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| Phil Fraundorf. Traveler-point dynamics. 2017. hal-01503971v2

**HAL Id: hal-01503971**

**<https://hal.science/hal-01503971v2>**

Preprint submitted on 17 Aug 2017 (v2), last revised 29 Mar 2023 (v5)

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# Traveler-point dynamics

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(Dated: August 17, 2017)

Locally-defined parameters chosen to be minimally frame-variant can be useful for describing motion in accelerated frames and in curved spacetimes. In particular the metric-equation's synchrony-free "traveler-point parameters", namely proper-time, plus 3-vectors proper-velocity and proper-force, are useful in curved spacetime because extended arrays of synchronized clocks (e.g. for local measurement of the denominator in  $\Delta x/\Delta t$ ) may be hard to find. Combined with a recognition of improper (geometric) forces, these same parameters can better prepare intro-physics students for their everyday world, as well as for the technological world e.g. of GPS systems where differential aging must be considered explicitly. The approach also suggests a traveler-perspective path to 3-vector simulations that are exact at any speed in flat spacetime, and will soon help frame on-line simulations of interstellar navigation including effects of gravitational fields.

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## I. INTRODUCTION

In an imagined BBC interview, a friendly electric charge was quoted as saying that: "I see magnetic fields interacting with charges moving around me all the time, but according to the phone accelerometer in my pocket, those magnetic fields *never* have any effect on me!" In this paper we show that our friendly electron's assertion makes sense! More importantly we show that the purely-electrostatic proper force felt by a traveling charge, and other "traveler-point parameters", serve as a common point of reference (just as does proper time along a world line) for discussion of object-trajectories by observers in all kinds of curved-spacetime situations, including the curved spacetime in which we live on earth.

In a larger context, general-relativity revealed a century ago why we can get by with using Newton's laws in spacetime so curvy that it's tough to jump higher

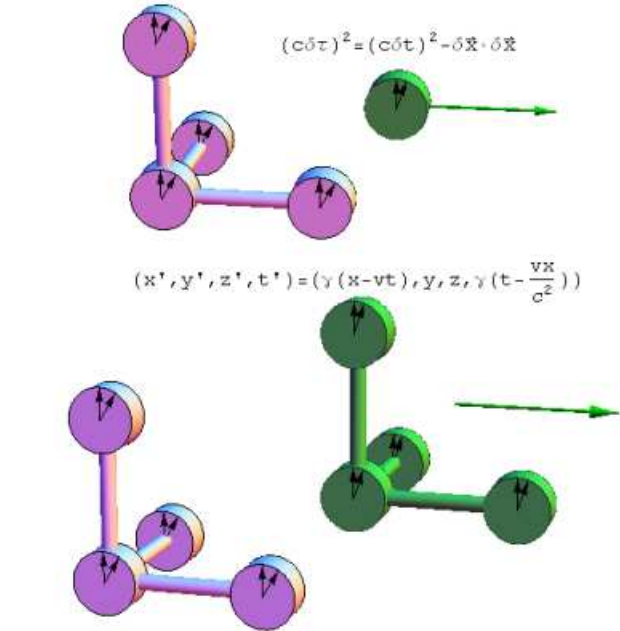


FIG. 1. Even one extended-frame with synchronized clocks may be more than we have in curved spacetime.

than a meter. It's because those laws work locally in all frames of reference<sup>1</sup>, provided we recognize that motion is generally affected by both proper and "geometric" (i.e. connection-coefficient) forces.

In that context, it's probably time to give introductory students the good news. Here we discuss a way to do this without telling them to measure time and distance in kilograms (for example), and without asking them to juggle more than one concurrent definition of simultaneity.

The metric equation of course avoids these things (cf. Fig. 1) by specifying locally-defined frame invariants like proper-time, and contenting itself with a single definition of simultaneity i.e. that associated with the set of book-keeper (or map) coordinates that one chooses to describe the spacetime metric. Hence the focus here is on a metric-

first, as distinct from transform-first, strategy for describing motion in accelerated frames, in curved spacetime, as well as at high speeds.

To facilitate this, the (proper<sup>1</sup> or Riemann-normal<sup>2</sup> coordinate-frame) variables discussed here are chosen to be minimally frame-variant, synchrony-free<sup>3</sup>, and inherently local so as to maximize their utility in accelerated frames and in curved spacetime. They are referred to as elements of a *traveler-point*, rather than a traveler-frame, dynamic in part to clarify that they involve distances with respect to the chosen map or bookkeeper frame, but that they are indexed to frame-invariant proper-time local to the traveler’s clock. Their lack of dependence on synchronized clocks arises because they involve neither distributed-clocks synchronized to the “map time” variable used in the metric-equation, nor an extended (e.g. tangent free-float) frame of yardsticks and clocks linked to a traveling object.

