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Asymptotic ensemble stabilizability of the Bloch equation

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Abstract

In this paper we are concerned with the stabilizability to an equilibrium point of an ensemble of non interacting half-spins. We assume that the spins are immersed in a static magnetic field, with dispersion in the Larmor frequency, and are controlled by a time varying transverse field. Our goal is to steer the whole ensemble to the uniform “down” position.

Two cases are addressed: for a finite ensemble of spins, we provide a control function (in feedback form) that asymptotically stabilizes the ensemble in the “down” position, generically with respect to the initial condition. For an ensemble containing a countable number of spins, we construct a sequence of control functions such that the sequence of the corresponding solutions pointwise converges, asymptotically in time, to the target state, generically with respect to the initial conditions.

The control functions proposed are uniformly bounded and continuous.

Keywords: ensemble controllability, quantum control

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1. Introduction

Ensemble controllability (also called *simultaneous controllability*) is a notion introduced in [1, 2, 3] for quantum system described by a family of parameter-dependent ordinary differential equations; it concerns the possibility of finding control functions that compensate the dispersion in the parameters and drive the whole family (ensemble) from some initial state to some prescribed target state.

Such an issue is motivated by recent engineering applications, such as, for instance, quantum control (see for instance [3, 4, 5, 6, 7] and references therein), distributed parameters systems and PDEs [8, 9, 10, 11, 12], and flocks of identical systems [13].

General results for the ensemble controllability of linear and nonlinear systems, in continuous and discrete time, can be found in the recent papers [14, 15, 16].

This paper deals with the simultaneous control of an ensemble of spin immersed on a magnetic field, where each spin is described by a magnetization vector $\mathbf{M} \in \mathbb{R}^3$, subject to the dynamics $\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \times \mathbf{B}(\mathbf{r}, t)$, where $\mathbf{B}(\mathbf{r}, t)$ is a magnetic field composed by a static component directed along the z -axis, and a time varying component on the xy -plane, called *radio-frequency (rf) field*, and γ denotes the gyromagnetic ratios of the spins. In this system, since all spins are controlled by the same magnetic field $\mathbf{B}(\mathbf{r}, t)$, spatial dispersion in the amplitude of the magnetic field gives rise to the following inhomogeneities in the dynamics: *rf inhomogeneity*, caused by dispersion in the radio-frequency field, and a spread in the Larmor frequency, given by dispersion of the static component of the field. This problem arises from in NMR spectroscopy (see references in [3, 4]).

The task of controlling such system is wide, multi-faceted and very rich, depending on the cardinality of the set of the spin to be controlled (and the topology of this set), on the particular notion of controllability addressed, and on the functional space where control functions live.

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The above-cited articles [1, 2, 3, 17] are concerned with both rf inhomogeneity and Larmor dispersion, with dispersion parameters that belong to some compact domain \mathcal{D} . The magnetization vector of the system is thus a function on \mathcal{D} , taking values in the unit sphere of \mathbb{C}^2 , and ensemble controllability has to be intended as convergence in the $L^\infty(\mathcal{D}, \mathbb{R}^3)$ -norm. The controllability result is achieved by means of Lie algebraic techniques coupled with adiabatic evolution, and holds for both bounded and unbounded controls.

In [4], the authors focus on systems subject to Larmor dispersions, and provide a complete analysis of controllability properties of the ensemble in different scenarios, such as: bounded/unbounded controls; finite time/asymptotic controllability; approximate/exact controllability in the $L^2(\mathcal{D}, \mathbb{R}^3)$ norm; boundedness/unboundedness of the set \mathcal{D} . In particular, results on exact local controllability with unbounded controls are provided.

In this paper we consider an ensemble of Bloch equations presenting Larmor dispersion, with frequencies belonging to some possibly unbounded subset $\mathcal{E} \subset \mathbb{R}$. Coupling a Lyapunov function approach with some tools of dynamical systems theory, we exhibit a control function (in feedback form) that approximately drives, asymptotically in time and generically with respect to the initial conditions, all spins to the “down” position. Two cases are addressed: if the set \mathcal{E} is finite, our strategy provides exact exponential stabilizability in infinite time, while in the case where \mathcal{E} is a countable collection of energies, our approach implies asymptotic pointwise convergence towards the target state.

The main advantages of this result lie in the fact that it works for generic initial conditions, and the control is in feedback form, so that it can be applied even in presence of some uncertainties in the knowledge of the parameters of the system. In a true quantum context, due to the impossibility of applying a feedback control that needs the exact knowledge of the position of the spin, this robustness feature is no more present. Nevertheless, we stress that feedback controls computed off-line and then applied in open-loop are a quite common tool in quantum control (see for instance [18] and references therein).

We finally remark that the control functions needed in this strategy are continuous and bounded, thus more easy to implement in practical situations.

The structure of the paper is the following: in Section 2 we state the problem in general form; in Section 3 we tackle the finite dimensional case, while in Section 4 we analyse the case of a countable family of systems. Section 5 is devoted to some numerical results.

2. Statement of the problem

We consider an ensemble of non-interacting spins immersed in a static magnetic field of strength $B_0(\mathbf{r})$, directed along the z -axis, and a time varying transverse field $(B_x(t), B_y(t), 0)$ (rf field), that we can control. Bloch equation for this system takes then the form

$$\frac{\partial \mathbf{M}}{\partial t}(\mathbf{r}, t) = \begin{pmatrix} 0 & -B_0(\mathbf{r}) & B_y(t) \\ B_0(\mathbf{r}) & 0 & -B_x(t) \\ -B_y(t) & B_x(t) & 0 \end{pmatrix} \mathbf{M}(\mathbf{r}, t) \quad (1)$$

(here for simplicity we set $\gamma = 1$). For more details, we mention the monograph [19].

Since the dependence on the spatial coordinate \mathbf{r} appears only in $B_0(\mathbf{r})$, we can represent $\mathbf{M}(\mathbf{r}, t)$ as a collection of time-dependent vectors $X_e(t) = (x_e(t), y_e(t), z_e(t))$, where $e = B_0(\mathbf{r})$, each one belonging to the unit sphere $S^2 \subset \mathbb{R}^3$ and subject to the law

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{z}_e \end{pmatrix} = \begin{pmatrix} 0 & -e & u_2 \\ e & 0 & u_1 \\ -u_2 & -u_1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \end{pmatrix}. \quad (2)$$

The Larmor frequencies e of the spins in the ensemble take value in some subset $\mathcal{E} \subset I$ of an

interval I ¹. Depending on the spatial distribution of the spins, \mathcal{E} could be a finite set, an infinite countable set, or an interval.

We are concerned with the following control problem:

(P) *Design a control function $\mathbf{u} : [0, +\infty) \rightarrow \mathbb{R}^2$ such that for every $e \in \mathcal{E}$ the solution of equation (2) is driven to $X_e = (0, 0, -1)$.*

To face this problem, we consider the product $\mathbf{S} = \prod_{e \in \mathcal{E}} S^2$, whose elements are the collections $\mathbf{X} = \{X_e\}_{e \in \mathcal{E}}$ such that $X_e \in S^2$ for every $e \in \mathcal{E}$. Depending on the structure of \mathcal{E} , \mathbf{X} can be a finite or an infinite countable collection of states $X_e \in S^2$, or a function $\mathbf{X} : \mathcal{E} \rightarrow S^2$ belonging to some functional space. The collection \mathbf{X} of magnetic moments evolves according to the equation

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \mathbf{u}), \quad (3)$$

where \mathbf{F} denotes the collection $\mathbf{F} = \{F_e\}_{e \in \mathcal{E}}$ of tangent vectors to S^2 , with $F_e(\mathbf{X}, \mathbf{u}) = \begin{pmatrix} 0 & -e & u_2 \\ e & 0 & u_1 \\ -u_2 & -u_1 & 0 \end{pmatrix} X_e$,

and $\mathbf{u} = (u_1, u_2)$.

Some remarks on the existence of solutions for equation (3) are in order, and will be provided case by case. Assuming that these issues are already fixed, we define the two states $\mathbf{X}^+ = \{X_e : X_e = (0, 0, 1) \forall e \in \mathcal{E}\}$ and $\mathbf{X}^- = \{X_e : X_e = (0, 0, -1) \forall e \in \mathcal{E}\}$, and rewrite the problem **(P)** as

(P') *Design a control function $\mathbf{u} : [0, +\infty) \rightarrow \mathbb{R}^2$ such that the solution of equation (3) is driven to $\mathbf{X} = \mathbf{X}^-$.*

We remark that the notion of convergence of $\mathbf{X}(\cdot)$ towards \mathbf{X}^- in problem **(P')** has to be specified case by case, depending on the structure of the set \mathcal{E} and on the topology of \mathbf{S} .

