}$ coincides with a cooperative Planner with the Principal.

Remark 5.1. *In fact, notice that $\lambda \in (0, 1) \rightarrow g(\lambda, z_{P_e})$ has a continuous extension when $\lambda \rightarrow 0$ or $\lambda \rightarrow 1$, which corresponds to weak Pareto optima (see [25, Theorem 3.1.1]). In this case, we set*

$$g(0) := g(0, z_{P_e}) = \frac{1}{2k^{11}} + \frac{1}{2k^{21}}, \quad g(1) := g(1, z_{P_e}) = \frac{1}{2k^{22}} + \frac{1}{2k^{12}}.$$

5.2.2 Pareto optimum with a cooperative Planner

It remains to the case where the Planner and the Principal cooperate by choosing $\lambda_1 = \frac{1}{2} = \lambda_2$.

The first order conditions (FOC) induces the following Pareto optima conditions

$$\begin{cases} z_{P_c}^{11} + z_{P_c}^{12} = 1 \\ z_{P_c}^{22} + z_{P_c}^{21} = 1 \\ a^{\lambda,\star}(z_{P_c}) = \begin{pmatrix} \frac{1}{k^{11}} & -\frac{1}{k^{12}} \\ -\frac{1}{k^{21}} & \frac{1}{k^{22}} \end{pmatrix} \\ U_0^P = -e^{-R_P T g(0.5)}, \end{cases}$$

with

$$g(0.5) := g(0.5, z_{P_c}^1, z_{P_c}^2) = \left(\frac{1}{2k^{11}} + \frac{1}{2k^{12}} \right) + \left(\frac{1}{2k^{22}} + \frac{1}{2k^{21}} \right).$$

We thus have the following optimal contracts for Agent 1 and Agent 2 respectively

$$\begin{aligned}\xi^{1,\star} &= R_0^1 + T \left(\frac{1}{2k^{11}} + \frac{1}{2k^{21}} \right) - T z_{P_c}^{11} \left(\frac{1}{k^{11}} + \frac{1}{k^{12}} \right) - T z_{P_c}^{21} \left(\frac{1}{k^{21}} + \frac{1}{k^{22}} \right) \\ &\quad + (z_{P_c}^{11} - \gamma)X_T^1 + (z_{P_c}^{21} + \gamma)X_T^2,\end{aligned}$$

and

$$\begin{aligned} \xi^{2,*} = & R_0^2 + T \left(\frac{1}{2k^{12}} + \frac{1}{2k^{22}} \right) - Tz_{Pc}^{12} \left(\frac{1}{k^{11}} + \frac{1}{k^{12}} \right) - Tz_{Pc}^{22} \left(\frac{1}{k^{21}} + \frac{1}{k^{22}} \right) \\ & + (z_{Pc}^{12} + \gamma)X_T^1 + (z_{Pc}^{22} - \gamma)X_T^2. \end{aligned}$$

Interpretations.

- (i) We find the first-best effort cf [11, Theorem 3.3.] which is both a Nash equilibrium in the first best model and a Pareto optimum with a cooperative Planner.
- (ii) We get in the case of a cooperative Planner infinitely number of optimal contracts which differs clearly form the classical case and results in [11].
- (iii) If $k^{11} \leq k^{22}$ and $k^{21} \leq k^{12}$, i.e. Agent 1 is more efficient that Agent 2 to manage his project and to help Agent 2 to manage his project. Take $z_{Pc}^{11} = z_{Pc}^{22} = 1$ each Agent strictly will receive the value of his own project. Fixed part is greater for Agent 1 than for Agent 2.
- (iv) If $k^{12} \geq k^{11}$, i.e. Agent 1 is more efficient with project 1 than Agent 2, the part of the project 1 given to Agent 1 is greater than the part of this project given to Agent 2 and depend of the relative performance of this Agent compared to the others. We get similar results with project 2.
- (v) Notice that for any Pareto coefficients, the optimal effort of each Agent is always a booster of each project. At the equilibrium, each Agent does not penalize the project of the others. This phenomenon indeed appears as soon as the appetence coefficients are different for Agent 1 and Agent 2 as emphasized in [11].
- (vi) Finally, notice that at the optimum, the appetence parameter γ does not impact the value function of the Principal and optimal efforts. More exactly, in this model the Principal proposes optimal contracts to the Agents which cancelled their ambitious in their works and such that ambitions of the Agents have no impact on the value of the Principal. Besides, we recover the suppression of the appetence for competitions of the Agent by the Principal since she penalizes each i th Agent with the amount $-\gamma(X_T^i - X_T^j)$, $i \neq j \in \{1, 2\}$, as in [11].

5.2.3 Comparison with the no-Planner model

We now turn to a comparison between a model in which a Planner manages the effort of the Agents and a model in which Agents are rational entities who aim at finding a Nash equilibrium. Using [11] and results of Section 3, the Nash equilibrium is given for any $z \in \mathcal{M}_2(\mathbb{R})$ by

$$e^*(z) = \begin{pmatrix} \frac{z^{11}}{k^{11} \frac{1}{21}} & -\frac{z^{12}}{k^{12}} \\ -\frac{z^{21}}{k^{21}} & \frac{z^{22}}{k^{22}} \end{pmatrix}$$

The problem of the Principal is then to solve

$$U_0^{P,NA} := \sup_{\xi^1, \xi^2} \mathbb{E}^* \left[-e^{-R_P(X_T \cdot \mathbf{1}_N - \xi^1 - \xi^2)} \right].$$

