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Abstract. In this article, we study the behavior of a population composed by rational individuals, which interact through a binary game. After deducing the kinetic description of the system, we study existence and uniqueness of the resulting equation. We then focus our attention on a linear game and provide its asymptotics in the high-frequency and vanishing payoff case. Several numerical simulations show the quantitative behavior of the model.

1. Introduction

The kinetic theory approach [7, 10] is particularly suitable for describing a population composed by identical rational individuals which evolve in time through pairwise interactions. In many situations, it is possible to describe the interactions by imposing a given interaction rule. Among the available results in the literature on the description of rational interacting agents, we can cite the early contributions by Helbing [12, 11, 13], and we refer to the review articles [6, 3] for further references.

However, the global features of the population often influence the individual behavior and hence the individual strategies cannot be defined by a simple *a priori* argument. This remark has been the starting point, in the early 2000's, of the Mean Field Games theory, which is the limit of non-cooperative games in very large populations, composed of interacting individuals, each of them having a small influence on the global behavior of the system. The first results have been obtained by Lasry and Lions [16, 17, 18, 19] and by Huang, Malhamé and Caines [15, 14]. We quote moreover the article [8], which proposes and studies a dynamic model for an ensemble of players in the game-theoretical sense defined in the phase-space of the system and [4, 5], which study a Boltzmann-like model for knowledge growth introduced in the literature by Lucas and Moll in [20].

The pertinence of Mean Field Games theory has been observed in many situations, such as – for example – the optimal production of oil, the study of wealth distribution in developed societies, the dynamics of social networks and crowd motion. In this theory, players act according to a unique principle (they are indistinguishable and have the same optimization criteria), have individually a minor (infinitesimal) influence and their strategy takes into account the behavior of the co-players through the total mass of the system (in this sense, the Mean Field Games approach is less sophisticated than a full N-players game).

In this article we introduce and study a new model for describing binary games which borrows some features of both approaches. The system is described at the kinetic level and, at the same

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time, the strategies of the binary game depend not only on the values of the exchange variables but they take into account the global behavior of the whole population via its distribution function.

More precisely, we suppose that the population is composed by rational individuals, which interact through a game. In particular, as usually done in Game Theory, we suppose to have:

- an infinite set of players,
- a set of possible actions made by the players,
- a set of game results or issues,
- an order relation between the results which expresses the preferences of a player.

The game we consider will be:

- binary: players interact only pairwise,
- zero-sum: the players' choices can neither increase nor decrease the available resources,
- simultaneous: both players move simultaneously,
- non-cooperative: players cannot form alliances, i.e. every player acts individually,
- with imperfect information: players do not know the moves previously made by all other players neither they know the precise features of the competitor.

In particular, the players do not know the exchange variable of their competitor in the game. They have only a (possibly partial) knowledge of the whole population density and, for this reason, they are induced to use a mixed strategy, by taking into account their own exchange variable and the known properties of the distribution profile of the population.

The main consequence of this lack of knowledge is given by the high nonlinear structure of the model, which implies some peculiar properties of the solutions.

Compared with standard Mean Field Games models, our class of models appears purely of kinetic type. However, the optimization process, typical of Mean Field Games, is hidden in the determination of the optimal strategy of each player in order to reach Nash equilibrium. In some particular case, the structure of the game allows the possibility of directly deducing the Nash equilibrium, see for example the model in Section 3, in these cases the models are purely of kinetic type.

The structure of the article is the following: in Section 2 we introduce a general framework for describing binary games by using the kinetic approach and we prove the existence and uniqueness of the solution. Then, in Section 3 we introduce a specific game with a linear interacting rule which promotes wealth redistribution. In Section 4, we prove some additional mathematical properties of this model and in 5, we rigorously deduce its quasi-invariant limit (see [2] for another example of quasi-invariant limit procedure in the framework of rational populations described by kinetic equations). Finally, in Section 6, we numerically study the solutions of the model and its long-time behavior.

2. A general framework for the kinetic modeling of binary games

In this section, we introduce a general kinetic system to describe a population of rational agents which interact between themselves through simultaneous binary non-cooperative games with imperfect information.

Let $t \in [0, T]$ be the time variable and $x \in \Omega \subseteq \mathbb{R}^d$ the exchange variable ($d \in \mathbb{N}$). The system of interacting rational individuals is described by a distribution function $f = f(t, x)$, defined on

$[0,T]\times\Omega$ having the following precise meaning: let $D\subseteq\Omega$ be a sub-domain, for all t, the integral

$$\int_D f(t,x)\,\mathrm{d}x$$

represents the number of individuals for which the exchange variable belongs to D. It is obvious that a reasonable hypothesis on f is that it is non-negative and such that $f(t,x)\in L^1(\Omega)$ for all $t\in\mathbb{R}$, or, more generally, that f is a non-negative measure.

Let us denote with $(x',x'_*)\in\Omega\times\Omega$ the exchange variables of two agents before an interaction. We suppose that they interact between themselves through a zero-sum binary game (whose choices are represented by 0 and 1 respectively), having the following payoff rule:

	1	0
1	$a_{11},-a_{11}$	$a_{10},-a_{10}$
0	$a_{01},-a_{01}$	$a_{00},-a_{00}$

where $a_{ij}=a_{ij}(x',x'_*)$, $i,j=0,1$, and all $a_{ij}:\Omega\times\Omega\to\mathbb{R}^d$, $a_{ij}\in C(\Omega\times\Omega)$: we hence suppose that the payoffs depend on the exchange variables of the two players before the interaction. This hypothesis implies that the payoffs vary with respect to the pre-interaction exchange variables, and hence the strategy of the players should be chosen accordingly to them. By the indistinguishability of players, we must have for all $i,j=0,1$,

$$a_{ij}(x',x'_*)=-a_{ji}(x'_*,x'),$$

in particular the strategies $(1,0)$ and $(0,1)$ must induce the same payoff rules. Moreover, we suppose that the players do not know the exchange variable of their competitor in the game. This implies that the agents have an incomplete information of the state of the game before playing.

