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Constructing confidence sets for the matrix completion problem

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Abstract

In the present note we consider the problem of constructing honest and adaptive confidence sets for the matrix completion problem. For the Bernoulli model with known variance of the noise we provide a realizable method for constructing confidence sets that adapt to the unknown rank of the true matrix.

Keywords: low rank recovery, confidence sets, adaptivity, matrix completion.

1 Introduction

In recent years, there has been a considerable interest in statistical inference for high-dimensional matrices. One particular problem is matrix completion where one observes only a small number $n \ll m_1 m_2$ of the entries of a high-dimensional $m_1 \times m_2$ matrix M_0 of unknown rank r ; it aims at inferring the missing entries. The problem of matrix completion comes up in many areas including collaborative filtering, multi-class learning in data analysis, system identification in control, global positioning from partial distance information and computer vision, to mention some of them. For instance, in computer vision, this problem arises as many pixels may be missing in digital images. In collaborative filtering, one wants to make automatic predictions about the preferences of a user by collecting information from many users. So, we have a data matrix where rows are users and columns are items. For each user, we have a partial list of his preferences. We would like to predict the missing ones in order to be able to recommend items that he may be interested in.

In general, recovery of a matrix from a small number of observed entries is impossible, but, if the unknown matrix has low rank, then accurate and even exact recovery is possible. In the noiseless setting, [6, 4, 3] established the following remarkable result: assuming that it satisfies a low coherence condition, M_0 can be recovered exactly by constrained nuclear norm minimization with high probability from only $n \gtrsim r(m_1 \vee m_2) \log^2(m_1 \vee m_2)$ entries observed uniformly at random.

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What makes low-rank matrices special is that they depend on a number of free parameters that is much smaller than the total number of entries. Taking the singular value decomposition of a matrix $A \in \mathbb{R}^{m_1 \times m_2}$ of rank r , it is easy to see that A depends upon $(m_1 + m_2)r - r^2$ free parameters. This number of free parameters gives us a lower bound for the number of observations needed to complete the matrix.

A situation, common in applications, corresponds to the noisy setting in which the few available entries are corrupted by noise. Noisy matrix completion has been extensively studied recently (e.g., [12, 15, 2, 8]). Here we observe a relatively small number of entries of a data matrix

$$Y = M_0 + E$$

where $M_0 = (M_{ij}) \in \mathbb{R}^{m_1 \times m_2}$ is the unknown matrix of interest and $E = (\varepsilon_{ij}) \in \mathbb{R}^{m_1 \times m_2}$ is a matrix of random errors. It is an important issue in applications to be able to say from the observations how well the recovery procedure has worked or, in the sequential sampling setting, to be able to give data-driven stopping rules that guarantee the recovery of the matrix M_0 at a given precision. This fundamental statistical question was recently studied in [7] where two statistical models for matrix completion are considered: the *trace regression model* and the *Bernoulli model* (for details see Section 1.1). In particular, in [7], the authors show that in the case of unknown noise variance, the information-theoretic structure of these two models is fundamentally different. In the trace regression model, even if only an upper bound for the variance of the noise is known, a honest and rank adaptive Frobenius-confidence set whose diameter scales with the minimax optimal estimation rate exists. In the Bernoulli model however, such sets do not exist.

Another major difference is that, in the case of known variance of the noise, [7] provides a realizable method for constructing confidence sets for the trace regression model whereas for the Bernoulli model only the existence of adaptive and honest confidence sets is demonstrated. The proof uses the duality between the problem of testing the rank of a matrix and the existence of honest and adaptive confidence sets. In particular, the construction in [7] is based on infimum test statistics which can not be computed in polynomial time for the matrix completion problem. The present note aims to close this gap and provides a realizable method for constructing confidence sets for the Bernoulli model.

1.1 Notation, assumptions and some basic results

We assume that each entry of Y is observed independently of the other entries with probability $p = n/(m_1 m_2)$. More precisely, if $n \leq m_1 m_2$ is given and B_{ij} are i.i.d. Bernoulli random variables of parameter p independent of the ε_{ij} 's, we observe

$$Y_{ij} = B_{ij} (M_{ij} + \varepsilon_{ij}), \quad 1 \leq i \leq m_1, 1 \leq j \leq m_2. \quad (1)$$

This model for the matrix completion problem is usually called the *Bernoulli model*. Another model often considered in the matrix completion literature is the trace regression model (e.g., [12, 15, 2, 10]). Let $k_0 = \text{rank}(M_0)$.

In many of the most cited applications of the matrix completion problem, such as recommendation systems or the problem of global positioning from the local distances, the noise is bounded but not necessarily identically distributed. This is the assumption which we adopt in the present paper. More precisely, we assume that the noise variables are independent, homoscedastic, bounded and centered:

Assumption 1. For any $(ij) \in [m_1] \times [m_2]$ we assume that $\mathbb{E}(\varepsilon_{ij}) = 0$, $\mathbb{E}(\varepsilon_{ij}^2) = \sigma^2$ and that there exists a positive constant $U > 0$ such that

$$\max_{ij} |\varepsilon_{ij}| \leq U.$$

Let $m = \min(m_1, m_2)$, $d = m_1 + m_2$. For any $l \in \mathbb{N}$ we set $[l] = \{1, \dots, l\}$. For any integer $0 \leq k \leq m$ and any $\mathbf{a} > 0$, we define the parameter space of rank k matrices with entries bounded by \mathbf{a} in absolute value as

$$\mathcal{A}(k, \mathbf{a}) = \{M \in \mathbb{R}^{m_1 \times m_2} : \text{rank}(M) \leq k, \|M\|_\infty \leq \mathbf{a}\}. \quad (2)$$

