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To cite this version:

Luigi de Pascale, Jean Louet. A study of the dual problem of the one-dimensional L-infinity optimal transport problem with applications. 2017. hal-01504249v2

HAL Id: hal-01504249

https://hal.science/hal-01504249v2

Preprint submitted on 4 Aug 2017

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A STUDY OF THE DUAL PROBLEM OF THE ONE-DIMENSIONAL L^∞-OPTIMAL TRANSPORT PROBLEM WITH APPLICATIONS

LUIGI DE PASCALE AND JEAN LOUET

Abstract. The Monge-Kantorovich problem for the $\mathcal{W}_\infty$ distance presents several peculiarities. Among them the lack of convexity and then of a direct duality. We study in dimension 1 the dual problem introduced by Barron, Bocea and Jensen in [2]. We construct a couple of Kantorovich potentials which is non trivial in the best possible way. More precisely, we build a potential which is non constant around any point that the restrictable, minimizing plan moves at maximal distance. As an application, we show that the set of points which are displaced at maximal distance by a "locally optimal" transport plan is shared by all the other optimal transport plans, and we describe the general structure of all the one-dimensional optimal transport plans.

1. Introduction

Given two probability measures μ, ν on $\mathbb{R}^d$, the infinite Wasserstein distance between μ and ν is defined as

$$\mathcal{W}_\infty(\mu,\nu) := \min_{\gamma\in\Pi(\mu,\nu)} \gamma - \operatorname*{ess.sup}_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|, \tag{1.1}$$

where $\Pi(\mu,\nu)$ denotes the set of positive measures on $\mathbb{R}^d$ whose first and second marginals are μ, ν respectively. This distance is the natural limit as $p \to \infty$ of the more common Kantorovich-Wasserstein distances $\mathcal{W}_p$.

Problem (1.1) above was first studied in [8] where it was observed that, in spite of its proximity with the $\mathcal{W}_p$ distances, it presents several peculiarities. The two most striking phenomena are the lack of linearity or even convexity of (1.1) with respect to γ (in fact the functional $\gamma \mapsto \|y-x\|_{L^\infty_\gamma(\mathbb{R}^{2d})}$ is only level-convex), and the non-uniqueness of the minimizer: indeed, one may guess that, from any optimal plan γ, any small perturbation of γ around a point (x,y) which is not moved at maximal distance will provide a new and different optimal transport plan. It is therefore necessary to find a suitable notion of *local solution* of (1.1). The following definition, introduced in [8], turned out to be the right one.

Definition 1.1. A transport plan $\gamma \in \Pi(\mu,\nu)$ is a *restrictable solution* of (1.1) if any positive and nonzero Borel measure γ' on $\mathbb{R}^{2d}$ that is majorized by γ is a solution to the problem

$$\inf_{\lambda\in\Pi(\mu',\nu')} \lambda - \operatorname*{ess.sup}_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|,$$

Date: August 4, 2017.
2000 *Mathematics Subject Classification.* 49Q20, 49K30, 49J45.
Key words and phrases. Monge-Kantorovich problem, optimal transport problem, cyclical monotonicity.

where $\mu' = (\pi_1)_\# \gamma'$ and $\nu' = (\pi_2)_\# \gamma'$.

It is likely that a small "local modification" of a restrictable solution will not be a restrictable solution anymore. From the analysis of the problem (1.1) as a limit of the classical optimal transport problem with cost $c(x,y) = |y-x|^p$ as $p \to +\infty$, it turns out that the notion of restrictable solution is equivalent to the one of *infinite cyclical monotonicity* of a transport plan, whose definition is the following:

Definition 1.2. A transport plan $\gamma \in \Pi(\mu,\nu)$ is ∞-cyclically-monotone (**∞-cm** in the paper) if, for all $n \in \mathbb{N}$ and $(x_0,y_0),\dots,(x_n,y_n) \in \mathrm{spt}(\gamma)$, it holds

$$\max_{1\le i\le n}\{|x_i - y_i|\} \le \max_{1\le i\le n}\{|x_i - y_{i+1}|\}, \tag{1.2}$$

where, as usual, we set $y_{n+1} = y_0$.

The infinite cyclical monotonicity is the L^∞ version of the c-cyclical monotonicity, which is a fundamental notion in the classical theory of optimal transportation.

The equivalence between the two definitions above is proven in [8], where it is also proven, under suitable assumptions, that **∞-cm** transport plans are induced by maps (see Theorems 3.2 and 4.4 therein, and let us also mention [17] where similar results are extended to more general cost functions). In fact, using the cyclical monotonicity to prove the existence of an optimal transport map was the first way to get around the absence of a satisfying duality theory for the problem (1.1). Indeed, as we said, the energy

$$\gamma \mapsto \|y-x\|_{L^\infty_\gamma}$$

is non-linear, and not even convex, with respect to the transport plan γ, which makes difficult to guess a dual formulation of (1.1), and this is why this alternative method using cyclical monotonicity together with a regularity property of the transport plan has been introduced. Later on, similar "duality free" methods allowed to prove existence of optimal transport maps for many non-standard cases of cost functions, including the Monge's distance cost for arbitrary norms (see for example [9, 10, 16, 4]).

However, looking for a (sort of) dual formulation of (1.1) is natural in order to try to find characterization of optimal transport plans: indeed, in the classical theory of optimal transportation, the duality method plays a crucial role not only in the proof of existence of an optimal transport map but also in the characterization of this map. Heuristically, the reasoning which leads to the characterization is the following: from the duality formula

$$\inf\left\{\int c\,\mathrm{d}\gamma \,:\, \gamma\in\Pi(\mu,\nu)\right\} = \sup\left\{\int \varphi\,\mathrm{d}\mu + \int \psi\,\mathrm{d}\nu \,:\, \varphi(x)+\psi(y)\le c(x,y)\right\},$$

one can deduce that, if γ and (φ,ψ) are respectively a minimizer and a maximizing pair for these problems, the condition $\varphi\oplus\psi\le c$ is saturated (in sense that equality holds) on the support of γ. Therefore, if c and φ are regular enough, one can deduce

$$\nabla_1 c(x,y) = \nabla\varphi(x) \quad \text{for } \gamma\text{-a.e. } (x,y)\in(\mathbb{R}^d)^2.$$

If the cost function c is such that this relationship may be inverted, one concludes that an optimal transport maps exists and is characterized by φ (namely, it is exactly the map $x\mapsto(\nabla_1 c(x,\cdot))^{-1}(\nabla\varphi(x))$). Following these arguments, Brenier [5, 6] proved the

existence of an optimal map for the quadratic cost and that this map is induced by a convex potential; it must then be a solution of the *Monge-Ampère equation*, which serves as basis for a whole regularity theory of optimal transport maps, *cf.* [20, Chapter 4]. Similar results have been generalized to a much larger class of cost functions, first for strictly convex costs with respect to the difference [14, 15, 7] and later on for those satisfying the so-called *twist condition*, see [13, 11].

These arguments cannot be reproduced for the cost functionals for which the existence of solutions (φ, ψ) is not known (see for instance [3, 18] for recent progresses in this direction in special cases) and even, for our L^∞ problem (1.1), the meaning of "dual formulation" is itself not clear. Yet, although the functional $\gamma \mapsto \|y - x\|_{L^\infty_\gamma}$ is not convex, it is still level-convex (in the sense that the level sets of the functional are convex sets and this, sometimes, goes under the name of quasi-convex). Then it is still possible to consider a (sort of) duality theory. This theory has been recently introduced and investigated by Barron, Bocea and Jensen in [2] where it is proven that the minimal value of (1.1) is equal to

$$\inf_{\lambda \geq 0} \left(\sup \left\{ \lambda + \int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\nu \, : \, \begin{array}{c} \varphi(x) + \psi(y) \leq 0 \\ \text{whenever } |y - x| \leq \lambda \end{array} \right\} \right), \tag{1.3}$$

and that the infimum with respect to λ in (1.3) is attained for $\lambda = \mathcal{W}_\infty(\mu, \nu)$ (see the next section for more details). It follows that $\mathcal{W}_\infty(\mu, \nu)$ is the smallest λ such that

$$\sup \left\{ \int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\nu \, : \, \varphi(x) + \psi(y) \leq 0 \text{ whenever } |y - x| \leq \lambda \right\} = 0.$$

Unfortunately, the supremum with respect to (φ, ψ) always admits a trivial solution $\varphi = \psi = 0$. It is therefore important to look for the "most interesting solutions" of this dual problem, and to study what information can be obtained from them about the optimal plans for (1.1).