Note that it is essential to guarantee the consistency of the exchange mechanism with the domain Ω: i.e. the exchange variables after the interaction must belong to the domain Ω itself. We will therefore suppose, from now on, that the domain Ω is closed with respect to the exchange mechanism.

If we denote with $(x_{ij},x_{ij*})\in\Omega\times\Omega$ for $i,j=0,1$, the values of the exchange variable of two agents after the interaction through the strategy (i,j), the exchange rules are the following: let $\tilde{\mathcal{F}}_{ij}=(\mathcal{F}_{ij},\mathcal{F}_{ij*}):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^d$ be the interaction function $(x_{ij},x_{ij*})=\tilde{\mathcal{F}}_{ij}(x',x'_*)$ defined as

$$\text{(1)}\qquad \begin{cases} x_{ij} = \mathcal{F}_{ij}(x',x'_*) = x'+a_{ij}(x',x'_*) \\ x_{ij*} = \mathcal{F}_{ij*}(x',x'_*) = x'_*-a_{ij}(x',x'_*). \end{cases}$$

Of course, by the indistinguishability of players, we have that, for all $i,j=0,1$

$$\text{(2)}\qquad \mathcal{F}_{ij}(x',x'_*)=x'+a_{ij}(x',x'_*)=x'-a_{ji}(x'_*,x')=\mathcal{F}_{ji*}(x'_*,x').$$

According to the closure property of the domain with respect to the collisional mechanism, we must have that for all $i,j=0,1$, $x_{ij},\ x_{ij*}\in\Omega$, i.e. $\tilde{\mathcal{F}}_{ij}(\Omega\times\Omega)\subseteq\Omega\times\Omega$. This hypothesis translates into the following condition on the Jacobian J_{ij} of (1):

$$J_{ij}:=\left|\frac{\partial(x_{ij},x_{ij*})}{\partial(x',x'_*)}\right|\leq 1$$

for all $i,j=0,1$. Here $|M|$, denotes the determinant of the square matrix M.

In order to avoid pathological situations, we suppose in addiction that the Jacobians are always strictly positive: $J_{ij}>0$ for all $i,j=0,1$. It will be then possible to invert $\tilde{\mathcal{F}}_{ij}$. We denote with $\tilde{\mathcal{F}}_{ij}^{-1}=(\mathcal{F}_{ij}^{-1},\mathcal{F}_{ij*}^{-1}):\tilde{\mathcal{F}}_{ij}(\Omega\times\Omega)\to\Omega\times\Omega$ the inverse function $(x'_{ij},x'_{ij*})=\tilde{\mathcal{F}}_{ij}^{-1}(x,x_*)$. According to this notation, for all $x',x'_*\in\Omega$, we have $(x',x'_*)=\tilde{\mathcal{F}}_{ij}^{-1}(\tilde{\mathcal{F}}_{ij}(x',x'_*))$ and for all $(x_{ij},x_{ij*})\in\tilde{\mathcal{F}}_{ij}(\Omega\times\Omega)$, we have $(x_{ij},x_{ij*})=\tilde{\mathcal{F}}_{ij}(\tilde{\mathcal{F}}_{ij}^{-1}(x_{ij},x_{ij*}))$.

The conservation of the exchange variable during the game is a consequence of the payoff table (it is indeed a zero-sum game):

$$x'+x'_*=x_{ij}+x_{ij*}\quad\forall i,j=0,1.$$

Since the payoffs a_{ij} depend on the values of the exchange variables, the choices between the strategies 0 and 1 depend also on the values of the exchange variable of each player. In this game, we suppose that the pre-interaction value of the exchange variable of one player is unknown to the other player: the individuals of the population are hence induced to use a mixed strategy, by taking into account their own exchange variable and the known features of the distribution profile of the population.

Let

$$\alpha:C([0,T];L^1(\Omega))\to C([0,T]\times\Omega),$$

be the player's probability of choosing the strategy labelled with 1. Hence α has the following properties

$$\alpha(f)(t,x)\geq 0\ \forall\ (t,x)\in[0,T]\times\Omega,\forall f\in C([0,T];L^1(\Omega)),\quad \|\alpha(f)\|_{C([0,T]\times\Omega)}\leq 1.$$

This probability obviously depends on the interaction rule between the agents and on the profile of the population. Moreover, it is optimally chosen as the Nash equilibrium in mixed strategies for the binary game described by (1). The precise structure of the operator α is a consequence of the payoff table. The Mean Field Games part of our problem is indeed hidden in the optimization process necessary to find the optimal strategy for each player and consequently the probability α.

Furthermore, we suppose that all the individuals have no preference in choosing their competitor. This is equivalent to requiring that the cross section, which governs the probability that two individuals interact, is a constant $\eta\in\mathbb{R}_+$. In this situation, this parameter can be interpreted as a simple rescaling of the time variable.

Finally, the model takes the following form:

$$\begin{cases}\dfrac{1}{\eta}\dfrac{\partial f}{\partial t}(t,x)=Q^+(f)(t,x)-f(t,x)\displaystyle\int_\Omega f(t,x_*)\,\mathrm{d}x_*\\ f(0,\cdot)=f^{\mathrm{in}}(\cdot)\end{cases},\tag{3}$$

where, the gain collisional operator is defined as

$$
\begin{aligned}
Q^+(f)(t,x) := &\int_\Omega \frac{1}{J_{11}} f(t,x'_{11}) f(t,x'_{11*}) \alpha(f)(t,x'_{11}) \alpha(f)(t,x'_{11*}) \chi_{\tilde{\mathcal{F}}_{11}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_* \\
&+ \int_\Omega \frac{1}{J_{10}} f(t,x'_{10}) f(t,x'_{10*}) \alpha(f)(t,x'_{10}) [1-\alpha(f)(t,x'_{10*})] \chi_{\tilde{\mathcal{F}}_{10}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_* \\
&+ \int_\Omega \frac{1}{J_{01}} f(t,x'_{01}) f(t,x'_{01*}) [1-\alpha(f)(t,x'_{01})] \alpha(f)(t,x'_{01*}) \chi_{\tilde{\mathcal{F}}_{01}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_* \\
&+ \int_\Omega \frac{1}{J_{00}} f(t,x'_{00}) f(t,x'_{00*}) [1-\alpha(f)(t,x'_{00})]\,[1-\alpha(f)(t,x'_{00*})] \chi_{\tilde{\mathcal{F}}_{00}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_*.
\end{aligned}
$$

We recall here that, for all $i,j=0,1$, $(x'_{ij},x'_{ij*}) = \tilde{\mathcal{F}}_{ij}^{-1}(x,x_*)$ are the values of the exchange variables of two agents before the interaction due to the strategy (i,j).