For constants $\beta \in (0, 1)$ and $c = c(\sigma, \mathbf{a}) > 0$ we have that

$$\inf_{\widehat{M}} \sup_{M_0 \in \mathcal{A}(k, \mathbf{a})} \mathbb{P}_{M_0, \sigma} \left(\frac{\|\widehat{M} - M_0\|_F^2}{m_1 m_2} > c \frac{kd}{n} \right) \geq \beta$$

where \widehat{M} is an estimator of M_0 (see, e.g., [11]). It has been also shown in [11] that an iterative soft thresholding estimator \widehat{M} satisfies with $\mathbb{P}_{M_0, \sigma}$ -probability at least $1 - 8/d$

$$\frac{\|\widehat{M} - M_0\|_F^2}{m_1 m_2} \leq C \frac{(\mathbf{a} + \sigma)^2 kd}{n} \quad \text{and} \quad \|M_0 - \widehat{M}\|_\infty \leq 2\mathbf{a} \quad (3)$$

for a constant $C > 0$. These lower and upper bounds imply that for the Frobenius loss the minimax risk for recovering a matrix $M_0 \in \mathcal{A}(k_0, \mathbf{a})$ is of order $\sqrt{\frac{(\sigma + \mathbf{a})^2 k_0 d m_1 m_2}{n}}$.

For $k \in [m]$ we set

$$r_k = C \frac{(\sigma + \mathbf{a})^2 dk}{n},$$

where C is the numerical constant in (3).

Let A, B be matrices in $\mathbb{R}^{m_1 \times m_2}$. We define the *matrix scalar product* as $\langle A, B \rangle = \text{tr}(A^T B)$. The trace norm of a matrix $A = (a_{ij})$ is defined as $\|A\|_* := \sum \sigma_j(A)$, the operator norm as $\|A\| := \sigma_1(A)$ and the Frobenius norm as $\|A\|_2^2 := \sum_i \sigma_i^2 = \sum_{i,j} a_{ij}^2$ where $(\sigma_j(A))_j$ are the singular values of A ordered decreasingly. $\|A\|_\infty = \max_{i,j} |a_{ij}|$ denotes the largest absolute value of any entry of A .

In what follows, we use symbols C, c for a generic positive constant, which is independent of n, m_1, m_2 , and may take different values at different places. We denote by $a \vee b = \max(a, b)$.

We use the following definition of honest and adaptive confidence sets:

Definition 1. Let $\alpha, \alpha' > 0$ be given. A set $C_n = C_n((Y_{ij}, B_{ij}), \alpha) \subset \mathcal{A}(m, \mathbf{a})$ is a *honest confidence set at level α for the model $\mathcal{A}(m, \mathbf{a})$* if

$$\liminf_n \inf_{M \in \mathcal{A}(m, \mathbf{a})} \mathbb{P}_M^n (M \in C_n) \geq 1 - \alpha.$$

Furthermore, we say that C_n is *adaptive for the sub-model $\mathcal{A}(k, \mathbf{a})$ at level α'* if there exists a constant $C = C(\alpha, \alpha') > 0$ such that

$$\sup_{M \in \mathcal{A}(k, \mathbf{a})} \mathbb{P}_M^n (\|C_n\|_2 > Cr_k) \leq \alpha'$$

while still retaining

$$\sup_{M \in \mathcal{A}(m, \mathbf{a})} \mathbb{P}_M^n (\|C_n\|_2 > Cr_m) \leq \alpha'.$$

2 A non-asymptotic confidence set for matrix completion problem

Let \widehat{M} be an estimator of M_0 based on the observations (Y_{ij}, B_{ij}) from the Bernoulli model (1) such that $\|\widehat{M}\|_\infty \leq \mathbf{a}$. Assume that for some $\beta > 0$ \widehat{M} satisfies the following risk bound:

$$\sup_{M_0 \in \mathcal{A}(k_0, \mathbf{a})} \mathbb{P} \left(\frac{\|\widehat{M} - M_0\|_2^2}{m_1 m_2} \leq C \frac{(\sigma + \mathbf{a})^2 k_0 d}{n} \right) \geq 1 - \beta. \quad (4)$$

We can take, for example, the thresholding, estimator considered in [11] which attains (4) with $\beta = 8/d$. Our construction is based on Lepski's method. We denote by \widehat{M}_k the projection of \widehat{M} on the set $\mathcal{A}(k, \mathbf{a})$ of matrices of rank k with sup-norm bounded by \mathbf{a} :

$$\widehat{M}_k \in \arg \min_{A \in \mathcal{A}(k, \mathbf{a})} \|\widehat{M} - A\|_2.$$

Set

$$S = \{k : \|\widehat{M} - \widehat{M}_k\|_2^2 \leq r_k\} \quad \text{and} \quad k^* = \min\{k \in S\}.$$

We will use $\widetilde{M} = \widehat{M}_{k^*}$ to center our confidence set and the residual sum of squares statistic \hat{r}_n :

$$\hat{r}_n = \frac{1}{n} \sum_{ij} (Y_{ij} - B_{ij} \widetilde{M}_{ij})^2 - \sigma^2. \quad (5)$$