3. Finite dimensional case

First of all, we consider the case in which the set \mathcal{E} is a finite collection of pairwise distinct energies, that is $\mathcal{E} = (e_1, \dots, e_p)$ such that $e_k \in I$, $k = 1, \dots, p$ and $e_k \neq e_j$ if $i \neq j$. We recall that the state space \mathbf{S} of the system is the finite product of p copies of S^2 .

Lemma 1. *Assume that all energy levels e_i are pairwise distinct. Let $\mathcal{I} = \{\mathbf{X} \in \mathbf{S} : x_{e_i} = y_{e_i} = 0 \forall i = 1, \dots, p\}$. Then every solution of the the control system (2) with control*

$$\begin{cases} u_1 = \sum_{i=1}^p y_{e_i} \\ u_2 = \sum_{i=1}^p x_{e_i} \end{cases} \quad (4)$$

tends to \mathcal{I} as $t \rightarrow +\infty$.

Proof. Consider the function $V(\mathbf{X}) = \sum_{i=1}^p z_{e_i}$, and let $\Xi(\cdot)$ be a solution of (3) with the control given in (4). We notice that $\dot{V}(\Xi(t)) = -(\sum_{i=1}^p x_{e_i})^2 - (\sum_{i=1}^p y_{e_i})^2$, therefore it is non-positive on the whole \mathbf{S} , and it is zero only on the set $\mathcal{M} = \{\mathbf{X} \in \mathbf{S} : \sum_{i=1}^p x_{e_i} = \sum_{i=1}^p y_{e_i} = 0\}$. We can then apply La Salle invariance principle to conclude that, for every initial condition, $\Xi(t)$ tends to the largest invariant subset of \mathcal{M} .

Consider a trajectory $\Xi(\cdot)$ entirely contained in \mathcal{M} . Since $\mathbf{u} = 0$, then for every i we have that

$$\Xi_i(t) = \begin{pmatrix} \cos(e_i t) & -\sin(e_i t) & 0 \\ \sin(e_i t) & \cos(e_i t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{e_i}(0) \\ y_{e_i}(0) \\ z_{e_i}(0) \end{pmatrix}.$$

¹for simplicity in the exposition, here we will consider the interval I bounded. The results holds true also if I is unbounded

By definition, for every $t \geq 0$ it holds $\sum_{i=1}^p x_{e_i}(t) = \sum_{i=1}^p y_{e_i}(t) = 0$. Differentiating these equalities $p - 1$ times and evaluating at $t = 0$ we obtain the two conditions

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ e_1 & e_2 & \dots & e_p \\ \vdots & \vdots & & \\ e_1^{p-1} & e_2^{p-1} & \dots & e_p^{p-1} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \dots & 1 \\ e_1 & e_2 & \dots & e_p \\ \vdots & \vdots & & \\ e_1^{p-1} & e_2^{p-1} & \dots & e_p^{p-1} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The determinant of the Vandermonde matrix here above is given by $\prod_{1 \leq i < j \leq p} (e_i - e_j)$, which is non-zero under the assumptions. Therefore the two equations are satisfied if and only if $\Xi(0) \in \mathcal{I}$. It is immediate to see that \mathcal{I} is the largest invariant subset of \mathcal{M} . \square

The set \mathcal{I} is composed by a collection of 2^p isolated points $\mathbf{Q}^k = (Q_1^k, \dots, Q_p^k)$, $k = 1, \dots, 2^p$, where $Q_j^k = (0, 0, \alpha_j^k)$ and $|\alpha_j^k| = 1$. These points are equilibria for the closed-loop system (2)-(4). We distinguish three cases

- if $\alpha_j^k = -1$ for every j , then $\mathbf{Q}^k = \mathbf{X}^-$, and it is an asymptotically stable equilibrium for the system;
- if $\alpha_j^k = 1$ for every j , then $\mathbf{Q}^k = \mathbf{X}^+$ is an unstable equilibrium for the system; in particular, \mathbf{X}^+ is a repeller;
- all other points in \mathcal{I} are neither attractor neither repellers, since each of these points is a saddle-point of V .

Lemma 2. *Let \mathcal{B} be the basin of attraction of \mathbf{X}^- . Then \mathcal{B} is an open neighbourhood of \mathbf{X}^- and there exists at least one $\mathbf{Q} \in \mathcal{I} \setminus \{\mathbf{X}^+, \mathbf{X}^-\}$ such that $\mathbf{Q} \in \partial\mathcal{B}$.*

Proof. Let us denote with ϕ^t the map that associates with each $\mathbf{X}_0 \in \mathcal{S}$ the solution at time t of the control system (3)-(4) with initial condition equal to \mathbf{X}_0 .

By definition, $\mathbf{X}_0 \in \mathcal{B}$ if for every neighbourhood \mathcal{V} of \mathbf{X}^- there exists a time $T_{\mathcal{V}, \mathbf{X}_0}$ such that $\phi^t(\mathbf{X}_0) \in \mathcal{V}$ for every $t > T_{\mathcal{V}, \mathbf{X}_0}$.

By asymptotic stability of the point \mathbf{X}^- , there exists a neighbourhood \mathcal{U} of \mathbf{X}^- which is contained in \mathcal{B} . Moreover, by continuity with respect to initial data, for every $T > 0$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{X} - \mathbf{X}'| < \delta$ then $|\phi^T(\mathbf{X}) - \phi^T(\mathbf{X}')| < \epsilon$. For every $\mathbf{X}_0 \in \mathcal{B}$ we can choose T and ϵ such that the ball of radius ϵ centred at $\phi^T(\mathbf{X}_0)$ intersected with \mathcal{S} is contained in \mathcal{U} . Then we can choose δ such that every for \mathbf{X}'_0 satisfying $|\mathbf{X}_0 - \mathbf{X}'_0| < \delta$, then $\phi^T(\mathbf{X}'_0)$ is contained in the ball of radius ϵ centred at $\phi^T(\mathbf{X}_0)$, that is, $\phi^T(\mathbf{X}'_0) \in \mathcal{U}$. By definition of \mathcal{B} , it holds $\mathbf{X}'_0 \in \mathcal{B}$.

Consider now a point $\mathbf{X} \in \partial\mathcal{B}$. Then there exists some $\mathbf{Q} \in \mathcal{I} \setminus (\mathbf{X}^- \cup \mathbf{X}^+)$ such that $\phi^t(\mathbf{X}) \rightarrow \mathbf{Q}$ as $t \rightarrow +\infty$. By continuity with respect to initial conditions, for every $\epsilon > 0$ there exists $\mathbf{X}' \in \mathcal{B}$ and $t > 0$ such that $|\phi^t(\mathbf{X}') - \mathbf{Q}| < \epsilon$, that is $\mathbf{Q} \in \partial\mathcal{B}$. \square

3.1. Linearised system

In order to study the structure of the basin of attraction \mathcal{B} , we linearise the system (3)-(4) around a point in $\mathbf{Q} \in \mathcal{I}$, and we study the corresponding eigenvalues. We will show below that the linearised system is always hyperbolic (when we consider its restriction to the tangent space to the collection of spheres).

The linearisation gives

$$\begin{pmatrix} \delta \dot{\mathbf{x}} \\ \delta \dot{\mathbf{y}} \\ \delta \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} K_{\mathbf{Q}} & -E & 0 \\ E & K_{\mathbf{Q}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \mathbf{x} \\ \delta \mathbf{y} \\ \delta \mathbf{z} \end{pmatrix} \quad (5)$$

where

$$K_{\mathbf{Q}} = \begin{pmatrix} z_1^{\mathbf{Q}} & z_1^{\mathbf{Q}} & \dots & z_1^{\mathbf{Q}} \\ z_2^{\mathbf{Q}} & z_2^{\mathbf{Q}} & \dots & z_2^{\mathbf{Q}} \\ \vdots & \vdots & & \vdots \\ z_p^{\mathbf{Q}} & z_p^{\mathbf{Q}} & \dots & z_p^{\mathbf{Q}} \end{pmatrix} \quad E = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e_p \end{pmatrix}$$

and $z_i^{\mathbf{Q}}$ is the value of the coordinate z_i at the point \mathbf{Q} . Set moreover $M_{\mathbf{Q}} = \begin{pmatrix} K_{\mathbf{Q}} & -E \\ E & K_{\mathbf{Q}} \end{pmatrix}$. Notice that we can write $K_{\mathbf{Q}} = \kappa_{\mathbf{Q}} \zeta^T$, where $\kappa_{\mathbf{Q}} = (z_1^{\mathbf{Q}}, \dots, z_p^{\mathbf{Q}})^T$ and $\zeta = (1, 1, \dots, 1)^T$, then $\text{rank} K_{\mathbf{Q}} = 1$.

In the following, with a little abuse of notation, we will remove the dependence on \mathbf{Q} from K , M , κ and its components, specifying it only when necessary.