We get

$$U_0^P = \sup_Z \mathbb{E}^* \left[-\mathcal{E} \left(-R_P \int_0^T (\mathbf{1}_2 - (Z_s^1 + Z_s^2)) \cdot dW_s^* \right) e^{-R_P \int_0^T g_{NA}(Z_t, R_P) dt} \right]$$

with

$$g_{NA}(z, R_P) = -\frac{R_P}{2} (z^{11} + z^{12} - 1)^2 - \frac{R_P}{2} (z^{22} + z^{21} - 1)^2 \\ + \left(\frac{z^{11}}{k^{11}} + \frac{z^{12}}{k^{12}} + \frac{z^{21}}{k^{21}} + \frac{z^{22}}{k^{22}} \right) - \frac{k^{11}}{2} \left| \frac{z^{11}}{k^{11}} \right|^2 - \frac{k^{21}}{2} \left| \frac{z^{21}}{k^{21}} \right|^2 - \frac{k^{22}}{2} \left| \frac{z^{22}}{k^{22}} \right|^2 - \frac{k^{12}}{2} \left| \frac{z^{12}}{k^{12}} \right|^2.$$

Noticing that g^{NA} is coercive and convex, one get from the first order conditions the following solutions

$$\begin{cases} z_{NA}^{11} = \frac{1+R_P k^{12}}{1+R_P(k^{12}+k^{11})} \\ z_{NA}^{12} = \frac{1+R_P k^{11}}{1+R_P(k^{12}+k^{11})} \\ z_{NA}^{22} = \frac{1+R_P k^{21}}{1+R_P(k^{21}+k^{22})} \\ z_{NA}^{21} = \frac{1+R_P k^{22}}{1+R_P(k^{21}+k^{22})} \\ e^*(z) = \begin{pmatrix} \frac{z_{NA}^{11}}{k^{11}} & -\frac{z_{NA}^{12}}{k^{12}} \\ -\frac{z_{NA}^{21}}{k^{21}} & \frac{z_{NA}^{22}}{k^{22}} \end{pmatrix} \\ U_0^{P,NA} = -e^{-R_P T} g_{NA}(z, R_P) \end{cases}$$

In this case, optimal contracts are given by

$$\xi^{1,*} = R_0^1 + T \left(\frac{1}{2k^{11}} + \frac{1}{2k^{21}} \right) - T z_{NA}^{11} \left(\frac{z_{NA}^{11}}{k^{11}} + \frac{z_{NA}^{12}}{k^{12}} \right) - T z_{NA}^{21} \left(\frac{z_{NA}^{21}}{k^{21}} + \frac{z_{NA}^{22}}{k^{22}} \right) \\ + (z_{NA}^{11} - \gamma) X_T^1 + (z_{NA}^{21} + \gamma) X_T^2$$

and

$$\xi^{2,*} = R_0^2 + T \left(\frac{1}{2k^{12}} + \frac{1}{2k^{22}} \right) - T z_{NA}^{12} \left(\frac{z_{NA}^{11}}{k^{11}} + \frac{z_{NA}^{12}}{k^{12}} \right) - T z_{NA}^{22} \left(\frac{z_{NA}^{21}}{k^{21}} + \frac{z_{NA}^{22}}{k^{22}} \right) \\ + (z_{NA}^{12} + \gamma) X_T^1 + (z_{NA}^{22} - \gamma) X_T^2.$$

The result below gives necessary and sufficient conditions for which the Nash equilibrium coincides with a Pareto equilibrium.

Proposition 5.1 (Suffisant and necessary conditions). *Assume that A is convex. The Nash equilibrium e^* is a Pareto optimum if and only if there exists $\lambda^* \in (0, 1)$ such that*

$$\begin{cases} \frac{1 + R_P k^{12}}{1 + R_P(k^{12} + k^{11})} = \frac{\lambda^* z_{Pe}^{11} + (1 - \lambda^*) z_{Pe}^{12}}{\lambda^*} \\ \frac{1 + R_P k^{22}}{1 + R_P(k^{21} + k^{22})} = \frac{\lambda^* z_{Pe}^{21} + (1 - \lambda^*) z_{Pe}^{22}}{\lambda^*} \\ \frac{1 + R_P k^{12}}{1 + R_P k^{11}} = \frac{1 + R_P k^{22}}{1 + R_P k^{21}} = \frac{1 - \lambda^*}{\lambda^*}. \end{cases}$$

In this case e^* coincides with Pareto optimum with parameter λ^* .

Proof. Since A is convex, we notice that $a \mapsto U_0^i(\xi, a)$ is concave since b is linear and k is convex with respect to a and by using comparison of BSDEs (see [5] for instance). Thus, finding a Pareto equilibrium is equivalent to find λ^* such that there exists a solution of the multi-objective problem from [25, Theorem 3.1.7].

Now, a Nash equilibrium e^* is Pareto optimal if and only if there exists $\lambda^* \in (0, 1)$ such that

$$\left\{ \begin{array}{l} \frac{1 + R_P k^{12}}{1 + R_P(k^{12} + k^{11})} = \frac{\lambda^* z_{Pe}^{11} + (1 - \lambda^*) z_{Pe}^{12}}{\lambda^*} \\ \frac{1 + R_P k^{11}}{1 + R_P(k^{12} + k^{11})} = \frac{\lambda^* z_{Pe}^{11} + (1 - \lambda^*) z_{Pe}^{12}}{(1 - \lambda^*)} \\ \frac{1 + R_P k^{22}}{1 + R_P(k^{21} + k^{22})} = \frac{\lambda^* z_{Pe}^{21} + (1 - \lambda^*) z_{Pe}^{22}}{\lambda^*} \\ \frac{1 + R_P k^{21}}{1 + R_P(k^{21} + k^{22})} = \frac{\lambda^* z_{Pe}^{21} + (1 - \lambda^*) z_{Pe}^{22}}{1 - \lambda^*} \end{array} \right.$$

By rewriting these conditions, we get the result. \square

Toy models in which Nash equilibrium are Pareto optimal.