Given $\alpha > 0$, let

$$\bar{z} = \frac{p}{256} \|M - \widetilde{M}\|_2^2 + z(UC^*)^2 dk^* \quad \text{and} \quad \xi_{\alpha, U} = 2U^2 \sqrt{\log(\alpha^{-1})} + \frac{4U^2 \log(\alpha^{-1})}{3\sqrt{n}}.$$

Here z is a sufficiently large numerical constant to be chosen later on and $C^* \geq 2$ is an universal constant in Corollary 3.12 [1]. We define the confidence set as follows:

$$C_n = \left\{ M \in \mathbb{R}^{m_1 \times m_2} : \frac{\|M - \widetilde{M}\|_2^2}{m_1 m_2} \leq 128 \left(\hat{r}_n + \frac{\mathbf{a}^2 z dk^* + \bar{z}}{n} + \frac{\xi_{\alpha, U}}{\sqrt{n}} \right) \right\}. \quad (6)$$

Theorem 1. *Let $\alpha > 0$, $d > 16$ and suppose that \widehat{M} attains the risk bound (4) with probability at least $1 - \beta$. Let C_n be given by (6). Assume that $\|M_0\|_\infty \leq \mathbf{a}$ and that Assumption 1 is satisfied. Then, for every $n \geq m \log(d)$, we have*

$$\mathbb{P}_{M_0}(M_0 \in C_n) \geq 1 - \alpha - \exp(-cd). \quad (7)$$

Moreover, with probability at least $1 - \beta - \exp(-cd)$

$$\frac{\|C_n\|_2^2}{m_1 m_2} \leq C \frac{(\sigma + \mathbf{a})^2 dk_0}{n}. \quad (8)$$

Theorem 1 implies that C_n is an honest and adaptive confidence set:

Corollary 1. *Let $\alpha > 0$, $d > 16$ and suppose that \widehat{M} attains the risk bound (4) with probability at least $1 - \beta$. Let C_n be given by (6). Assume that Assumption 1 is satisfied. Then, for $n \geq m \log(d)$, C_n is a $\alpha + \exp(-cd)$ honest confidence set for the model $\mathcal{A}(m, \mathbf{a})$ and adapts to every sub-model $\mathcal{A}(k, \mathbf{a})$, $1 \leq k \leq m$, at level $\beta + \exp(-cd)$.*

Proof of Theorem 1. We consider the following sets

$$\mathcal{C}(k, \mathbf{a}) = \left\{ A \in \mathbb{R}^{m_1 \times m_2} : \|A\|_\infty \leq \mathbf{a}, \|M_0 - A\|_2^2 \geq \frac{256(\mathbf{a} \vee U)^2 zd}{p} \text{ and } \text{rank}(A) \leq k \right\}$$

and write

$$\mathcal{C} = \cup_{k=1}^m \mathcal{C}(k, \mathbf{a}). \quad (9)$$

When $\|M_0 - \widetilde{M}\|_2^2 \leq \frac{256(\mathbf{a} \vee U)^2 zd}{p}$ we have that $M_0 \in C_n$. So, we only need to consider the case $\|M_0 - \widetilde{M}\|_2^2 \geq \frac{256(\mathbf{a} \vee U)^2 zd}{p}$. In this case we have that $\widetilde{M} \in \mathcal{C}$. We introduce the observation operator \mathcal{X} defined as follows,

$$\mathcal{X} : \mathbb{R}^{m_1 \times m_2} \rightarrow \mathbb{R}^{m_1 \times m_2} \quad \text{with} \quad \mathcal{X}(A) = (B_{ij} a_{ij})_{ij}.$$

and set $\|A\|_{L_2(\Pi)}^2 = \mathbb{E}\|\mathcal{X}(A)\|_2^2 = p\|A\|_2^2$. We can decompose

$$\hat{r}_n = n^{-1}\|\mathcal{X}(\widetilde{M} - M_0)\|_2^2 + 2n^{-1}\langle \mathcal{X}(E), M_0 - \widetilde{M} \rangle + n^{-1}\|\mathcal{X}(E)\|_2^2 - \sigma^2.$$

Then we can bound the probability $\mathbb{P}_{M_0}(M_0 \notin C_n)$ by the sum of the following probabilities:

$$I := \mathbb{P}_{M_0} \left(\frac{\|\widetilde{M} - M_0\|_{L_2(\Pi)}^2}{128} > \|\mathcal{X}(\widetilde{M} - M_0)\|_2^2 + z\mathbf{a}^2 dk^* \right),$$

$$II := \mathbb{P}_{M_0} \left(-2\langle \mathcal{X}(E), M_0 - \widetilde{M} \rangle > \bar{z} \right),$$

$$III := \mathbb{P}_{M_0} \left(-\|\mathcal{X}(E)\|_2^2 + n\sigma^2 > \sqrt{n}\xi_{\alpha,U} \right).$$

By Lemma 1, the first probability is bounded by $8\exp(-4d)$ for $z \geq (27C^*)^2$. For the second term we use Lemma 7 which implies that $II \leq \exp(-cd)$ for $z \geq 6240$. Finally, for the third term, Bernstein's inequality implies

$$\mathbb{P} \left\{ -\|\mathcal{X}(E)\|_2^2 + n\sigma^2 > t \right\} \leq \exp \left(-\frac{t^2}{2\sigma^2 n U^2 + \frac{2}{3} U^2 t} \right).$$

Taking $t = 2U^2 \sqrt{n \log(\alpha^{-1})} + \frac{4}{3} U^2 \log(\alpha^{-1})$ we get that $III \leq \alpha$ by definition of $\xi_{\alpha,U}$. This completes the proof of (7).