In order to compute the eigenvalues of the matrix M , we consider the complexification of system (5), that is we set $\xi = \delta \mathbf{x} + i \delta \mathbf{y}$, observing that $\dot{\xi} = (K + iE)\xi$. It is easy to see that ℓ is an eigenvalue of M if and only if it is also either an eigenvalue of $(K + iE)$ or an eigenvalue of $(K - iE)$, that is, the spectrum of M is equal to the union of the spectra of $(K + iE)$ and $(K - iE)$.

Properties In the following, we will use the following properties of block matrices

(P1) Let M be the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. If A is invertible, then $\det M = \det(A) \det(D - CA^{-1}B)$. If D is invertible, then $\det M = \det(D) \det(A - BD^{-1}C)$.

(P2) Let A be an invertible matrix of size n , and x, y two n -dimensional vectors. Then $\det(A + xy^T) = \det(A)(1 + y^T A^{-1}x)$.

Lemma 3. *The matrices $(K + iE)$ and $(K - iE)$ are invertible.*

Proof. Assume that E is invertible. Then $\det(K + iE) = \det(iE) \det(\mathbb{1} - iE^{-1}K)$; since $-iE^{-1}K = -iE^{-1}\kappa\zeta^T = \tilde{\kappa}\zeta^T$, by **(P2)** we have that $\det(\mathbb{1} + \tilde{\kappa}\zeta^T) = (\mathbb{1} + \zeta^T \tilde{\kappa}) \neq 0$, since $\zeta^T \tilde{\kappa}$ is purely imaginary.

If E is not invertible, up to permutations and relabelling we assume that $e_1 = 0$. We suitably add or subtract the first row of K to all other ones, in order to get that

$$\det(K + iE) = \det \begin{pmatrix} z_1 & z_1 & \dots & z_1 \\ 0 & ie_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & ie_p \end{pmatrix} = (i)^{p-1} z_1 e_2 \dots e_p.$$

The same arguments prove that $(K - iE)$ is invertible. □

Lemma 4. *For every $i = 1, \dots, p$ the matrices $(K + iE + ie_i \mathbb{1})$, $(K + iE - ie_i \mathbb{1})$, $(K - iE + ie_i \mathbb{1})$ and $(K - iE - ie_i \mathbb{1})$ are invertible.*

Proof. By contradiction, assume that ie_i is an eigenvalue of $K + iE$, that is there exists $v \in \mathbb{C}^p$ such that $(K + iE)v = ie_i v$, that is $Kv = (i(e_i - e_1)v_1, \dots, i(e_i - e_p)v_p)^T$; since $Kv = \kappa(\zeta^T v)$ and all the components of κ are different from zero, this implies that $\zeta^T v = 0$ and therefore $Kv = 0$. Then $i(E - e_i \mathbb{1})v = 0$, that is $v_j = 0$ for every $j \neq i$, therefore $(Kv)_l = z_l \sum_{k=1}^p v_k = z_l v_i$ for every l . Since $Kv = 0$, then $v = 0$. Then ie_i cannot be an eigenvalue of $K + iE$.

Assume now that $-ie_i$ is an eigenvalue of $K + iE$. Analogously, we have that there exists $v \in \mathbb{C}^p$ such that $(Kv)_j = -i(e_i + e_j)v_j$ for every j . But as above $(Kv)_j = z_j(\sum_{l=1}^p v_l)$. If there is some j such that $e_i + e_j = 0$, reasoning as above we can prove that $v = 0$. If not, we can write $v_j = \frac{iz_j}{e_i + e_j} \sum_{l=1}^p v_l$ for every j . Then $\sum_{j=1}^p v_j = i \sum_{j=1}^p \frac{z_j}{e_i + e_j} \sum_{l=1}^p v_l$, that is $i \sum_{j=1}^p \frac{z_j}{e_i + e_j} = 1$, which is a contradiction, since all z_{e_i} and e_j are real.

Analogous computations prove that also $(K - iE - ie_i \mathbb{1})$ and $(K - iE + ie_i \mathbb{1})$ are invertible for every $i = 1, \dots, p$. □

Remark 1. In particular, 0 and $\pm ie_i$, $i = 1, \dots, p$, are not eigenvalues of M .

We are now ready to prove the hyperbolicity of every fixed point of the linearised system (5).

Proposition 1. For every $\mathbf{Q} \in \mathcal{I}$, all eigenvalues of the matrix $M_{\mathbf{Q}}$ have non-zero real part.

Proof. Assume by contradiction that $i\ell$, $\ell \in \mathbb{R}$, is an eigenvalue of M relative to the eigenvector $(X, Y)^T$ (where $X, Y \in \mathbb{C}^p$), and assume that $X + iY \neq 0$ (if this is the case, then $X - iY \neq 0$ and we can repeat the same argument below with $(K - iE)$). Then $X + iY$ is an eigenvector of $(K + iE)$ relative to $i\ell$, and we remark that, by Lemma 4, ℓ is different from every of the e_i . Let $\chi, \eta \in \mathbb{R}^p$ be the real vectors such that $X + iY = \chi + i\eta$. Straight computations show that

$$\begin{cases} K\chi - E\eta = -\ell\eta \\ K\eta + E\chi = \ell\chi \end{cases} \Rightarrow \begin{cases} (E - \ell\mathbb{1})\eta = K\chi = (\zeta^T\chi)\kappa \\ (E - \ell\mathbb{1})\chi = -K\eta = -(\zeta^T\eta)\kappa. \end{cases}$$

Since $(E - \ell\mathbb{1})$ is invertible, we have that

$$\begin{aligned} \chi &= -(\zeta^T\eta)(E - \ell\mathbb{1})^{-1}\kappa \\ \eta &= (\zeta^T\chi)(E - \ell\mathbb{1})^{-1}\kappa, \end{aligned}$$

that is, η and χ are parallel and $X + iY = (a + ib)(E - \ell\mathbb{1})^{-1}\kappa$, for some real coefficients a, b . Then $(E - \ell\mathbb{1})^{-1}\kappa$ is a real eigenvector of $(K + iE)$ relative to $i\ell$, which implies that $K(E - \ell\mathbb{1})^{-1}\kappa = 0$ and $E(E - \ell\mathbb{1})^{-1}\kappa = \ell(E - \ell\mathbb{1})^{-1}\kappa$, which is possible only if κ is null. \square

Lemma 5. Let ℓ be an eigenvalue of $K_{\mathbf{Q}} + iE$. Then the following equality holds

$$\sum_{j=1}^p \frac{z_j^{\mathbf{Q}}(\lambda + i(\mu - e_j))}{\lambda^2 + (e_j - \mu)^2} = 1, \quad (6)$$

where λ and μ denote respectively the real and the imaginary part of ℓ . In particular, all the eigenvalues of $K_{\mathbf{X}^-} + iE$ have positive real part and all the eigenvalues of $K_{\mathbf{X}^+} + iE$ have negative real part.

Proof. Thank to property **(P2)** and the fact that ℓ is not an eigenvalue of iE , it holds

$$\begin{aligned} \det(K + iE - \ell\mathbb{1}) &= \det(iE - \ell\mathbb{1})(1 + \zeta^T(iE - \ell\mathbb{1})^{-1}\kappa) \\ &= \det(iE - \ell\mathbb{1}) \left(1 + \sum_{j=1}^p \frac{z_j}{ie_j - \ell}\right). \end{aligned}$$

\square

Analogous computations show that every eigenvalue $\ell = \lambda + i\mu$ of $K_{\mathbf{Q}} - iE$ satisfy the equation

$$\sum_{j=1}^p \frac{z_j^{\mathbf{Q}}(\lambda - i(\mu - e_j))}{\lambda^2 + (e_j - \mu)^2} = 1.$$

Let us now consider the linearised flow in the tangent space to \mathcal{S} at some $\mathbf{Q} \in \mathcal{I}$. First of all, we notice that $T_{\mathbf{Q}}\mathcal{S} = \{(\delta\mathbf{x}, \delta\mathbf{y}, 0)\}$ for every $\mathbf{Q} \in \mathcal{I}$, therefore the linearisation of the flow ϕ^t on $T_{\mathbf{Q}}\mathcal{S}$ can be represented by the matrix M . In particular, Proposition 1 implies that each $\mathbf{Q} \in \mathcal{I}$ is a hyperbolic equilibrium for the flow ϕ^t (restricted to \mathcal{S}).

Moreover, it is possible to see that, for every $\mathbf{Q} \neq \mathbf{X}^-$, then at least two eigenvalues of $D_{\mathbf{Q}}\phi^t$ have positive real part. Indeed, if this were not the case, we will obtain a contradiction with the fact none of the $\mathbf{Q} \in \mathcal{I} \setminus \{\mathbf{X}^-\}$ is a local minimum of the Lyapunov function, that is, none of these equilibria is stable.