## II. PROPER TIME

Proper-time is the time elapsed locally along the world-line of any physical clock (broadly defined). It is also the frame-invariant time-interval  $\delta\tau$  that shows up in metric equations of the form  $(c\delta\tau)^2 = \Sigma_\mu \Sigma_\nu g_{\mu\nu} \delta X_\mu \delta X_\nu$ , which in the flat-space Cartesian case takes the Pythagorean form  $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2 - (\delta y)^2 - (\delta z)^2$ , where bookkeeper or map coordinates are found on the right-hand side of the equation, and  $c$  is the “lightspeed” spacetime constant for converting e.g. seconds into meters.

When simultaneity is defined by a flat-spacetime bookkeeper coordinate frame, time-elapsing on traveling clocks is always “dilated” (e.g. spread out), since from the metric equation the differential-aging or Lorentz factor  $\gamma \equiv dt/d\tau \geq 1$ . Expressions for this in terms of velocity will be discussed below.

We won’t discuss the utility of proper-time  $\tau$  in detail here because it is widely used now even by introductory textbook authors<sup>4</sup>. However it’s worth pointing out that early treatments of special relativity were so focused on the equivalence between frames, each with their own extended set of synchronized clocks, that proper-time (if discussed<sup>5</sup>) was generally a late-comer to the discussion, and of course used only for “rest frame” (i.e. unaccelerated) travelers<sup>6</sup>.

Moving yardsticks also have a “proper-length” associated with them, which can be useful too. However this is a “non-local” quantity not defined by the metric equation, the concept of “rigid body” is only an approximation in spacetime, and length contraction itself involves three separate events as distinct from the two involved in time-dilation. Hence we don’t include proper-length in a collection of those traveler-point parameters which show promise of wide-ranging usefulness in accelerated frames and curved spacetime.

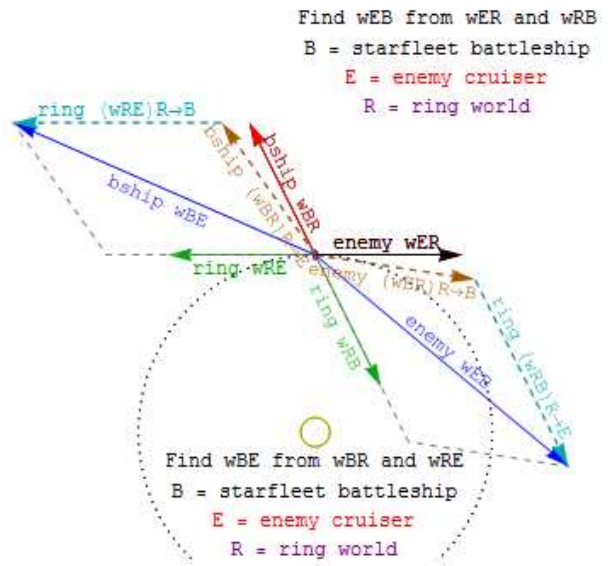


FIG. 2. Proper-velocity (momentum/mass) vector addition for a sci-fi puzzler, in which your starfleet battle-cruiser B drops in from hyperspace heading away from a nearby star R, only to find the enemy ship E heading off in another direction. Note that the blue resultants (proper-velocities of enemy relative to battleship and vice-versa) are “precessed” relative to the simple vector sum since B and E use different definitions of simultaneity.

## III. PROPER VELOCITY

Proper-velocity<sup>7</sup>  $\vec{w} \equiv d\vec{x}/d\tau$  is the rate at which bookkeeper or map 3-vector position  $\vec{x}$  is changing per unit-time elapsed locally on traveler clocks. Because it is proportional to momentum (i.e.  $\vec{p} = m\vec{w}$ ), unlike coordinate velocity  $\vec{v} \equiv d\vec{x}/dt$  it has no upper limit. Also unlike coordinate velocity, its measurement does not depend on map-clock readings (synchronized or not) along that world line. Proper-velocity is also represented by the space-like components of a traveling object’s velocity 4-vector.

An interesting thought problem for future engineers (as well as teachers) might be to ask if speed-limit signs, in a “Mr. Tompkins” world<sup>8</sup> with a much slower light-speed spacetime constant, would use coordinate-speed or proper-speed. Four criteria to consider might involve the measure’s connection to momentum, kinetic-energy, and to reaction-times (after the “warning photons” arrive) for both the driver, and a pedestrian who is tempted to cross the street.

The time-like component of that 4-vector is  $c\gamma$ , where  $\gamma \equiv dt/d\tau$  is the also-useful (but not synchrony free) Lorentz differential-aging or “speed of map-time” factor mentioned above. In flat spacetime, from the metric equation it is easy to show that  $\gamma = w/v = \sqrt{1 + (w/c)^2} \geq 1$ .