The loss part of the collisional operator takes the simpler form $f(t,x)\int_\Omega f(t,x_*)\,\mathrm{d}x_*$ since the probability of an agent to play is equal to one.

Note that, the general structure of a kinetic model, that describes a set of identical players interacting between themselves through simultaneous binary non-cooperative games with imperfect information, shows a gain term with an intricate dependency with respect to the unknown (much more complex than the standard Boltzmann equation, although the equivalent of the cross section is uniformly bounded).

Equation (3) can be rewritten in a distributional form with respect to the exchange variable. To this end, let us define the duality form, for any $F\in L^1(\Omega)$,

$$
\langle F\,,\,\varphi\rangle := \int_\Omega F(x)\varphi(x)\,\mathrm{d}x, \qquad \text{for all } \varphi\in C(\Omega). \tag{4}
$$

We have that

$$
\begin{cases}
\dfrac{1}{\eta}\dfrac{\mathrm{d}}{\mathrm{d}t}\langle f\,,\,\varphi\rangle - \langle Q^+(f),\varphi\rangle + \langle f\,,\,\varphi\rangle \displaystyle\int_\Omega f(t,x)\,\mathrm{d}x = 0 \\[2ex]
\langle f(0,\,\cdot\,)\,,\,\varphi\rangle = \displaystyle\int_\Omega f^{\mathrm{in}}(x)\varphi(x)\,\mathrm{d}x
\end{cases} \tag{5}
$$

for all $\varphi\in C(\Omega)$, with $f^{\mathrm{in}}\in L^1(\Omega)$, where

$$
\begin{aligned}
\langle Q^+(f),\varphi\rangle = &\int_{\Omega\times\Omega} f(t,x) f(t,x_*) \alpha(f)(t,x) \alpha(f)(t,x_*) \varphi(x_{11})\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x) f(t,x_*) \alpha(f)(t,x) [1-\alpha(f)(t,x_*)] \varphi(x_{10})\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x) f(t,x_*) [1-\alpha(f)(t,x)] \alpha(f)(t,x_*) \varphi(x_{01})\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x) f(t,x_*) [1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*))] \varphi(x_{00})\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

In the computations above, by an abuse of notation, with x_{ij} we mean $x_{ij} = \mathcal{F}_{ij}(x, x_*)$ for all $i, j = 0, 1$.

Note that, taking $\varphi(x) = 1$ for all $x \in \Omega$, we have

$$\int_\Omega Q^+(f)(t,x)\,\mathrm{d}x = \left(\int_\Omega f(t,x)\,\mathrm{d}x\right)^2 \quad \text{for all } t \in [0,T]. \tag{6}$$

The model guarantees the conservation of the total number of agents. This property, in analogy with physical notations, will also be indicated as the conservation of the total mass.

Proposition 2.1. *Let $f = f(t,x)$ be a solution of (3), with a nonnegative initial condition $f^{\mathrm{in}} \in L^1(\Omega)$. Then we have*

$$\int_\Omega f(t,x)\ \mathrm{d}x = \|f^{\mathrm{in}}\|_{L^1(\Omega)} =: \rho \qquad \textit{for all } t \in [0,T].$$

Proof. The proof is a direct consequence of Equation (5) used with the test function $\varphi(x) = 1$ for all $x \in \Omega$. Recall also Equation (6). □

Another peculiar feature of the system, which is a direct consequence of the zero-sum payoff rule, is the total conservation of the exchange variable:

Proposition 2.2. *Let $f = f(t,x)$ be a solution of (3), with a nonnegative initial condition $f^{\mathrm{in}} \in L^1(\Omega)$. Then we have*

$$\int_\Omega x f(t,x)\ \mathrm{d}x = \|x f^{\mathrm{in}}\|_{L^1(\Omega)} \qquad \textit{for all } t \in [0,T].$$

Proof. The proof can be obtained by means of the equivalent distributional formulation of $Q^+(f)$, where we stress the use of the change of variables (1):

$$\begin{aligned}
\langle Q^+(f), \varphi\rangle = &\int_{\Omega\times\Omega} f(t,x) f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)\varphi(\mathcal{F}_{11}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x) f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)]\varphi(\mathcal{F}_{10}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x) f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)\varphi(\mathcal{F}_{01}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x) f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)]\varphi(\mathcal{F}_{00}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}$$

Inverting the role of x and x_*, and exploiting the symmetry of the first and the last integral, we have

$$
\begin{aligned}
2\langle Q^+(f),\varphi\rangle = &\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{11}(x,x_*))+\varphi(\mathcal{F}_{11}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)]\varphi(\mathcal{F}_{10}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x_*)[1-\alpha(f)(t,x)]\varphi(\mathcal{F}_{10}(x_*,x))\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)\varphi(\mathcal{F}_{01}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x_*)]\alpha(f)(t,x)\varphi(\mathcal{F}_{01}(x_*,x))\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{00}(x,x_*))+\varphi(\mathcal{F}_{00}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

Hence

$$
\begin{aligned}
2\langle Q^+(f),\varphi\rangle = &\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{11}(x,x_*))+\varphi(\mathcal{F}_{11}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{10}(x,x_*))+\varphi(\mathcal{F}_{01}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{01}(x,x_*))+\varphi(\mathcal{F}_{10}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{00}(x,x_*))+\varphi(\mathcal{F}_{00}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

Then, we use the symmetry of our model given by (2),

$$
\begin{aligned}
2\langle Q^+(f),\varphi\rangle = &\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{11}(x,x_*))+\varphi(\mathcal{F}_{11*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{10}(x,x_*))+\varphi(\mathcal{F}_{10*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{01}(x,x_*))+\varphi(\mathcal{F}_{01*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{00}(x,x_*))+\varphi(\mathcal{F}_{00*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

