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Phil Fraundorf. Traveler-point dynamics: A 3-vector bridge to curved spacetimes. 2022. hal-01503971v4

HAL Id: hal-01503971

<https://hal.science/hal-01503971v4>

Preprint submitted on 29 Apr 2022 (v4), last revised 29 Mar 2023 (v5)

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Traveler-point dynamics: A 3-vector bridge to curved spacetimes

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(Dated: April 29, 2022)

Locally-defined parameters chosen to be minimally frame-variant can be useful for describing motion in accelerated frames and in curved spacetimes. In particular the metric-equation's synchrony-free "traveler-point parameters", like scalar proper-time and the 3-vectors: proper-velocity and proper-acceleration, are useful in curved spacetime because extended arrays of synchronized clocks (e.g. for local measurement of the denominator in $\Delta x/\Delta t$) may be hard to find. Combined with a recognition of improper (geometric) forces with approximate Lagrangian $L \equiv K - U$, these same parameters can further prepare intro-physics students for "flat-patch" engineering in curved spacetimes e.g. via $\Delta L \simeq (\gamma - 1)mc^2$, of which Newton's approximation to gravity is the oldest example. This strategy also suggests an "accelerated-traveler perspective" on high speed trips that are exact at any speed in flat spacetime, and approximate in curved settings e.g. for computing gravitational trajectories at high speed.

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The problem with the former, of course, is that minimally frame-variant quantities like proper time, 3-vector proper velocity⁴, and 3-vector proper acceleration⁵ are inherently local to one traveler's perspective, hence we refer to them as "traveler-point" quantities. The advantage is that they are either frame-invariant (in the case of proper time and the magnitude of proper-acceleration) or synchrony-free⁶ (i.e. do not require an extended network of synchronized clocks which is often unavailable in curved space time and accelerated frames).

In a larger context, general-relativity revealed a century ago why we can get by with using Newton's laws in spacetime so curvy that it's tough to jump higher than a meter. It's because those laws work locally in all frames of reference^{7,8}, provided we recognize that motion is generally affected by both proper and "geometric" (i.e. accelerometer-invisible connection-coefficient) forces. The emergent "metric-first" approach to exploring curved spacetimes with calculus⁹ can choose to restrict *all measurement* to locally-flat patches of spacetime in this context, which can then be sewn together by a global metric^{10,11}.

I. INTRODUCTION

In an excellent 1968 article¹, Robert W. Brehme discussed the advantage of teaching relativity with four-vectors. This is not done for intro-physics courses. One reason may be that spacetime's 4-vector symmetry is broken *locally* into (3+1)D, so that e.g. engineers working in curved spacetimes² (including that here on earth) will naturally want to describe time-intervals in seconds and space-intervals e.g. in meters. In this note, we therefore explore in depth an idea about use of minimally frame-variant scalars and 3-vectors, suggested by many but perhaps first hinted at in Percy W. Bridgman's 1928 ruminations³ about the promise of defining 3-vector velocity using traveler proper time instead of an "extended" time system.

This opens the door to engineering use of spatial 3-vectors and temporal scalars at any speed, and locally in curved spacetimes, for engineers who have little interest in ignoring the local (3+1)D break in our global 4-vector symmetry which gives rise to fundamental differences in the way that space and time are experienced. The metric equation also contents itself with a single definition of extended simultaneity (cf. Fig. 1) i.e. that associated with the set of book-keeper (or map) coordinates with which one chooses to describe one's local patch (for measurement) and one's global spacetime metric (e.g. for formally defining extended simultaneity). Avoiding the Lorentz transform's need for two frames with synchronized clocks in special relativity¹² of course opens the door to simple treatments of accelerated motion in flat spacetime as well.

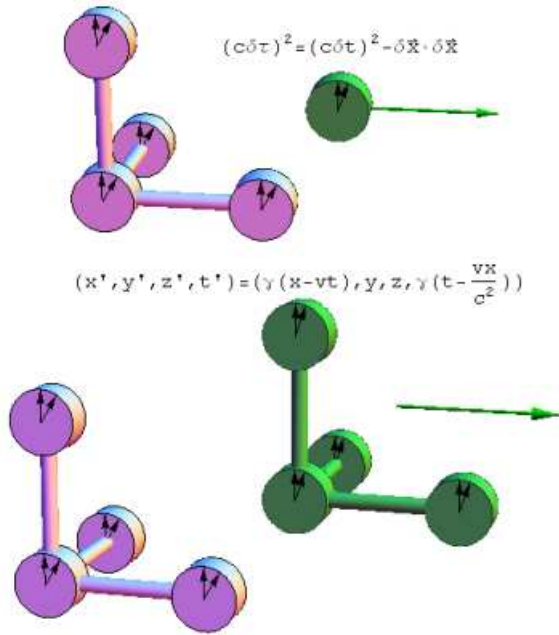


FIG. 1. Even one extended-frame with synchronized clocks may be more than we have in curved spacetime.

II. PROPER TIME

Proper-time is the time elapsed locally along the world-line of any physical clock (broadly defined). It is also the frame-invariant time-interval $\delta\tau$ that shows up in metric equations of the form $(c\delta\tau)^2 = \Sigma_\mu \Sigma_\nu g_{\mu\nu} \delta X_\mu \delta X_\nu$, which in the flat-space Cartesian case takes the Pythagorean form $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2 - (\delta y)^2 - (\delta z)^2$, where book-keeper or map coordinates are found on the right-hand side of the equation, and c is the “lightspeed” spacetime constant for converting e.g. seconds into meters.

When simultaneity is defined by a flat-spacetime book-keeper coordinate frame, time-elapsed on traveling clocks is always “dilated” (e.g. spread out), since from the metric equation the differential-aging or Lorentz factor $\gamma \equiv dt/d\tau \geq 1$. Expressions for this in terms of velocity will be discussed below.