In gravitationally-curved Schwarzschild spacetime

with “far-coordinate” map distances and times, one gets instead from the metric for non-radial proper-velocity  $w_\phi$  that  $\gamma \equiv dt/d\tau = \gamma_r \sqrt{1 + (w_\phi/c)^2} \geq 1$ , where the gravitational time-dilation factor is  $\gamma_r \equiv 1/\sqrt{1 - (r_s/r)} \geq 1$  and  $r \geq r_s$  is distance from the center of a mass  $M$  object whose Schwarzschild (event-horizon) radius  $r_s = 2GM/c^2$ . Global positioning system satellites have to consider both of these factors in compensating for the rate of time’s passage in their orbits around earth.

For unidirectional velocity addition, the advantage of a collider over an accelerator is easily seen because  $w_{AC} = \gamma_{AB}\gamma_{BC}(v_{AB} + v_{BC})$ . In other words, the Lorentz-factors multiply even though the coordinate-velocities (which add) never exceed  $c$ . Proper velocity’s direct connection to momentum makes it more relevant to speed-limits e.g. in worlds with a reduced value for lightspeed, and its direct connection to traveler time makes it more relevant to passengers and crew when planning long high-speed trips.

More generally, unlike coordinate-velocities, proper velocities as “momenta per unit mass” add as 3-vectors, in analogy to the classical prescription  $\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$ , provided that one reframes  $\vec{v}_{AB}$  in terms of the sum velocity’s metric ( $B \rightarrow C$ ) and  $\vec{v}_{BC}$  in terms of the sum velocity’s clock ( $B \rightarrow A$ ). In other words  $\vec{w}_{AC} = \vec{w}_{A,B:C} + \vec{w}_{B:A,C}$ . Here the second term is increased only in magnitude to correct for differences between B and A’s clock i.e.  $\vec{w}_{B:A,C} = \gamma_{AB}\vec{w}_{BC}$ , while the first term increases in magnitude and is “Thomas precessed” toward the second because of the shift from B to C’s map-frame and thus metric distance and simultaneity i.e.  $\vec{w}_{A,B:C} = \vec{w}_{AB\perp}\vec{w}_{BC} + \gamma_{BC}\vec{w}_{AB\parallel}\vec{w}_{BC}$ . Hence vector diagrams, and even the original low-speed equation for velocity-addition (almost), survive intact when proper-velocity is used. Because proper-velocities are synchrony free, these relationships should hold up *locally* in curved spacetime and accelerated frames as well.

This would open the door to new settings for those relative motion problems (that students dislike so much). For example, suppose that a starfleet battle-cruiser drops out of hyperspace in the orbital plane of a ringworld, traveling at a proper-velocity of 1 [lightyear/traveler-year] obliquely away from the ringworld’s star. An enemy cruiser drops out of hyperspace nearby at the same time, traveling 1 [ly/ty] in the rotation-direction of the ringworld’s orbit. What is the proper-velocity (magnitude in [ly/ty] and direction) of the enemy cruiser relative to the starfleet ship, or vice-versa? The vector solution to this is illustrated in the Fig. 2.

#### IV. PROPER ACCELERATION

The proper-acceleration 3-vector  $\vec{\alpha}_o$  is the acceleration felt by accelerometers (broadly defined) that are moving with an object<sup>9</sup> relative to the tangent free-float (or geodesic as a generalization of “inertial”) frame. Acceleration in curved space time with respect to bookkeeper

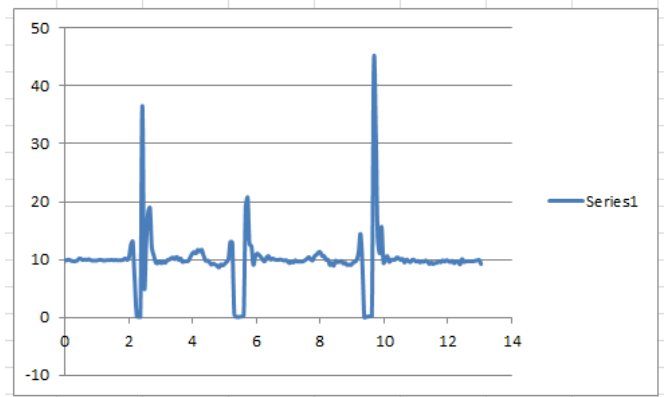


FIG. 3. Accelerometer data from a phone dropped and caught three times. During the free-fall segments, the accelerometer reading drops to zero because gravity is a geometric force (like centrifugal) which acts on every ounce of the phone’s structure, and is hence not detected. The positive spikes occur when the fall is arrested as the falling phone is caught (also by hand) before it hit the floor.

coordinates is instead described by the net-acceleration 4-vector  $\{cd\gamma/d\tau, \vec{\alpha}\} \equiv dU^\lambda/d\tau$ . Both of these 4-vectors have frame-invariant magnitudes.