The aim of this paper is to provide, in the one-dimensional case, a solution of the dual problem (1.3) which is non constant as much as possible. More precisely, if (φ, ψ) is a solution of (1.3) for $\lambda = \mathcal{W}_\infty$ and γ is a fixed optimal transport plan for (1.1), one can at least formally notice that φ is locally constant around any point x which is sent by γ at smaller distance that $\mathcal{W}_\infty$ (see below a more precise statement in Prop. 2.2). In the present paper, we construct a pair (φ, ψ) of solutions of (1.3) (see Theorem 3.6) from a fixed infinite cyclically monotone transport plan $\overline{\gamma}$, and such that φ is a BV function whose derivative is exactly supported on the set of points which are moved by $\overline{\gamma}$ at maximal distance: in this sense, our solution has a maximal set of variation. As an application, we will prove that there exists a set of points of the support of the source measure which are displaced at maximal distance by all the optimal transport plans for (1.3); in particular, this set is shared by all the infinitely cyclically monotone transport plans. Moreover, we prove that any optimal transport plan for (1.1) must exactly coincide with the infinitely cyclically monotone one on the set of points which are displaced at maximal distance. The proofs of both these results are based on the properties of the non trivial solution (φ, ψ) that we construct for the dual problem.

The potentials we construct only carry informations on the points which are moved at maximal distance by all the optimal plans. This fact is not surprising: indeed, as we

said, the points which are moved at non-maximal distance by a transport plan are in some sense "not relevant", since the plan can be modified on a small region around such a point without loosing the optimality. It is then natural that the characterizations provided by this dual formulation only impacts the region where modifications could alter the optimality of a transport plans. In other words, the "zone where φ carries some information" (i.e. the zone where φ is not locally constant) is exactly the "minimal set of maximal displacement of optimal plans", so that our result is nearly optimal in that sense.

Acknowledgements. The research of the first author is part of the project 2010-A2TFX2 *Calcolo delle Variazioni* funded by the Italian Ministry of Research, and has been partially financed by the *Fondi di ricerca di ateneo* of the University of Pisa. The second author acknowledges the support of PGMO project MACRO, funded by EDF and *Fondation Mathématique Jacques Hadamard*, and of the *Laboratoire Ypatia des Sciences Mathématiques* (LYSM). He also acknowledges the hospitality of Universities of Pisa and of Florence during several research visits where part of this work has been done.

2. Notations and the dual problem

In this section, we quickly collect all the notations and known facts of measure theory and optimal transportation that we will use throughout the paper.

Let X and Y be two Polish spaces, and μ, ν be two positive measures on X, Y whose total masses are finite and equal. We denote $\Pi(\mu,\nu)$ the set of *transport plans* from μ to ν, that is, the set of positive measures on $X\times Y$ satisfying

$$\text{for any Borel sets } A\subset X \text{ and } B\subset Y,\ \gamma(A\times Y)=\mu(A) \text{ and } \gamma(X\times B)=\nu(B);$$

recall that this constraint can be reformulated as

$$\text{for any } (u,v)\in C_b(X)\times C_b(Y), \qquad \text{and} \quad \begin{aligned} \iint u(x)\,\mathrm{d}\gamma(x,y) &= \int u\,\mathrm{d}\mu \\ \iint v(y)\,\mathrm{d}\gamma(x,y) &= \int v\,\mathrm{d}\nu. \end{aligned}$$

In our settings, $X=Y=\mathbb{R}^d$ (and $d=1$ along almost the whole paper) and the "primal" problem that we consider is the minimization of the supremal functional (1.1) above. We will denote by $\mathcal{O}_\infty(\mu,\nu)$ the set of its minimizers. The definition of an *infinitely cyclically monotone transport plan* has been recalled in the introduction, see Definition 1.2. We will use by simplicity the abbreviation "**∞-cm** plan"; recall that, from [8, Theorems 3.2 and 4.4], we know that at least one such plan exists provided μ,ν are both compactly supported in $\mathbb{R}^d$, and that these plans are exactly those which are *restrictable solutions* of (1.1), in sense given by Definition 1.1.

The following "dual problem" was introduced in Theorem 2.3 and Remark 2.4 of [2].

Theorem 2.1 (Duality formula for the L^∞-optimal transport problem)**.** *Let $\mu,\nu\in\mathcal{P}(\mathbb{R}^d)$ be compactly supported. For any $\lambda>0$, denote by $\mathcal{U}_\lambda$ the set of couples $(\varphi,\psi)\in$*

$L^1_\mu \times L^1_\nu$ such that, for μ-a.e. x and ν-a.e. y, the inequality $\varphi(x) + \psi(y) \leq 0$ holds whenever $|y - x| \leq \lambda$. Then,

$$\mathcal{W}_\infty(\mu,\nu) = \inf_{\lambda \geq 0} \left(\sup \left\{ \lambda + \int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\nu \, : \, (\varphi,\psi) \in \mathcal{U}_\lambda \right\} \right),$$

and the infimum with respect to λ is attained for $\lambda = \mathcal{W}_\infty(\mu,\nu)$.

For shorter notations we introduce $\lambda_C := \mathcal{W}_\infty(\mu,\nu)$ where C stands for critical. As a consequence

$$\max \left\{ \int \varphi \, \mathrm{d}\mu + \int \psi \, \mathrm{d}\nu \, : \, (\varphi,\psi) \in \mathcal{U}_{\lambda_C} \right\} = 0 \tag{2.1}$$

and we observe that, given an optimal transport plan γ for the primal problem (1.1), a pair (φ,ψ) is optimal for the dual problem (2.1) if and only if the equality $\varphi(x)+\psi(y) = 0$ holds for γ-a.e. (x,y). Such functions φ,ψ will be then called *Kantorovich potentials.* We notice that among the maximizers for problem (2.1) above there are always $\varphi \equiv 0$ and $\psi \equiv 0$. The duality conditions impose a strong constraint on the variability of the Kantorovich potentials which make difficult to find non trivial couples. We explain this in the next Proposition.

Proposition 2.2. *Let $(\varphi,\psi) \in L^1_\mu \times L^1_\nu$ be Kantorovich potentials and let γ be an optimal transport plan. Then μ is concentrated on a set L such that if $\overline{x} \in L$ has the property*

$$\max_{(\overline{x},y)\in \mathrm{spt}(\gamma)} |\overline{x} - y| = \lambda < \lambda_C, \tag{2.2}$$

then there exists $\varepsilon > 0$ such that φ is constant in $B(\overline{x},\varepsilon) \cap L$

Proof. First we observe that since the distance is continuous and $\mathrm{spt}(\gamma)$ is closed, there exists $\varepsilon > 0$ such that for every $x \in B(\overline{x},\varepsilon)$

$$\max_{(x,y)\in \mathrm{spt}(\gamma)} |x - y| < \lambda + \frac{\lambda_C - \lambda}{2} < \lambda_C.$$

Denote by $D_c := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |x - y| \leq \lambda_C\}$, and let $\mathcal{N}_\mu$ and $\mathcal{N}_\nu$ be such that $\mu(\mathcal{N}_\mu) = \nu(\mathcal{N}_\nu) = 0$ and

$$\varphi(x) + \psi(y) \leq 0 \text{ on } D_c \setminus (\mathcal{N}_\mu \times \mathbb{R}^d \cup \mathbb{R}^d \times \mathcal{N}_\nu).$$

Denote by Γ the set on which γ is concentrated and such that

$$\varphi(x) + \psi(y) = 0 \text{ for all } (x,y) \in \Gamma.$$

Finally let $\hat{\Gamma} := \Gamma \setminus (\mathcal{N}_\mu \times \mathbb{R}^d \cup \mathbb{R}^d \times \mathcal{N}_\nu)$. We will prove that $L = \pi_1(\hat{\Gamma})$ has the desired property.

