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Smooth finite elements

seamless handling of incompressibility, distorted and polygonal meshes; links with equilibrium methods

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RÉSUMÉ. A définir par la commande `\resume{...}`

ABSTRACT. We present a method which we believe can serve as a mid-way solution between computationally expensive meshfree methods and mesh-distortion-sensitive finite element methods. The method is almost insensitive to mesh distortion and is naturally suited to polygonal meshes. More precisely, we show results concerning the application of strain smoothing or stabilized conforming nodal integration to the finite element method. The technique was coined the Smooth Finite Element Method (SFEM) by G.R. Liu et al. We show that some versions of the method completely suppress locking in incompressible elasticity and elasto-plasticity. Additionally, locking can be suppressed in the case of plate and shell formulations. More importantly perhaps, these smooth finite elements helped discover a remarkable link between displacement and equilibrium finite elements.

MOTS-CLÉS : A définir par la commande `\motsclés{...}`

KEYWORDS: Smooth finite elements; locking; incompressibility; plates; shells; equilibrium elements; displacement elements; plasticity; selective integration.

1. Introduction

The stabilized conforming nodal integration finite element method, based on strain smoothing stabilization, presented recently as the Smooth Finite Element Method (SFEM) by (Liu *et al.*, 2006) is studied theoretically and numerically in (Nguyen-Xuan *et al.*, 2007e). The integration of the stiffness matrix is performed on the boundaries of subcells in each finite element. A rigorous variational framework based on the Hu-Washizu assumed strain variational form was developed in (Nguyen-Xuan *et al.*, 2007e).

We prove that smooth finite element solutions are in a space bounded by the standard finite element solution (infinite number of subcells) and an equilibrium finite element solution (one subcell) (Nguyen-Xuan *et al.*, 2007d).

It is remarkable that the lower the number of cells, the more accurate the dual unknown and the less accurate the primal unknown, a characteristic of equilibrium finite elements. The converse effect is noted for high numbers of subcells where the dual unknown is not as accurate as the primal unknown, characteristic of standard finite element solutions.

We show numerically that the one-cell smoothed four-noded quadrilateral finite element does not lock, has a convergence rate of 2.0 in the energy norm for problems with smooth solutions. For problems with singular solutions, this element always converges faster than standard finite elements (Nguyen-Xuan *et al.*, 2007e).

Additionally, selective cell-wise smoothing can be constructed, where the deviatoric and volumetric parts of the strain field are approximated with a smooth finite element approximation based on a different number of subcells (Nguyen-Xuan *et al.*, 2007a).

Smooth finite elements are Chen *et al.* also remarkable means to suppress locking in plate (Nguyen-Xuan *et al.*, 2007b) and shell (Nguyen-Xuan *et al.*, 2007c) formulations.

This communication only presents the key ingredients behind smooth finite element theory and concentrates on a summary of the principal features of the method, and how it may be applied to engineering analysis. The interested readers are referred to the articles in the bibliography for more detail.

2. Basic idea

The strain-smoothing method (SSM) was proposed in (Chen *et al.*, 2001). A strain smoothing stabilization is created to compute the nodal strain as the divergence of a spatial average of the strain field. This strain smoothing avoids evaluating derivatives of mesh-free shape functions at nodes and thus eliminates defective modes. We show in this contribution how such strain smoothing approaches may be useful in a finite element context. Strain smoothing can be seen as a stabilized conforming nodal

integration method, as in Galerkin mesh-free methods applied to the finite element method. The smooth strain field at an arbitrary point \mathbf{x}_C is written as

$$\tilde{\varepsilon}_{ij}^h(\mathbf{x}_C) = \int_{\Omega^h} \varepsilon_{ij}^h(\mathbf{x}) \Phi(\mathbf{x} - \mathbf{x}_C) d\Omega \quad [1]$$

where Φ is a smoothing function that generally satisfies the following properties (Yoo *et al.*, 2004)

$$\Phi \geq 0 \quad \text{and} \quad \int_{\Omega^h} \Phi d\Omega = 1 \quad [2]$$