To illustrate the difference in curved spacetime, try running an accelerometer app on your phone and performing a hold, drop, catch sequence as shown in Fig. 3. Because accelerometers only detect (frame-invariant) net proper-forces  $\Sigma \vec{F}_o \equiv m\vec{\alpha}_o$ , like the upward force of your hand when holding the phone in place before the drop, nothing is detected when the phone is released even through the net acceleration 4-vector in that case has a vertically downward component of  $g$ .

The proper-acceleration 4-vector also has a null time-component from the traveler’s point of view, and hence becomes a spatial 3-vector with directions that are defined only locally in the frame of the traveling object (much as is the magnitude of the proper-time elapsed on traveling object clocks). In flat spacetime, the two 4-vectors are one and the same.

In flat spacetime, coordinate-acceleration  $\vec{a} \equiv d^2\vec{x}/dt^2 = \vec{\alpha}_{\parallel\vec{w}}/\gamma^3 + \vec{\alpha}_{\perp\vec{w}}/\gamma^2$  scales differently with proper-acceleration components parallel and perpendicular to the direction of our traveling object’s proper-velocity  $\vec{w}$ . If we stick to (1+1)D i.e. unidirectional motion, this yields some wonderfully simple integrals for constant proper-acceleration, namely  $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$ , where “hyperbolic velocity angle” or rapidity  $\eta = \sinh^{-1}[w/c] = \tanh^{-1}[v/c]$ .

These reduce to the familiar conceptual-physics relationships  $a = \Delta v/\Delta t = \frac{1}{2}\Delta(v^2)/\Delta x$  at low speed. However they allow beginning students to explore interstellar constant proper-acceleration round-trip problems, almost as easily as one does low-speed constant acceleration problems.

## V. PROPER FORCE

The general-relativistic equation of motion may be written in force-units as  $mDU^\lambda/d\tau - m\Gamma_{\mu\nu}^\lambda U^\mu U^\nu = mdU^\lambda/d\tau$  in the form “net proper 4-vector force” + “net geometric 4-vector force” = mass times the “net 4-vector acceleration”. Notation is standard in that  $m$  is frame-invariant mass, upper case  $D$  denotes the covariant derivative of 4-velocity component  $U^\lambda$ ,  $\Gamma_{\mu\nu}^\lambda$  denotes one of  $4 \times 4 \times 4 = 64$  connection coefficients defined in terms of metric tensor derivatives, and repeated indices in products like  $\Gamma_{\mu\nu}^\lambda U^\mu U^\nu$  are implicitly summed over all (spatial and temporal) values of those indices.

The foregoing is a 4-vector equation, which in “frame-invariant” traveler-point terms (i.e. when projected into 3-space along the traveler’s time direction) may be reduced to the locally-valid 3-vector equation  $\Sigma\vec{F}_o + \Sigma\vec{F}_g = m\vec{a}$ . The first term corresponds to the net proper force (equal to  $m\vec{a}_o$ ) detectable by “on-board” accelerometers, the second to the spatial part of the net geometric force which might be locally made to vanish by choice of a “free-float” frame, while the last corresponds to the spatial part of the net acceleration 4-vector.

For example, if we imagine a stationary charge  $q$  held in place by some combination of electromagnetic forces in a Schwarzschild (non-spinning spherical mass) potential, the net proper force is the Lorentz 4-vector  $qF_\beta^\lambda U^\beta = q\gamma\{\vec{E} \cdot \vec{v}/c, \vec{E} + \vec{v} \times \vec{B}\}$ , which in traveler-point form<sup>10</sup> for charge  $q$  becomes  $\Sigma\vec{F}_o = q\vec{E}' = q(\vec{E}_{\parallel\vec{w}} + \gamma\vec{E}_{\perp\vec{w}} + \vec{w} \times \vec{B})$ , where  $\vec{E}$  and  $\vec{B}$  are the bookkeeper-frame field vectors, and  $\vec{E}'$  is only the electric field seen by charge  $q$ . The magnetic field  $\vec{B}'$  seen by the charge indeed has no effect on its motion, so that in traveler-point terms such problems become electrostatic even in curved spacetime!

The net geometric force, on the other hand, is obtained by summing the 13 out of 64 non-zero connection coefficients (8 of 40 with independent values) for a Schwarzschild object of mass  $M$  and “far-coordinate” radius  $r$ , to get  $\Sigma\vec{F}_g = -GMm\hat{r}/r^2$ . If we wish to support the object in place so that net acceleration 3-vector  $\vec{a}$  is zero, then we require an E-field directed in the radial direction for which upward proper force  $qE' = GMm/r^2$ . One nice thing about discussing this force-balance in traveler-point terms is that all observers, those using far-time in a Schwarzschild potential as well as e.g. those passing by in a star ship accelerating at 1 gee, will be talking about the same thing in terms of time-intervals and forces experienced by the traveler, albeit with position coordinates of their own choosing.