To prove (8), using Lemma 1 and Lemma 7, we can bound the square Frobenius norm diameter of our confidence set C_n defined in (6) as follows:

$$\frac{\|C_n\|_2^2}{m_1 m_2} \lesssim \frac{\|\widetilde{M} - M_0\|_2^2}{m_1 m_2} + \left(r_{k^*} + \frac{\xi_{\alpha,U}}{\sqrt{n}} \right).$$

This bound holds on an event of probability at least $1 - \exp(-cd)$. Now we restrict to the event where \widetilde{M} attains the risk bound in (4) which happens with probability at least $1 - \beta$. On this event, $M_0 \in \mathcal{A}(k_0, \mathbf{a})$ implies $\|\widetilde{M} - \widetilde{M}_{k_0}\|_2^2 \leq r_{k_0}$. So, $k_0 \in \mathcal{S}$ and $k^* \leq k_0$. Now, the triangle inequality and $r_{k^*} \leq r_{k_0}$ imply that on the intersection of those two events we have that

$$\|\widetilde{M} - M_0\|_2^2 \lesssim m_1 m_2 (r_{k_0} + r_{k^*}) \lesssim m_1 m_2 r_{k_0}.$$

This, together with the definition of $\xi_{\alpha,U}$ and the condition $n \leq m_1 m_2$, completes the proof of (8). \square

A Technical Lemmas

Lemma 1. *With probability larger than $1 - 8 \exp(-4d)$ we have that*

$$\sup_{A \in \mathcal{C}} \frac{\left| \|\mathcal{X}(M_0 - A)\|_2 - \|M_0 - A\|_{L_2(\Pi)} \right| - \frac{7}{8} \|M_0 - A\|_{L_2(\Pi)}}{\mathbf{a} \sqrt{\text{rank}(A)d}} \leq 27 C^*$$

where C^* is an universal numerical constant and \mathcal{C} is defined in (9).

Proof. We have that

$$\begin{aligned} & \mathbb{P} \left(\sup_{A \in \mathcal{C}} \frac{\left| \|\mathcal{X}(M_0 - A)\|_2 - \|M_0 - A\|_{L_2(\Pi)} \right| - \frac{7}{8} \|M_0 - A\|_{L_2(\Pi)}}{\mathbf{a} \sqrt{\text{rank}(A)d}} \geq 27 C^* \right) \\ & \leq \sum_{k=1}^{k_0} \underbrace{\mathbb{P} \left(\sup_{A \in \mathcal{C}(k, \mathbf{a})} \left| \|\mathcal{X}(M_0 - A)\|_2 - \|M_0 - A\|_{L_2(\Pi)} \right| - \frac{7}{8} \|M_0 - A\|_{L_2(\Pi)} \geq 27 C^* \mathbf{a} \sqrt{kd} \right)}_{\text{I}}. \end{aligned} \quad (10)$$

In order to upper bound I, we use a standard peeling argument. Let $\alpha = 7/6$ and $\nu^2 = \frac{470 \mathbf{a}^2 z d}{p}$. For $l \in \mathbb{N}$ set

$$S_l = \{A \in \mathcal{C}(k, \mathbf{a}) : \alpha^l \nu \leq \|A - M_0\|_2 \leq \alpha^{l+1} \nu\}.$$

Then

$$\begin{aligned} \text{I} & \leq \sum_{l=1}^{\infty} \mathbb{P} \left(\sup_{A \in S_l} \left| \|\mathcal{X}(M_0 - A)\|_2 - \|M_0 - A\|_{L_2(\Pi)} \right| \geq 27 C^* \mathbf{a} \sqrt{kd} + \frac{7}{8} \alpha^l \mathbf{a} \sqrt{470 z d} \right) \\ & \leq \sum_{l=1}^{\infty} \underbrace{\mathbb{P} \left(\sup_{A \in \mathcal{C}(k, \mathbf{a}, \alpha^{l+1} \nu)} \left| \|\mathcal{X}(M_0 - A)\|_2 - \|M_0 - A\|_{L_2(\Pi)} \right| \geq 27 C^* \mathbf{a} \sqrt{kd} + \frac{7}{8} \alpha^l \mathbf{a} \sqrt{470 z d} \right)}_{\text{II}} \end{aligned}$$

where $\mathcal{C}(k, \mathbf{a}, T) = \{A \in \mathcal{C}(k, \mathbf{a}) : \|M_0 - A\|_2 \leq T\}$. The following Lemma gives an upper bound on II:

Lemma 2. *Consider the following set of matrices*

$$\mathcal{C}(k, \mathbf{a}, T) = \{A \in \mathcal{C}(k, \mathbf{a}) : \|M_0 - A\|_2 \leq T\}$$

and set

$$Z_T = \sup_{A \in \mathcal{C}(k, \mathbf{a}, T)} \left| \|\mathcal{X}(M_0 - A)\|_2 - \|M_0 - A\|_{L_2(\Pi)} \right|.$$

Then, we have that

$$\mathbb{P} \left(Z_T \geq \frac{3}{4} \sqrt{p} T + 27 C^* \mathbf{a} \sqrt{kd} \right) \leq 4e^{-c_1 p T^2 / \mathbf{a}^2}$$

with $c_1 \geq (512)^{-1}$.