The smoothed discretized gradient operator $\tilde{\mathbf{B}}$ is defined by

$$\tilde{\varepsilon}^h = \tilde{\mathbf{B}}_C \mathbf{q} \quad [3]$$

The smoothed element stiffness matrix for element e is computed by

$$\tilde{\mathbf{K}}^e = \sum_{C=1}^{nc} \int_{\Omega_C} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C d\Omega = \sum_{C=1}^{nc} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C A_C \quad [4]$$

where nc is the number of the smoothing cells of the element. Here, the integrands are constant over each Ω_C and the non-local strain displacement matrix reads

$$\tilde{\mathbf{B}}_{Ci} = \frac{1}{A_C} \int_{\Gamma_C} \begin{pmatrix} N_i n_x & 0 \\ 0 & N_i n_y \\ N_i n_y & N_i n_x \end{pmatrix} d\Gamma = \frac{1}{A_C} \int_{\Gamma_C} \mathbf{n}^T N_i(\mathbf{x}) d\Gamma \quad \forall i = 1, 2, 3, 4 \quad [5]$$

Introducing (5) into (4), the smoothed element stiffness matrix is evaluated along boundary of the smoothing cells of the element :

$$\tilde{\mathbf{K}}^e = \sum_{C=1}^{nc} \frac{1}{A_C} \left(\int_{\Gamma_C} \mathbf{n}^T \mathbf{N}(\mathbf{x}) d\Gamma \right)^T \mathbf{D} \left(\int_{\Gamma_C} \mathbf{n}^T \mathbf{N}(\mathbf{x}) d\Gamma \right) \quad [6]$$

3. Application to incompressible elasto-plasticity

Consider a thick cylinder under pressure, the material is assumed to follow von Mises plasticity. Hill (Hill, 1950) shows the exact solution for this problem. The yield stress is $\sigma_y = 24$, Poisson's ratio $\nu = 0.49999$, and the internal pressure p is varied

between 8 and 20. We denote the plastic radius, i.e. the value of r below which the cylinder undergoes plastic deformation by c . The return mapping algorithm (Simo *et al.*, 1999) combined with four-noded smooth finite elements is employed to simulate the plastic behavior. Here we evaluate the development of the plastic domain in the case of isotropic hardening plasticity.

Figures 1 present elastic-plastic results for the thick cylinder in three cases : a) perfectly plastic material, $H = 0$, b) hardening case, $H = E/3$ and c) $H = 2E/3$. The numerical results show that the development of the plastic domain slows down with increasing hardening factor, H . Also, the proposed method agrees well with the exact solution (Hill, 1950) and other published results (Owen *et al.*, 1980, Commend *et al.*, 2001).