Theorem 1. *There exists an open dense set $\mathbf{G}^p \subset \mathbf{S}$ such that for every $\mathbf{X}_0 \in \mathbf{G}^p$ the solution of the control system (2)-(4) with initial condition \mathbf{X}_0 tends asymptotically to \mathbf{X}^- , with exponential velocity.*

Proof.

Consider $\mathbf{Q} \in \mathcal{I} \setminus (\mathbf{X}^+ \cup \mathbf{X}^-)$. From Proposition 1 we know that the restriction of $D\phi^t$ to $T_{\mathbf{Q}}\mathbf{S}$ satisfies the following properties: there exists a splitting of the tangent space $T_{\mathbf{Q}}\mathbf{S} = E_{\mathbf{Q}}^- \oplus E_{\mathbf{Q}}^+$ such that

- there exists $\rho_+ > 1$ such that $\|D\phi^{-t}|_{E_{\mathbf{Q}}^+}\| \leq \rho_+^{-t}$ and $\dim E_{\mathbf{Q}}^+ \geq 2$
- there exists $\rho_- < 1$ such that $\|D\phi^t|_{E_{\mathbf{Q}}^-}\| \leq \rho_-^t$ and $\dim E_{\mathbf{Q}}^- \geq 2$.

Then we can apply Hadamard-Perron Theorem [20] and conclude that there exist two \mathcal{C}^1 -smooth injectively immersed submanifolds $W_{\mathbf{Q}}^s, W_{\mathbf{Q}}^u \subset \mathbf{S}$ such that

$$W_{\mathbf{Q}}^s = \{X \in \mathbf{S} : \text{dist}(\phi^t(X), \mathbf{Q}) \rightarrow 0 \text{ as } t \rightarrow +\infty\} \quad \text{and} \quad T_{\mathbf{Q}}W_{\mathbf{Q}}^s = E_{\mathbf{Q}}^- \quad (7)$$

$$W_{\mathbf{Q}}^u = \{X \in \mathbf{S} : \text{dist}(\phi^{-t}(X), \mathbf{Q}) \rightarrow 0 \text{ as } t \rightarrow +\infty\} \quad \text{and} \quad T_{\mathbf{Q}}W_{\mathbf{Q}}^u = E_{\mathbf{Q}}^+. \quad (8)$$

We recall that every point in \mathbf{S} asymptotically reaches \mathcal{I} , under the action of the flow ϕ^t . Therefore, the set of all points that do not asymptotically reach \mathbf{X}^- is

$$\mathcal{Q} = \bigcup_{\mathbf{Q} \in \mathcal{I} \setminus \{\mathbf{X}^+ \cup \mathbf{X}^-\}} W_{\mathbf{Q}}^s \cup \{\mathbf{X}^+\}.$$

Set $\mathbf{G}^p = \mathbf{S} \setminus \mathcal{Q}$, and notice that \mathcal{Q} is a finite union of smooth manifolds of codimension at least 2. This implies that its complement is dense.

Let \mathbf{X}_k be a sequence in \mathcal{Q} , converging to some $\bar{\mathbf{X}} \in \mathbf{S}$. By continuity with respect to initial condition, for every $\epsilon > 0$ and every $T > 0$ there exists \bar{k} such that if $k \geq \bar{k}$, then $|\phi^T(\mathbf{X}_k) - \phi^T(\bar{\mathbf{X}})| \leq \epsilon$, which implies, by smoothness of the Lyapunov function V , that $|V(\phi^T(\mathbf{X}_k)) - V(\phi^T(\bar{\mathbf{X}}))| \leq L\epsilon$, for some $L > 0$.

Since for every t and every k it holds $V(\phi^t(\mathbf{X}_k)) \geq \bar{V}$, where $\bar{V} = \min_{\mathcal{I} \setminus \{\mathbf{X}^-\}} V$, then we can conclude that for every $\epsilon > 0$ and $T > 0$ we can find \bar{k} such that $V(\phi^T(\bar{\mathbf{X}})) \geq V(\phi^T(\mathbf{X}_{\bar{k}})) - \epsilon \geq \bar{V} - \epsilon$. Then

$$\phi^t(\bar{\mathbf{X}}) \rightarrow \mathcal{I} \setminus \{\mathbf{X}^-\},$$

that is

$$\bar{\mathcal{Q}} \subset \bigcup_{\mathbf{Q} \in \mathcal{I} \setminus \{\mathbf{X}^-\}} W_{\mathbf{Q}}^s = \mathcal{Q}.$$

□

4. Countable case

4.1. Existence of solutions

Let us now assume that $\mathcal{E} = \{e_k\}_{k \in \mathbb{N}}$ is sequence of pairwise distinct elements contained in I .

The state of the system is represented by the sequence $\mathbf{X} = \{X_{e_k}\}_k$, with $X_k \in \mathbb{S}^2$, and the state space is the countable product $\mathbf{S} = \prod_{k=1}^{\infty} \mathbb{S}_{e_k}^2$.

Before trying to solve the problem (**P'**), it is necessary to discuss its well-posedness. To do this, we embed \mathbf{S} into the metric space $\prod_{k=1}^{\infty} \mathbb{R}^3$ endowed with the distance function d defined by

$$d(\mathbf{X}, \mathbf{X}') = \sum_{k=1}^{\infty} w_k |X_k - X'_k|,$$

where $|X_k - X'_k|^2 = |x_{e_k} - x'_{e_k}|^2 + |y_{e_k} - y'_{e_k}|^2 + |z_{e_k} - z'_{e_k}|^2$ and $\{w_k\}_{k \in \mathbb{N}}$ is a positive monotone sequence such that the series $\sum_{k \in \mathbb{N}} w_k$ converges. Without loss of generality, here below we put $w_k = 2^{-k}$.

Lemma 6 (Completeness). $(\prod_{k=1}^{\infty} \mathbb{R}^3, \mathbf{d})$ is a Banach space.

Proof. The proof of the completeness is standard: let $(\mathbf{X}^n)^n$ be a fundamental sequence in $\prod_{k=1}^{\infty} \mathbb{R}^3$. By definition, for every $\epsilon > 0$ there exists $N(\epsilon) > 0$ such that if $n, m > N(\epsilon)$, then $\mathbf{d}(\mathbf{X}^n, \mathbf{X}^m) \leq \epsilon$, and in particular $|X_k^n - X_k^m| \leq 2^k \epsilon$ for every k . In particular, for every k $(X_k^n)^n$ converges to some $\bar{X}_k \in \prod_{k=1}^{\infty} \mathbb{R}^3$ as n tends to infinity; we define $\bar{\mathbf{X}} = \{\bar{X}_n\}_n$. Let us fix $M > 0$. By definition, if $n, m > N(\epsilon)$, then $\sum_{k=1}^M 2^{-k} |X_k^n - X_k^m| \leq \epsilon$, therefore

$$\sum_{k=1}^M 2^{-k} |\bar{X}_k - X_k^n| = \lim_m \sum_{k=1}^M 2^{-k} |X_k^m - X_k^n| \leq \epsilon.$$

Since this is true for each M , if $n > N(\epsilon)$ we have $\mathbf{d}(\bar{\mathbf{X}}, \mathbf{X}^n) = \lim_M \sum_{k=1}^M 2^{-k} |\bar{X}_k - X_k^n| \leq \epsilon$. This completes the proof. \square

We notice that \mathcal{S} is a proper connected subset of the unit sphere in the Banach space $(\mathbf{\Pi}^{\infty} \mathbb{R}^3, \mathbf{d})$.

Remark 2. By standard arguments, it is easy to prove that, for every $-\infty < a < b < +\infty$, $\mathcal{C}([a, b], \mathbf{\Pi}^{\infty} \mathbb{R}^3)$ is a Banach space with respect to the sup norm

$$\|\mathbf{f}\|_{\mathcal{C}([a, b], \mathbf{\Pi}^{\infty} \mathbb{R}^3)} = \sup_{t \in [a, b]} \left| \sum_{k=1}^{\infty} 2^{-k} |f_k(t)| \right|.$$

We now consider the feedback control $\mathbf{u} = (u_1, u_2)$, defined by

$$\begin{cases} u_1 = \sum_{k=1}^{\infty} 2^{-k} y_{e_k} \\ u_2 = \sum_{k=1}^{\infty} 2^{-k} x_{e_k} \end{cases} \quad (9)$$

and we plug it into the control system (3). The resulting autonomous dynamical system on \mathcal{S} is well defined, as the following result states.

Theorem 2. The Cauchy problem $\dot{\Xi} = \mathbf{F}(\Xi, \mathbf{u})$ with initial condition in \mathcal{S} is well-defined.