- The Nash equilibrium is Pareto efficient with $\lambda^* = \frac{1}{2}$ if and only if $R_P = 0$ (the Principal is risk-neutral).
- Assume that $k^{11} = k^{21}$ and $k^{22} = k^{12}$, i.e. the cost of the effort of the Agent i is the same towards the project i or the project j . In this case, we necessarily have

$$\lambda^* = \frac{1 + R_P k^{11}}{2 + R_P(k^{11} + k^{22})} \in (0, 1).$$

Then, the Nash equilibrium $e^*(\xi)$ coincides with a Pareto optimum with parameter λ^* if and only if

$$\frac{1 + R_P k^{22}}{1 + R_P(k^{22} + k^{11})} = \frac{(1 - \lambda^*)^2 k^{22} + (1 - \lambda^*) \lambda^* k^{11}}{|\lambda^*|^2 k^{11} + (1 - \lambda^*)^2 k^{22}}.$$

We now turn to a last interesting results remaining to say that the Principal can benefit from the action of a Planner to broke the Stackelberg game between her and her Agents.

Proposition 5.2 (Advantage of adding a Planner). *There exists a (non-empty) set of parameters $\Lambda \subset (0, 1)$ such that $U_0^{P,NA} \leq U_0^P(\lambda)$, for any $\lambda \in \Lambda$, i.e. adding a Planner can improve the value of the Principal.*

Moreover, $U_0^{P,NA} > U_0^P(0)$ and $U_0^{P,NA} > U_0^P(1)$. In other words, a Planner choosing weak Pareto optima does not improve the value function of the Principal compared to a no-Planner model.

Proof. For any $z \in \mathcal{M}_2(\mathbb{R})$, notice that $g_{NA}(z_{NA}, R_P) \leq g(0.5)$. Thus,

$$U_0^{PNA} = -e^{-R_P T g_{NA}(z, R_P)} \leq -e^{-R_P g(0.5)} = U_0^P\left(\frac{1}{2}\right).$$

Thus, by using the continuity with respect to λ of $\lambda \mapsto -e^{-R_P g(\lambda)}$, we deduce that there exists a set $\Lambda \subset (0, 1)$ such that $\frac{1}{2} \in \Lambda$ and $U_0^{PNA} \leq U_0^P(\lambda)$ for any $\lambda \in \Lambda$.

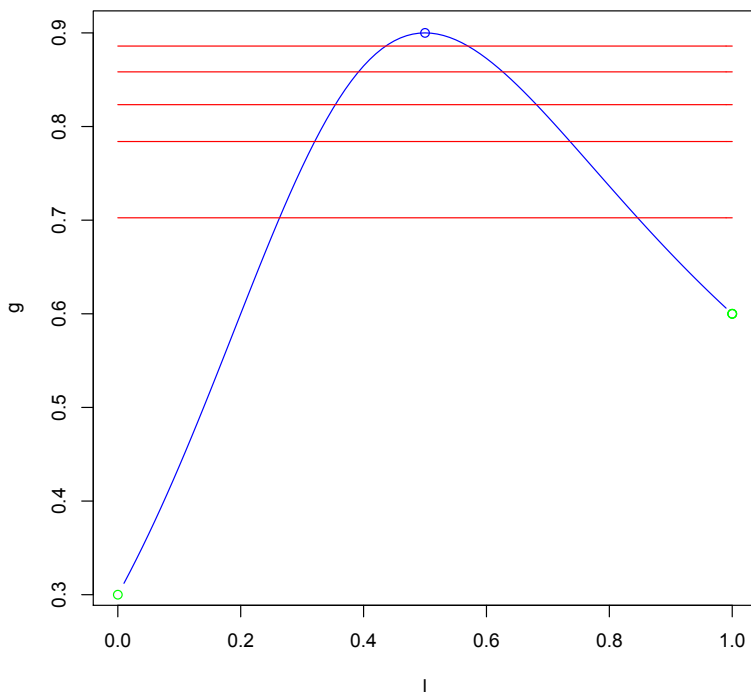
A tedious computation shows that $R > 0 \mapsto g_{NA}(z_{NA}, R)$ is decreasing, and

$$\begin{aligned} \lim_{R \rightarrow +\infty} g_{NA}(z_{NA}, R) &= \frac{k^{12}}{k^{11}+k^{12}} \left(1 - \frac{k^{12}}{2(k^{11} + k^{12})} \right) + \frac{k^{11}}{k^{12}} \left(1 - \frac{k^{11}}{2(k^{11} + k^{12})} \right) \\ &+ \frac{k^{22}}{k^{22}+k^{21}} \left(1 - \frac{k^{22}}{2(k^{22} + k^{21})} \right) + \frac{k^{21}}{k^{22}} \left(1 - \frac{k^{21}}{2(k^{22} + k^{21})} \right) \end{aligned}$$

Thus, by using Remark 5.1, easy computations directly give $\lim_{R \rightarrow +\infty} g_{NA}(z_{NA}, R) > \frac{1}{2k^{11}} + \frac{1}{2k^{21}} = g(0)$ and $\lim_{R \rightarrow +\infty} g_{NA}(z_{NA}, R) > \frac{1}{2k^{12}} + \frac{1}{2k^{22}} = g(1)$. \square

We illustrate this proposition with the Figure 1 below. Notice that $\lambda \mapsto U_0^P(\lambda)$ has the same monotonicity than $\lambda \mapsto g(\lambda, z_{Pe \text{ or } Pc})$. Thus comparing $g(\lambda, z_{Pe \text{ or } Pc})$ with $g_{NA}(R, z_{NA})$ is enough to compare the value functions of the Principal with and without a Planner. We notice in particular that the length of the set Λ is decreasing with the risk aversion of the Principal.