Lemma 2 implies that $\text{II} \leq 8 \exp(-c_1 p \alpha^{2l} \nu^2 / \mathbf{a}^2)$ and we obtain

$$\text{I} \leq 8 \sum_{l=1}^{\infty} \exp(-c_1 p \alpha^{2l+2} \nu^2 / \mathbf{a}^2) \leq 8 \sum_{l=1}^{\infty} \exp(-2c_1 p \nu^2 \log(\alpha) l / \mathbf{a}^2)$$

where we used $e^x \geq x$. We finally compute for $\nu^2 = 188\mathbf{a}^2 z d p^{-1}$

$$I \leq \frac{8 \exp(-2c_1 p \nu^2 \log(\alpha)/\mathbf{a}^2)}{1 - \exp(-2c_1 p \nu^2 \log(\alpha)/\mathbf{a}^2)} \leq 16 \exp(-376 c_1 z d \log(7/6)) \leq \exp(-5d)$$

where we take $z \geq (27C^*)^2$. Using (10) and $d \geq \log(m)$ we get the statement of Lemma 1. \square

Proof of Lemma 2. This proof is close to the proof of Theorem 1 in [15]. We start by applying the discretization argument. Let $\{G_\delta^1, \dots, G_\delta^{N(\delta)}\}$ be a δ -covering of $\mathcal{C}(k, \mathbf{a}, T)$ given by Lemma 3. Then, for any $A \in \mathcal{C}(k, \mathbf{a}, T)$ there exists some index $i \in \{1, \dots, N(\delta)\}$ and a matrix Δ with $\|\Delta\|_2 \leq \delta$ such that $A = G_\delta^i + \Delta$. Using the triangle inequality we have

$$\left| \|M_0 - A\|_{L_2(\Pi)} - \|\mathcal{X}(M_0 - A)\|_2 \right| \leq \left| \|\mathcal{X}(M_0 - G_\delta^i)\|_2 - \|M_0 - G_\delta^i\|_{L_2(\Pi)} \right| + \|\mathcal{X}\Delta\|_2 + \sqrt{p}\delta.$$

Lemma 3 implies that $\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)$ where

$$\mathcal{D}_\delta(k, \mathbf{a}, T) = \{A \in \mathbb{R}^{m_1 \times m_2} : \|A\|_\infty \leq \mathbf{a}, \|A\|_2 \leq \delta \text{ and } \|A\|_* \leq \sqrt{k}T\}.$$

Then,

$$Z_T \leq \max_{i=1, \dots, N(\delta)} \left| \|\mathcal{X}(M_0 - G_\delta^i)\|_2 - \|M_0 - G_\delta^i\|_{L_2(\Pi)} \right| + \sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \|\mathcal{X}\Delta\|_2 + \sqrt{p}\delta.$$

Now we take $\delta = T/8$ and use Lemma 6 and Lemma 5 to get

$$Z_T \leq \sqrt{p}\delta + 8C^* \mathbf{a} \sqrt{k}d + 19\mathbf{a} C^* \sqrt{2kd} + \sqrt{p}T/2 + \sqrt{p}\delta \leq 27C^* \mathbf{a} \sqrt{k}d + 6\sqrt{p}T/8.$$

with probability at least $1 - 8 \exp\left(-\frac{pT^2}{512\mathbf{a}^2}\right)$. \square

Lemma 3. *Let $\delta = T/8$. There exists a set of matrices $\{G_\delta^1, \dots, G_\delta^{N(\delta)}\}$ with $N_\delta \leq \left(\frac{18T}{\delta}\right)^{2(d+1)k}$ and such that*

(i) *For any $A \in \mathcal{C}(k, \mathbf{a}, T)$ there exists a $G_\delta^A \in \{G_\delta^1, \dots, G_\delta^{N(\delta)}\}$ satisfying*

$$\|A - G_\delta^A\|_2 \leq \delta \text{ and } (A - G_\delta^A) \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T).$$

(ii) *Moreover, $\|G_\delta^j - M_0\|_\infty \leq 2\mathbf{a}$ and $\|G_\delta^j - M_0\|_2 \leq 2T$ for any $j = 1, \dots, N_\delta$.*

Proof. We use the following result (see Lemma 3.1 in [5] and Lemma A.2 in [17]):

Lemma 4. *Let $S(k, T) = \{A \in \mathbb{R}^{m_1 \times m_2} : \text{rank}(A) \leq k \text{ and } \|A\|_2 \leq T\}$. Then, there exists an ϵ -net $\bar{S}(k, T)$ for the Frobenius norm obeying*

$$|\bar{S}(k, T)| \leq (9T/\epsilon)^{(m_1+m_2+1)k}.$$

Let $S_{M_0}(k, T) = \{A \in \mathbb{R}^{m_1 \times m_2} : \text{rank}(A) \leq k \text{ and } \|A - M_0\|_2 \leq T\}$ and take a $X_0 \in \mathcal{C}(k, \mathbf{a}, T)$. We have that $S_{M_0}(k, T) - X_0 \subset S(2k, 2T)$. Let $\bar{S}(2k, 2T)$ be an δ -net given by Lemma 4. Then, for any $A \in S_{M_0}(k, T)$ there exists a $\bar{G}_\delta^A \in \bar{S}(2k, 2T)$ such that $\|A - X_0 - \bar{G}_\delta^A\|_2 \leq \delta$. Let $G_\delta^j = \Pi(\bar{G}_\delta^j) + X_0$ for $j = 1, \dots, |\bar{S}(2k, 2T)|$ where Π is the projection operator under Frobenius norm into the set $\mathcal{D}(2k, 2\mathbf{a}, 2T) = \{A \in \mathbb{R}^{m_1 \times m_2} : \|A\|_\infty \leq 2\mathbf{a}, \text{ and } \|A\|_* \leq 2\sqrt{2k}T\}$. Note