The net proper-force  $\Sigma\vec{F}_o$  that our cell-phones measure (which is generally NOT a rate of momentum change) is frame-invariant in magnitude as is proper time, with a 3-vector direction presumably defined in traveler terms. This makes it useful<sup>11</sup> for comparing electromagnetic forces in different frames (e.g. to see how magnetic attraction becomes purely electrostatic), and makes curved

spacetime (like that we experience on earth’s surface as illustrated in Fig. 4) a bit simpler to understand than we might have imagined in a world where time passes differently according to your location as well as your rate of travel.

Thus Einstein’s general relativity, far from invalidating Newton, revealed that the classical laws work *locally* in all frames (including accelerated-frames in curved-spacetime) provided that, in addition to proper-forces, we recognize geometric (i.e. connection-coefficient) forces like gravity and inertial-forces (acceleration “gees”, centrifugal, etc.) that “act on every ounce” of an object’s being. These highly frame-dependent geometric forces (esp. when the metric is diagonal) link to differential-aging ( $\gamma \equiv dt/d\tau$ ) with “well-depth” as well as kinetic energies (e.g. in rotating habitats, accelerating spaceships<sup>12</sup>, and gravity on earth<sup>2</sup>) that go something like  $(\gamma - 1)mc^2$ .

To give you a taste of the other traveler-point variables in classic terms, imagine a traveler with book-keeper coordinates  $\vec{x}[t]$  from the vantage point of a “free-float” or inertial frame in flat spacetime with no geometric forces, so that the equation of motion predicts that the net proper-force  $\Sigma\vec{F}_o$  alone will equal  $m\vec{a}$ . Then differential-aging factor  $\gamma \equiv dt/d\tau = \sqrt{1 + (w/c)^2}$ , proper-velocity  $\vec{w} \equiv d\vec{x}/d\tau = \gamma\vec{v}$  where coordinate-velocity  $\vec{v} \equiv d\vec{x}/dt$ , kinetic energy  $K = (\gamma - 1)mc^2$ , momentum  $\vec{p} = m\vec{w}$ , rate of energy change  $dK/d\tau = m\vec{a} \cdot \vec{w}$ , and the rate of momentum change is  $d\vec{p}/d\tau = m\vec{a} + (\gamma - 1)m\vec{a}_{\parallel\vec{w}}$ . As Tony French suggested in his classic text, coordinate-acceleration  $\vec{a} \equiv d^2\vec{x}/dt^2$  is simply related to neither energy nor momentum, and so mainly serves to approximate the net-acceleration  $\vec{a}$  at low speeds (e.g. as measured by your phone in the absence of geometric forces).

## VI. DISCUSSION

The foregoing is targeted toward university physics teachers, but needless to say uses concepts which may be unfamiliar to some. In that context, in Appendix A we outline an introduction to this material that might be useful for introductory students as well as teachers. In particular, we address there possibilities for an “after-the-fact” traveler-point relativity section, as well as some ways this material might be introduced from the very beginning.

The traveler-point proper coordinate perspective on the importance of specifying “which clock”, and on the distinction between proper and geometric forces with help from your cell-phone, can be helpful for students at the very beginning of their engineering physics education. It also provides “Newtonian-like” tools for the 3-vector addition of proper-velocities and momenta, as well as for proper-force/acceleration analysis at any speed using simultaneity defined by a free-float bookkeeper frame. In curved spacetimes the best bet for this reference frame may be the tangent free-float<sup>13</sup> frame (or Fermi-normal<sup>2</sup>) coordinate system, like for instance the “shell frame” that