We won’t discuss the utility of proper-time τ in detail here because it is widely used now even by introductory textbook authors¹³. However it’s worth pointing out that early treatments of special relativity were so focused on the equivalence between frames, each with their own extended set of synchronized clocks, that proper-time (if discussed¹⁴) was generally a late-comer to the discussion, and of course used only for “rest frame” (i.e. unaccelerated) travelers¹².

Moving yardsticks also have a “proper-length” associated with them, which can be useful too. However this is a “non-local” quantity not defined by the metric equation, the concept of “rigid body” is only an approximation in spacetime, and length contraction itself involves three separate events as distinct from the two involved in

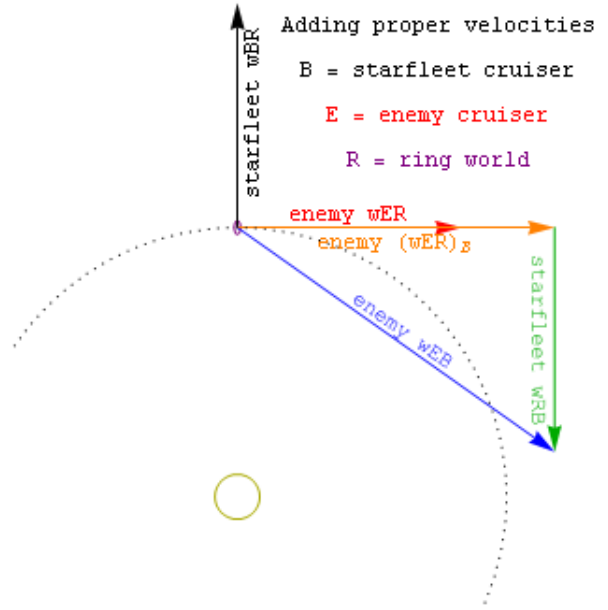


FIG. 2. Proper-velocity addition for a sci-fi puzzler, in which your starfleet battle-cruiser drops in from hyperspace heading away from a nearby star, only to find the enemy ship heading off in another direction.

time-dilation. Hence we don’t include proper-length in a collection of those traveler-point parameters which show promise of wide-ranging usefulness in accelerated frames and curved spacetime.

III. PROPER VELOCITY

Proper-velocity¹⁵ $\vec{w} \equiv d\vec{x}/d\tau$ is the rate at which book-keeper or map 3-vector position \vec{x} is changing per unit-time elapsed locally on traveler clocks. Because it is proportional to momentum (i.e. $\vec{p} = m\vec{w}$), unlike coordinate velocity $\vec{v} \equiv d\vec{x}/dt$ it has no upper limit. Also unlike coordinate velocity, its measurement does not depend on map-clock readings (synchronized or not) along that world line. Proper-velocity is also represented by the space-like components of a traveling object’s velocity 4-vector.

An interesting thought problem for future engineers (as well as teachers) might be to ask if speed-limit signs, in a “Mr. Tompkins” world¹⁶ with a much slower light-speed spacetime constant, would use coordinate-speed or proper-speed. Four criteria to consider might involve the measure’s connection to momentum, kinetic-energy, and to reaction-times (after the “warning photons” arrive) for both the driver, and a pedestrian who is tempted to cross the street.

The time-like component of that 4-vector is $c\gamma$, where $\gamma \equiv dt/d\tau$ is the also-useful (but not synchrony free) Lorentz differential-aging or “speed of map-time” factor mentioned above. In flat spacetime, from the met-

ric equation it is easy to show that $\gamma = w/v = \sqrt{1 + (w/c)^2} \geq 1$. This works better at high speeds than the version in terms of coordinate speed because there is no difference in the denominator between numbers very near to 1.

This same proper velocity form also emerges directly for circular orbits around a spherically symmetric mass, as the exact azimuthal motion factor in the Schwarzschild time-dilation equation $\gamma \equiv dt/d\tau = \gamma_r \sqrt{1 + (w_\phi/c)^2}$, where the gravitational time-dilation factor is $\gamma_r \equiv 1/\sqrt{1 - (r_s/r)} \geq 1$ and $r \geq r_s$ is distance from the center of a mass M object whose Schwarzschild radius $r_s = 2GM/c^2$. Global positioning system satellites have to consider both of these factors in compensating for the rate of time's passage in their orbits around earth.

For unidirectional velocity addition, the advantage of a collider over an accelerator is easily seen because $w_{AC} = \gamma_{AB}\gamma_{BC}(v_{AB} + v_{BC})$. In other words, the Lorentz-factors multiply even though the coordinate-velocities (which add) never exceed c . Proper velocity's direct connection to momentum makes it more relevant to speed-limits e.g. in worlds with a reduced value for lightspeed, and its direct connection to traveler time makes it more relevant to passengers and crew when planning long high-speed trips.

More generally, unlike coordinate-velocities, proper velocities add vectorially provided that one rescales (in magnitude only) the "out of frame" component. In other words $\vec{w}_{AC} = \gamma_{AC}\vec{v}_{AC} = (\vec{w}_{AB})_C + \vec{w}_{BC}$. Here C 's view of the out-of-frame proper-velocity $(\vec{w}_{AB})_C$ is in the same direction as \vec{w}_{AB} but rescaled in magnitude by a factor of $(\gamma_{BC} + (\gamma_{AB} - 1)\vec{w}_{BC} \cdot \vec{w}_{AB}/w_{AB}^2) \geq 0$. Hence vector diagrams, and even the original low-speed equation for velocity-addition (almost), survive intact when proper-velocity is used.

This would open the door to new settings for those relative motion problems (that students dislike so much). For example, suppose that a starfleet battle-cruiser drops out of hyperspace in the orbital plane of a ringworld, traveling at a proper-velocity of 1[lightyear/traveler-year] radially away from the ringworld's star. An enemy cruiser drops out of hyperspace nearby at the same time, traveling 1[ly/ty] in the rotation-direction of the ringworld's orbit, and in a direction perpendicular to the starfleet cruiser's radial-trajectory. What is the proper-velocity (magnitude in [ly/ty] and direction) of the enemy cruiser relative to the starfleet ship? The vector solution to this is illustrated in the Fig. 2.