We prove that if $\overline{x} \in L$ is such that $\max_{(\overline{x},y)\in \mathrm{spt}(\gamma)} |\overline{x} - y| = \lambda < \lambda_C$ then for all $x \in B(\overline{x}, \lambda_C - \lambda) \cap L$

$$\varphi(x) \leq \varphi(\overline{x}),$$

i.e. $\overline{x}$ is a local maximum for φ on the set L which has full μ-measure. Since $\overline{x} \in L$ there exists $\overline{y}$ such that $(\overline{x},\overline{y}) \in \hat{\Gamma}$ (so $\overline{x} \notin \mathcal{N}_\mu$ nor $\overline{y} \notin \mathcal{N}_\nu$). Let $\tilde{x} \in L \cap B(\overline{x}, \lambda_C - \lambda)$, we have $|\tilde{x} - \overline{y}| \leq \lambda_C$ and again $\tilde{x} \notin \mathcal{N}_\mu$ and $\overline{y} \notin \mathcal{N}_\nu$ it follows that

$$\varphi(\tilde{x}) + \psi(\overline{y}) \leq 0$$

and then

$$\varphi(\tilde{x}) \leq -\psi(\overline{y}) = \varphi(\overline{x}).$$

If, in the first step of the proof, we choose $\varepsilon < \dfrac{\lambda_C - \lambda}{4}$ and we take $\tilde{x} \in B(\overline{x}, \varepsilon) \cap L$, by symmetry, we also have

$$\varphi(\overline{x}) \leq \varphi(\tilde{x})$$

which concludes the proof. □

3. Construction of non-trivial potentials

In this section, we construct, in the one-dimensional case, a couple of Kantorovich potentials (φ, ψ) which are not locally constant on the largest possible set. Our assumptions on the data are that μ, ν are two probability measures on $\mathbb{R}$ with compact support and without atoms.

We will use the following notations:

- as before, we denote by λ_C the optimal value of the problem (1.1), and by $\mathcal{O}_\infty$ the set of optimizers;
- for any $\gamma \in \mathcal{O}_\infty$, we introduce the sets

$$\mathcal{M}_\gamma^+ := \{(x,y) \in \mathrm{spt}(\gamma) \,:\, y - x = \lambda_C\},$$

$$\mathcal{M}_\gamma^- := \{(x,y) \in \mathrm{spt}(\gamma) \,:\, y - x = -\lambda_C\},$$

$$\mathcal{M}_\gamma := \mathcal{M}_\gamma^+ \cup \mathcal{M}_\gamma^-$$

$$\text{and} \quad M_\gamma^+ = \pi_1(\mathcal{M}_\gamma^+),\ M_\gamma^- = \pi_1(\mathcal{M}_\gamma^-),\ M_\gamma = \pi_1(\mathcal{M}_\gamma),$$

  where π_1 is the projection on the first variable ($\pi_1(x,y) = x$ for any $(x,y) \in \mathbb{R}^2$). We notice that M_γ^+, M_γ^- are, respectively, the sets of points which are moved by γ at maximal distance to the right or to the left. We will then call M_γ the *Maximal displacement set of* γ.

We also observe that $\mathcal{M}_\gamma^+$, $\mathcal{M}_\gamma^-$, $\mathcal{M}_\gamma$ and M_γ^+, M_γ^-, M_γ are compact subsets of $\mathbb{R}^2$ and $\mathbb{R}$ respectively; moreover, the sets M_γ^+ and M_γ^- are never simultaneously empty (and so is not M_γ).

Remark 3.1. Let $A \subset \mathbb{R}$ be a compact set a point $\bar{x} \in A$ will be called *right-extreme point of* A (resp. *left-extreme point* of A) if there exists $\delta > 0$ such that the interval $]\bar{x}, \bar{x} + \delta[$ (resp. $]\bar{x} - \delta, \bar{x}[$) does not intersect A. We notice that since A is closed, the set $\mathbb{R} \setminus A$ may be written as

$$\mathbb{R} \setminus A = \bigcup_i]a_i, b_i[$$

where the union is taken on an at most countable set of indexes i. In particular, the left-extreme and right-extreme points are all part of the a_i, b_i, so that such points are at most countably many: since μ and ν have no atoms, the sets of left-extreme or right-extreme points of any closed set have always zero mass for μ, ν.

3.1. **Properties of the maximal displacement set of ∞-cm plans.** In this paragraph, we fix an **∞-cm** plan $\bar\gamma$. Actually, in the one-dimensional context, it can be proven that such a γ is necessarily induced by a monotone map T, in sense that

$$\gamma = (\mathrm{id}\times T)_\#\mu \quad \text{where } T \text{ is non-decreasing and } T_\#\mu = \nu$$

and that such a map T is unique up to μ-negligible sets, see [19, Chapter 2]; the support of $\overline{\gamma}$ is then exactly the set of points

$$\left\{\left(x, \lim_{\substack{t\to x\\ t<x}} T(t)\right) \,:\, x\in \operatorname{spt}\mu\right\}\bigcup\left\{\left(x, \lim_{\substack{t\to x\\ t>x}} T(t)\right) \,:\, x\in \operatorname{spt}\mu\right\}$$

for a well-chosen representative of T. However, since the only property that we really use is the formula (1.2) itself, we will not enter into more details concerning the characterization of $\operatorname{spt}\overline{\gamma}$.

We can then prove some additional properties of the maximal displacement set.

Lemma 3.2. *For any $\bar x\in M^+_{\overline{\gamma}}$, the intersection $]\bar x, \bar x+2\lambda_C[\cap M^-_{\overline{\gamma}}$ is empty. Similarly, if $\bar x\in M^-_{\overline{\gamma}}$, then $]\bar x-2\lambda_C, \bar x[\cap M^+_{\overline{\gamma}}=\emptyset$.*

Proof. We give a proof of the first case, being the second analogous. Assume by contradiction that there exists some $z\in M^-_{\overline{\gamma}}$ such that $\bar x<z<\bar x+2\lambda_C$. Both $(z, z-\lambda_C)$ and $(\bar x, \bar x+\lambda_C)$ belong to the support of $\overline{\gamma}$, so that the **∞-cm** property implies

$$\lambda_C = \max\Big(|(z-\lambda_C)-z|, |(\bar x+\lambda_C)-\bar x|\Big) \le \max\Big(|(z-\lambda_C)-\bar x|, |(\bar x+\lambda_C)-z|\Big). \quad (3.1)$$

On the other hand, the fact that $\bar x<z<\bar x+2\lambda_C$ implies immediately that the right-handside in (3.1) is smaller than λ_C and this is a contradiction. □

Lemma 3.3. *The set $M^+_{\bar\gamma}\cap M^-_{\bar\gamma}$ is finite.*

Proof. Since $M^+_{\bar\gamma}\cap M^-_{\bar\gamma}$ is included in the support of μ, which is bounded, it is sufficient to prove that all of its points are isolated, and this is true since, by Lemma 3.2, above, if $\bar x\in M^+_\gamma\cap M^-_\gamma$, then $(M^+_\gamma\cap M^-_\gamma)\cap]\bar x, \bar x+2\lambda_C[=\emptyset$ and $(M^+_\gamma\cap M^-_\gamma)\cap]\bar x-2\lambda_C, \bar x[=\emptyset$. □

3.2. **Construction of a non-trivial potential.** In this paragraph, starting from an **∞-cm** optimal transport plan $\bar\gamma$, we introduce a couple (φ,ψ) of non-trivial solutions of the dual problem (2.1). The couple enjoys the property that the points around which φ is not locally constant are exactly those that $\bar\gamma$ moves at maximal distance i.e. those of $M_{\overline{\gamma}}$. We will need the following lemma of measure theory.