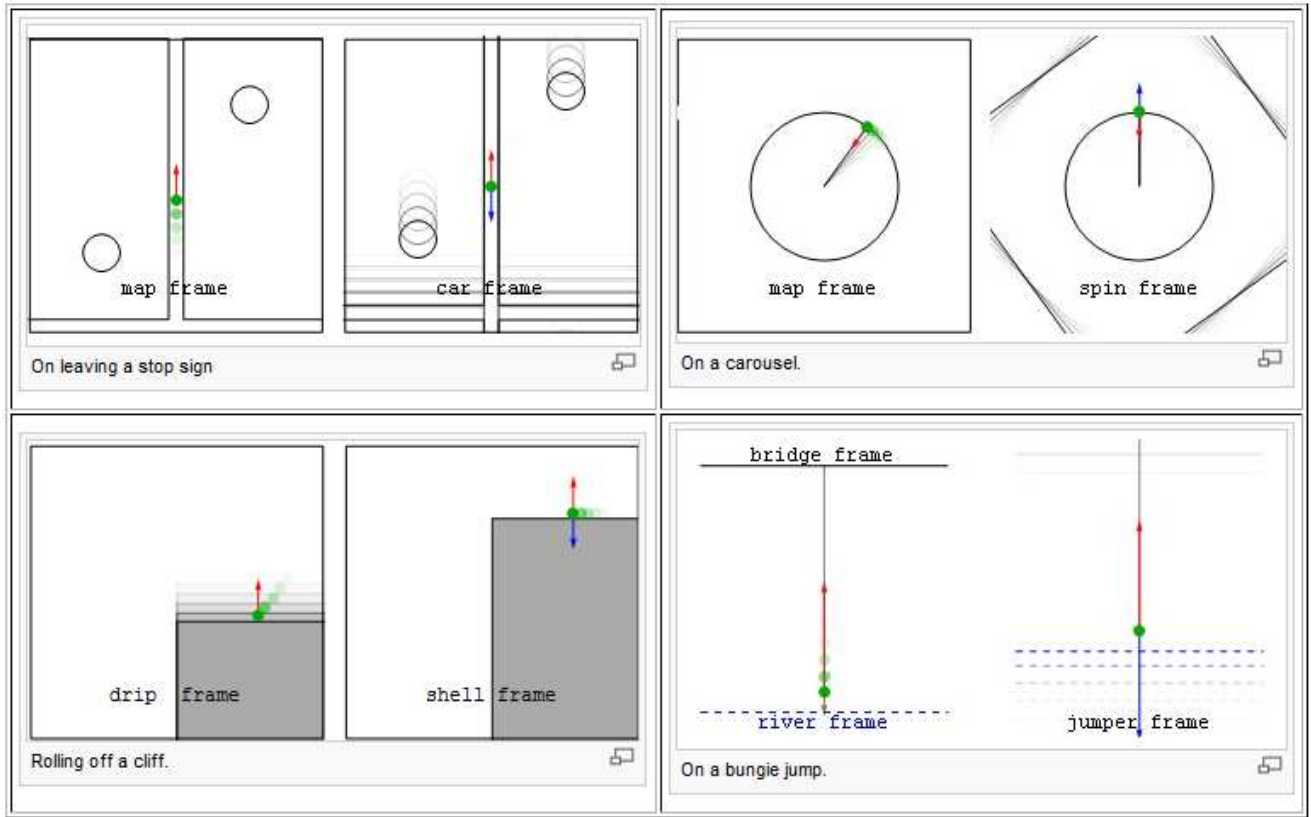


FIG. 4. Two views of proper (red) and geometric (dark blue or brown) forces in some everyday settings.

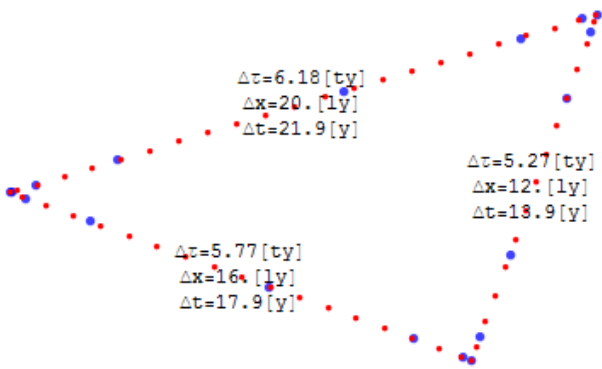


FIG. 5. Years elapsed in proper time (blue) and map time (red) on a  $16+12+20 = 48$  lightyear 1-g acceleration round trip lasting about 17.2 traveler years and 53.7 map years.

we use for our inertial reference here on earth.

This traveler-point perspective perhaps does not do much for rigorous simplification of “geometric-force” analysis in curved spacetime, because e.g. the geodesic equation (i.e. without proper-forces) for arbitrary coordinate-systems is not reducible to purely 3-vector form. However, the electromagnetic example discussed above would make it quite straightforward e.g. to define a “gravitational proper-force” of the form  $mG_{\mu\nu}^{\lambda}U^{\mu}U^{\nu}$ .

This proper-force approximation for including gravito-magnetic effects *at any speed* might, for example, come in handy for making 3-vector simulators of interstellar travel, e.g. using the [celestiaproject.net](http://celestiaproject.net) database, which would work pretty well in the vicinity of main-sequence stars and planets, as well as between stars.

In that case, interesting extreme-gravity effects might have to be patched into the algorithms after the fact, e.g. with help from a mean-field estimate of gravitational  $dt/d\tau$ . Regardless of whether one uses geometric-force gravity<sup>14</sup> or its proper-force approximation, metric-first simulations of accelerated motion between and around stars using simultaneity defined by bookkeeper far-time, but which are predicted on actions taken by a single traveler on their proper-time clock (cf. Fig. 5), will likely become quite robust in the days ahead.

In summary, traveler-point parameters can help open the door: (a) to introductory students wanting to explore the consequences of life in accelerated frames and curved spacetime here on earth, as well as (b) to engineers wanting to apply Newton’s laws locally in environments where high speeds and differential aging must be taken into account.