IV. PROPER ACCELERATION

The felt or proper-acceleration 3-vector $\vec{\alpha}$ is defined as the acceleration felt by accelerometers (broadly defined) that are moving with an object¹⁷ relative to the tangent free-float (or geodesic as a generalization of "inertial") frame. In flat spacetime its magnitude is the frame-invariant magnitude of the net-acceleration 4-

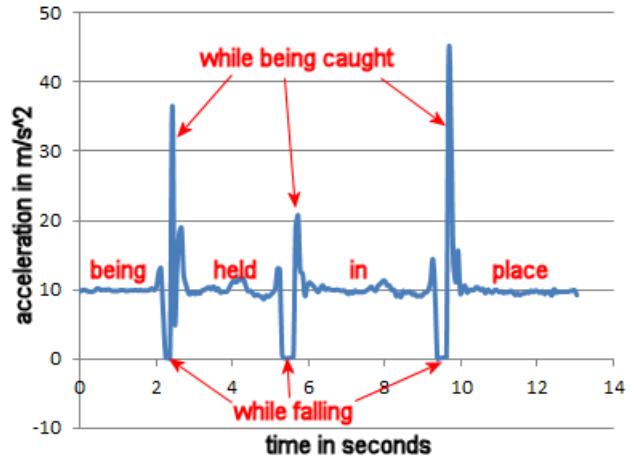


FIG. 3. Accelerometer data from a phone dropped and caught three times. During the free-fall segments, the accelerometer reading drops to zero because gravity is a geometric force (like centrifugal) which acts on every ounce of the phone's structure, and is hence not detected. The positive spikes occur when the fall is arrested as the falling phone is caught (also by hand) before it hit the floor.

vector $A^\lambda \equiv \frac{\delta U^\lambda}{\delta \tau} \equiv \frac{\delta^2 X^\lambda}{\delta \tau^2}$. This four-vector has a null time-component in the frame of the accelerated traveler, from whose "traveler-point" perspective its 3-vector direction is defined.

In flat spacetime, this 3-vector is simply related to the rate of proper-velocity change through $\vec{\alpha} = \frac{\delta \vec{w}}{\delta t} \Big|_{\vec{w}} + \gamma \frac{\delta \vec{w}}{\delta t} \Big|_{\perp \vec{w}}$, which in the unidirectional case reduces to $\vec{\alpha} = \frac{\delta \vec{w}}{\delta t}$. This link between proper acceleration and proper velocity will be important when we look at traveler-point dynamics in the next section.

To illustrate what happens in curved spacetime (and accelerated frames), try running an accelerometer app on your phone and performing a hold, drop, catch sequence as shown in Fig. 3. Because accelerometers only detect proper-accelerations, like that due to the upward force of your hand when holding the phone in place before the drop, nothing is detected when the phone is released even though the net acceleration 4-vector in that case has a vertically downward component of g .

Thus in curved spacetimes and accelerated frames, understanding motion requires a recognition of geometric (connection-coefficient) effects which thankfully (following Newton) can often be approximated locally as gravitational and inertial forces. The bottom line is that in flat spacetime motion is explained by the forces that cause proper acceleration, but as soon as we move to curved spacetimes and accelerated frames (even here on earth) a second kind of cause comes into play. As we'll see, these geometric effects act on every ounce of their target, vanish locally from the perspective of a comoving free-float frame, and are therefore undetected by on-board accelerometers.