that as $\mathcal{D}(2k, 2\mathbf{a}, 2T)$ is convex and closed, Π is non-expansive in Frobenius norm. For any $A \in \mathcal{C}(k, \mathbf{a}, T) \subset S_{M_0}(k, T)$, we have that $A - X_0 \in \mathcal{D}(2k, 2\mathbf{a}, 2T)$ which implies

$$\|A - X_0 - \Pi(\bar{G}_\delta^A)\|_2 = \|\Pi(A - X_0 - \bar{G}_\delta^A)\|_2 \leq \|A - \bar{G}_\delta^A - X_0\|_2 \leq \delta$$

and we have that $(A - \Pi(\bar{G}_\delta^A) - X_0) \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)$ which completes the proof of (i) of Lemma 3. To prove (ii), note that by the definition of Π we have that $\|G_\delta^j - M_0\|_\infty = \|\Pi(\bar{G}_\delta^j) + X_0 - M_0\|_\infty = \|\Pi(\bar{G}_\delta^j + X_0 - M_0)\|_\infty \leq 2\mathbf{a}$ and $\|G_\delta^j - M_0\|_2 \leq 2T$. \square

Lemma 5. *Let $\delta = T/8$ and assume that $n \geq m \log(m)$. We have that with probability at least $1 - 4 \exp\left(-\frac{pT^2}{512\mathbf{a}^2}\right)$*

$$\sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \|\mathcal{X}\Delta\|_2 \leq 19\mathbf{a}C^*\sqrt{2kd} + \sqrt{p}T/2.$$

Proof. Let $X_T = \sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \|\mathcal{X}\Delta\|_2$. We use the following Talagrand's concentration inequality :

Theorem 2. *Suppose that $f : [-1, 1]^N \rightarrow \mathbb{R}$ is a convex Lipschitz function with Lipschitz constant L . Let Ξ_1, \dots, Ξ_N be independent random variables taking value in $[-1, 1]$. Let $Z := f(\Xi_1, \dots, \Xi_n)$. Then for any $t \geq 0$,*

$$\mathbb{P}(|Z - \mathbb{E}(Z)| \geq 16L + t) \leq 4e^{-t^2/2L^2}.$$

For a proof see [16] and [8]. Let $f(x_{11}, \dots, x_{m_1 m_2}) := \sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \sqrt{\sum_{(i,j)} x_{ij}^2 \Delta_{ij}^2}$. It is easy to see that $f(x_{11}, \dots, x_{m_1 m_2})$ is a Lipschitz function with Lipschitz constant $L = 2\mathbf{a}$. Indeed,

$$\begin{aligned} & |f(x_{11}, \dots, x_{m_1 m_2}) - f(z_{11}, \dots, z_{m_1 m_2})| \\ &= \left| \sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \sqrt{\sum_{(i,j)} x_{ij}^2 \Delta_{ij}^2} - \sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \sqrt{\sum_{(i,j)} z_{ij}^2 \Delta_{ij}^2} \right| \\ &\leq \sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \sqrt{\sum_{(i,j)} (x_{ij} - z_{ij})^2 \Delta_{ij}^2} \leq 2\mathbf{a}\|x - z\|_2 \end{aligned}$$

where $x = (x_{11}, \dots, x_{m_1 m_2})$ and $z = (z_{11}, \dots, z_{m_1 m_2})$. Now, Theorem 2 implies

$$\mathbb{P}(X_T \geq \mathbb{E}(X_T) + 32\mathbf{a} + t) \leq 4 \exp\left(-\frac{t^2}{8\mathbf{a}^2}\right). \quad (11)$$

Next, we bound the expectation $\mathbb{E}(X_T)$. Applying Jensen's inequality, a symmetrization argument and the Ledoux-Talagrand contraction inequality (see, e.g., [13]) we get

$$\begin{aligned} (\mathbb{E}(X_T))^2 &\leq \mathbb{E} \left(\sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \sum_{(i,j)} B_{ij} \Delta_{ij}^2 \right) \leq \mathbb{E} \left(\sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \sum_{(i,j)} B_{ij} \Delta_{ij}^2 - \mathbb{E}(B_{ij} \Delta_{ij}^2) \right) + p\delta^2 \\ &\leq 2\mathbb{E} \left(\sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \left| \sum_{(i,j)} \epsilon_{ij} B_{ij} \Delta_{ij}^2 \right| \right) + p\delta^2 \leq 8\mathbf{a} \mathbb{E} \left(\sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} \left| \sum_{(i,j)} \epsilon_{ij} B_{ij} \Delta_{ij} \right| \right) + p\delta^2 \\ &= 8\mathbf{a} \mathbb{E} \left(\sup_{\Delta \in \mathcal{D}_\delta(2k, 2\mathbf{a}, 2T)} |\langle \Sigma_R, \Delta \rangle| \right) + p\delta^2 \leq 16\mathbf{a}\sqrt{2k} T \mathbb{E}(\|\Sigma_R\|) + p\delta^2 \end{aligned}$$