## ACKNOWLEDGMENTS

Thanks to the late Bill Shurcliff (1909-2006), David G. Messerschmitt, and David Neustadter for helpful in-

sights on minimizing frame-variance, on acceleration scaling relations<sup>15</sup>, and on velocity addition algebra<sup>16</sup>.

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## Appendix A: possible course notes

The sections to follow might be dropped into the introductory physics schedule as the usual “relativity section”, near the end of the course, or may be rearranged for piecemeal discussion earlier so that the Newtonian models, that students will be working with, are framed as approximations from the start.

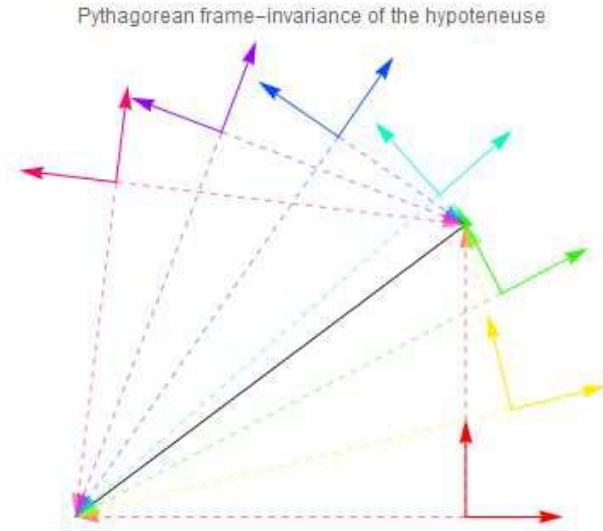


FIG. 6. Pythagorean frame-invariance of the hypotenuse (in black), from the metric equation  $(\delta h)^2 = (\delta x)^2 + (\delta y)^2$  for seven different Cartesian coordinate systems, whose unit-vectors are shown in color. Although none of these coordinate systems agrees on the coordinates of the black line’s endpoints, all agree that the length of the hypotenuse is 5.

### 1. spacetime version of Pythagoras’ theorem

Time is local to a given clock, and simultaneity is determined by your choice of reference frame. Although Maxwell’s equations on electromagnetism were “informed” to this reality in the mid 1800’s, humans really didn’t start to get the picture until the early 1900’s. But how might one deal with this quantitatively?

Start with the (1+1)D flat-space **metric equation**, namely  $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2$  where  $x$  and  $t$  are position and time coordinates associated with your reference “map-frame” of yardsticks and synchronized clocks. The quantity  $\tau$  is the **proper-time** elapsed on the clocks of a traveling observer whose **map-position**  $x$  may be written as a function of **map-time**  $t$ . As usual  $c$  is the spacetime constant (literally the number of meters in a second) which is traditionally referred to as **lightspeed** because it equals the speed of light in a vacuum.

The term on the left in the metric equation is referred to as a **frame-invariant**. Just like a given hypotenuse (cf. Fig. 6) can be expressed in terms of a bunch of different xy coordinate systems, all of which agree on its length, so a given proper-time interval e.g. on a traveling

object’s clock, can be expressed in terms of many different “bookkeeper” reference-frames, all of which will also agree on its duration even if they can’t agree on which of two spatially-separated events happened first.

This equation is seriously powerful. As Einstein illustrated, if one tweaks the “unit” coefficients of the terms on the right by only “one part per billion”, we find ourselves in a gravitational field like that on earth where a fall of only a few meters can do you in.

## 2. traveler-point kinematics

From the foregoing, it is easy to define a **proper-velocity**  $\vec{w} \equiv d\vec{x}/d\tau = \gamma\vec{v}$ , where  $\vec{v} \equiv d\vec{x}/dt$  is **coordinate-velocity** as usual, and **speed of map-time** or “differential-aging factor”  $\gamma \equiv dt/d\tau = \sqrt{1 + (w/c)^2} = 1/\sqrt{1 - (v/c)^2} \geq 1$ . This last relation tells us that *when simultaneity is defined by a network of synchronized map clocks*, a moving traveler’s clock will always run slow. These relationships follow directly from the metric equation itself.

Thus having a new time-variable  $\tau$  also gives us some new ways to measure rate of travel. Proper-velocity, as we’ll see, has no upper limit and is related to conserved-quantity **momentum** by the simple vector relation  $\vec{p} = m\vec{w}$  where  $m$  is our traveler’s frame-invariant rest-mass. The upper limit of  $c \geq dx/dt$  on coordinate-velocity results simply from the fact that momentum (and  $dx/d\tau$ ) have to be finite. Speed of map-time  $\gamma$ , on the other hand, relates to **total energy** by  $E = \gamma mc^2$ , and to **kinetic energy** by  $K = (\gamma - 1)mc^2$ .