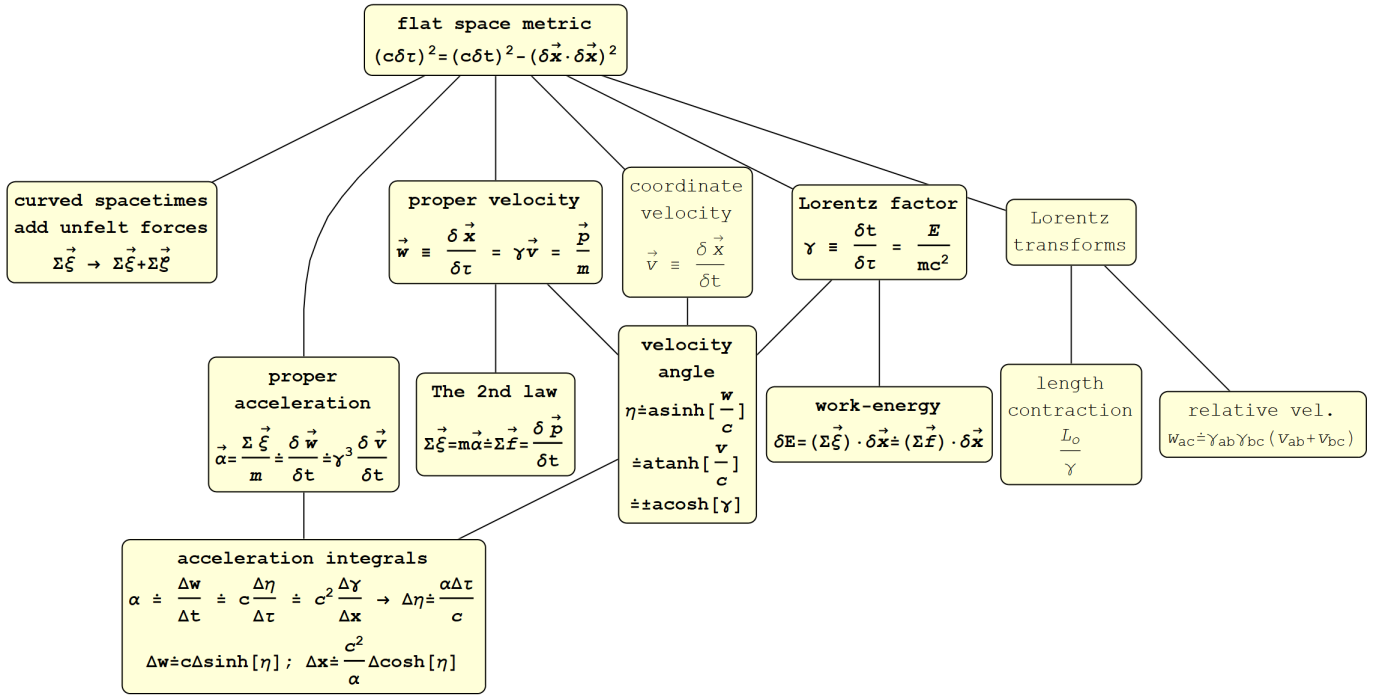


FIG. 4. Metric-1st equations of (1+1)D motion. The dot over the equals sign means this works only for unidirectional motion. The Lorentz transform section at far right is the only place which requires more than one map (or bookkeeper) reference frame of yardsticks and synchronized clocks.

Returning to flat spacetime for the moment, let's also consider the equations of constant proper-acceleration (cf. Fig. 4). In flat spacetime, it is proper and not coordinate acceleration that can be (and normally is) held constant.

If we stick to (1+1)D i.e. unidirectional motion, constant proper acceleration yields some wonderfully simple integrals of motion, namely $\alpha = \Delta w / \Delta t = c \Delta \eta / \Delta \tau = c^2 \Delta \gamma / \Delta x$, where “hyperbolic velocity angle” or rapidity $\eta = \sinh^{-1}[w/c] = \tanh^{-1}[v/c]$. These are approximated by the familiar intro-physics relationships $a = \Delta v / \Delta t = \frac{1}{2} \Delta(v^2) / \Delta x$ at low speed, but also allow beginning students to explore interstellar constant proper-acceleration round-trip problems almost as easily as one does low-speed constant acceleration problems.

In (3+1)D the integrals are also analytical, but messy and perhaps best expressed¹⁸ in terms of a characteristic time τ_o equal to $\sqrt{(\gamma_o + 1)/2c/\alpha}$, which connects to a hyperbolic velocity angle τ/τ_o . Here γ_o is the “aging factor at turnaround”, which for unidirectional motion yields the (1+1)D rapidity discussed above.

V. CAUSES OF MOTION

We've discussed a robust strategy for describing motion (kinematics) at any speed plus locally in curved spacetime & accelerated frames, using (3+1)D parameters which are either frame-invariant or synchrony free. The causes of motion (dynamics) on the other hand are

traditionally described by a connection between net 3-vector force as the cause, and the second time derivative of position where e.g. at low speed we normally write $\Sigma \vec{F} = m \delta^2 \vec{x} / \delta t^2$.

To extend this (3+1)-vector analysis to high speeds and curved spacetimes we simply follow the tradition already established with gravity, but add some new notation. For instance, we might denote the *proper forces* (with frame-invariant magnitudes which are felt by on-board accelerometers) with the greek letter xi ($\vec{\xi}$). For inertial frames in flat spacetime, then, $\Sigma \vec{\xi} \equiv m \vec{a} = m \delta^2 \vec{x} / \delta \tau^2$.

In the “local patch” of curved spacetimes and accelerated frames, we can often approximate the effects of metric connection coefficients on the equation of motion by imagining geometric forces which: (a) are unfelt by on-board accelerometers, (b) act on every ounce of an object's being, and (c) disappear when seen from the vantage point of a local “free-float frame”. We denote these *geometric forces* (like gravity and inertial forces such as centrifugal) with the greek letter zeta ($\vec{\zeta}$). Locally then we can write $\Sigma \vec{\xi} + \Sigma \vec{\zeta} \simeq m \delta^2 \vec{x} / \delta \tau^2$.

Although these force expressions work well for tracking map position versus time, energies, accelerometer readings and differential aging, etc., in the special case of high speed non-unidirectional motion a kinematic correction is needed when looking at momentum transfers (e.g. in collisions). This is because rates of proper velocity change (proportional to momentum change) are frame variant

even in flat spacetime when proper acceleration is not. As a result $\delta\vec{p}/\delta t$ is equal instead to a net sum of *apparent forces* $\Sigma\vec{f} = \delta\vec{p}/\delta t = m\delta\vec{w}/\delta t$, where $\vec{f} = \vec{\xi}_{\parallel\vec{w}} + (1/\gamma)\vec{\xi}_{\perp\vec{w}}$, depends on that object's proper velocity \vec{w} .

The idea that rates of momentum change depend on one's frame of reference may seem odd, but rates of energy change ($\delta E/\delta t = \Sigma\vec{\xi} \cdot \vec{v} \rightarrow \Sigma\vec{f} \cdot \vec{v}$) are frame-variant even at low speeds. This in spite of the fact that when I asked an AAPT audience in 1998 for a show of hands by those who thought that classical rates of an object's energy change are frame-variant, the only hand to go up in the audience was that of Edwin F. Taylor (that year's Oersted Medal winner).

As mentioned above, in curved spacetimes the move from 4-vector to local (3+1)D dynamics is normally done by the "geometric force" approximation, in which motion is seen as caused by adding geometric to proper forces. These relations are inspired by the general-relativistic 4-vector equation of motion, written in force-units as $mDU^\lambda/d\tau - m\Gamma_{\mu\nu}^\lambda U^\mu U^\nu = mDU^\lambda/d\tau$, which may be described as "net proper 4-vector force" + "net geometric 4-vector force" = mass times the "net 4-vector acceleration". As usual here m is frame-invariant mass, upper case D denotes the covariant derivative of 4-velocity component U^λ , $\Gamma_{\mu\nu}^\lambda$ denotes one of $4 \times 4 \times 4 = 64$ connection coefficients defined in terms of metric tensor derivatives, and repeated indices in products like $\Gamma_{\mu\nu}^\lambda U^\mu U^\nu$ are implicitly summed over all (spatial and temporal) values of those indices.