Proof. In order to apply the standard techniques, we need our solution space to be a Banach space. Therefore, we consider the Cauchy problem on $\mathbf{\Pi}^{\infty} \mathbb{R}^3$

$$\begin{cases} \dot{\Xi} = \tilde{\mathbf{F}}(\Xi) \\ \Xi(0) = \Xi^0, \end{cases} \quad (10)$$

where $\tilde{\mathbf{F}} = \{\tilde{F}_k\}_{k=1}^{\infty}$ is the vector field on $\mathbf{\Pi}^{\infty} \mathbb{R}^3$ defined by

$$\tilde{F}_k(\mathbf{X}) = e_k A \psi(X_{e_k}) + \varphi \left(\sum_{j=0}^{\infty} 2^{-j} x_{e_j} \right) B \psi(X_{e_k}) + \varphi \left(\sum_{j=0}^{\infty} 2^{-j} y_{e_j} \right) C \psi(X_{e_k}),$$

where $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are the cut-off functions

$$\varphi(x) = \begin{cases} x & \text{if } |x| \leq b \\ b & \text{if } |x| \geq b \end{cases} \quad \psi(\mathbf{w}) = \begin{cases} \mathbf{w} & \text{if } |\mathbf{w}| \leq a \\ a \frac{\mathbf{w}}{|\mathbf{w}|} & \text{if } |\mathbf{w}| \geq a \end{cases},$$

for some real numbers $a, b > 1$. Proceeding as in the proof of the Cauchy-Lipschitz (Picard-Lindelöf) Theorem, we can prove that there exists an interval I_0 containing 0 such that the Cauchy problem (10) admits a unique solution, continuous on I_0 .

We notice that, if the initial condition belongs to \mathcal{S} , then the solution of (10) belongs to \mathcal{S} for all $t \in I_0$. Moreover, by computations we can prove that the solution arising from an initial condition in \mathcal{S} is well defined for all $t \in \mathbb{R}$. Finally, we observe that $\tilde{\mathbf{F}}|_{\mathcal{S}} = \mathbf{F}$, therefore the solutions of (10) with initial condition in \mathcal{S} coincide with the solutions of the equation $\dot{\Xi} = \mathbf{F}(\Xi, \mathbf{u}(\Xi))$ with the same initial condition. \square

Proposition 2. *The solutions of the Cauchy problem (10) depend smoothly on initial conditions.*

Proof. Let $\Xi(\cdot), \Xi'(\cdot)$ be the solutions of (10) with initial condition equal to Ξ^0 and $\Xi^{0'}$, respectively, and let $\xi(t) = \|\Xi(t) - \Xi'(t)\|$. It is easy to see that

$$|\xi(t)| \leq |\xi(0)| + \int_0^t \alpha |\xi(s)| ds,$$

where α is a constant depending on the cut-off parameters a and b . By Gronwall inequality we get that $|\xi(t)| \leq e^{\alpha t} |\xi(0)|$. \square

4.2. Pointwise convergence to \mathbf{X}^-

Let us consider the function $V : \mathbf{S} \rightarrow \mathbb{R}$ defined by $V(\mathbf{X}) = \sum_{k=1}^{\infty} 2^{-k} z_{e_k}$. It is easy to see that its time derivative along the integral curves of the vector field $\mathbf{F}(\mathbf{X}, \mathbf{u}(\mathbf{X}))$ satisfies $\dot{V} = -(\sum_{k=1}^{\infty} 2^{-k} x_{e_k})^2 - (\sum_{k=1}^{\infty} 2^{-k} y_{e_k})^2 \leq 0$. In order to conclude about the stability of these trajectories by means of a La Salle-type argument, we need to prove that \mathbf{S} is compact. To do that, let us first recall the following definition (see for instance [21]).

Definition 1. *The product topology \mathcal{T} on \mathbf{S} is the coarsest topology that makes continuous all the projections $\pi_k : \mathbf{S} \rightarrow S_{e_k}^2$.*

By Tychonoff's Theorem, any product of compact topological spaces is compact with respect to the product topology ([21]). This in particular implies that \mathbf{S} is compact with respect to \mathcal{T} .

As we will see just below, the product topology is equivalent to topology induced by the distance \mathbf{d} , so \mathbf{S} is compact with respect to the latter.

Lemma 7. *Let us denote with $\mathcal{T}_{\mathbf{d}}$ the topology on \mathbf{S} induced by \mathbf{d} . We have that $\mathcal{T} = \mathcal{T}_{\mathbf{d}}$.*

Proof. By definition, $\mathcal{T} \subset \mathcal{T}_{\mathbf{d}}$. If we prove that the open balls (that are a basis for $\mathcal{T}_{\mathbf{d}}$) are open with respect to \mathcal{T} , then $\mathcal{T}_{\mathbf{d}} \subset \mathcal{T}$ and we get the result.

Let $N > 0$ and let us define the function $\mathbf{d}^N : \mathbf{S} \times \mathbf{S} \rightarrow [0, 1]$ as $\mathbf{d}^N(\mathbf{X}, \mathbf{X}') = \sum_{k=1}^N 2^{-k} |\mathbf{X}_k - \mathbf{X}'_k|$. It is easy to prove that \mathbf{d}^N is continuous with respect to \mathcal{T} ; indeed, the restriction $\mathbf{d}|_{\prod_{i=1}^N S_{e_i}^2}$ is obviously continuous with respect to the product topology on $\prod_{i=1}^N S_{e_i}^2$, and for every open interval $(a, b) \subset [0, 1]$ we have that $(\mathbf{d}^N)^{-1}(a, b) = (\mathbf{d}|_{\prod_{i=1}^N S_{e_i}^2})^{-1}(a, b) \times \prod_{i>N} S_{e_i}^2$, which is open with respect to \mathcal{T} .

The sequence $\{\mathbf{d}^N\}^N$ converges uniformly to \mathbf{d} . Indeed, for every $\mathbf{X}, \mathbf{X}' \in \mathbf{S}$ we have that

$$|\mathbf{d}^N(\mathbf{X}, \mathbf{X}') - \mathbf{d}(\mathbf{X}, \mathbf{X}')| = \left| \sum_{k \geq N+1} 2^{-k} |X_{e_k} - X'_{e_k}| \right| \leq 2^{-N}.$$

Then \mathbf{d} is continuous with respect to \mathcal{T} , and this completes the proof. \square

Thanks to previous Lemma, we can conclude that \mathbf{S} is compact with respect to \mathbf{d} . In particular, we are able to prove a version of La Salle invariance principle holding for the equation $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \mathbf{u}(\mathbf{X}))$. We also remark that the Lyapunov function V is continuous with respect to \mathcal{T} .

Proposition 3 (Adapted La Salle). *Let us consider the set $\mathcal{M} = \{\mathbf{X} \in \mathbf{S} : \dot{V}(\mathbf{X}) = 0\}$, where we use the notation $\dot{V}(\mathbf{X}) = \frac{d}{dt} V(\phi^t(\mathbf{X}))|_{t=0}$, and ϕ^t denotes the flow associated with the dynamical system $\dot{\mathbf{X}} = (\mathbf{X}, \mathbf{u}(\mathbf{X}))$. Let \mathcal{I} be the largest subset of \mathcal{M} which is invariant for the flow ϕ^t . Then for every $\mathbf{X} \in \mathbf{S}$ we have that $\phi^t(\mathbf{X}) \rightarrow \mathcal{I}$ as $t \rightarrow +\infty$.*

Proof. The proof of this proposition relies on the compactness of \mathbf{S} with respect to the topology $\mathcal{T}_{\mathbf{d}}$, and follows standard arguments.

Let $m = \min_{\mathbf{X} \in \mathbf{S}} V(\mathbf{X})$, and fix $\mathbf{X}^0 \in \mathbf{S}$. By continuity of V , there exists $a = \lim_{t \rightarrow +\infty} V(\phi^t(\mathbf{X}^0))$, $a \geq m$.

Let $\Omega_{\mathbf{X}^0} = \{\mathbf{X} \in \mathbf{S} : \exists (t_n)_n \rightarrow +\infty : \phi^{t_n}(\mathbf{X}^0) \rightarrow \mathbf{X}\}$ denote the ω -limit set issued from \mathbf{X}^0 ; notice that $\Omega_{\mathbf{X}^0}$ is non-empty, since \mathbf{S} is compact, therefore for every sequence $(t_n)_n \rightarrow +\infty$ there

exists a subsequence $(t_{n_k})_k$ such that $\phi^{t_{n_k}}(\mathbf{X}^0)$ converges to some point in \mathbf{S} . It is easy to see that $\Omega_{\mathbf{X}^0}$ is invariant for the flow ϕ^t and therefore, since by continuity $V|_{\Omega_{\mathbf{X}^0}} = a$, we obtain that $\dot{V}(\mathbf{X}) = 0$ for every $\mathbf{X} \in \Omega_{\mathbf{X}^0}$. This implies that $\Omega_{\mathbf{X}^0} \subset \mathcal{M}$.