Lemma 3.4. *There exists two positive measures ρ^+, ρ^- on $\mathbb{R}$, having finite mass and such that, denoting by $\rho=\rho^+-\rho^-$, the following properties are satisfied:*

(i) $\operatorname{spt}\rho^+ = M^+_{\overline{\gamma}}$, $\operatorname{spt}\rho^- = M^-_{\overline{\gamma}}$ *and* $\operatorname{spt}\rho = M_{\overline{\gamma}}$*;*

(ii) *for any point $\overline{x}$ of $M^+_{\overline{\gamma}}$ which is a left-extreme or a right-extreme of $M^+_{\overline{\gamma}}$, we have $\rho(\{\bar x\})>0$.*

Proof. By lemma 3.3, the set $M^+_{\overline{\gamma}}\cap M^-_{\overline{\gamma}}$ is finite, we denote by $\{z_1,\dots,z_N\}$ the (possibly empty) set of its elements. We also select two at most countable (and also possibly empty) families $(x_i)_i$, $(y_j)_j$ of points of $M^+_{\overline{\gamma}}\setminus M^-_{\overline{\gamma}}$, and of $M^-_{\overline{\gamma}}\setminus M^+_{\overline{\gamma}}$ which are dense

in $M^+_{\overline{\gamma}} \setminus M^-_{\overline{\gamma}}$ and $M^-_{\overline{\gamma}} \setminus M^+_{\overline{\gamma}}$ respectively. Moreover, by Remark 3.1, the left-isolated (resp. right-isolated) points of $M^+_{\overline{\gamma}}$ are at most countably many, so we can assume that all of them are part of the family $(x_i)_i$ or $(z_k)_k$. It is then straightforward to check that the measures

$$\rho^+ := \sum_i 2^{-i}\delta_{x_i} + \sum_{k=1}^N \delta_{z_k} \quad \text{and} \quad \rho^- := \sum_j 2^{-j}\delta_{y_j} + \sum_{k=1}^N 2\delta_{z_k}$$

satisfy the required property. □

Define, now, a candidate pair of Kantorovich potentials.

Definition 3.5. For $x, y \in \mathbb{R}$, we define

$$\varphi_r(x) := -\rho((-\infty, x]) \text{ and } \psi_r(y) := \inf\big\{ -\varphi_r(x) : x \in [y - \lambda_C, y + \lambda_C]\big\}.$$

Theorem 3.6. *The functions φ_r and ψ_r defined above satisfy the following properties:*

(i) *φ_r has bounded variation (in particular, it has a left-limit and a right-limit at any point) and is right-continuous;*

(ii) *the support of $(\varphi_r')^-$ is exactly $M^+_{\overline{\gamma}}$, and the support of φ_r' is exactly $M_{\overline{\gamma}}$;*

(iii) *for any point $\bar{x} \in M^+_{\overline{\gamma}}$ which is left-isolated or right-isolated in $M^+_{\overline{\gamma}}$, we have*

$$\lim_{\substack{x \to \bar{x} \\ x < \bar{x}}} \varphi(x) > \varphi_r(\bar{x});$$

(iv) *the couple (φ_r, ψ_r) is a Kantorovich potential, i.e. a maximizer of* (2.1).

In some sense, Property (ii) of Theorem 3.6 means that the potential φ_r is "optimal with respect to $\overline{\gamma}$": indeed, Prop. 2.2 suggests that, for any potential φ having bounded variation, the support of φ' should not be larger than M_γ, as φ should be locally constant anywhere else. Moreover, we will that the smallest possible M_γ among the optimal plans γ is actually achieved by the **∞-cm** transport plans (*cf.* Th. 4.2 below).

Proof of Theorem 3.6. By definition φ_r is the cumulative distribution function of the measure $-\rho$, then it is a BV function whose distributional derivative is the measure $-\rho$; this and the definition of ρ classically imply the properties (i) and (ii). As for (iii), let $\bar{x}$ be an isolated point of $M^+_{\overline{\gamma}}$. Then, for any $x < \bar{x}$,

$$\varphi_r(\bar{x}) - \varphi_r(x) = -\rho(]x, \bar{x}]) = -\rho(\{\bar{x}\}) - \rho(]x, \bar{x}[).$$

By Property (ii) of Lemma 3.4 we have $\rho(\{\bar{x}\}) > 0$ while the other term in the last equality vanishes as $x \to \bar{x}$, which proves (iii).

It remains to show that (φ_r, ψ_r) is a pair of Kantorovich potentials. First of all, from the definition of ψ_r, it follows that

$$\text{for any } (x, y) \in \mathbb{R}^2 \text{ with } |y - x| \le \lambda_C, \quad \varphi_r(x) + \psi_r(y) \le 0.$$

This proves that (φ_r, ψ_r) is an admissible couple for the dual problem. We moreover claim that

$$\varphi_r(\bar{x}) + \psi_r(\bar{y}) = 0 \text{ for } \overline{\gamma}\text{-a.e. } (x, y); \tag{3.2}$$

more precisely, we will prove that, if $(\bar{x},\bar{y}) \in \operatorname{spt}\mu \times \operatorname{spt}\nu$ is such that (3.2) does not hold, then either $\bar{x}$ belongs to an at most countable subset of $\operatorname{spt}\mu$ or $\bar{y}$ belongs to an at most countable subset of $\operatorname{spt}\nu$. This will be enough to prove (iv) since it implies that

$$\int \varphi_r \,\mathrm{d}\mu + \int \psi_r \,\mathrm{d}\nu = \int (\varphi_r(x) + \psi_r(y))\,\mathrm{d}\overline{\gamma}(x,y) = 0,$$

so that (φ_r, ψ_r) is optimal.

Let $(\bar{x},\bar{y})$ be a point of $\operatorname{spt}(\overline{\gamma})$ such that $\varphi_r(\bar{x}) + \psi_r(\bar{y}) < 0$. Then:

$$-\varphi_r(\bar{x}) > \psi_r(\bar{y}) = \inf\big\{ -\varphi_r(x) \,:\, x \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]\big\}.$$

We deduce that there exists $z \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]$ such that $\varphi_r(z) > \varphi_r(\bar{x})$, and we distinguish the cases $z < \bar{x}$ and $z > \bar{x}$.

First case: $z < \bar{x}$. In this case, the definition of z reads

$$\varphi_r(\bar{x}) - \varphi_r(z) = -\rho(]z,\bar{x}]) < 0.$$

In particular, there exists $z' \in]z,\bar{x}]$ which belongs to $M^+_{\overline{\gamma}}$, so that $(z', z' + \lambda_C) \in \operatorname{spt}(\overline{\gamma})$. Then, notice that

$$\max\big\{|\bar{y} - \bar{x}|, |(z' + \lambda_C) - z'|\big\} = \lambda_C \tag{3.3}$$

and consider

$$\max\big\{|\bar{y} - z'|,\ |(z' + \lambda_C) - \bar{x}|\big\}. \tag{3.4}$$

By the cyclical monotonicity of $\bar{\gamma}$, the max in (3.4) must be at least λ_C too, meaning that either $|\bar{y} - z'|$ or $|(z' + \lambda_C) - \bar{x}|$ is larger or equal to λ_C. Now:

- From the fact that $z' \in (z, \bar{x}]$ and that $z, \bar{x} \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]$, we have

  $$-\lambda_C < z' - \bar{y} \le \bar{x} - \bar{y} \le \lambda_C.$$

  Therefore, the inequality $|\bar{y} - z'| \ge \lambda_C$ is only possible if $z' = \bar{x} = \bar{y} + \lambda_C$. This implies that $\bar{x}$ belongs to $M^+_{\overline{\gamma}} \cap M^-_{\overline{\gamma}}$, which is finite by Lemma 3.3.
- On the other hand, using again that $z' \in]z,\bar{x}]$ and $z, \bar{x} \in [\bar{y} - \lambda_C, \bar{y} + \lambda_C]$, we observe

  $$-\lambda_C \le \bar{y} - \bar{x} < z' + \lambda_C - \bar{x} \le \lambda_C.$$

  The inequality $|(z' + \lambda_C) - \bar{x}| \ge \lambda_C$ can then only hold if $\bar{x} = z'$. In this case, both $(\bar{x},\bar{y})$ and $(\bar{x}, \bar{x} + \lambda_C)$ belong to the support of $\overline{\gamma}$. As a consequence of **∞-cm** condition, for every $(x, y') \in \operatorname{spt}\overline{\gamma}$ with $\bar{y} < y' < \bar{x} + \lambda_C$ it must hold $x = \bar{x}$ (in fact, if $x < \bar{x}$ the couples (x, y') and $(\bar{x},\bar{y})$ would violate the condition while if $\overline{x} < x$ the bad couples would be (x, y') and $(\bar{x}, \bar{x} + \lambda_C)$). It follows that

  $$\begin{aligned} 0 = \mu(\{\bar{x}\}) &= \bar{\gamma}(\{\bar{x}\} \times \mathbb{R}) \\ &\ge \bar{\gamma}(\{\bar{x}\} \times]\bar{y}, \bar{x} + \lambda_C[) \\ &= \bar{\gamma}(\mathbb{R} \times]\bar{y}, \bar{x} + \lambda_C[) \\ &= \nu(]\bar{y}, \bar{x} + \lambda_C[). \end{aligned}$$

  Then $\bar{y}$ is a right-isolated point of $\operatorname{spt}\nu$ which, as discussed in Remark 3.1, are at most countably many.

Second case: $z > \bar{x}$. This case is treated in a very similar way as the previous one. First of all, we start by deducing from the inequality $\varphi_r(z) > \varphi_r(\bar{x})$ that there exists $z' \in (\bar{x}, z]$ such that $(z', z' - \lambda_C) \in \mathrm{spt}(\overline{\gamma})$. Then, by the $\boldsymbol{\infty}$**-cm** condition,

$$\lambda_C = \max\big\{|\bar{y} - \bar{x}|, |(z' - \lambda_C) - z'|\big\} \leq \max\big\{|\bar{y} - z'|, |(z' - \lambda_C) - \bar{x}|\big\}, \tag{3.5}$$

so that either $|\bar{y} - z'|$ or $|z' - \lambda_C - \bar{x}|$ is at least equal to λ_C. On the other hand, the fact that $z' \in (\bar{x}, z]$ with $\bar{y} - \lambda_C \leq \bar{x} \leq z \leq \bar{y} + \lambda_C$ enforces

$$-\lambda_C < z' - \lambda_C - \bar{x} \leq \bar{y} - \bar{x} \leq \lambda_C. \tag{3.6}$$

$$\text{and} - \lambda_C < z' - \bar{y} \leq \lambda_C \tag{3.7}$$

Therefore, the condition (3.5) is only satisfied in the two following cases:

- there is equality in the two last inequalities of (3.6): in this case, we have simultaneously $z' = \bar{y} + \lambda_C$ and $\bar{x} = \bar{y} - \lambda_C$. Therefore, $(\bar{y} - \lambda_C, \bar{y})$ and $(\bar{y} + \lambda_C, \bar{y})$ both belong to the support of $\overline{\gamma}$: this means that $\bar{y} \in M^+_{\tilde{\gamma}} \cap M^-_{\tilde{\gamma}}$, where $\tilde{\gamma}$ is the "symmetric plan" of $\overline{\gamma}$, that is
$$\tilde{\gamma} = \tau_{\#}\overline{\gamma} \quad \text{with } \tau(x, y) = (y, x).$$
The transport plan $\tilde{\gamma}$ (which belongs to $\Pi(\nu, \mu)$) being itself $\boldsymbol{\infty}$**-cm** , Lemma 3.3 also applies, yielding finiteness of $M^+_{\tilde{\gamma}} \cap M^-_{\tilde{\gamma}}$. Therefore, the equality case in (3.6) is only possible for a finite number of $\bar{y}$.
- If there is equality in the last inequality of (3.7), we deduce that both $(\bar{x}, \bar{y})$ and $(\bar{y} + \lambda_C, \bar{y})$ belong to the support of $\overline{\gamma}$, with $\bar{x} < \bar{y} + \lambda_C$. As in the previous case, we conclude that $\bar{x}$ is a boundary point of one of the connected components of $\mathbb{R} \setminus (\mathrm{spt}\,\mu)$, which are at most countably many thanks to Remark 3.1, concluding the proof of (iv). □

4. Application: existence of a minimal set of maximal displacements

In this section, we will use the pair of Kantorovich potentials provided by Theorem 3.6 to get some informations on general optimal transport plans for (1.1) (not necessarily $\boldsymbol{\infty}$**-cm**). Since any optimal plan γ is concentrated on the set

$$\Big\{(x, y) \in \mathbb{R}^2 \,:\, \varphi_r(x) + \psi_r(y) = 0 \text{ and } |y - x| \leq \lambda_C\Big\},$$

it is natural to start by collecting some properties of this set. This is the goal of the following proposition: roughly speaking, it suggests that, if $\bar{x}$ belongs to $M^+_{\overline{\gamma}}$, its only possible "image" by an optimal transport plan γ is $\bar{x} + \lambda_C$.

Proposition 4.1. *For any x, denote by*

$$B_r(x) := \Big\{y \in [x - \lambda_C, x + \lambda_C] \,:\, \varphi_r(x) + \psi_r(y) = 0\Big\}.$$

Let $\bar{x} \in M^+_{\overline{\gamma}}$. Then, for any x such that $\bar{x} \leq x < \bar{x} + 2\lambda_C$, we have $B_r(x) \subseteq [\bar{x} + \lambda_C, x + \lambda_C]$. In particular, the set $B_r(\bar{x})$ is reduced to $\{\bar{x} + \lambda_C\}$.

Proof. Let us fix $\bar{x} \in M^+_{\overline{\gamma}}$ and select $x \in [\bar{x}, \bar{x} + 2\lambda_C[$. By definition, all the elements of $B_r(x)$ are at most equal to $x + \lambda_C$, thus we only have to prove that $B_r(x)$ does not contain any y which is smaller than $\bar{x} + \lambda_C$. Consider then y with $x - \lambda_C \leq y < \bar{x} + \lambda_C$,

and let us prove that the equality $\varphi_r(x)+\psi_r(y)=0$ can not hold. We observe that, by Lemma 3.2, if $x>\bar{x}$ then $\rho^-(]\bar{x},x])=0$, so that

$$\varphi_r(\bar{x})\geq\varphi_r(x) \tag{4.1}$$

which is of course also true when $x=\bar{x}$. On the other hand, by definition of ψ_r and since $[y-\lambda_C,\bar{x}[\subset[y-\lambda_C,y+\lambda_C]$, we also have

$$\psi_r(y)\leq\inf\Big\{-\varphi_r(z)\,:\,y-\lambda_C\leq z<\bar{x}\Big\}. \tag{4.2}$$

From (4.1) and (4.2), it is clear that proving that

$$\exists z\in[y-\lambda_C,\bar{x}[,\ \varphi_r(z)>\varphi_r(\bar{x}) \tag{4.3}$$

would be enough to conclude that $\psi_r(y)<-\varphi_r(x)$. We prove (4.3) by separating two cases.