The traveler-point approach expresses this in terms of a locally-useful set of 3-vector and scalar relations, like $\Sigma\vec{\xi} + \Sigma\vec{\zeta} \simeq m\delta^2\vec{x}/\delta\tau^2$ and $\delta E \simeq (\Sigma\vec{\xi} + \Sigma\vec{\zeta}) \cdot \delta\vec{x}$. The scalar part connects to energy conservation, while the vector part connects to momentum conservation *locally* through the flat spacetime connection to frame-variant forces \vec{f} discussed above. Both the "geometric force" approximation which gives rise to the $\vec{\zeta}$ vectors, and the connection of "felt plus geometric" forces to rates of momentum change, rely on the general assertions: (a) that spacetime is "locally flat" even in curved spacetimes and accelerated frames, and (b) that the proper acceleration 4-vector is purely spacelike (i.e. a spatial 3-vector) from the perspective of the traveling object. The locality requirement, of course, may be more severe as spacetime curvature increases.

For example, if we imagine a stationary charge q held in place by some combination of electromagnetic forces in a Schwarzschild (non-spinning spherical mass) potential, the net proper force is the Lorentz 4-vector $qF_\beta^\alpha U^\beta = q\gamma\{\vec{E} \cdot \vec{v}/c, \vec{E} + \vec{v} \times \vec{B}\}$. In traveler-point form¹⁹ for charge q this becomes $\Sigma\vec{\xi} = q\vec{E}' = q(\vec{E}_{\parallel\vec{w}} + \gamma\vec{E}_{\perp\vec{w}} + \vec{w} \times \vec{B})$, where \vec{E} and \vec{B} are the bookkeeper-frame field vectors, and \vec{E}' is only the electric field seen by charge q .

The magnetic field \vec{B}' seen by the charge has no effect on its motion, so that in traveler-point terms such problems become purely electrostatic even in curved spacetime! As an aside in this context, the above relations may

be used to show how frame-variant proper forces can in general be decomposed into "static" and "kinetic" components, $\vec{f}_s = \gamma^3 m\vec{a} = \vec{\xi}_{\parallel\vec{w}} + \gamma\vec{\xi}_{\perp\vec{w}}$ and $\vec{f}_k = (\frac{1}{\gamma} - \gamma)\vec{\xi}_{\perp\vec{w}}$, which are analog to the electrostatic and magnetic forces (respectively) associated with electromagnetic fields.

The net geometric force, on the other hand, is obtained by summing the 13 out of 64 non-zero connection coefficients (8 of 40 with independent values) for a Schwarzschild object of mass M and "far-coordinate" radius r , to get $\Sigma\vec{\zeta} = -GMm\hat{r}/r^2$. If we wish to support the object in place so that net acceleration 3-vector \vec{a} is zero, then we require an E-field directed in the radial direction for which upward proper force $qE' = GMm/r^2$. One nice thing about discussing this force-balance in traveler-point terms is that all observers, those using far-time in a Schwarzschild potential as well as e.g. those passing by in a star ship accelerating at 1 gee, will be talking about the same thing in terms of time-intervals and forces experienced by the traveler, albeit with position coordinates of their own choosing.

The net proper-force that our cell-phones measure is frame-invariant in magnitude as is proper time, with a 3-vector direction presumably defined in traveler terms. This makes it useful²⁰ for comparing electromagnetic forces in different frames (e.g. to see how magnetic attraction becomes purely electrostatic), and makes curved spacetime (like that we experience on earth's surface as illustrated in Fig. 5) a bit simpler to understand than we might have imagined in a world where time passes differently according to your location as well as your rate of travel.

Thus Einstein's general relativity, far from invalidating Newton, revealed that the classical laws work *locally* in all frames (including accelerated-frames in curved-spacetime) provided that, in addition to proper-forces, we recognize geometric (i.e. connection-coefficient) forces like gravity and inertial-forces (acceleration "gees", centrifugal, etc.) that "act on every ounce" of an object's being.

These highly frame-dependent geometric forces also have effective potentials ΔU linked to the differential-aging (generalized Lorentz) factor $\gamma \equiv dt/d\tau$ (equal to $\sqrt{1/g_{00}}$ for motionless objects even when the metric is non-diagonal) via $L_\tau - L_t = (\gamma - 1)mc^2$ where Lagrangian $L \equiv K - U$, e.g. in rotating habitats, accelerating spaceships²¹, and gravity on earth⁸. Potential energies associated with proper forces instead involve delocalized interactions linked to the total rest mass-energy m of a composite system, and do not involve differential aging.

VI. DISCUSSION

The foregoing, summarized in Fig. 4, is targeted toward university physics teachers, but uses concepts which may be unfamiliar to some. In that context, in Appendix A we provide notes that may provide shortcuts for intro-physics "relativity sections", and we are work-