Let us now prove that $\Omega_{\mathbf{X}^0}$ is compact. Consider a sequence $(\mathbf{X}_k)_k$ contained in $\Omega_{\mathbf{X}^0}$; by compactness of \mathbf{S} , it converges, up to subsequences, to some $\bar{\mathbf{X}} \in \mathbf{S}$ (we relabel the indexes). By definition, for every k there exist a sequence $(t_{k_n})_n \rightarrow +\infty$ such that $\lim_n \phi^{t_{k_n}}(\mathbf{X}^0) = \mathbf{X}_k$. Moreover, it is possible to define a divergent sequence $(\tau_k)_k$ such that $d(\phi^{\tau_k}(\mathbf{X}^0), \mathbf{X}_k) \leq 1/2k$ for every k . Fix $\epsilon > 0$ and choose some $\bar{k} \geq 1/\epsilon$ such that $d(\bar{\mathbf{X}}, \mathbf{X}_k) \leq \epsilon/2$ for $k \geq \bar{k}$ (possibly taking a suitable subsequence). Then for $k \geq \bar{k}$ we have that $d(\bar{\mathbf{X}}, \phi^{\tau_k}(\mathbf{X}^0)) \leq d(\bar{\mathbf{X}}, \mathbf{X}_k) + d(\phi^{\tau_k}(\mathbf{X}^0), \mathbf{X}_k) \leq \epsilon$. This means that $\bar{\mathbf{X}} \in \Omega_{\mathbf{X}^0}$, that is $\Omega_{\mathbf{X}^0}$ is compact.

Finally, let us assume, by contradiction, that there exist an open neighbourhood U of $\Omega_{\mathbf{X}^0}$ in \mathbf{S} and a sequence $(t_n)_n \rightarrow +\infty$ such that $\phi^{t_n}(\mathbf{X}^0) \in \mathbf{S} \setminus U$ for every n . By compactness of \mathbf{S} , ϕ^{t_n} converges up to subsequences to some $\bar{\mathbf{X}} \in \mathbf{S} \setminus U$. But by definition $\bar{\mathbf{X}} \in \Omega_{\mathbf{X}^0}$, then we have a contradiction.

Let us now set $\mathcal{I} = \cup_{\mathbf{X} \in \mathbf{S}} \Omega_{\mathbf{X}}$. By construction, it is an invariant subset contained in \mathcal{M} . \square

By definition, $\mathcal{M} = \{\mathbf{X} \in \mathbf{S} : \sum_{k=1}^{\infty} 2^{-k} x_{e_k} = \sum_{k=1}^{\infty} 2^{-k} y_{e_k} = 0\}$. Now we look for its largest invariant subset. Let $\mathbf{X}^0 \in \mathcal{M}$; with the same argument than above, we can see that $\phi^t(\mathbf{X}^0) = \{(x_{e_k}(t), y_{e_k}(t), z_{e_k}(t))\}_k$ with

$$x_{e_k}(t) = \cos(e_k t) x_{e_k}^0 - \sin(e_k t) y_{e_k}^0 \quad y_{e_k}(t) = \sin(e_k t) x_{e_k}^0 + \cos(e_k t) y_{e_k}^0 \quad z_{e_k}(t) = z_{e_k}^0.$$

If \mathbf{X}^0 belongs to an invariant subset of \mathcal{M} , then $\sum_{k=1}^{\infty} 2^{-k} x_{e_k}(t) = \sum_{k=1}^{\infty} 2^{-k} y_{e_k}(t) = 0$ for every t . Let us consider the two functions

$$\begin{aligned} f(t) &= \sum_{k=1}^{\infty} 2^{-k} x_{e_k}(t) = \sum_{k=1}^{\infty} 2^{-k} (\cos(e_k t) x_{e_k}^0 - \sin(e_k t) y_{e_k}^0) \\ g(t) &= \sum_{k=1}^{\infty} 2^{-k} y_{e_k}(t) = \sum_{k=1}^{\infty} 2^{-k} (\sin(e_k t) x_{e_k}^0 + \cos(e_k t) y_{e_k}^0). \end{aligned}$$

It is easy to see that both $f(t)$ and $g(t)$ are uniform limits of trigonometric polynomials, therefore they are almost periodic functions ([22]). By immediate computations, we can prove that the Fourier series for $f(t)$ and $g(t)$ are respectively

$$f(t) \sim \sum_{k=1}^{\infty} \left(\frac{x_{e_k}^0 + i y_{e_k}^0}{2} \right) e^{i e_k t} + \left(\frac{x_{e_k}^0 - i y_{e_k}^0}{2} \right) e^{-i e_k t} \quad g(t) \sim \sum_{k=1}^{\infty} \left(\frac{y_{e_k}^0 - i x_{e_k}^0}{2} \right) e^{i e_k t} + \left(\frac{y_{e_k}^0 + i x_{e_k}^0}{2} \right) e^{-i e_k t}$$

By the properties of the Fourier series of almost periodic functions (see [22]), f and g are identically zero if and only if all the coefficients in their Fourier series are all null, that is $x_{e_k}^0 = y_{e_k}^0 = 0$ for every k . Then the largest invariant subset of \mathcal{M} is $\mathcal{I} = \{\mathbf{X} : x_{e_k} = y_{e_k} = 0, |z_{e_k}| = 1 \forall k\}$.

Applying Proposition 3 to these facts, we get the following result.

Corollary 1. *Let $\mathbf{X}^0 \in \mathbf{S}$, and let $\mathbf{X}(t) = \phi^t(\mathbf{X}^0)$, with the usual notation $\mathbf{X}(t) = \{X_{e_k}(t)\}_k$ and $X_{e_k}(t) = (x_{e_k}(t), y_{e_k}(t), z_{e_k}(t))$. Then*

$$\begin{aligned} \lim_{t \rightarrow +\infty} x_{e_k}(t) &= \lim_{t \rightarrow +\infty} y_{e_k}(t) = 0 \\ \left| \lim_{t \rightarrow +\infty} z_{e_k}(t) \right| &= 1 \end{aligned}$$

for every $k \geq 1$.

Remark 3. *It is easy to see that every point $\mathbf{X} \in \mathcal{I}$ is an accumulation point for the set \mathcal{I} , but \mathcal{I} is not dense in \mathbf{S} . In particular, also \mathbf{X}^- is an accumulation point for \mathcal{I} .*

In the following, for every $N \geq 1$ we consider the *truncated feedback control* $\mathbf{u}^N = (u_1^N, u_2^N)$ with $u_1^N = \sum_{k=1}^N 2^{-k} y_{e_k}$ and $u_2^N = \sum_{k=1}^N 2^{-k} x_{e_k}$, and we call $\Xi^N(\cdot)$ the solution of the equation $\dot{\Xi}^N = \mathbf{F}(\Xi^N, \mathbf{u}^N)$. We remark that, for initial conditions in \mathcal{S} , the solution of the Cauchy problem exists and remains in \mathcal{S} for all t . As above, we use the notations $\Xi^N(\cdot) = \{\Xi_{e_k}^N(\cdot)\}_k$ with $\Xi_{e_k}^N = \{(x_{e_k}^N(\cdot), y_{e_k}^N(\cdot), z_{e_k}^N(\cdot))\}_k$.

Applying the same arguments as in Section 3, we can prove the following result.

Proposition 4. *For every $N > 0$, there exists an open dense set $A \subset \mathcal{S}$ such that for every $\mathbf{X} \in A$ the solution $\Xi^N(\cdot)$ of the equation $\dot{\Xi}^N = \mathbf{F}(\Xi^N, \mathbf{u}^N)$ with initial condition equal to \mathbf{X} has the following asymptotic behaviour:*

$$\lim_{t \rightarrow +\infty} x_{e_k}^N(t) = 0 \quad \lim_{t \rightarrow +\infty} y_{e_k}^N(t) = 0 \quad (11)$$

$$\lim_{t \rightarrow +\infty} z_{e_k}^N(t) = -1 \quad (12)$$

for $1 \leq k \leq N$.

Proof. The proof relies on the fact that the restriction of Ξ^N to the first N components obeys to the dynamical system (2)-(4). Then we can apply Theorem 1 and conclude that there exists an open dense subset A' of $\prod_{k=1}^N S^2$ such that for every $\mathbf{X} \in \mathcal{S}$ with $\{X_{e_1}, \dots, X_{e_N}\} \in A'$ the solution $\Xi^N(\cdot)$ of the equation $\dot{\Xi}^N = \mathbf{F}(\Xi^N, \mathbf{u}^N)$ with initial condition equal to \mathbf{X} satisfies the behaviour described in (11), independently on the value of $\{X_k\}_{k \geq N+1}$. \square

Proposition 4 leads to the asymptotic pointwise convergence of the trajectories of (\mathbf{X}, \mathbf{u}) to \mathbf{X}^- , according to the following definition:

Definition 2. *The sequence of functions $\{\Xi_k\}_k$, with $\Xi_k(\cdot) : \mathbb{R} \rightarrow \mathcal{S}$ for every k , converges asymptotically pointwise to the point $\mathbf{X} \in \mathcal{S}$ if for every $\epsilon > 0$ there exist an integer $\bar{N} > 0$ such that for every $N \geq \bar{N}$ there exists a time $t = t(N, \epsilon)$ such that if $t \geq t(N, \epsilon)$ then $\mathbf{d}(\Xi_N(t), \mathbf{X}) \leq \epsilon$.*

We can then state the following result.