- *First case:* *$\bar{x}$ is a left-isolated point of $M^+_{\overline{\gamma}}$.* In this case, Property (iii) of Theorem 3.6 ensures
$$\lim_{\substack{z\to\bar{x}\\ z<\bar{x}}}\varphi_r(z)>\varphi_r(\bar{x})$$
which is clearly enough to find the desired z.
- *Second case:* *$\bar{x}$ is not a left-isolated point of $M^+_{\overline{\gamma}}$.* Let us select some $\tilde{x}\in]\bar{x}-\lambda_C,\bar{x}[\cap M^+_{\overline{\gamma}}$. By applying Lemma 3.2 to $\tilde{x}$, we know that $]\tilde{x},\bar{x}]\cap M^-_{\overline{\gamma}}=\emptyset$. From this fact and since $\bar{x}$ is a left-cluster point of $M^+_{\overline{\gamma}}$, we deduce
$$\rho^+(]\tilde{x},\bar{x}])>0\quad\text{and}\quad\rho^-(]\tilde{x},\bar{x}])=0$$
which implies $\varphi_r(\tilde{x})>\varphi_r(\bar{x})$, as expected. □

We are now able to state the next main result of this paper, which in particular asserts the existence of a "minimal set of maximal displacements", which is shared by all the optimal transport plans:

Theorem 4.2. *Let $\gamma,\overline{\gamma}\in\mathcal{O}_\infty$ be optimal transport plans and assume that $\overline{\gamma}$ satisfies the* **∞-cm** *-condition. Then*

$$M^+_{\overline{\gamma}}\subseteq M^+_\gamma\quad\text{and}\quad M^-_{\overline{\gamma}}\subseteq M^-_\gamma.$$

In other words, there are two compact sets M^+ and M^- which are "minimal sets of maximal displacement for optimal plans", in sense that, for any optimal plan γ,

$$M^+\subseteq M^+_{\overline{\gamma}}\quad\text{and}\quad M^-\subseteq M^-_{\overline{\gamma}}$$

with equality if γ is an **∞-cm** *plan (and in particular, all the* **∞-cm** *plans have same set of maximal displacements).*

Proof. First we consider some consequences of Prop. 4.1 for γ.

Step I. For any $\bar{x}\in M^+_{\overline{\gamma}}$, the quarter of plane $Q:=[\bar{x},+\infty)\times(-\infty,\bar{x}+\lambda_C]$ has zero mass for γ. To prove this result, we first observe that, by the optimality of γ and of (φ_r,ψ_r) it follows that γ is concentrated on the set

$$B:=\Big\{(x,y)\in\mathbb{R}^2\,:\,|y-x|\leq\lambda_C\text{ and }\varphi_r(x)+\psi_r(y)=0\Big\}$$

which can be written as $\Big\{(x,y)\,:\,y\in B_r(x)\Big\}$. Now, for $x\geq\bar{x}$ and $y\in B_r(x)$ we have:

- if $x<\bar{x}+2\lambda_C$, then Prop. 4.1 ensures that $y\geq\bar{x}+\lambda_C$;
- if $x\geq x+2\lambda_C$, the same holds since $|y-x|\leq\lambda_C$.

This exactly means that B has no intersection with the interior of the quarter of plane Q, which has therefore zero mass for γ; since moreover the lines $\{\bar{x}\}\times\mathbb{R}$ and $\mathbb{R}\times\{\bar{x}+\lambda_C\}$ have also zero mass for γ (because μ and ν are non atomic), the Step I is proven.

Step II: if there exists $\bar{x}\in M^+_{\overline{\gamma}}\setminus M^+_\gamma$, then such an $\bar{x}$ satisfies $\mu([\bar{x},\bar{x}+\delta])=\nu([\bar{x}+\lambda_C-\delta,\bar{x}+\lambda_C])=0$ for $\delta>0$ small enough. Let $\bar{x}$ be such a point; since $\bar{x}\notin M^+_\gamma$, we can select $\delta>0$ such that the square with side-length 2δ and centered in the point $(\bar{x},\bar{x}+\lambda_C)$ has zero γ-measure. Without loss of generality we may also assume that $0<\delta<\lambda_C$. Since μ is the first marginal of γ, we then have

$$\mu([\bar{x},\bar{x}+\delta])=\gamma([\bar{x},\bar{x}+\delta]\times\mathbb{R}).$$

Since $\bar{x}\in M^+_{\overline{\gamma}}$ the result of Step I applies and gives $\gamma([\bar{x},\bar{x}+\delta]\times(-\infty,\bar{x}+\lambda_C])=0$, so that

$$\gamma([\bar{x},\bar{x}+\delta]\times\mathbb{R})=\gamma([\bar{x},\bar{x}+\delta]\times[\bar{x}+\lambda_C,+\infty)).$$

Moreover, since γ is an optimal transport plan, we have $|y-x|\leq\lambda_C$ for γ-a.e. (x,y); in particular, for γ-a.e. (x,y) with $x\leq\bar{x}+\delta$, we have $y\leq\bar{x}+\delta+\lambda_C$. Consequently,

$$\gamma([\bar{x},\bar{x}+\delta]\times[\bar{x}+\lambda_C,+\infty))=\gamma([\bar{x},\bar{x}+\delta]\times[\bar{x}+\lambda_C,\bar{x}+\lambda_C+\delta]),$$

which is zero since, by assumption on $\bar{x}$ and δ, the square $[\bar{x}-\delta,\overline{x}+\delta]\times[\bar{x}+\lambda_C-\delta,\overline{x}+\lambda_C+\delta]$ has zero mass for γ. The three last equalities imply that $\mu([\bar{x},\bar{x}+\delta])=0$. The proof of the other equality is similar, and may be described as follows:

$$\begin{aligned}\nu([\bar{x}+\lambda_C-\delta,\bar{x}+\lambda_C])&=\gamma(\mathbb{R}\times[\bar{x}+\lambda_C-\delta,\bar{x}+\lambda_C])\text{ since }\nu=(\pi_2)_\#\gamma\\&=\gamma((-\infty,\bar{x}]\times[\bar{x}+\lambda_C-\delta,\bar{x}+\lambda_C])\text{ by Step I}\\&=\gamma([\bar{x}-\delta,\bar{x}]\times[\bar{x}+\lambda_C-\delta,\bar{x}+\lambda_C])\text{ since }|y-x|\leq\lambda_C\ \gamma\text{-a.e.}\\&=0\text{ by assumption.}\end{aligned}$$

Step III: $M^+_{\overline{\gamma}}\subset M^+_\gamma$. By contradiction, assume that there exists $\bar{x}\in M^+_{\overline{\gamma}}$ which does not belong to M^+_γ. Since $\overline{\gamma}$ is concentrated on the half-space $\Big\{(x,y)\in\mathbb{R}^2\,:\,y\leq x+\lambda_C\Big\}$, we notice that

$$\overline{\gamma}([\bar{x}-\delta,\bar{x}]\times[\bar{x}+\lambda_C,\bar{x}+\lambda_C+\delta])=0$$

because, in this square, only the point $(\bar{x},\bar{x}+\lambda_C)$ satisfies $|y-x|\leq\lambda_C$ and $\overline{\gamma}$ has no atom (because μ and ν don't).

From this property, and from the fact that $\overline{\gamma}$ gives no mass to any horizontal or vertical line, it follows

$$\begin{aligned}\overline{\gamma}([\overline{x}-\delta,\overline{x}+\delta]\times[\bar{x}+\lambda_C-\delta,\overline{x}+\lambda_C+\delta])=\ &\overline{\gamma}([\bar{x},\bar{x}+\delta]\times[\overline{x}+\lambda_C-\delta,\overline{x}+\lambda_C+\delta])\\&+\overline{\gamma}([\overline{x}-\delta,\overline{x}]\times[\overline{x}+\lambda_C-\delta,\bar{x}+\lambda_C]).\end{aligned}\tag{4.4}$$

But, from the result of Step II, it follows that

$$\begin{aligned}\overline{\gamma}([\bar{x},\bar{x}+\delta]\times[\bar{x}+\lambda_C\pm\delta]) &\leq \overline{\gamma}([\bar{x},\bar{x}+\delta]\times\mathbb{R})\\ &= \mu([\bar{x},\bar{x}+\delta])\\ &= 0\end{aligned}$$

and that, similarly, $\overline{\gamma}([\bar{x}-\delta,\bar{x}]\times[\bar{x}-\lambda_C-\delta,\bar{x}-\lambda_C])$ is zero. The equality (4.4) therefore implies that the open square with side-length 2δ and with center $(\bar{x},\bar{x}+\lambda_C)$ has zero mass for $\overline{\gamma}$, which gives the desired contradiction and concludes Step III.