Figure 1: Comparison between $l \in (0, 1) \mapsto g(l, z_{Pe \text{ or } Pc})$ (see the blue curve and the blue point for $l = \frac{1}{2}$) and $g_{NA}(R, z_{NA})$ for $R \in \{0.1, 0.25, 0.5, 1, 50\}$ (see the horizontal red curves) by adding the weak Pareto optima for $l \in \{0, 1\}$ (see the green points). Cost Parameters chosen are $k^{11} = 2$, $k^{22} = 5$, $k^{12} = 1$, $k^{21} = 10$. In this case for any $R > 0$, we have $g_{NA}(z_{NA}, R) \geq \lim_{R \rightarrow +\infty} g_{NA}(z_{NA}, R) = 0.7 > \max(g(0), g(1)) = g(1) = 0.6$. Thus, the set of Pareto optima Λ ensuring a better value for the Principal compared to the case without adding a Planner is composed by all the $\lambda \in (0, 1)$ such that the blue curve is above the red curve with corresponding risk-aversion parameter $R > 0$.



6 Conclusion

In this paper, we have proved that in a linear-quadratic model there exists configurations such that Nash equilibria are Pareto efficient (see Proposition 5.1). We also have seen that it is sometimes advantageous from the Principal point of view, to add a Planner in the model who imposes Pareto optimal actions to the Agents, compared to the case in which the Agents are rational, non-cooperative, and compute themselves their best-reaction effort (see Proposition 5.2). On the contrary, if the Planner chooses a weak Pareto optimum, then the value of the Principal is always worse than a no-Planner model. We would like to propose some extensions to this work for the path of future researches which might also be worth investigating.

First extension will consist in the studying of classical exponential utilities for the Agents. The main issue of this work is indeed that we assume that Agents are risk neutral in the application in order to make all the computations. In this case, we do not fit with separable utilities and the scalarization method proposed to solve the problem does not work. An other approach which could be considered should be to use other methods to find Pareto optima, see for instance [25], more tractable for exponential utilities.

The second extension is directly linked to Remark 4.1 since the natural question which is not considered in the present work is to wonder what is happening if the Planner is not exogenous in the sense that he is also hired by the Principal. In this case, the Planner has to be seen as a *consulting mediator* which aims at finding Pareto optima and which is remunerated by the Principal. This required to modify the classical Stackelberg-type approach by adding this third player in the game.

7 Acknowledgments

The author is very grateful to François Delarue for having suggested to do this study during a discussion and for his advices together with Dylan Possamai for his general advices in the writing of this paper. The author also thanks Miquel Oliu-Barton and Patrick Beißner for discussions on welfare economics and Pareto efficiency.

References

- [1] D. Acemoglu and A. Simsek. Moral hazard and efficiency in general equilibrium with anonymous trading *MIT Department of Economics Working Paper No. 10-8*, 2010.
- [2] R. Arnott and J. Stiglitz. The welfare economics of moral hazard. In *Risk, information and insurance*, pages 91–121. Springer, 1991.
- [3] K. J Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, pages 265–290, 1954.
- [4] P. Briand and Y. Hu. BSDE with quadratic growth and unbounded terminal value. *Probability Theory and Related Fields*, 136(4):604–618, 2006.

- [5] P. Briand and Y. Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probability Theory and Related Fields*, 141(3-4):543–567, 2008.
- [6] J. Cvitanović, D. Possamaï, and N. Touzi. Moral hazard in dynamic risk management. *Management Science*, to appear, 2014.
- [7] J. Cvitanović, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–agent problems. *arXiv preprint arXiv:1510.07111v3*, 2017.
- [8] J. Cvitanović and J. Zhang. *Contract theory in continuous–time models*. Springer, 2012.
- [9] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.
- [10] R. Élie, T. Mastrolia, and D. Possamaï. A tale of a principal and many many agents. *arXiv preprint arXiv:1608.05226*, 2016.
- [11] R. Élie and D. Possamaï. Contracting theory with competitive interacting agents. *arXiv preprint arXiv:1605.08099*, 2016.
- [12] A. Feertchak and G. Poingt. Présidentielle : faut-il remettre en cause les 3% de déficit public? *Le Figaro*, March 2017.
- [13] B. Fofana. La règle des 3% de déficit des états est-elle un "non-sens"? *Libération*, February, 2017.
- [14] C. Frei and G. Dos Reis. A financial market with interacting investors: does an equilibrium exist? *Mathematics and financial economics*, 4(3):161–182, 2011.
- [15] J. Harter and A. Richou. A stability approach for solving multidimensional quadratic BSDEs. *arXiv preprint arXiv:1606.08627*, 2016.
- [16] B. Holmström. Moral hazard in teams. *The Bell Journal of Economics*, 13(2):324–340, 1982.
- [17] B. Holmström and P. Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica*, 55(2):303–328, 1987.
- [18] Y. Hu, P. Imkeller, M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.
- [19] N. Kazamaki. *Continuous exponential martingales and BMO*. Springer, 2006.
- [20] M. Khan. How the european central bank became the real villain of greece’s debt drama. *The Telegraph*, May 2015.
- [21] H.K. Koo, G. Shim, and J. Sung. Optimal multi–agent performance measures for team contracts. *Mathematical Finance*, 18(4):649–667, 2008.
- [22] N. KA Läufer. Die maastricht-kriterien: Fug oder unfug. *Im Internet unter: <http://www.uni-konstanz.de/FuF/wiwi/lauffer/lecture2/kriterien-text.html>*, 2008.
- [23] A. Mas-Colell, M. Dennis Whinston, J. R Green, et al. *Microeconomic theory*, volume 1. Oxford university press New York, 1995.
- [24] T. Mastrolia, D. Possamaï, and A. Réveillac. On the Malliavin differentiability of BSDEs. *Annales de l’institut Henri Poincaré, Probabilités et Statistiques (B)*, to appear, 2014.