Theorem 3. *There exists a residual set $\mathcal{G} \subset \mathcal{S}$ such that for every $\mathbf{X} \in \mathcal{G}$ there exists a sequence $\{\mathbf{u}^k\}_k$ of controls such that the sequence $\{\Xi_k\}_k$ of solutions of the equation $\dot{\Xi}_k = \mathbf{F}(\Xi_k, \mathbf{u}_k)$ with initial condition equal to \mathbf{X} converges asymptotically pointwise to \mathbf{X}^- .*

Proof. Let $\mathbf{G}^N \subset \prod_{k=1}^N S_k^2$ be the set of “good initial conditions” for the N -dimensional system, as defined in Theorem 1, and let us define $\widehat{\mathbf{G}}^N = \mathbf{G}^N \times \prod_{k \geq N+1} S^2 \subset \mathcal{S}$. Proposition 4 states that the solution of the truncated system $\dot{\Xi}^N = \mathbf{F}(\Xi^N, \mathbf{u}^N)$ with initial condition in $\widehat{\mathbf{G}}^N$ has the limit (11). Since $\widehat{\mathbf{G}}^N$ is an open dense subset of \mathcal{S} for every N , and \mathcal{S} has the Baire property ([21]), then $\mathcal{G} = \bigcap_N \widehat{\mathbf{G}}^N$ is a dense subset of \mathcal{S} .

Let $\mathbf{X} \in \mathcal{G}$ and fix $\epsilon > 0$. For some integer N such that $2^{-N+1} < \epsilon$, consider the truncated feedback \mathbf{u}^N , defined as above, and the corresponding trajectory Ξ^N with $\Xi^N(0) = \mathbf{X}$. Since $\mathbf{X} \in \widehat{\mathbf{G}}^N$, by Proposition 4 there exists a time $t = t(N, \epsilon)$ such that for $t \geq t(N, \epsilon)$ it holds $\sum_{k=1}^N 2^{-k} |X_{e_k}^N(t) - (0, 0, -1)^T| \leq \epsilon/2$, then, since $\sum_{k \geq N+1} 2^{-k} |X_{e_k}^N(t) - (0, 0, -1)^T| \leq 2^{-N} \leq \epsilon/2$, we get that $\mathbf{d}(\mathbf{X}^N(t), \mathbf{X}^-) \leq \epsilon$ for $t \geq t(N, \epsilon)$. \square

5. Closed-loop simulations

Let \mathcal{E} be a collection of $N = 30$ randomly chosen points contained in the interval $[1, 4]$, and we consider N randomly chosen initial conditions $X_e(0)$ with $z_e(0) \in [0.8, 1]$ and $|X_e(0)| = 1$. We perform closed-loop simulation of the dynamical system (2) with feedback control $u_1(t) = \sum_{k=1}^N y_{e_k}(t)$ and $u_2(t) = \sum_{k=1}^N x_{e_k}(t)$, up to a final time $T = 20000$.

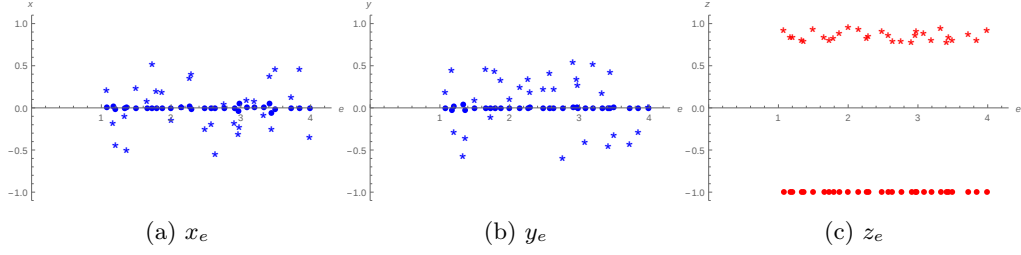


Figure 1: Initial and final states with respect to different values of frequencies (stars denote the initial point, bullets the final point).

In Figure 1 we show the convergence to the target point of the collections X_e , $e \in \mathcal{E}$.

Figure 2 plots the time evolution of the feedback control function, while in Figures 3a and 3b we plot respectively the values of the last coordinate $z_e(t)$, for all $e \in \mathcal{E}$, and of the Lyapunov function $V(t)$, normalized by N .

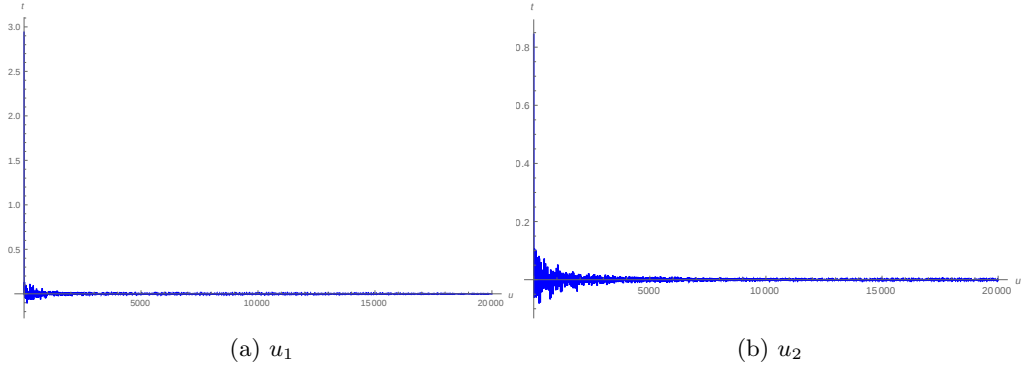


Figure 2: Time evolution of the control function

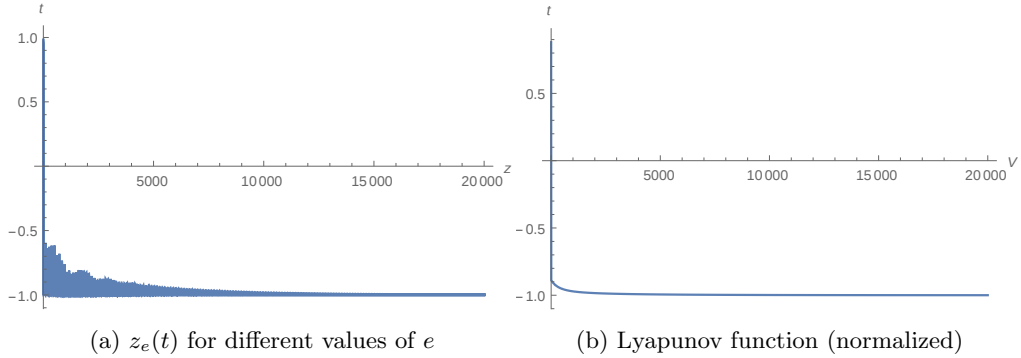


Figure 3: Time evolution of z_e and the Lyapunov function

We then take the same collection \mathcal{E} as before, and we consider N randomly chosen initial conditions $X_e(0)$ with $z_e(0) \in [0.8, 1]$ and $|X_e(0)| = 1$. We now perform closed-loop simulation of the dynamical system (2) with feedback control $u_1(t) = \sum_{k=1}^N w_k y_{e_k}(t)$ and $u_2(t) = \sum_{k=1}^N w_k x_{e_k}(t)$, with $w_k = (1.1)^{-k}$, up to a final time $T = 20000$. The purpose of this new run is to visualise the influence of the weights w_k on the convergence of the systems. As we can see from Figure 6a, the weights slow down the convergence of the systems (this cannot be seen from Figure 6b, since the slower components in the Lyapunov function are multiplied by a small weight).

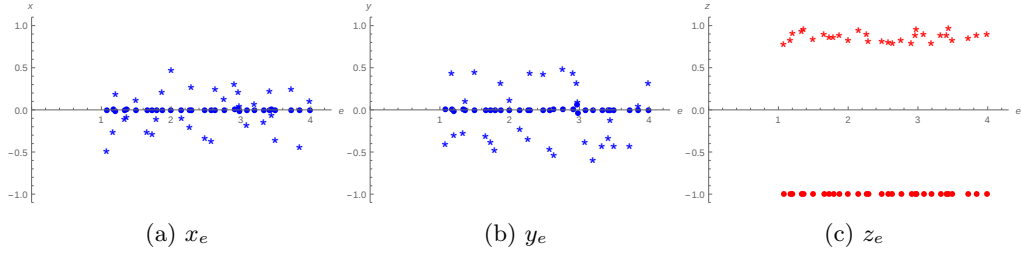


Figure 4: Initial and final states with respect to different values of frequencies (stars denote the initial point, bullets the final point).