By setting $\varphi(s) = s$ and using (1) , we obtain

$$\begin{aligned}
2\langle Q^+(f), x\rangle &= \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[x+x_*]\,\mathrm{d}x\,\mathrm{d}x_* \\
&\quad + \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)][x+x_*]\,\mathrm{d}x\,\mathrm{d}x_* \\
&\quad + \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)[x+x_*]\,\mathrm{d}x\,\mathrm{d}x_* \\
&\quad + \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][x+x_*]\,\mathrm{d}x\,\mathrm{d}x_* \\
&= \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[x+x_*]\,\mathrm{d}x\,\mathrm{d}x_* = 2\int_{\Omega\times\Omega} f(t,x)f(t,x_*)x\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}$$

Hence

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\langle f\,,\,x\rangle = \langle Q^+(f), x\rangle - \langle f, x\rangle \int_\Omega f = 0.$$

□

The previous result allows to prove the following existence theorem:

Theorem 2.3. *Let f^{in} a nonnegative function of class $L^1(\Omega)$. Let $T > 0$ be a given constant and suppose that α is such that for all $f_1, f_2 \in C([0,T]; L^1(\Omega))$,*

$$\|\alpha(f_1) - \alpha(f_2)\|_{C([0,T]\times\Omega)} \le C\|f_1 - f_2\|_{C([0,T];L^1(\Omega))} \tag{7}$$

for $C \in \mathbb{R}_+$.

Then, the Cauchy problem (3) admits a unique nonnegative solution in $C^1([0,T]; L^1(\Omega))$.

Proof. First of all, we transform the Cauchy problem (3) in integral form, using the classical Duhamel formula. We have indeed that

$$f(t,x) = f^{\mathrm{in}}(x)e^{-\eta\rho t} + \eta\int_0^t e^{-\eta\rho(t-s)}Q^+(f)(s,x)\,\mathrm{d}s := \mathcal{A}(f)(t,x), \tag{8}$$

here we used the fact that $\int_\Omega f(t,x)\,\mathrm{d}x = \rho$ for all $t \in [0,T]$. We then apply a contraction argument to the previous equation.

Let us define

$$F := \{\psi \in C([0,T]; L^1(\Omega)) \,:\, \|\psi(t,\cdot)\|_{L^1(\Omega)} = \rho\ \forall t \in [0,T],\ \psi \ge 0\}.$$

F is a Banach space, being a closed subset of $C([0,T]; L^1(\Omega))$. We have to prove that that $\mathcal{A}$ maps F into F. First of all, let us observe that, due to the integral form (8), the operator $\mathcal{A}$ maps nonnegative functions into nonnegative functions.

Then, for all $f \in F$, and all $t \in [0,T]$,

$$\begin{aligned}
\int_\Omega |\mathcal{A}(f)(t,x)|\,\mathrm{d}x &= e^{-\eta\rho t}\int_\Omega f^{\mathrm{in}}(x)\,\mathrm{d}x + \eta\int_0^t e^{-\eta\rho(t-s)}\int_\Omega Q^+(f)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&= e^{-\eta\rho t}\rho + \eta\rho^2\int_0^t e^{-\eta\rho(t-s)}\,\mathrm{d}s = \rho,
\end{aligned}$$

where we used (6) and the fact that $\int_\Omega f(t,x)\,\mathrm{d}x = \rho$ for all $t \in [0,T]$. Hence

$$\|\mathcal{A}(f)(t,\cdot)\|_{L^1(\Omega)} = \rho \quad \forall t \in [0,T] \quad \text{and} \quad \|\mathcal{A}(f)\|_{C([0,T];L^1(\Omega))} = \max_{t\in[0,T]} \|\mathcal{A}(f)(t,\cdot)\|_{L^1(\Omega)} = \rho.$$

Let us now prove that, under suitable conditions on T, $\mathcal{A}$ is a contraction in F. Indeed, let $f_i \in F$ for $i = 1, 2$. We compute

$$\begin{aligned}
&\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{C([0,T];L^1(\Omega))} = \\
&= \max_{t\in[0,T]} \eta \int_\Omega \left| \int_0^t e^{-\eta\rho(t-s)} \left(Q^+(f_1)(s,x) - Q^+(f_2)(s,x) \right) \mathrm{d}s \right| \mathrm{d}x \\
&\le \max_{t\in[0,T]} \eta \int_\Omega \int_0^t \left| Q^+(f_1)(s,x) - Q^+(f_2)(s,x) \right| \mathrm{d}s\,\mathrm{d}x \\
&\le \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) \alpha(f_i)(s,x') \alpha(f_i)(s,x'_*) \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s \\
&\quad + \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) \alpha(f_i)(s,x') [1 - \alpha(f_i)(s,x'_*)] \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s \\
&\quad + \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) [1 - \alpha(f_i)(s,x')] \alpha(f_i)(s,x'_*) \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s \\
&\quad + \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) [1 - \alpha(f_i)(s,x')] \left[1 - \alpha(f_i)(s,x'_*)\right] \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s,
\end{aligned}$$

where in the last inequality we performed the change of variable $(x, x_*) \to (x', x'_*)$. In the following we will omit the prime notation.