where $\{\epsilon_{ij}\}$ is i.i.d. Rademacher sequence, $\Sigma_R = \sum_{(i,j)} B_{ij} \epsilon_{ij} X_{ij}$ with $X_{ij} = e_i(m_1) e_j^T(m_2)$ and $e_k(l)$ are the canonical basis vectors in \mathbb{R}^l . Lemma 4 in [11] and $n \geq m \log(m)$ imply that

$$\mathbb{E} \|\Sigma_R\| \leq C^* \sqrt{pd} \quad (12)$$

where $C^* \geq 2$ is an universal numerical constant. Using (12), $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, $2xy \leq x^2 + y^2$ and $\delta = T/8$ we compute

$$\mathbb{E}(X_T) \leq 4 \left(\mathbf{a} C^* \sqrt{2kpdT} \right)^{1/2} + \sqrt{p}\delta \leq 16\mathbf{a} C^* \sqrt{2kd} + 3\sqrt{p}T/8.$$

Taking in (11) $t = \sqrt{p}T/8$ we get the statement of Lemma 5. \square

Lemma 6. *Let $\delta = T/8$, $d > 16$ and $(G_\delta^1, \dots, G_\delta^{N(\delta)})$ be the collection of matrices given by Lemma 3. We have that*

$$\max_{k=1, \dots, N(\delta)} \left| \|\mathcal{X}(M_0 - G_\delta^k)\|_2 - \|M_0 - G_\delta^k\|_{L_2(\Pi)} \right| \leq \sqrt{p}\delta + 8C^* \mathbf{a} \sqrt{kd}$$

with probability at least $1 - 4 \exp\left(-\frac{p\delta^2}{8\mathbf{a}^2}\right)$.

Proof. For any fixed $A \in \mathbb{R}^{m_1 \times m_2}$ satisfying $\|A\|_\infty \leq 2\mathbf{a}$ we have that

$$\|\mathcal{X}A\|_2 = \sqrt{\sum_{ij} B_{ij} A_{ij}^2} = \sup_{\|u\|_2=1} \sum_{ij} (u_{ij} B_{ij} A_{ij}).$$

Then we can apply Theorem 2 with $f(x_{11}, \dots, x_{m_1 m_2}) := \sup_{\|u\|_2=1} \sum_{ij} (u_{ij} x_{ij})$ to get

$$\mathbb{P} \left(\left| \|\mathcal{X}A\|_2 - \mathbb{E}\|\mathcal{X}A\|_2 \right| > t + 32\mathbf{a} \right) \leq 4 \exp \left\{ -\frac{t^2}{8\mathbf{a}^2} \right\}. \quad (13)$$

On the other hand let $Z = \sup_{\|u\|_2=1} \sum_{ij} (u_{ij} B_{ij} A_{ij})$. Applying Corollary 4.8 from [14] we get that $\text{Var } Z = \|A\|_{L_2(\Pi)}^2 - (\mathbb{E}\|\mathcal{X}A\|_2)^2 \leq 16^2 \mathbf{a}^2$ which together with (13) implies

$$\mathbb{P} \left(\left| \|\mathcal{X}A\|_2 - \|A\|_{L_2(\Pi)} \right| > t + 48\mathbf{a} \right) \leq 4 \exp \left\{ -\frac{t^2}{8\mathbf{a}^2} \right\}. \quad (14)$$

Now Lemma 6 follows from Lemma 3, (14) with $t = \sqrt{p}\delta + 5C^* \mathbf{a} \sqrt{kd}$ and the union bound. \square

Lemma 7. *We have that*

$$\sup_{A \in \mathcal{C}} \frac{|\langle \mathcal{X}(E), A - M_0 \rangle| - \frac{1}{256} \|M_0 - A\|_{L_2(\Pi)}^2}{d \text{rank}(A)} \leq 6240(UC^*)^2.$$

with probability larger than $1 - \exp(-cd)$ with $c \geq 0,003$

Proof. Following the lines of the proof of Lemma 1 with $\alpha = \sqrt{65/64}$ and $\nu^2 = \frac{252(\mathbf{a} \vee U)^2 zd}{p}$ we get

$$\begin{aligned} & \mathbb{P} \left(\sup_{A \in \mathcal{C}} \frac{|\langle \mathcal{X}(E), A - M_0 \rangle| - \frac{1}{256} \|M_0 - A\|_{L_2(\Pi)}^2}{d \operatorname{rank}(A)} \geq 6240(UC^*)^2 \right) \\ & \leq \sum_{k=1}^{k_0} \sum_{l=1}^{\infty} \mathbb{P} \left(\sup_{A \in \mathcal{C}(k, a, \alpha^{l+1} \nu)} |\langle \mathcal{X}(E), A - M_0 \rangle| \geq 6240(UC^*)^2 dk + \frac{p\alpha^{2l}\nu^2}{256} \right) \\ & \leq 4 \sum_{k=1}^{k_0} \sum_{l=1}^{\infty} \exp \left(-\frac{c_2 p \alpha^{2l+2} \nu^2}{(\mathbf{a} \vee U)^2} \right) \leq \exp(-cd) \end{aligned}$$

where we use the following lemma:

Lemma 8. *Consider the following set of matrices*

$$\mathcal{C}(k, a, T) = \{A \in \mathcal{C}(k, a) : \|M_0 - A\|_2 \leq T\}$$

and set

$$\tilde{Z}_T = \sup_{A \in \mathcal{C}(k, a, T)} |\langle \mathcal{X}(E), A - M_0 \rangle|.$$

We have that

$$\mathbb{P} \left(\tilde{Z}_T \geq 6240(C^*U)^2 dk + pT^2/260 \right) \leq 4 \exp(-pT^2/c_2(\mathbf{a} \vee U)^2)$$

with $c_2 \leq 12(1560)^2$

□

Proof of Lemma 8. Fix a $X_0 \in \mathcal{C}(k, a, T)$. For any $A \in \mathcal{C}(k, a, T)$, we set $\Delta = A - X_0$ and we have that $\operatorname{rank}(\Delta) \leq 2k$ and $\|\Delta\|_2 \leq 2T$. Then using $|\langle \mathcal{X}(E), A - M_0 \rangle| \leq |\langle \mathcal{X}(E), X_0 - M_0 \rangle| + |\langle \mathcal{X}(E), \Delta \rangle|$ we get

$$\tilde{Z}_T \leq |\langle \mathcal{X}(E), X_0 - M_0 \rangle| + \sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} |\langle \mathcal{X}(E), \Delta \rangle|$$

where

$$\mathcal{T}(2k, 2a, 2T) = \{A \in \mathbb{R}^{m_1 \times m_2} : \|A\|_{\infty} \leq 2a, \|A\|_2 \leq 2T \text{ and } \operatorname{rank}(A) \leq 2k\}.$$

Bernstein's inequality and $\|X_0 - M_0\|_2 \leq T$ imply that

$$\mathbb{P} \{ |\langle \mathcal{X}(E), X_0 - M_0 \rangle| > t \} \leq 2 \exp \left(-\frac{t^2}{2\sigma^2 p T^2 + \frac{4}{3} U \mathbf{a} t} \right).$$

Taking $t = pT^2/520$ we get

$$\mathbb{P} \{ |\langle \mathcal{X}(E), X_0 - M_0 \rangle| > pT^2/520 \} \leq 2 \exp \left(-\frac{pT^2}{c_2(a \vee U)^2} \right). \quad (15)$$

On the other hand, Lemma 9 implies that with probability at least $1 - 2 \exp \left(-\frac{pT^2}{c_3(a \vee U)^2} \right)$

$$\sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} |\langle \mathcal{X}(E), \Delta \rangle| \leq 6240(C^*U)^2 kd + pT^2/520$$

which together with (15) implies the statement of Lemma 8. □

Lemma 9. Assume that $n \geq m \log(m)$. We have that with probability at least $1 - 2 \exp\left(-\frac{pT^2}{(a \vee U)^2}\right)$

$$\sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} |\langle \mathcal{X}(E), \Delta \rangle| \leq 6240(C^*U)^2 kd + pT^2/520$$

where c_3 is a numerical constant.

Proof. Let $\tilde{X}_T = \sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} |\langle \mathcal{X}(E), \Delta \rangle| = \sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} \langle \mathcal{X}(E), \Delta \rangle$. First we bound the expectation $\mathbb{E}(\tilde{X}_T)$:

$$\begin{aligned} \mathbb{E}(\tilde{X}_T) &\leq \mathbb{E} \left(\sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} \left| \sum_{(i,j)} \varepsilon_{ij} B_{ij} \Delta_{ij} \right| \right) = \mathbb{E} \left(\sup_{\Delta \in \mathcal{T}(2k, 2a, 2T)} |\langle \Sigma, \Delta \rangle| \right) \\ &\leq 2\sqrt{2k} T \mathbb{E}(\|\Sigma\|) \end{aligned}$$

where $\Sigma = \sum_{(i,j)} B_{ij} \varepsilon_{ij} X_{ij}$ with $X_{ij} = e_i(m_1) e_j^T(m_2)$ and $e_k(l)$ are the canonical basis vectors in \mathbb{R}^l . Using $n \geq m \log(m)$ Lemma 4 in [11] and Corollary 3.3 in [1] imply that

$$\mathbb{E} \|\Sigma\| \leq C^* U \sqrt{pd}. \quad (16)$$

where $C^* \geq 2$ is an universal numerical constant. Using (16) we get

$$\mathbb{E}(\tilde{X}_T) \leq 2C^* U \sqrt{2kpd} T \leq 3120(C^*U)^2 kd + pT^2/1560. \quad (17)$$

Now we use Theorem 3.3.16 in [9] (see also Theorem 8.1 in [7]) to obtain

$$\begin{aligned} \mathbb{P} \left(\tilde{X}_T \geq \mathbb{E}(\tilde{X}_T) + t \right) &\leq \exp \left(-\frac{t^2}{4U \mathbf{a} \mathbb{E}(\tilde{X}_T) + 4\sigma^2 pT^2 + 9U \mathbf{a} t} \right) \\ &\leq \exp \left(-\frac{t^2}{8\mathbf{a} U^2 C^* \sqrt{2kpd} T + 4\sigma^2 pT^2 + 9U \mathbf{a} t} \right) \end{aligned} \quad (18)$$

Taking in (18) $t = pT^2/1560 + 2UC^* \sqrt{2kpd} T$, together with (17) we get the statement of Lemma 9. □

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