*At this point, a variety of familiar time-dilation and relativistic energy/momentum topics might be covered as examples. If there is added time, one path to take is to introduce length-contraction, velocity addition, and Doppler effect with or without Lorentz transforms, historical notes on lightspeed measurement, etc. In what follows, we instead push the alignment with traveler-point concepts and the Newtonian treatment of kinematics and mechanics a bit further.*

It is also conceptually interesting to note that **proper-acceleration**, in turn, is simply the vector-acceleration detected by a *cell-phone accelerometer* in the traveler’s pocket. This quantity, after a couple of proper-time derivatives, pops up on the left side of the metric equation as a frame-invariant as did “hypotenuse” and proper-time.

For unidirectional motion in flat-spacetime (i.e. using map-coordinates in an inertial frame), proper-acceleration  $\vec{\alpha}_o = \gamma^3\vec{a}$ , where  $\vec{a} \equiv d^2\vec{x}/dt^2$  is the usual **coordinate-acceleration**. These relations also yield a few simple integrals for “constant” proper-acceleration, namely  $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$ .

That 2nd equality involves “hyperbolic velocity-angle” or **rapidity**  $\eta = \sinh^{-1}[w/c] = \tanh^{-1}[v/c]$ , so that  $\gamma = \cosh[\eta]$ . The first and third equalities reduce to the familiar conceptual-physics relationships  $a = \Delta v/\Delta t =$

$\frac{1}{2}\Delta(v^2)/\Delta x$  at low speed. However they allow beginning students to explore interstellar constant proper-acceleration round-trip problems, almost as easily as they do problems involving projectile trajectories on earth.

## 3. dynamics in flat spacetime

The net proper-force may in general be written as  $\Sigma\vec{F}_o \equiv m\vec{\alpha}_o$ . In flat (and unaccelerated) spacetime coordinate-systems, all forces are proper, and proper-acceleration  $\vec{\alpha}_o$  equals the **net-acceleration**  $\vec{a}$  observed *by a traveler*. Under a constant net proper-force, we can therefore expect constant net-acceleration.

At high speeds the constant proper-force equations are messier because of that pesky  $\gamma$  in equations like  $\Delta w = \Delta(\gamma v) = \alpha_o\Delta t$ . The low speed approximation (namely  $\Delta v = a\Delta t$ ) is therefore a bit simpler to deal with, and works fine for speeds well below  $c$ .

## 4. dynamics in curved spacetime

In both curved spacetime *and in accelerated frames*, Newton’s equations still work (at least locally) provided that we recognize the existence of non-proper or geometric forces, like gravity as well as inertial forces that arise in accelerated frames. Newton’s law for causes of motion “in the neighborhood of a traveling object” then takes the 3-vector form  $\Sigma\vec{F}_o + \Sigma\vec{F}_g = m\vec{a}$ , where  $\vec{a}$  is the net-acceleration actually observed by the traveler.

The reason that your cell-phone’s accelerometer can’t see gravity (or centrifugal force), even when gravity is causing a net-acceleration downward, is that these are geometric forces. Geometric forces ( $\Sigma\vec{F}_g$ ), which act on “every ounce” of an object, result from being in an accelerated frame or in curved spacetime. Accelerometers can only detect the result of net proper-forces ( $\Sigma\vec{F}_o$ ), i.e. one’s proper-acceleration  $\vec{\alpha}_o$ .

It’s traditional to approximate gravity on the surface of the earth as simply a proper-force that is proportional to mass  $m$ , for which  $\vec{F} = m\vec{g}$  where  $\vec{g}$  is a downward vector with a magnitude of about  $9.8 \text{ [m/s}^2]$ . This works quite well for most applications. However unlike geometric-forces, proper-forces are not associated with positional time-dilation like that which must be figured into GPS system calculations.

Just as kinetic energy in flat spacetime is related via  $(dt/d\tau - 1)mc^2$  to the faster passage of map-time ( $t$ ) with respect to traveler time ( $\tau$ ) when simultaneity is defined by the map-frame, so is the *position-dependent* potential-energy well-depth of some geometric-forces in accelerating frames and curved spacetime. Thus  $(dt/d\tau - 1)mc^2$  also describes the potential-energy well-depth for mass- $m$  travelers located: (i) at radius  $r$  from the axis in a habitat rotating with angular velocity  $\omega$ , which in the low-speed limit reduces to the classical value  $\frac{1}{2}m\omega^2r^2$ ;



(ii) a distance  $L$  behind the leading edge of a spaceship undergoing constant proper acceleration  $\alpha$ , which in the small  $L$  limit reduces to the classical value  $m\alpha L$ ; and

(iii) on the radius  $R$  surface of a mass  $M$  planet, which in the  $R \gg 2GM/c^2$  limit reduces to the classical value of  $GMm/R$  where  $G$  is the gravitational constant.