Note that, we can bound all the terms in the previous inequality reasoning in the same way. Let us show, for example, how to bound the first one. We have that

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x) f_i(s,x_*) \alpha(f_i)(s,x) \alpha(f_i)(s,x_*) \right| \mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^T \int_{\Omega\times\Omega} f_1(s,x_*) \alpha(f_1)(s,x) \alpha(f_1)(s,x_*) \left| f_1(s,x) - f_2(s,x) \right| \mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\Omega\times\Omega} f_2(s,x) \alpha(f_1)(s,x) \alpha(f_1)(s,x_*) \left| f_1(s,x_*) - f_2(s,x_*) \right| \mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\Omega\times\Omega} f_2(s,x) f_2(s,x_*) \alpha(f_1)(s,x_*) \left| \alpha(f_1)(s,x) - \alpha(f_2)(s,x) \right| \mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\Omega\times\Omega} f_2(s,x) f_2(s,x_*) \alpha(f_2)(s,x) \left| \alpha(f_1)(s,x_*) - \alpha(f_2)(s,x_*) \right| \mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq T\left[\|f_1\|_{C([0,T];L^1(\Omega))}+\|f_2\|_{C([0,T];L^1(\Omega))}\right]\|f_1-f_2\|_{C([0,T];L^1(\Omega))}\\
&\quad+2\|f_2\|_{C([0,T];L^1(\Omega))}\int_0^T\int_\Omega f_2(s,x)\,|\alpha(f_1)(s,x)-\alpha(f_2)(s,x)|\,\mathrm{d}x\,\mathrm{d}s\\
&\leq T\left[\|f_1\|_{C([0,T];L^1(\Omega))}+\|f_2\|_{C([0,T];L^1(\Omega))}\right]\|f_1-f_2\|_{C([0,T];L^1(\Omega))}\\
&\quad+2\|f_2\|_{C([0,T];L^1(\Omega))}\int_0^T\|f_2(s,\,\cdot\,)\|_{L^1(\Omega)}\|\alpha(f_1)-\alpha(f_2)\|_{C([0,T]\times\Omega)}\,\mathrm{d}s\\
&\leq T\left[\|f_1\|_{C([0,T];L^1(\Omega))}+\|f_2\|_{C([0,T];L^1(\Omega))}+2C\|f_2\|^2_{C([0,T];L^1(\Omega))}\right]\|f_1-f_2\|_{C([0,T];L^1(\Omega))}
\end{aligned}
$$

The other terms can be estimated in a similar way. Consequently, we deduce that,

$$\|\mathcal{A}(f_1)-\mathcal{A}(f_2)\|_{C([0,T];L^1(\Omega))}\leq 4\eta T\left(2\rho+2C\rho^2\right)\|f_1-f_2\|_{C([0,T];L^1(\Omega))}$$

where we used the fact that $f_1,\,f_2\in F$. Therefore, for

$$T<\frac{1}{8\eta\rho\,(1+C\rho)},$$

Banach's Fixed Point Theorem guarantees the existence of a unique fixed point, and hence a solution of the Cauchy problem (3) in $C([0,T];L^1(\Omega))$.

In addition, it is immediate to see, from the Cauchy problem (3) and formula (6), that

$$\|f_t\|_{C([0,T];L^1(\mathbb{R}_+))}\leq 2\eta\|f\|^2_{C([0,T];L^1(\mathbb{R}_+))}.$$

Hence $f_t\in C([0,T];L^1(\mathbb{R}_+))$. By a standard bootstrap argument we can extend the existence of the solution to all $T>0$ fixed since the bound on T depends only on the initial condition and time evolution is mass preserving (Proposition 2.1).

□

3. A linear game

Up to now, we have considered a quite general binary game. However, it is clear that many qualitative and quantitative properties of the equation (such as, for example, the concentration effects or the asymptotic behavior) heavily depend on the precise rules of the game. For this reason, we now focus our attention on a specific game with a linear interacting rule which promotes wealth redistribution.

Let $0\leq\varepsilon<\frac{1}{2}$ and $(x',x'_*)\in\mathbb{R}_+\times\mathbb{R}_+$ be the wealths of two agents before an interaction. Our agents interact between themselves through a binary game, having the following payoff rule:

	1	0
1	$0,0$	$0,0$
0	$0,0$	$\varepsilon(x'_*-x'),\varepsilon(x'-x'_*)$

Table 1

It means that, if we denote with $(x, x_*) \in \mathbb{R}_+ \times \mathbb{R}_+$ the wealths of two agents after the interaction, the exchange rules are the following: in the case they play $(0, 0)$, we have

$$\begin{cases} x = (1-\varepsilon)x' + \varepsilon x'_* \\ \\ x_* = \varepsilon x' + (1-\varepsilon)x'_*. \end{cases} \tag{9}$$

These relationships can be inverted in the following way:

$$\begin{cases} x' = \dfrac{1-\varepsilon}{1-2\varepsilon}x - \dfrac{\varepsilon}{1-2\varepsilon}x_* \\ \\ x'_* = -\dfrac{\varepsilon}{1-2\varepsilon}x + \dfrac{1-\varepsilon}{1-2\varepsilon}x_*. \end{cases} \tag{10}$$

Note that the existence of a pre-collisional non-negative pair (x', x'_*) generated by a non-negative post-collisional pair (x, x_*) is not guaranteed, unless we suppose that

$$\frac{\varepsilon}{1-\varepsilon}x \leq x_* \leq \frac{1-\varepsilon}{\varepsilon}x.$$

When the agents play $(1, 1)$, $(1, 0)$ or $(0, 1)$, the interaction rule is simply the identity:

$$\begin{cases} x = x' \\ \\ x_* = x'_*. \end{cases} \tag{11}$$

The Jacobian J of (9) is easily computable and it is always strictly positive (remember $0 \leq \varepsilon < \frac{1}{2}$) and contracting:

$$0 < J := \left| \frac{\partial(x, x_*)}{\partial(x', x'_*)} \right| = (1-2\varepsilon) \leq 1.$$

As a consequence of the payoff table we have wealth's conservation during the game, i.e. we are considering a zero-sum game:

$$x + x_* = x' + x'_*.$$

We can therefore see this game as a particular case of the model described in Section 2, we just need to define the probability function α, which determine the optimal strategy. It is clear that the optimal strategy of the two players will depend on their relative wealth: indeed, the strategy 0 is advantageous if and only if the player is poorer than his competitor. On the other hand, the richer player is always forced to adopt the strategy 1. As in the general case, we suppose that the pre-interaction wealths are unknown to the other player. Players have to use then a mixed strategy, by taking into account their own wealth and knowledge on the the distribution profile of the population. In this example, we suppose that the distribution profile f is not completely known, but that each agent only knows how many players are poorer than him. In other terms, if a given individual has wealth $x \in \mathbb{R}_+$, he knows the cumulative distribution function

$$(t, x) \mapsto \int_0^x f(t, \xi)\, \mathrm{d}\xi.$$

Let

$$m_0 := \int_{\mathbb{R}_+} f^{\mathrm{in}}(x)\, \mathrm{d}x,$$

where f^{in} is a nonnegative function of class $L^1(\mathbb{R}_+)$. We know from Section 2 that $\forall t \in [0,T]$, $T>0$ fixed,

$$\int_{\mathbb{R}_+} f(t,x)\,\mathrm{d}x = m_0.$$

If a player has wealth $x' \geq 0$, then the probability of meeting a poorer player with wealth $0 \leq x'_* \leq x'$ at a time $t \in [0,T]$ is given by

$$\alpha(f)(t,x') := \mathbb{P}(x' > x'_* \,|(t,x',x'_*)) = \frac{1}{m_0}\int_0^{x'} f(t,\xi)\,\mathrm{d}\xi.$$