- [25] K. Miettinen. *Nonlinear multiobjective optimization*, volume 12. Springer Science & Business Media, 2012.
- [26] J.A. Mirrlees. Notes on welfare economics, information and uncertainty. In M.S. Balch, D.L. McFadden, and S.Y. Wu, editors, *Essays on economic behavior under uncertainty*, pages 243–261. Amsterdam: North Holland, 1974.
- [27] J.A. Mirrlees. The optimal structure of incentives and authority within an organization. *The Bell Journal of Economics*, 7(1):105–131, 1976.
- [28] L. Panaccione et al. Pareto optima and competitive equilibria with moral hazard and financial markets. *The BE Journal of Theoretical Economics*, 7(1):1–21, 2007.
- [29] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.
- [30] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In B.L. Rozovskii and R.B. Sowers, editors, *Stochastic partial differential equations and their applications. Proceedings of IFIP WG 7/1 international conference University of North Carolina at Charlotte, NC June 6–8, 1991*, volume 176 of *Lecture notes in control and information sciences*, pages 200–217. Springer, 1992.
- [31] J. Peet. Maastricht follies. *Economist*, 347(8063), 1998.
- [32] D. Possamai, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. *arXiv preprint arXiv:1510.08439*, 2015.
- [33] E. C Prescott and R. M Townsend. Pareto optima and competitive equilibria with adverse selection and moral hazard. *Econometrica: Journal of the Econometric Society*, pages 21–45, 1984.
- [34] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.
- [35] Y. Sannikov. A continuous–time version of the principal–agent problem. *The Review of Economic Studies*, 75(3):957–984, 2008.
- [36] H K Scheller. The european central bank. *History, Role and Functions*. ECB, 2004.
- [37] H. Mete Soner, N. Touzi, and J. Zhang. Wellposedness of second order backward sdes. *Probability Theory and Related Fields*, 153(1-2):149–190, 2012.
- [38] J. Sung. Lectures on the theory of contracts in corporate finance: from discrete–time to continuous–time models. *Com2Mac Lecture Note Series*, 4, 2001.
- [39] N. Touzi. *Optimal stochastic control, stochastic target problems, and backward SDE*, volume 29. Springer Science & Business Media, 2012.
- [40] L. Walras. *Éléments d’économie politique pure; ou, Théorie de la richesse sociale*. F. Rouge, 1896.
- [41] H. Xing and G. Žitković. A class of globally solvable Markovian quadratic BSDE systems and applications. *arXiv preprint arXiv:1603.00217*, 2016.

A The mathematical model and general notations

In this section we set all the notations and spaces used in this paper in the paragraphs "General notations" and "Spaces" respectively. We also defined mathematically the considered model in Paragraph "General model and definitions" and we put the general assumptions which are supposed to be satisfied in all the paper.

General notations. Let \mathbb{R} be the set of reals. Let m and n be two positive integers. We denote by $\mathcal{M}_{m,n}(\mathbb{R})$ the set of matrices with m rows and n columns, and simplify the notations when $m = n$, by using $\mathcal{M}_n(\mathbb{R}) := \mathcal{M}_{n,n}(\mathbb{R})$. We denote by $\text{Id}_n \in \mathcal{M}_n(\mathbb{R})$ the identity matrix of order n . For any $M \in \mathcal{M}_{m,n}(\mathbb{R})$, we define $M^\top \in \mathcal{M}_{n,m}(\mathbb{R})$ as the usual transpose of the matrix M . We will always identify \mathbb{R}^n with $\mathcal{M}_{n,1}(\mathbb{R})$. For any matrix $M \in \mathcal{M}_n(\mathbb{R})$ and for any $1 \leq i \leq n$ we denote by $M^{:,i}$ its i th column and similarly we denote by $M^{i,:}$ its i th row. Besides, for any $x \in \mathbb{R}^n$, we denote its coordinates by x^1, \dots, x^n . We denote by $\|\cdot\|$ the Euclidian norm on \mathbb{R}^n , when there is no ambiguity on the dimension. The associated inner product between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is denoted by $x \cdot y$. We also denote by $\mathbf{1}_n$ the n -dimensional vector $(1, \dots, 1)^\top$. Similarly, for any $x \in \mathbb{R}^n$, we define for any $i = 1, \dots, n$, $x^{-i} \in \mathbb{R}^{n-1}$ as the vector x without its i th component, that is to say $x^{-i} := (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)^\top$. For any $(a, \tilde{a}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and any $i = 1, \dots, n$, we define the following n -dimensional vector

$$a \otimes_i \tilde{a} := (\tilde{a}^1, \dots, \tilde{a}^{i-1}, a, \tilde{a}^i, \dots, \tilde{a}^{n-1}).$$

For any $(\alpha, \tilde{M}) \in \mathbb{R} \times \mathcal{M}_{n,n-1}(\mathbb{R})$, and any $i = 1, \dots, n$, we define the following n -dimensional matrix

$$\alpha \otimes_i \tilde{M} := (\tilde{M}^{:,1}, \dots, \tilde{M}^{:,i-1}, \alpha, \tilde{M}^{:,i}, \dots, \tilde{M}^{:,n-1}).$$