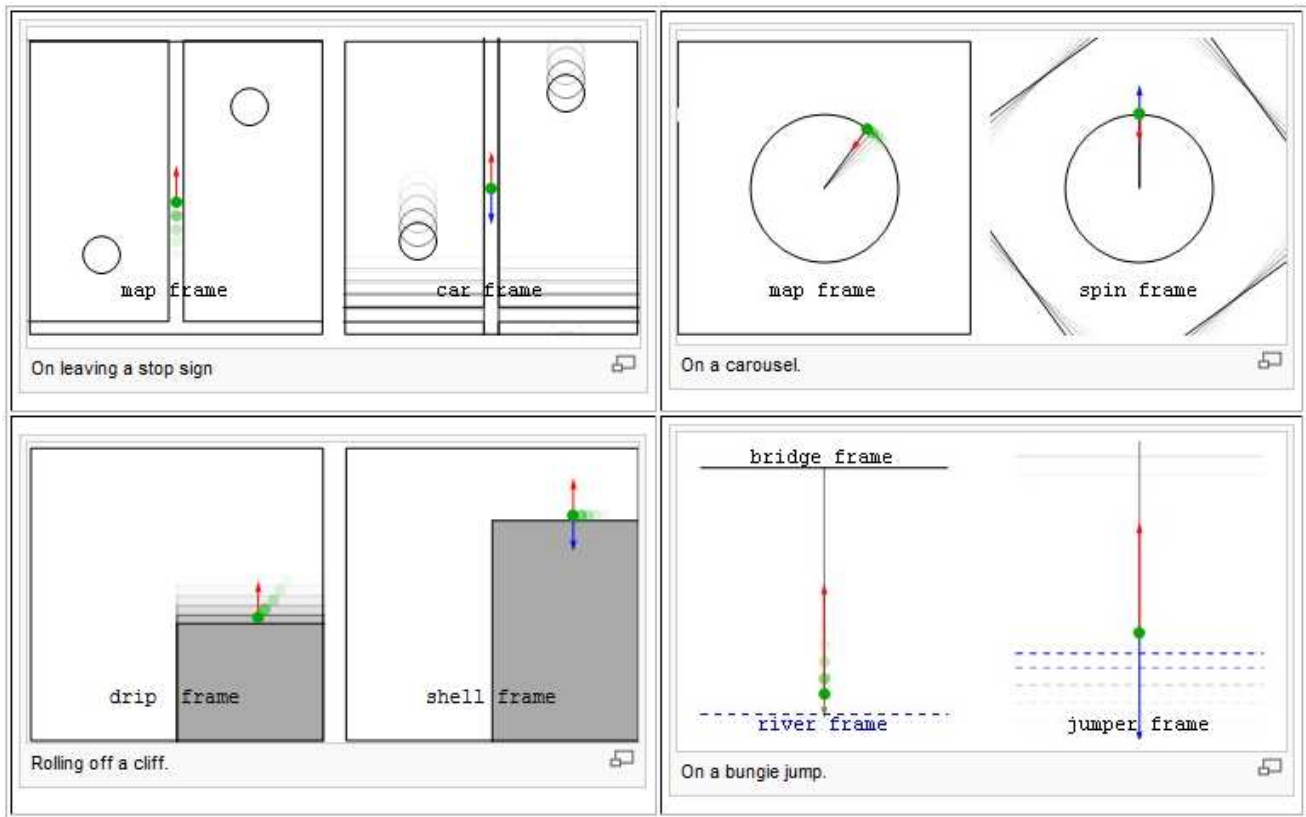


FIG. 5. Two views of proper (red) and geometric (dark blue or brown) forces in some everyday settings.

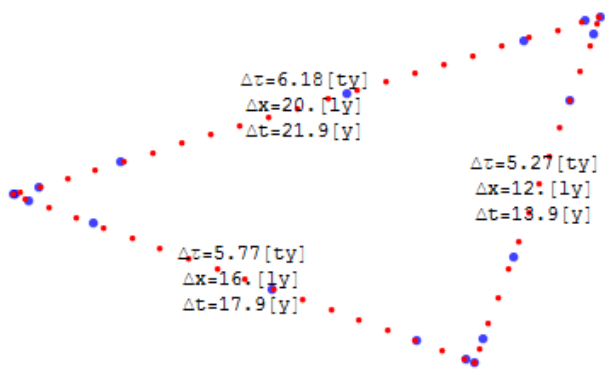


FIG. 6. Years elapsed in proper time (blue) and map time (red) on a $16+12+20 = 48$ lightyear 1-gee acceleration round trip lasting about 17.2 traveler years and 53.7 map years.

ing on a one-class intro to Newtonian variables as useful approximations²².

The traveler-point proper coordinate perspective on the importance of specifying “which clock”, and on the distinction between proper and geometric forces with help from your cell-phone, can be helpful for students at the very beginning of their engineering physics education. It also provides “Newtonian-like” tools for the 3-vector addition of proper-velocities and momenta, as well as for

proper-force/acceleration analysis at any speed using simultaneity defined by a free-float bookkeeper frame. In curved spacetimes the best bet for this reference frame may be the tangent free-float⁹ frame (or Fermi-normal⁸) coordinate system, like for instance the “shell frame” that we use for our inertial reference here on earth.

This traveler-point perspective perhaps does not do much for rigorous simplification of “geometric-force” analysis in curved spacetime, because e.g. the geodesic equation (i.e. without proper-forces) for arbitrary coordinate-systems is not reducible to purely 3-vector form. However, the electromagnetic example discussed above would make it quite straightforward e.g. to define a “gravitational proper-force” of the form $mG_{\mu\nu}^{\lambda}U^{\mu}U^{\nu}$. This proper-force approximation for including gravitomagnetic effects *at any speed* might, for example, come in handy for making 3-vector simulators of interstellar travel, e.g. using the celestiaproject.net database, which would work pretty well in the vicinity of main-sequence stars and planets, as well as between stars.

In that case, interesting extreme-gravity effects might have to be patched into the algorithms after the fact, e.g. with help from a mean-field estimate of gravitational $dt/d\tau$. Regardless of whether one uses geometric-force gravity²³ or its proper-force approximation, metric-first simulations of accelerated motion between and around stars using simultaneity defined by bookkeeper far-time,

but which are predicted on actions taken by a single traveler on their proper-time clock (cf. Fig. 6), will likely become quite robust in the days ahead.

In summary, traveler-point parameters can help open the door: (a) to introductory students wanting to explore the consequences of life in accelerated frames and curved spacetime here on earth, as well as (b) to engineers wanting to apply Newton’s laws locally in environments where high speeds and differential aging must be taken into account.

ACKNOWLEDGMENTS

Thanks to: Matt Wentzel-Long for his ideas, the late Bill Shurcliff (1909-2006) for his counsel on “minimally-variant” approaches, Phil Asquith for calling the Bridgman, Bondi & Brehme papers to our attention, plus D. G. Messerschmitt & D. Neustadter for their interest.