Step IV: $M^-_{\overline{\gamma}}\subset M^-_\gamma$. Consider the "opposite transport plans"

$$\begin{cases}\overline{\gamma}_{\rm neg}:=\chi_\#\overline{\gamma}\\ \gamma_{\rm neg}:=\chi_\#\gamma\end{cases}\qquad \text{where } \chi(x,y)=(-x,-y).$$

It is clear that $\overline{\gamma}_{\rm neg},\gamma_{\rm neg}$ have same marginals which are atomless and compactly supported (they are $\mathrm{d}\mu(-x)$ and $\mathrm{d}\nu(-y)$), have same maximal displacement which is still equal to λ_C and that $\overline{\gamma}_{\rm neg}$ is still an **∞-cm** plan, and that

$$x\in M^+_{\overline{\gamma}_{\rm neg}}\iff(-x)\in M^-_{\overline{\gamma}}\quad\text{and}\quad x\in M^+_{\gamma_{\rm neg}}\iff(-x)\in M^-_\gamma.$$

We can then apply the result if Steps I-III to the plans $\overline{\gamma}_{\rm neg}$ and $\gamma_{\rm neg}$, which concludes the proof. □

We conclude this paper with a description of the general structure of an optimal transport plans for (1.1). This result is very similar to the one obtained in [12, Prop. 3.1] in the L^1 context. The optimal plans are constrained to coincide with the graph of the translation of $\pm\lambda_C$ on the critical regions $M^\pm_{\overline{\gamma}}$ while on any maximal interval $I\subset\mathbb{R}\setminus M_{\overline{\gamma}}$, they are free to "fill" the whole region $\{|y-x|\leq\lambda_C\}\cap(I\times I\pm\lambda_C)$ provided the marginal constraint is satisfied.

In order to make precise this description, let us introduce some additional notations. Let $a<b$ be the lower and upper bounds of $\operatorname{spt}\mu$ and $c<d$ be those of $\operatorname{spt}\nu$. We can then write

$$]a,b[\setminus M_{\overline{\gamma}}=\bigcup_{i\in I}]a_i,b_i[$$

where I is an at most countable set and, for each i, $]a_i,b_i[$ is a maximal open interval fully included in $]a,b[\setminus M_{\overline{\gamma}}$. In particular, all the points a_i,b_i belong to $M_{\overline{\gamma}}\cup\{a,b\}$.

We then set, for each i,

$$c_i=\begin{cases}c & \text{if } a_i=a,\\ a_i+\lambda_C & \text{if } a_i\in M^+_{\overline{\gamma}} \text{ and } a_i\neq a,\\ a_i-\lambda_C & \text{if } a_i\in M^-_{\overline{\gamma}},\ a_i\notin M^+_{\overline{\gamma}} \text{ and } a_i\neq a\end{cases}$$

and similarly

$$d_i=\begin{cases}d & \text{if } b_i=b,\\ b_i-\lambda_C & \text{if } b_i\in M^-_{\overline{\gamma}} \text{ and } b_i\neq b,\\ b_i+\lambda_C & \text{if } b_i\in M^+_{\overline{\gamma}},\ b_i\notin M^-_{\overline{\gamma}} \text{ and } b_i\neq b.\end{cases}$$

Roughly speaking, each c_i (resp. d_i) is "the image" of a_i (resp. b_i) by the infinitely cyclically monotone transport plan $\overline{\gamma}$. In particular, they satisfy the following property.

Lemma 4.3. *The intervals* $]c_i, d_i[$, $i \in I$, *are mutually disjoint. Moreover, for any* i, $M^+_{\overline{\gamma}} + \lambda_C$ *and* $M^-_{\overline{\gamma}} - \lambda_C$ *have also empty intersection with* $]c_i, d_i[$.

Proof. Fix $i \neq j \in I$; assume for instance that $a_i < b_i \leq a_j < b_j$. It is then enough to prove that $d_i \leq c_j$. From the definition of d_i, c_j, it turns out that the only case we need to treat is

$$d_i = b_i + \lambda_C \quad \text{and} \quad c_j = a_j - \lambda_C.$$

This case corresponds to to

$$a_j \neq a,\ a_j \in M^-_{\overline{\gamma}} \setminus M^+_{\overline{\gamma}},\ b_i \neq b, \in M^+_{\overline{\gamma}} \setminus M^-_{\overline{\gamma}}$$

and in particular implies that $b_i < a_j$ (otherwise these equal points would belong to $M^+_{\overline{\gamma}} \cap M^-_{\gamma}$). From the fact that the points $(b_i, b_i + \lambda_C)$ and $(a_j, a_j - \lambda_C)$ both belong to the support of the $\boldsymbol{\infty}$**-cm** plan $\overline{\gamma}$, we then deduce

$$|b_i + \lambda_C - a_j| \geq \lambda_C. \tag{4.5}$$

Assume now that $d_i > c_j$, *i.e.* that $b_i + \lambda_C > a_j - \lambda_C$. Then, since moreover $b_i - a_j > 0$, we deduce that the inequality (4.5) does not hold, a contradiction.

As for the second point, let $i \in I$, consider $\bar{y} \in (c_i, d_i)$ and let us prove that $\bar{x} := \bar{y} - \lambda_C$ does not belong to $M^+_{\overline{\gamma}}$. By definition of (a_i, b_i), this is clearly true if $a_i < \bar{x} < b_i$, thus we only need to consider the cases $\bar{x} \leq a_i$ and $\bar{x} \geq b_i$.

Assume first that $\bar{x} \leq a_i$. If this is an equality, this implies $a_i \in M^+_{\overline{\gamma}}$ thus $c_i = a_i + \lambda_C$ (this is also true in this case $a_i = a \in M^+_{\overline{\gamma}}$): since $\bar{y} = \bar{x} + \lambda_C = a_i + \lambda_C > c_i$, this is impossible. If now the inequality $\bar{x} \leq a_i$ is strict, then one has

$$\bar{x} = \bar{y} - \lambda_C < a_i \quad \text{and} \quad c_i < \bar{y}$$

from which we deduce $c_i - \lambda_C < a_i$. In particular, we are in the cases "$a_i = a$" or "$a_i \in M^-_{\overline{\gamma}} \setminus M^+_{\overline{\gamma}}$" of the definition of c_i. If $a_i = a$, the inequality $\bar{x} < a_i$ leads to $\bar{x} \notin \operatorname{spt} \mu$ so that $\bar{x} \notin M^+_{\overline{\gamma}}$, as expected; if now $a_i \in M^-_{\overline{\gamma}} \setminus M^+_{\overline{\gamma}}$, then from Lemma 3.2, we deduce

$$(a_i - 2\lambda_C, a_i) \cap M^+_{\overline{\gamma}} = \emptyset.$$

Since $\bar{x} < a_i$, $c_i < \bar{y}$, $c_i = a_i - \lambda_C$ and $\bar{y} = \bar{x} + \lambda_C$, it is easy to see that $\bar{x} \in (a_i - 2\lambda_C, a_i)$, so that it cannot belong to $M^+_{\overline{\gamma}}$.

In the case $\bar{x} \geq b_i$ we have

$$\bar{x} \geq b_i, \quad \bar{x} = \bar{y} - \lambda_C \quad \text{and} \quad \bar{y} < d_i$$

from which we deduce $d_i > b_i + \lambda_C$. But from the definition of d_i, we can see that (b_i, d_i) must belong to the support of $\overline{\gamma}$ so that $d_i \leq b_i + \lambda_C$, which is a contradiction.

The last part of Lemma 4.3, namely the fact that $\bar{y} + \lambda_C$ never belongs to $M^-_{\overline{\gamma}}$ for any $\bar{y} \in (c_i, d_i)$, can be proven in a very similar way. □

Having at hand the last notations and the result of Lemma 4.3, we can now state the structural result which characterizes all the one-dimensional optimal plans for (1.1).