For any Banach space $(E, \|\cdot\|_E)$, let f be a map from $E \times \mathbb{R}^n$ into \mathbb{R} . For any $x \in E$, we denote by $\nabla_a f(x, a)$ the gradient of $a \mapsto f(x, a)$, and we denote by $\Delta_{aa} f(x, a)$ the Hessian matrix of $a \mapsto f(x, a)$ and similarly we denote by $\nabla_x f(x, a)$ the gradient of $x \mapsto f(x, a)$, and we denote by $\Delta_{xx} f(x, a)$ the Hessian matrix of $x \mapsto f(x, a)$. Finally, we denote by $U_A^{(-1)}$ the inverse function of any $U_A : E \rightarrow \mathbb{R}$ if it exists.

Spaces. For any finite dimensional normed space $(E, \|\cdot\|_E)$, $\mathcal{P}(E)$ (resp. $\mathcal{P}_r(E)$) will denote the set of E -valued, \mathbb{F} -adapted processes (resp. \mathbb{F} -predictable processes) and for any $p \geq 1$ we set

$$\begin{aligned} \mathcal{E}(E) &:= \left\{ Y \in \mathcal{P}(E), \text{ càdlàg, such that for all } p \geq 1, \mathbb{E} \left[\exp \left(p \sup_{t \in [0, T]} \|Y_t\|_E \right) \right] < +\infty \right\}, \\ \mathcal{S}^\infty(E) &:= \left\{ Y \in \mathcal{P}(E), \text{ càdlàg, such that } \|Y\|_{\mathcal{S}^\infty(E)} := \sup_{t \in [0, T]} \|Y_t\|_E < +\infty \right\}, \\ \mathcal{H}^p(E) &:= \left\{ Z \in \mathcal{P}_r(E), \|Z\|_{\mathcal{H}^p(E)}^p := \mathbb{E} \left[\left(\int_0^T \|Z_t\|_E^2 dt \right)^{p/2} \right] < +\infty \right\} \\ \mathcal{H}_{\text{BMO}}^p(E) &:= \left\{ Z \in \mathcal{H}^p(E), \exists C > 0, \forall \tau \in \mathcal{T}, \mathbb{E} \left[\int_\tau^T \|Z_s\|^2 ds \middle| \mathcal{F}_\tau \right] \leq C^2, \mathbb{P} - a.s. \right\}. \end{aligned}$$

$\mathcal{H}_{\text{BMO}}^p(E)$ is the so-called set of predictable processes Z such that the stochastic integral is a BMO-martingale. We refer to [19] for more details on this space and properties related to this theory. We also denote $\mathcal{E}(M)$ the classical Doleans-Dade exponential of the martingale M .

General model and definitions. We assume that the following technical assumption holds in this paper

Assumption A.1. For any $i = 1, \dots, N$ and any $(t, x) \in [0, T] \times \mathbb{R}^N$, the maps $a \in \mathbb{R}^N \mapsto b^i(t, x, a)$ is continuously differentiable, the maps $(t, x) \in [0, T] \times \mathbb{R}^N \mapsto b^i(t, x, a)$ is \mathbb{F} -progressive and measurable for any $a \in \mathbb{R}^N$ and we assume that there exists a positive constant C such that for any $(t, x, a) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$

$$|b^i(t, x, a)| \leq C(1 + \|x\| + \|a\|), \quad \|\nabla_a b^i(t, x, a)\| \leq C. \quad (\text{A.1})$$

For any $i = 1, \dots, N$ the map $(t, x) \mapsto k^i(t, x, a)$ is \mathbb{F} -progressive and measurable. Moreover, the map $a \mapsto k^i(t, x, a)$ is increasing, convex and continuously differentiable for any $(t, x) \in [0, T] \times \mathbb{R}^N$. Assume also that there exists $\kappa > 0, l \geq 1, \underline{m}, m > 0$ such that

$$\frac{l+m}{\underline{m}+1-l} \vee \frac{\underline{m}+2-l}{\underline{m}+1-l} \leq 2, \quad (\text{A.2})$$

and for any $(t, x, a) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$

$$0 \leq k^i(s, x, a) \leq C \left(1 + \|x\| + \|a\|^{l+m} \right),$$

$$\|\nabla_a k^i(s, x, a)\| \geq \kappa \|a\|^m, \quad \text{and } \overline{\lim}_{|a| \rightarrow \infty} \frac{k^i(s, x, a)}{\|a\|} = +\infty.$$

Remark A.1. In the classical linear-quadratic framework, $l = m = \underline{m} = 1$ and Condition (A.2) holds. This condition will be fundamental to prove Theorem 4.1 (see the Appendix C).