Note that it is easy to verify that the probability operator α defined satisfies

$$\alpha : C([0,T];L^1(\mathbb{R}_+)) \to C([0,T]\times\mathbb{R}_+),$$

$$\alpha(f)(t,x) \geq 0 \;\forall\; (t,x) \in [0,T]\times\mathbb{R}_+, \forall f \in C([0,T];L^1(\mathbb{R}_+)), \quad \|\alpha(f)\|_{C([0,T]\times\mathbb{R}_+)} \leq 1$$

and the hypothesis (7). Hence, when two individuals with wealth x' and x'_* interact at a time $t \in [0,T]$, the probability to use the strategy $(0,0)$ is given by

$$\begin{aligned}\mathbb{P}((0,0)\,|(t,x',x'_*)) &= \left(1-\frac{1}{m_0}\int_0^{x'} f(t,\xi)\,\mathrm{d}\xi\right)\left(1-\frac{1}{m_0}\int_0^{x'_*} f(t,\xi)\,\mathrm{d}\xi\right)\\ &= \frac{1}{m_0^2}\left(\int_{x'}^{+\infty'} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right).\end{aligned}$$

By considering the whole population, its time evolution is hence described by a collisional equation of the following type:

$$\frac{\partial f}{\partial t}(t,x) = \eta\, Q^+(f) - \eta f \int f\,\mathrm{d}x,$$

where

$$\begin{aligned}Q^+(f)(t,x) := &\int_{\mathbb{R}_+} \alpha(f)(t,x)\alpha(f)(t,x_*)f(t,x)f(t,x_*)\,\mathrm{d}x_*\\ &+\int_{\mathbb{R}_+} (1-\alpha(f)(t,x))\,\alpha(f)(t,x_*)f(t,x)f(t,x_*)\,\mathrm{d}x_*\\ &+\int_{\mathbb{R}_+} \alpha(f)(t,x)\,(1-\alpha(f)(t,x_*))\,f(t,x)f(t,x_*)\,\mathrm{d}x_*\\ &+\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x} \big(1-\alpha(f)(t,x')\big)\,\big(1-\alpha(f)(t,x'_*)\big)\,\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_*.\end{aligned}$$

Note that $0 \leq \varepsilon < \frac{1}{2}$ ensures $\frac{\varepsilon}{1-\varepsilon}x \leq \frac{1-\varepsilon}{\varepsilon}x$. As in the general case, the cross section $\eta \in \mathbb{R}_+$ is a parameter which governs the probability that two individuals interact. Even here, we suppose that η is a constant.

Using the definition of α, we can rewrite our equation as

$$\frac{\partial f}{\partial t}(t,x) = Q(f)(t,x),$$

where

$$(12)\quad Q(f)(t,x) := \frac{\eta}{m_0^2}\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_* - \frac{\eta}{m_0^2}\int_{\mathbb{R}_+}\left(\int_{x}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)f(t,x_*)\,\mathrm{d}x_*.$$

4. Mathematical properties of the model

In this section we briefly prove some mathematical features of the equation introduced in the previous section. Therefore, we study the Cauchy problem

$$(13)\qquad \begin{cases} \frac{\partial f}{\partial t} = Q(f) \\ \\ f(0,x) = f^{\mathrm{in}}(x) \in L^1(\mathbb{R}_+) \end{cases},$$

where $Q(f)$ is defined by (12).

Remark 4.1. *Since the Cauchy problem* (13) *is a particular formulation of the Cauchy problem* (3) *for the binary game* (9)*, the conservation of mass*

$$m_0 := \int_0^{+\infty} f(t,x)\,\mathrm{d}x = \int_0^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x$$

and momentum

$$m_1 := \int_0^{+\infty} xf(t,x)\,\mathrm{d}x = \int_0^{+\infty} xf^{\mathrm{in}}(x)\,\mathrm{d}x$$

can be proven using the weak formulation of the problem as in Propositions 2.1 *and* 2.2*. Moreover, Theorem* 2.3 *holds true, and the Cauchy problem* (13) *admits a unique nonnegative solution in* $C^1([0,T];L^1(\mathbb{R}_+))$ *for all* $T>0$*.*

Remark 4.2. *Since* $f \in C^1([0,T];L^1(\mathbb{R}_+))$*, we have that*

$$\int_{\mathbb{R}_+}\left(\frac{1}{m_0}\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)\,\mathrm{d}x = -\frac{1}{2m_0}\int_{\mathbb{R}_+}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\mathrm{d}x = \frac{m_0}{2},$$

and the operator Q *can be written also in the equivalent form*

$$(14)\qquad Q(f) = \frac{\eta}{m_0^2}\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_* - \frac{\eta}{2}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x).$$

However, even if this expression can seem simpler than (12)*, we will keep the latter form because of its symmetric structure, which will make easier many computations.*

Remark 4.3. *Let us note that, due to the regularity of the solution* f*, we have that its spatial primitive, defined as the function* $(t,y)\mapsto\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi$ *for all* $(t,y)\in[0,T]\times\mathbb{R}_+$ *belongs to* $\in C^1([0,T];W^{1,1}(\mathbb{R}_+))$*.*

Let us, therefore, compute its time derivative.