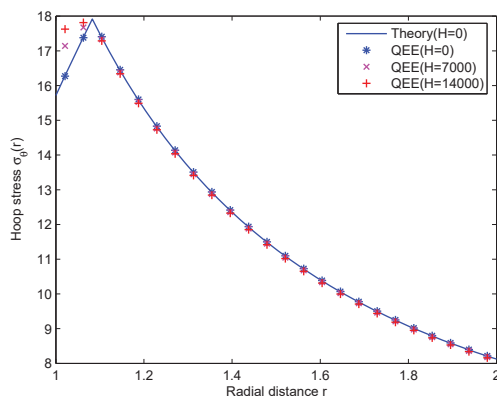
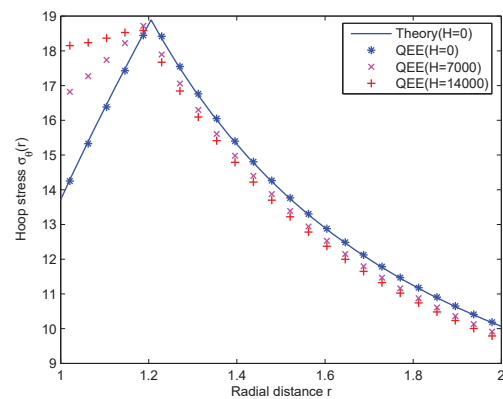
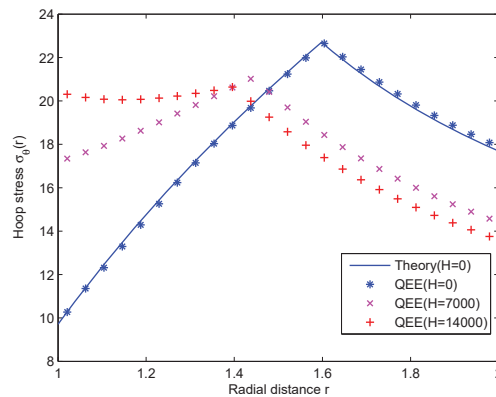
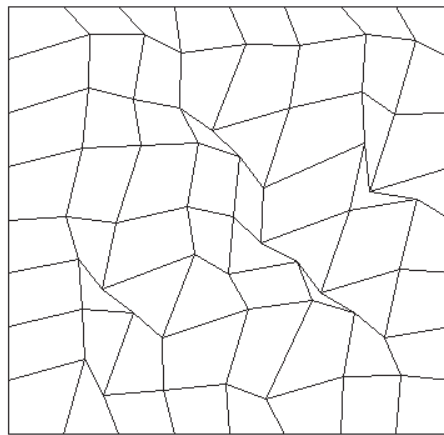
(a) $p = 12$ (b) $p = 14$ (c) $p = 18$

Figure 1. View of the hoop stress for the perfectly plastic behavior : (a) $p = 12$; (b) $p = 14$; and (c) $p = 18$. QEE stands for quasi-equilibrium element (see (Nguyen-Xuan *et al.*, 2007d) where it is proven equivalent to the one-subcell SFEM).

4. Application to plate formulations

A clamped plate subjected to a center point load $F = 16.3527$ with geometry and material parameters : length $L = 100$, thickness $t = 1$, Young's modulus $E = 10^4$, Poisson's ratio $\nu = 0.3$. Only a quarter of the plate is modelled with a mesh of 8×8 elements. To study the effect of mesh distortion on the results, the position of interior nodes is perturbed.

The results of our presented method are more accurate than those of the Q4-R element and the MITC4 element, especially for extremely distorted meshes. Here, the MISC1 element gives the best result. However, this element contains two zero-energy modes. In simple problems, these hourglass modes can be automatically eliminated by the boundary conditions. However, this is not in general the case. Otherwise, the MISC2, MISC3 and MISC4 elements retain a sufficient rank of the element stiffness matrix and give excellent results.



(a)

Figure 2. An example of a distorted mesh for the square plate.

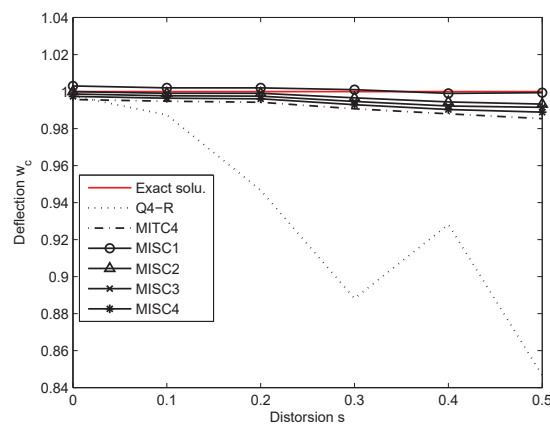


Figure 3. The center deflection with influence of mesh distortion for a clamped square plate subjected to a concentrated load

5. Conclusions

Strain smoothing in finite elements yields superconvergent elements alleviates shortcomings related to mesh distortion, singularities, locking in incompressible, plate and shell formulations, allows polygonal meshes and contour integration and applies to linear and non-linear problems. It seems clear that the contour integration, stability and robustness of this formulation will lend itself to important improvements of the extended finite element methods.

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