Assumption (G). For any $p \geq 1$ and for any $1 \leq i \leq N$,

$$\mathbb{E} \left[\exp \left(p \Gamma^i(X_T) \right) \right] < +\infty.$$

In order to define a probability \mathbb{P}^a equivalent to \mathbb{P} such that $W^a := W - \int_0^T \Sigma(s, X_s)^{-1} b(s, X_s, a_s) ds$ is a Brownian motion under \mathbb{P}^a , we need to introduce the set \mathcal{A} of admissible effort. An \mathbb{F} -adapted and $\mathcal{M}_N(A)$ -valued process α is said admissible if $\left(\mathcal{E} \left(\int_0^t b(s, X_s, \alpha_s) \cdot \Sigma(s, X_s)^{-1} dW_s \right) \right)_{t \in [0, T]}$ is an \mathbb{F} -martingale and if for the same l, m appearing in Assumption A.1 we have for any $1 \leq i \leq N$ and any $p \geq 1$

$$\mathbb{E} \left[\exp \left(p \int_0^T \|\alpha_s^{i,i}\|^{l+m} ds \right) \right] < +\infty.$$

We also define for any \mathbb{R}^{N-1} -valued and \mathbb{F} -adapted process $a^{i,-i}$ the set $\mathcal{A}^i(a^{i,-i})$ by

$$\mathcal{A}^i(a^{i,-i}) := \{ \alpha, \mathbb{R}^N\text{-valued and } \mathbb{F}\text{-adapted s.t. } \alpha \otimes_i a^{i,-i} \in \mathcal{A} \}. \quad (\text{A.3})$$

Now, in order to use the theory of quadratic BSDE and apply the result of for instance [4, 5], we must define the set of admissible contracts \mathcal{C} as the set of \mathcal{F}_T -measurable and \mathbb{R}^N -valued random variable ξ satisfying the constrain (2.3) and such that

$$\mathbb{E} \left[\exp \left(p \sum_{i=1}^N |U_A^i(\xi^i)| \right) \right] < +\infty.$$

B Multidimensional quadratic growth BSDEs: application of the results in [15]

Under Assumption A.1, from [10, Lemma 4.1], for any $(s, x, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$ there exists

$$a_{NA}^*(s, x, z) \in \arg \max_{a \in \mathcal{A}^i(a^{:, -i})} \{b(s, x, a \otimes_i a^{:, -i}) \cdot z - k^i(s, x, a)\},$$

satisfying

$$\|a_{NA}^*(s, x, z)\| \leq C_a \left(1 + \|z\|^{\frac{1}{\underline{m}+1-l}}\right).$$

The aim of this section is to provide conditions ensuring that Theorem 3.1 holds, by giving a set of admissible contracts for which there exists a solution to the multi-dimensional BSDE (3.3). We consider the set of admissible contract

$$\mathcal{C}_{NA} := \left\{ \xi \in \mathcal{C}, \xi \in \mathbb{D}^{1,2}, \|D\xi\|_{\mathcal{S}^\infty(\mathbb{R}^N)} < +\infty \right\}.$$

We besides assume

(Db) The map $x \mapsto b^i(s, x, a)$ is continuously differentiable such that there exists $C_b > 0$ with

$$\|\nabla_x b^i(s, x, a)\| \leq C_b,$$

(Dk) The map $x \mapsto k^i(s, x, a)$ is continuously differentiable such that there exists $C_k > 0$ with

$$\|\nabla_x k^i(s, x, a)\| \leq C_k(1 + \|a\|^{l+m}),$$

and moreover

$$\|\nabla_a k^i(s, x, a)\| \leq C_k \left(1 + |a|^{l+m-1}\right),$$

(Da) The map $x \mapsto a^*(s, x, z)$ is continuously differentiable such that there exists $C_a > 0$ with

$$|(a^*)^{:,i}(s, x, z)| + |(a^*)^{j,:}(s, x, z)| \leq C_a(1 + \|z\|^{\frac{1}{\underline{m}-1+l}}).$$

Thus,

$$\begin{aligned} \nabla_x f_{NA}^{*,i}(t, x, z) &= \sum_{j=1}^N \left[(\nabla_x b^j(t, x, (a^*)^{j,:}) + \nabla_a b^j(t, x, (a^*)^{j,:}) \nabla_x (a^*)^{j,:}) z^{j,i} \right. \\ &\quad \left. - \nabla_x k^i(t, x, (a_{NA}^*)^{:,i}) - \nabla_a k^i(t, x, (a_{NA}^*)^{:,i}) \nabla_x (a^*)^{j,:} \right]. \end{aligned}$$

Lemma B.1. *Under Assumptions A.1, **(Db)**, **(Dk)**, **(Da)** and assume that $l + m \leq 2(\underline{m} - 1 + l)$, we have for any $(s, x, z) \in [0, T] \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R})$*

$$\|\nabla_x f_{NA}^{*,i}(t, x, z)\| \leq C_f(1 + \|z\|^2),$$

with $C_f := C_b + 2CC_a + 2C_k N(1 + |C_a|^{l+m-1}(1 + C_a))$.

Proof. We compute directly

$$\begin{aligned}
\|\nabla_x f_{NA}^{*,i}(t, x, z)\| &\leq \sum_{j=1}^N \left[(\|\nabla_x b^j(t, x, (a^*)^{j,\cdot})\| + \|\nabla_a b^j(t, x, (a^*)^{j,\cdot}) \nabla_x (a^*)^{j,\cdot}\|) |z^{j^i}| \right. \\
&\quad \left. + \|\nabla_x k^i(t, x, (a_{NA}^*)^{i,\cdot})\| + \|\nabla_a k^i(t, x, (a_{NA}^*)^{i,\cdot}) \nabla_x (a^*)^{j,\cdot}\| \right] \\
&\leq C_b \sum_{j=1}^N |z^{j^i}| + CC_a (1 + \|z\|^{\frac{1}{\underline{m}-1+l}}) \sum_{j=1}^N |z^{j^i}| + C_k N (1 + |C_a|^{l+m} (1 + \|z\|^{\frac{l+m}{\underline{m}-1+l}})) \\
&\quad + C_k N \left(1 + |C_a|^{l+m-1} (1 + \|z\|^{\frac{l+m-1}{\underline{m}-1+l}}) \right) \\
&\leq C_b \|z\| + CC_a (1 + \|z\|^{\frac{1}{\underline{m}-1+l}}) \|z\| + C_k N (1 + |C_a|^{l+m} (1 + \|z\|^{\frac{l+m}{\underline{m}-1+l}})) \\
&\quad + C_k N \left(1 + |C_a|^{l+m-1} (1 + \|z\|^{\frac{l+m-1}{\underline{m}-1+l}}) \right) \\
&\leq C_b + 2CC_a + 2C_k N (1 + |C_a|^{l+m-1} (1 + C_a)) \\
&\quad + \|z\|^2 \left(C_b + 2CC_a + C_k N |C_a|^{l+m-1} (1 + C_a) \right) \\
&\leq C_f (1 + \|z\|^2).
\end{aligned}$$