Theorem 4.4. *For each i, denote by*

$$\mu_i := \mu\lfloor[a_i, b_i] \quad \text{and} \quad \nu_i = \nu\lfloor[c_i, d_i].$$

Then, for each i, μ_i and ν_i have same total mass. Moreover, the optimal plans γ for the problem (1.1) *are exactly those which can be written as*

$$\gamma = (\mathrm{id} \times (\mathrm{id} + \lambda_C))_\#(\mu\lfloor M_{\overline{\gamma}}^+) + (\mathrm{id} \times (\mathrm{id} - \lambda_C))_\#(\mu\lfloor M_{\overline{\gamma}}^-) + \sum_i \gamma_i \tag{4.6}$$

where, for each i, γ_i is a transport plan from μ_i to ν_i such that $|y - x| \le \lambda_C$ for γ_i-a.e. (x, y).

Proof. Let $\gamma \in \mathcal{O}_\infty(\mu, \nu)$, we will prove that it satisfies (4.6). Again we divide the proof in several steps.

Step I: the measure $\gamma\lfloor(M_{\overline{\gamma}}^+ \times \mathbb{R})$ (resp. $\gamma\lfloor(M_{\overline{\gamma}}^- \times \mathbb{R})$) is concentrated on the line $\{y = x + \lambda_C\}$ (resp. $\{y = x - \lambda_C\}$). Let $(\bar{x}, \bar{y})$ be such that

$$\text{for any } \varepsilon > 0,\ \gamma\Big(([\bar{x} - \varepsilon, \overline{x} + \varepsilon] \cap M_{\overline{\gamma}}^+) \times [\bar{y} \pm \varepsilon]\Big) > 0,$$

we prove that, for γ-a.e. such $(\bar{x}, \bar{y})$, $\bar{y}$ is necessarily equal to $\bar{x} + \lambda_C$. Assume by contradiction that $\bar{y} < \bar{x} + \lambda_C$ (the converse inequality can not hold since $\gamma \in \mathcal{O}_\infty(\mu, \nu)$); without loss of generality, we may assume that $\bar{x}$ is not a left-isolated point of $M_{\overline{\gamma}}^+$, since such points are countably many. Select then $\tilde{x} \in M_{\overline{\gamma}}^+$ such that

$$0 < \bar{x} - \tilde{x} < (\bar{x} + \lambda_C) - \bar{y}.$$

Since $\tilde{x} \in M_{\overline{\gamma}}^+$ by Th. 4.2 implies that

$$\gamma([\tilde{x}, +\infty) \times (-\infty, \tilde{x} + \lambda_C]) = 0.$$

In particular, for $\varepsilon > 0$ such that $\bar{x} - \varepsilon > \tilde{x}$ and $\bar{y} < (\tilde{x} + \lambda_C) - \varepsilon$, we obtain $\gamma([\bar{x} - \varepsilon, \overline{x} + \varepsilon] \times [\bar{y} - \varepsilon, \overline{y} + \varepsilon]) = 0$, in contradiction withthe starting point. This proves that $\gamma\lfloor(M_{\overline{\gamma}}^+ \times \mathbb{R})$ is concentrated on the line $\{y = x + \lambda_C\}$; the same for $\gamma\lfloor(M_{\overline{\gamma}}^- \times \mathbb{R})$ can then be obtained by considering the "opposite transport plans" $\overline{\gamma}_{\mathrm{neg}}, \gamma_{\mathrm{neg}}$, as in Step IV of the proof of Theorem 4.2.

Step II: for any i, the plan $\gamma\lfloor([a_i, b_i] \times \mathbb{R})$ is concentrated on $[a_i, b_i] \times [c_i, d_i]$. Let $i \in I$, first we prove that

$$\gamma\lfloor([a_i, +\infty) \times \mathbb{R}) \text{ is concentrated on } [a_i, +\infty) \times [c_i, +\infty). \tag{4.7}$$

By definition of c_i, this is obvious if $a_i = a$; moreover, in the case $c_i = a_i - \lambda_C$, we have for γ-a.e. (x, y) with $x \ge a_i$:

$$y \ge x - \lambda_C \ge a_i - \lambda_C = c_i.$$

Then we need to look at the case $c_i = a_i + \lambda_C$, which corresponds to $a_i \in M_{\overline{\gamma}}^+$, and this, again, follows from the first step of Th. 4.2.

To conclude we have to prove that

$$\gamma\lfloor((-\infty, b_i] \times \mathbb{R}) \text{ is concentrated on } (-\infty, b_i] \times (-\infty, d_i]. \tag{4.8}$$

This is again obvious in the cases $b_i = b$ and $d_i = b_i + \lambda_C$, we thus have only to consider the case $d_i = b_i - \lambda_C$, which enforces $b_i \in M_{\overline{\gamma}}^-$. Again, it suffices to consider the

negative transport plans $\gamma_{\rm neg}, \overline{\gamma}_{\rm neg}$: observing then that $-b_i \in M^+_{\overline{\gamma}_{\rm neg}}$, we apply Prop. 4.1 to $\gamma_{\rm neg}, \overline{\gamma}_{\rm neg}$ to deduce

$$\gamma_{\rm neg}([-b_i, +\infty) \times (-\infty, -d_i]) = 0$$

which proves (4.8).

Step III: conclusion. Let us define

$$\gamma^- := \gamma\lfloor(M^-_{\overline{\gamma}} \times \mathbb{R}),\ \gamma^+ := \gamma\lfloor(M^+_{\overline{\gamma}} \times \mathbb{R}) \text{ and } \gamma_i := \gamma\lfloor([a_i, b_i] \times \mathbb{R}) \text{ for each } i.$$

It is clear that $\gamma = \gamma^- + \gamma^+ + \sum_i \gamma_i$, and that γ^-, γ^+ have $\mu\lfloor M^-_{\overline{\gamma}}, \mu\lfloor M^+_{\overline{\gamma}}$ as respective first marginal. Moreover, Step I proves that γ^- (resp. γ^+) is the transport plan induced by $\mathrm{id} - \lambda_C$ (resp. $\mathrm{id} + \lambda_C$). Therefore, it remains to prove that, for each i, γ_i has μ_i, ν_i as marginals. Let us then fix $i \in I$; again the fact that $(\pi_1)_\# \gamma_i = \mu_i$ for each i is a direct consequence of the definition of γ_i. Moreover, γ_i is dominated by γ and, by Step II, is concentrated on $(a_i, b_i) \times (c_i, d_i)$: therefore

$$(\pi_2)_\# \gamma_i \le \Big((\pi_2)_\# \gamma\Big)\lfloor(c_i, d_i) = \nu_i.$$

Assume now that this last inequality is strict: in other words, there exists a Borel set $B \subset (c_i, d_i)$ such that

$$\gamma_i(\mathbb{R} \times B) < \nu(B).$$

By Lemma 4.3, B does not meet any (c_j, d_j) with $j \neq i$ and neither meets $M^+_{\overline{\gamma}} + \lambda_C$, $M^-_{\overline{\gamma}} - \lambda_C$. Therefore the result of Step I also implies

$$\gamma^-(\mathbb{R} \times B) = \gamma^+(\mathbb{R} \times B) = \gamma_j(\mathbb{R} \times B) = 0 \quad \text{for any } j \neq i.$$

Adding all of these equalities, we obtain $\gamma(\mathbb{R} \times B) < \nu(B)$, which is impossible since $\gamma \in \Pi(\mu, \nu)$.

This proves that any optimal plan satisfies (4.6). In particular, this implies that μ_i and ν_i have same total mass for any i, and that

$$\mu\lfloor M^+_{\overline{\gamma}} + \mu\lfloor M^-_{\overline{\gamma}} + \sum_i \mu_i = \mu, \quad \nu\lfloor(M^+_{\overline{\gamma}} + \lambda_C) + \nu\lfloor(M^-_{\overline{\gamma}} - \lambda_C) + \sum_i \nu_i = \nu.$$

For any plan satisfying (4.6) and belonging to $\Pi(\mu, \nu)$, the confinement betwen the two lines $y = x + \lambda_C$ and $y = x - \lambda_C$ inplies the optimality, concluding the proof. □

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