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Appendix A: possible course notes

The sections to follow might be dropped into the introductory physics schedule as the usual “relativity section”, near the end of the course, or may be rearranged for piecemeal discussion earlier so that the Newtonian models, that students will be working with, are framed as approximations from the start.

1. spacetime version of Pythagoras’ theorem

Time is local to a given clock, and simultaneity is determined by your choice of reference frame. Although Maxwell’s equations on electromagnetism were “informed” to this reality in the mid 1800’s, humans really didn’t start to get the picture until the early 1900’s. But how might one deal with this quantitatively?

Start with the (1+1)D flat-space **metric equation**, namely $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2$ where x and t are position and time coordinates associated with your reference “map-frame” of yardsticks and synchronized clocks. The quantity τ is the **proper-time** elapsed on the clocks of a traveling observer whose **map-position** x may be written as a function of **map-time** t . As usual c is the spacetime constant (literally the number of meters in a second) which is traditionally referred to as **lightspeed** because it equals the speed of light in a vacuum.

The term on the left in the metric equation is referred to as a **frame-invariant**. Just like a given hypotenuse (cf. Fig. 7) can be expressed in terms of a bunch of different xy coordinate systems, all of which agree on its length, so a given proper-time interval e.g. on a traveling

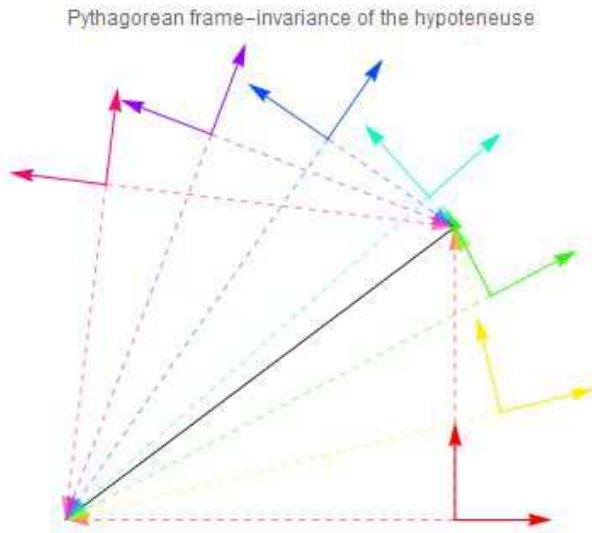


FIG. 7. Pythagorean frame-invariance of the hypotenuse (in black), from the metric equation $(\delta h)^2 = (\delta x)^2 + (\delta y)^2$ for seven different Cartesian coordinate systems, whose unit-vectors are shown in color. Although none of these coordinate systems agrees on the coordinates of the black line's endpoints, all agree that the length of the hypotenuse is 5.

object's clock, can be expressed in terms of many different "bookkeeper" reference-frames, all of which will also agree on its duration even if they can't agree on which of two spatially-separated events happened first.

This equation is seriously powerful. As Einstein illustrated, if one tweaks the "unit" coefficients of the terms on the right by only "one part per billion", we find ourselves in a gravitational field like that on earth where a fall of only a few meters can do you in.

2. traveler-point kinematics

A (2+1)D acceleration puzzler schematic of a starfleet battlecruiser traveling upward which, on seeing light from an enemy spaceship dropping out of hyperspace (dotted red circle), accelerates rightward in order to intercept. The enemy trajectory is in gray, while the starfleet cruiser trajectory is in black. Proper-time intervals are marked on both trajectories in 2 traveler-month intervals. Units for distance are lightyears, for proper-velocity vectors (green) are lightyears/traveler-year, and for proper-acceleration (red) is in lightyears/year² or (to first order) "gees". The battlecruiser then reverses acceleration direction in order to (eventually) recover its original trajectory, amazingly about four months of ship time ahead of schedule following a detour that took 4 years!

From the foregoing, it is easy to define a **proper-velocity** $\vec{w} \equiv d\vec{x}/d\tau = \gamma\vec{v}$, where $\vec{v} \equiv d\vec{x}/dt$ is **coordinate-velocity** as usual, and **speed of map-time** or "differential-aging factor" $\gamma \equiv dt/d\tau =$

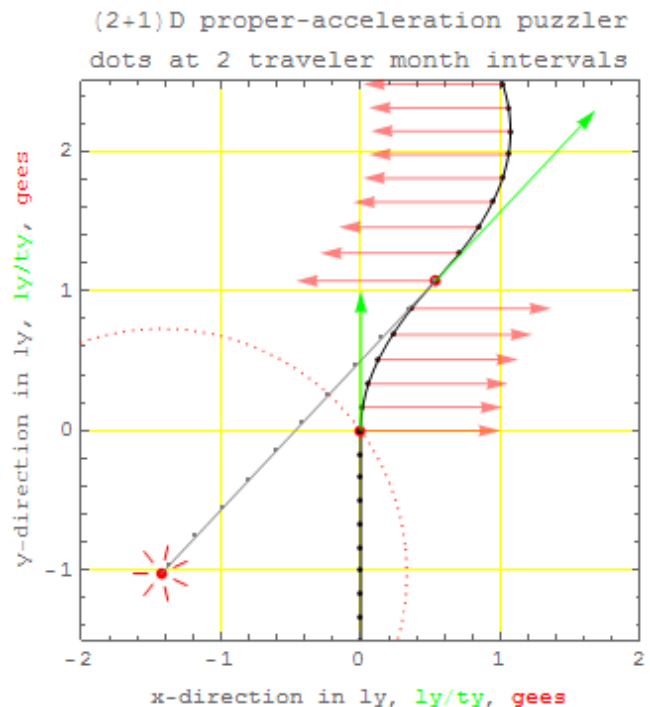


FIG. 8. A (2+1)D acceleration puzzler schematic of a starfleet battlecruiser traveling upward which, on seeing light from an enemy spaceship dropping out of hyperspace (dotted red circle), accelerates rightward in order to intercept. The enemy trajectory is in gray, while the starfleet cruiser trajectory is in black. Proper-time intervals are marked on both trajectories in 2 traveler-month intervals. Units for distance are lightyears, for proper-velocity vectors (green) are lightyears/traveler-year, and for proper-acceleration (red) is in lightyears/year. The battlecruiser then reverses acceleration direction in order to (eventually) recover its original trajectory, amazingly about four months of ship time **ahead** of schedule following a detour that took 4 years. A detour that saved time, at least for the traveler!