□

We now introduce a localisation of f^* defined by $f_M^*(s, x, z) := f^*(s, x, \rho_M(z))$ where ρ_M satisfies projection properties on a ball centred on $0_{N,N}$ with radius M . We denote by (Y^M, Z^M) the unique solution of BSDE (3.3) with generator f_M^* , which fits the classical Lipschitz BSDE framework. For all $p > 1$ we set the following assumption

(BMO,p) There exists a positive constant K such that

$$(i) \quad KC_f C_p' < \frac{1}{2},$$

$$(ii) \quad \sup_{M \in \mathbb{R}^M} \left\| \int_0^T (Z_s^M)^\top \Sigma_s dW_s \right\|_{\text{BMO}} \leq K.$$

As a direct¹⁰ consequence of [15, Theorem 3.1] and Lemma B.1 we have the following proposition

Proposition B.1. *Let $p > 1$ and let Assumptions A.1 (Db), (Dk), (Da), (BMO,p) be true with $l + m \leq 2(\underline{m} - 1 + l)$, then for any $\xi \in C_{NA}$, the N -dimensional quadratic BSDE (3.3) has a unique solution in the sense of Definition 3.1.*

¹⁰After having discussed with Jonathan Harter, assumption **(Df,b)**, **(iii)** in [15] is not a *canonical* assumption for the proof of [15, Theorem 3.1]. Indeed, in this paper, the authors needs this assumption only to ensure that the BSDE with projector f_M admits Malliavin differentiable solution as an application of [9]. However, as explained in [24, Section 5 and Section 6] this assumption can be removed in the Markovian case. The author thanks Jonathan Harter for this clarification.

C Technical proofs of Section 4

First recall that under Assumption A.1, from [10, Lemma 4.1], for any $(s, x, z, \lambda) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times (0, 1)^N$ there exists

$$a^*(s, x, z, \lambda) \in \operatorname{argmax}_{a \in \mathcal{M}_N(A)} \left\{ b(s, x, a) \cdot z - k^\lambda(s, x, a) \right\},$$

satisfying

$$\|a^*(s, x, z, \lambda)\| \leq C_a \left(1 + \|z\|^{\frac{1}{m+1-l}} \right). \quad (\text{C.1})$$

We now turn to the proofs of the main results in Section 4

Proof of Lemma 4.1. The fact that BSDE (4.3) admits a unique solution $(Y^{\lambda, a}, Z^{\lambda, a}) \in \mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ is a direct consequence of [5] using the definition of \mathcal{C} , \mathcal{A} and Assumptions A.1 and **(G)** in Appendix A. Now by changing the probability in BSDE (4.3) and by taking the conditional expectation we directly get $Y_t^{\lambda, a} = u_t^\lambda(\xi, a)$. \square

Proof of Theorem 4.1. Using (C.1) together with Condition (A.2), the definition of \mathcal{C} , \mathcal{A} and Assumptions A.1 and **(G)**, we obtain from (for instance) [5] the existence and uniqueness of a solution $(Y^\lambda, Z^\lambda) \in \mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ of BSDE (4.4). Now, by using a comparison theorem for BSDE (4.3) (see for instance [5, Theorem 5]), we deduce that $Y_0^\lambda = u_0^\lambda(\xi)$ and that any a^* in $\mathcal{A}^*(X, Z^\lambda, \lambda)$ is Pareto optimal from Proposition 4.1. \square

Proof of Proposition 4.2. Let $(Y^\lambda, Z^\lambda) \in \mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ be the solution of BSDE (4.4) from Theorem 4.1. The fact that BSDE (4.5) admits a unique solution (Y^i, Z^i) in $\mathcal{E}(\mathbb{R}) \times \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^N)$ is a direct consequence of the definitions of \mathcal{C} and \mathcal{A} . Changing the probability by using Girsanov's theorem, we directly get

$$Y_0^i = U_0^i(\xi, a^*(s, X_s, Z_s^\lambda, \lambda)).$$

Denote by $\mathcal{Y}^\lambda := \sum_{i=1}^N Y^i$ and $\mathcal{Z}^\lambda := \sum_{i=1}^N Z^i$, we notice that the pair of process $(\mathcal{Y}^\lambda, \mathcal{Z}^\lambda)$ is solution of

$$\mathcal{Y}_t^\lambda = U_A^\lambda(\xi) + \int_t^T \left(b(s, X_s, a^*(s, X_s, Z_s^\lambda, \lambda)) \cdot \mathcal{Z}_s^\lambda - k^\lambda(s, X_s, (a^*(s, X_s, Z_s^\lambda, \lambda))) \right) ds - \int_t^T \mathcal{Z}_s^\lambda \cdot dX_s.$$

Using the uniqueness of the solution (Y^λ, Z^λ) of BSDE (4.4), we deduce that $\mathcal{Y}^\lambda = Y^\lambda$ and $\mathcal{Z}^\lambda = Z^\lambda$. \square