In Figure 4 we show the convergence to the target point of the collections X_e , $e \in \mathcal{E}$.

As above, in Figure 5 we plot the time evolution of the feedback control function, in Figures 6a $z_e(t)$, and in 6b the Lyapunov function $V(t)$, normalized by N .

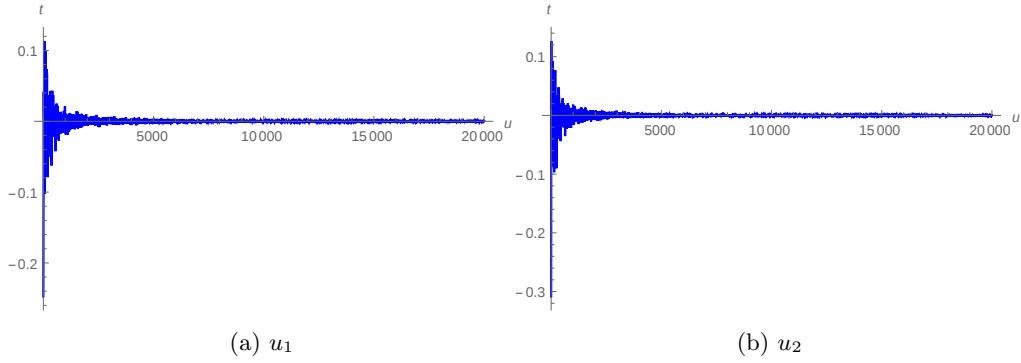


Figure 5: Time evolution of the control function

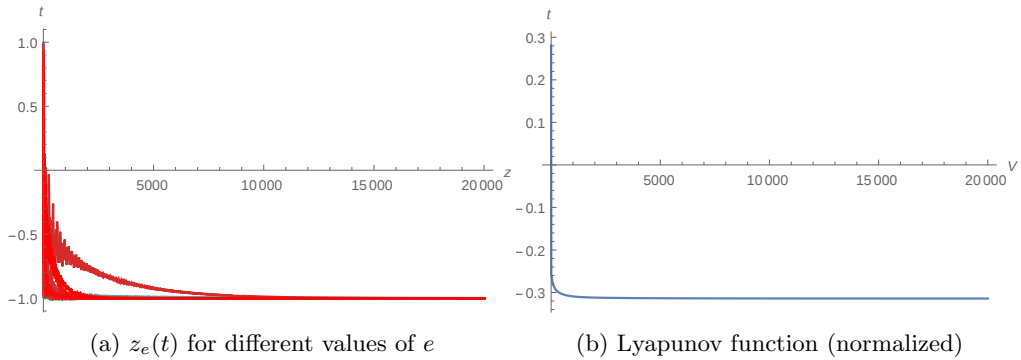


Figure 6: Time evolution of z_e and the Lyapunov function

6. Conclusions

In this paper, we have investigated the stabilization of an ensemble of spins to the uniform state $-1/2$ (represented by the state $(0, 0, -1)$ in the Bloch sphere); in particular, we provided a feedback control that stabilizes a generic initial condition to the target state, asymptotically in time.

If we allow unbounded control functions, it is well known that we can find a control that approximately steer the uniform state $(0, 0, 1)$ of the ensemble to the target state, with arbitrary

precision, and also for a continuous bounded ensemble. Indeed, taking $u_1 \equiv u_2 \equiv M$, where $M \gg \sup \mathcal{E}$, it is easy to see that for every $e \in \mathcal{E}$ the solution at time $T = \pi/\sqrt{u_1^2 + u_2^2}$ of equation (2) with initial condition $X(0) = (0, 0, 1)$ satisfies $|X(T) - (0, 0, -1)| \leq \epsilon$, with $\epsilon = \mathcal{O}(eT)$. This in particular implies strong convergence in the $L^2(\mathcal{E})$ -norm. We remark indeed that, in [18], impulsive controls are needed to achieve a stabilizability in the case where the interval \mathcal{E} is allowed to be unbounded.

In this paper, we provide a stabilizability result that applies bounded controls.

As for the case where \mathcal{E} is an interval, and $\mathbf{X} \in L^2(\mathcal{E}, S^2)$, the question addressed in [4] about controllability of the system by means of bounded controls is still left open. Preliminary simulations suggest that our approach could be generalized also in this last case, but the problem is very hard technically, and it will make the subject of further investigations of the authors.

Appendix A. Estimates on eigenvalues (given by A. O. Remizov)

In this section, we provide some estimates on the eigenvalues of the matrix $M = \begin{pmatrix} K & -E \\ E & K \end{pmatrix}$, with $E = \text{diag}\{e_1, \dots, e_N\}$ and $K = \kappa_{\mathbf{Q}} \zeta^T$, with $\kappa_{\mathbf{Q}}^T = (z_1^{\mathbf{Q}}, \dots, z_N^{\mathbf{Q}})$ and $\zeta^T = (w_1, \dots, w_N)$.

Proposition 5. *Let $\ell = \lambda + i\mu$ be an eigenvalue of M . Then*

$$|\lambda| \geq \min \left\{ \frac{m}{10}, \frac{\epsilon}{MN}, \frac{\epsilon m}{A(N-1)}, \frac{\epsilon^2 m^2}{A^2 M \gamma} \right\},$$

where $\gamma = 4N(N-1)^2$, $m = \min_k |w_k z_k^{\mathbf{Q}}|$, $M = \max_k |w_k z_k^{\mathbf{Q}}|$, $A = \max_k |w_k z_k^{\mathbf{Q}}(\mu - e_k)|$, $\epsilon = \min_{j \neq k} |e_j - e_k|$.

Proof. First of all, we recall that for every eigenvalue $\lambda + i\mu$, the following equalities hold:

$$\sum_k \frac{w_k z_k \lambda}{\lambda^2 + (e_k - \mu)^2} = 1 \quad \sum_k \frac{w_k z_k (\mu - e_k)}{\lambda^2 + (e_k - \mu)^2} = 0.$$

Assume by contradiction that there is at least an eigenvalue $\lambda + i\mu$ of M such that $\lambda < \min\{\frac{m}{10}, \frac{\epsilon}{MN}, \frac{\epsilon m}{A(N-1)}, \frac{\epsilon^2 m^2}{A^2 M \gamma}\}$, and put $\widehat{e}_k = \mu - e_k$.

By simple computations we can show that there exists at least one j such that $\left| \frac{w_j z_j}{\lambda^2 + \widehat{e}_j^2} \right| \geq \frac{1}{N|\lambda|}$ and $\left| \frac{w_j z_j \widehat{e}_j}{\lambda^2 + \widehat{e}_j^2} \right| \leq \frac{(N-1)A}{\epsilon} = C$.

In particular, this implies that

$$\frac{\lambda^2}{|\widehat{e}_j|} \geq \frac{1}{C} - |\widehat{e}_j| \geq \frac{1}{C} - \sqrt{NM|\lambda|} = \frac{1}{2C}$$

so that $|\widehat{e}_k| \leq 2C\lambda^2$

Let us now focus on the term $\frac{w_j z_j}{\lambda^2 + \widehat{e}_j^2} = \frac{1}{\lambda} - \sum_{k \neq j} \frac{w_k z_k}{\lambda^2 + \widehat{e}_k^2}$. From

$$\left| \frac{w_j z_j}{\lambda^2 + \widehat{e}_j^2} \right| \geq \frac{m}{\lambda^2 + \widehat{e}_j^2} \geq \frac{m}{\lambda^2(1 + 4C^2\lambda^2)}$$

and

$$\left| \frac{1}{\lambda} - \sum_{k \neq j} \frac{w_k z_k}{\lambda^2 + \widehat{e}_k^2} \right| \leq \frac{1}{|\lambda|} + (N-1) \frac{M}{\widehat{e}_j^2} \leq \frac{1}{|\lambda|} + (N-1) \frac{M}{\epsilon}$$

we obtain that

$$\frac{|\lambda|(1 + 4C^2\lambda^2)}{m} \geq \frac{\epsilon}{\epsilon + M(N-1)|\lambda|} \geq \frac{N}{2N-1} > \frac{1}{2}.$$

This is a contradiction, since

$$\frac{|\lambda|(1 + 4C^2\lambda^2)}{m} < \frac{|\lambda|}{m} \left(1 + 4C \frac{\epsilon^2 m^2}{A^2(N-1)^2} \right) < \frac{5|\lambda|}{m} < \frac{1}{2}.$$

□

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