$\sqrt{1 + (w/c)^2} = 1/\sqrt{1 - (v/c)^2} \geq 1$. This last relation tells us that *when simultaneity is defined by a network of synchronized map clocks*, a moving traveler's clock will always run slow. These relationships follow directly from the metric equation itself.

Thus having a new time-variable τ also gives us some new ways to measure rate of travel. Proper-velocity, as we'll see, has no upper limit and is related to conserved-quantity **momentum** by the simple vector relation $\vec{p} = m\vec{w}$ where m is our traveler's frame-invariant rest-mass. The upper limit of $c \geq dx/dt$ on coordinate-velocity results simply from the fact that momentum (and $dx/d\tau$) have to be finite. Speed of map-time γ , on the other hand, relates to **total energy** by $E = \gamma mc^2$, and to **kinetic energy** by $K = (\gamma - 1)mc^2$.

At this point, a variety of familiar time-dilation and relativistic energy/momentum topics might be covered as examples. If there is added time, one path to take is to introduce length-contraction, velocity addition, and

Doppler effect with or without Lorentz transforms, historical notes on lightspeed measurement, etc. In what follows, we instead push the alignment with traveler-point concepts and the Newtonian treatment of kinematics and mechanics a bit further.

It is also conceptually interesting to note that **proper-acceleration**, in turn, is simply the vector-acceleration detected by a *cell-phone accelerometer* in the traveler’s pocket. This quantity, after a couple of proper-time derivatives, pops up on the left side of the metric equation as a frame-invariant as did “hypotenuse” and proper-time.

For unidirectional motion in flat-spacetime (i.e. using map-coordinates in an inertial frame), proper-acceleration $\vec{\alpha}_o = \gamma^3 \vec{a}$, where $\vec{a} \equiv d^2\vec{x}/dt^2$ is the usual **coordinate-acceleration**. These relations also yield a few simple integrals for “constant” proper-acceleration, namely $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$.

That 2nd equality involves “hyperbolic velocity-angle” or **rapidity** $\eta = \sinh^{-1}[w/c] = \tanh^{-1}[v/c]$, so that $\gamma = \cosh[\eta]$. The first and third equalities reduce to the familiar conceptual-physics relationships $a = \Delta v/\Delta t = \frac{1}{2}\Delta(v^2)/\Delta x$ at low speed. However they allow beginning students to explore interstellar constant proper-acceleration round-trip problems, almost as easily as they do problems involving projectile trajectories on earth.

3. dynamics in flat spacetime

The net proper-force may in general be written as $\Sigma\vec{F}_o \equiv m\vec{\alpha}_o$. In flat (and unaccelerated) spacetime coordinate-systems, all forces are proper, and proper-acceleration $\vec{\alpha}_o$ equals the **net-acceleration** $\vec{\alpha}$ observed *by a traveler*. Under a constant net proper-force, we can therefore expect constant net-acceleration.

At high speeds the constant proper-force equations are messier because of that pesky γ in equations like $\Delta w = \Delta(\gamma v) = \alpha_o \Delta t$. The low speed approximation (namely $\Delta v = a\Delta t$) is therefore a bit simpler to deal with, and works fine for speeds well below c .

4. dynamics in curved spacetime

In both curved spacetime *and in accelerated frames*, Newton’s equations still work (at least locally) provided that we recognize the existence of non-proper or geometric forces, like gravity as well as inertial forces that arise in accelerated frames. Newton’s law for causes of motion “in the neighborhood of a traveling object” then takes the 3-vector form $\Sigma\vec{F}_o + \Sigma\vec{F}_g = m\vec{\alpha}$, where $\vec{\alpha}$ is the net-acceleration actually observed by the traveler.

The reason that your cell-phone’s accelerometer can’t see gravity (or centrifugal force), even when gravity is causing a net-acceleration downward, is that these are geometric forces. Geometric forces ($\Sigma\vec{F}_g$), which act on “every ounce” of an object, result from being in an accelerated frame or in curved spacetime. Accelerometers can only detect the result of net proper-forces ($\Sigma\vec{F}_o$), i.e. one’s proper-acceleration $\vec{\alpha}_o$.

It’s traditional to approximate gravity on the surface of the earth as simply a proper-force that is proportional to mass m , for which $\vec{F} = m\vec{g}$ where \vec{g} is a downward vector with a magnitude of about 9.8 [m/s²]. This works quite well for most applications. However unlike geometric-forces, proper-forces are not associated with positional time-dilation like that which must be figured into GPS system calculations.

Just as kinetic energy in flat spacetime is related via $(dt/d\tau - 1)mc^2$ to the faster passage of map-time (t) with respect to traveler time (τ) when simultaneity is defined by the map-frame, so is the *position-dependent* potential-energy well-depth of some geometric-forces in accelerating frames and curved spacetime. Thus $(dt/d\tau - 1)mc^2$ also describes the potential-energy well-depth for mass- m travelers located: (i) at radius r from the axis in a habitat rotating with angular velocity ω , which in the low-speed limit reduces to the classical value $\frac{1}{2}m\omega^2 r^2$; (ii) a distance L behind the leading edge of a spaceship undergoing constant proper acceleration α , which in the small L limit reduces to the classical value $m\alpha L$; and (iii) on the radius R surface of a mass M planet, which in the $R \gg 2GM/c^2$ limit reduces to the classical value of GMm/R where G is the gravitational constant.