Proposition 4.4. *For $T > 0$, let $f \in C^1([0,T]; L^1(\mathbb{R}_+))$ be the unique solution of the Cauchy problem (13), with $f^{\text{in}}(x) \in L^1(\mathbb{R}_+)$ and $0 \leq \varepsilon < \frac{1}{2}$. Then, for all $t \in [0,T]$, the function $(t,y) \mapsto \int_y^{+\infty} f(t,x)\,\mathrm{d}x$ satisfies*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2-\frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x.\end{aligned}$$

Proof. The equation satisfied by $\int_y^{+\infty} f(t,x)\,\mathrm{d}x$ can be recovered performing a change of variable as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_y^{+\infty}\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ &-\frac{\eta}{m_0^2}\int_y^{+\infty}\int_{\mathbb{R}_+}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ =&\frac{\eta}{m_0^2}\int_0^{\frac{y}{1-\varepsilon}} f(t,x')\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_{-\frac{1-\varepsilon}{\varepsilon}x'+\frac{y}{\varepsilon}}^{+\infty}\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x'_*)\,\mathrm{d}x'_*\,\mathrm{d}x'\\ &+\frac{\eta}{m_0^2}\int_{\frac{y}{1-\varepsilon}}^{+\infty} f(t,x')\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_0^{+\infty}\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x'_*)\,\mathrm{d}x'_*\,\mathrm{d}x'\\ &-\frac{\eta}{m_0^2}\int_y^{+\infty}\int_{\mathbb{R}_+}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ &-\frac{\eta}{m_0^2}\int_y^{\frac{y}{1-\varepsilon}} f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_0^{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x_*)\,\mathrm{d}x_*\mathrm{d}x.\end{aligned}$$

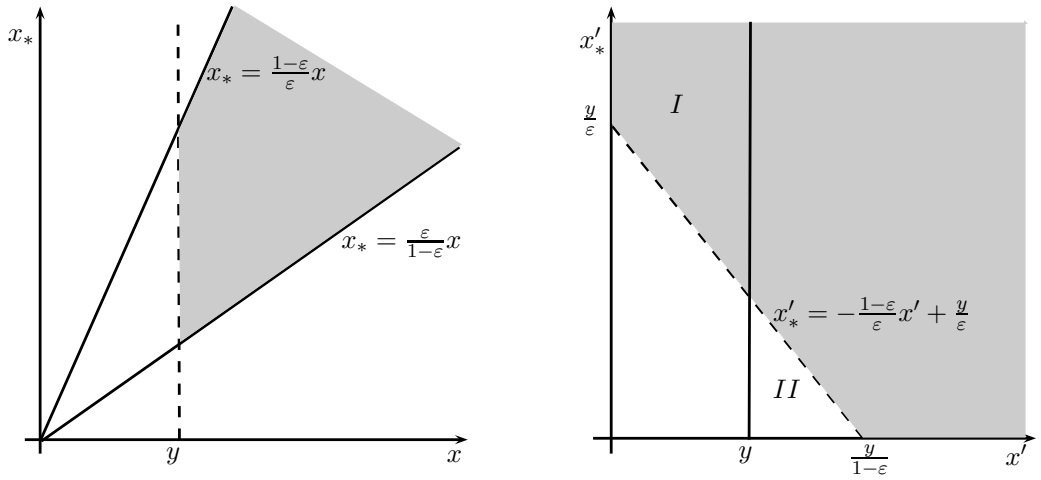

FIGURE 1. Change of variables and integration areas I and II

Exchanging the order of integration in the last term, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right) f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ &- \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_y^{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right) f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2-\frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x,\end{aligned}$$

were we used the fact that $f(t,x)\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi = -\frac{1}{2}\frac{\partial}{\partial x}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2$ and the convention $\int_{+\infty}^{+\infty} f(t,\xi)\,\mathrm{d}\xi = 0$. □

Proposition 4.5. *For $T > 0$, let $f \in C^1([0,T];L^1(\mathbb{R}_+))$ be the unique solution of the Cauchy problem* (13)*, with $f^{\mathrm{in}}(x) \in L^1(\mathbb{R}_+)$ and $0 \le \varepsilon < \frac{1}{2}$. Suppose f^{in} has compact support $E_0 \subset [x_m, x_M]$, where $x_m = \min\{x \in \mathbb{R}_+ \,|\, f^{\mathrm{in}}(x) \neq 0\}$, $x_M = \max\{x \in \mathbb{R}_+ \,|\, f^{\mathrm{in}}(x) \neq 0\}$. Then $f(t,\cdot)$ has compact support $E_t \subset [x_m, x_M]$ for all $t \in [0,T]$.*

Proof. We claim that, for all $y \in \mathbb{R}_+$,

$$\int_y^{+\infty} f(t,x)\,\mathrm{d}x \le \int_y^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0 t}{2}} \quad \text{for all } t \in [0,T].$$

Indeed, since for $x \in [0,y]$, $-\frac{1-\varepsilon}{\varepsilon}x + \frac{y}{\varepsilon}$ varies in $[y, y/\varepsilon]$ and $-\frac{\varepsilon}{1-\varepsilon}x + \frac{y}{1-\varepsilon}$ in $[y, y/(1-\varepsilon)]$, we have that

$$\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi \le \int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi \quad \text{and} \quad \int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi \le \int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi$$

Moreover we can bound for all $x \in \mathbb{R}_+$ the integral $\int_x^{+\infty} f(t,\xi)$ as

$$\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi \le m_0.$$

Hence, from Proposition 4.4,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\
&\left. + \frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2 - \frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x\\
\le& \frac{\eta}{2m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\mathrm{d}x\\
\le& \frac{\eta}{2m_0}\int_0^y f(t,x)\,\mathrm{d}x\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\\
\le& \frac{\eta m_0}{2}\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Using Gronwall's inequality we obtain the claim

$$\int_y^{+\infty} f(t,x)\,\mathrm{d}x \le \int_y^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0 t}{2}} \quad \text{for all } t \in [0,T].$$

Now let $y \in \mathbb{R}_+$ be such that $y \ge x_M$, i.e. $\int_y^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x = 0$, then the above inequality implies $\int_y^{+\infty} f(t,x)\,\mathrm{d}x = 0$. Hence, $f(t,x) = 0$ for all $x \in [y,+\infty[$ and for all $t \in [0,T]$. This shows that $E_t \subset [0, x_M]$ for all $t \in [0,T]$.

We claim now that, for all $y \in \mathbb{R}_+$,

$$\int_0^y f(t,x)\,\mathrm{d}x \le \int_0^y f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0}{2}t} \quad \text{for all } t \in [0,T].$$

Indeed, we have

$$\frac{d}{dt}\int_0^y f(t,x)\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x,$$

Hence, from Proposition 4.4,

$$\begin{aligned}\frac{d}{dt}\int_0^y f(t,x)\,\mathrm{d}x =& -\frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2-\frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x\\ \le&\frac{\eta}{2m_0^2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\mathrm{d}x\\ \le&\frac{\eta m_0}{2}\int_0^y f(t,x)\,\mathrm{d}x.\end{aligned}$$

Using again Gronwall's inequality, we obtain the claim

$$\int_0^y f(t,x)\,\mathrm{d}x \le \int_0^y f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0}{2}t} \quad \text{for all } t\in[0,T].$$

Now let $y\in\mathbb{R}_+$ be such that $y\le x_m$, i.e. $\int_0^y f^{\mathrm{in}}(x)\,\mathrm{d}x=0$, then the above inequality implies $\int_0^y f(t,x)\,\mathrm{d}x=0$. Hence, $f(t,x)=0$ for all $x\in[0,y]$ and for all $t\in[0,T]$. This shows that $E_t\subset[x_m,+\infty[$ for all $t\in[0,T]$.

Therefore, $E_t\subset[x_m,x_M]$ for all $t\in[0,T]$. □

5. The quasi-invariant limit

In this section we suppose that $\varepsilon\ll 1/2$ and show how to derive the first-order approximated model starting from (3).

We scale the space and the time variable with a scale parameter of order ε, and we consider the rescaled family of kinetic equations for the unknown $f^\varepsilon=f^\varepsilon(t,x)$, defined for all $0\le\varepsilon<1/2$,

$$\begin{cases}\varepsilon\dfrac{\partial}{\partial t}f^\varepsilon(t,x)=Q(f^\varepsilon)(t,x)\\[2ex] f^\varepsilon(0,x)=f^{\mathrm{in}}(x),\end{cases}\tag{15}$$

where $f^{\mathrm{in}}\in L^1(\mathbb{R}_+)$, is nonnegative, having its support contained in $E:=[0,x_M]\subset\mathbb{R}_+$. The following result holds.

Theorem 5.1. *Let $\{f^\varepsilon\}_{\varepsilon\in(0,1/2)}$ be a family of solutions of the Cauchy problems (15), supplemented with the same nonnegative initial condition $f^{\mathrm{in}}\in L^1(\Omega)$, with support contained in $E:=[0,x_M]\subset\mathbb{R}_+$. Then, in the limit $\varepsilon\to 0^+$, the family $\{f^\varepsilon\}_{\varepsilon\in(0,1/2)}$ weakly* converges in $\mathcal{M}([0,T]\times\mathbb{R}_+)$ to a function $f\in L^1([0,T];\mathcal{M}(\mathbb{R}_+))$, which is a distributional solution of*

$$\begin{cases}\dfrac{\partial}{\partial t}f(t,x)=\dfrac{\eta}{m_0^2}\dfrac{\partial}{\partial x}\left[\displaystyle\int_{\mathbb{R}_+}\left(\int_x^{+\infty}\mathrm{d}f(t,\xi)\right)\left(\int_{x_*}^{+\infty}\mathrm{d}f(t,\xi)\right)(x-x_*)f(t,x)\,\mathrm{d}f(t,x_*)\right]\\[2ex] f(0,x)=f^{\mathrm{in}}(x).\end{cases}\tag{16}$$

Proof. Note that, rescaling in ε does not affect the regularity of f^ε and the conservation properties of the equation. Hence, in particular, for all $0 < \varepsilon < 1/2$, $f^\varepsilon \in C^1([0,T];L^1(\mathbb{R}_+))$ for a fixed $T > 0$ and for all $t \in [0,T]$

$$m_0 := \int_{\mathbb{R}_+} f^\varepsilon(t,\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}_+} f^{\mathrm{in}}(\xi)\,\mathrm{d}\xi,$$

where we used the fact that the initial condition f^{in} is the same for all ε. Moreover, due to Proposition 4.5, we have that $f^\varepsilon(t,\cdot)$ has a compact support contained in E for all $t \in [0,T]$.

In order to show convergence as $\varepsilon \to 0$, we pass to the weak formulation of the kinetic equation. Let $\varphi = \varphi(t,x)$ be a function of class $C_c^\infty([0,T)\times\mathbb{R}_+)$. Using the regularity of f^ε and the definition of $Q(f)$, we have

$$\begin{aligned}
&-\varepsilon\int_0^T\int_{\mathbb{R}_+}\frac{\partial}{\partial t}\varphi(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_{\mathbb{R}_+}\varphi(0,x)f^{\mathrm{in}}(x)\,\mathrm{d}x = \varepsilon\int_0^T\int_{\mathbb{R}_+}\varphi(t,x)\frac{\partial}{\partial t}f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
=&\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+\times\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)f^\varepsilon(t,x_*)\varphi(t,x+\varepsilon(x_*-x))\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+\times\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)f^\varepsilon(t,x_*)\varphi(t,x)\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We now develop the test function in ε with respect to x

$$\varphi(t,x+\varepsilon(x_*-x)) = \varphi(t,x) + \varepsilon(x_*-x)\frac{\partial}{\partial x}\varphi(t,x) + O(\varepsilon^2).$$

Since

$$\left|\int_0^T\int_{\mathbb{R}_+\times\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)f^\varepsilon(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t\right| \le T(m_0)^4$$

we have that

$$\int_0^T\int_{\mathbb{R}_+\times\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)f^\varepsilon(t,x_*)O(\varepsilon^2)\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t = O(\varepsilon^2)$$

Hence, dividing by ε, we obtain

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}_+}\frac{\partial}{\partial t}\varphi(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}_+}\varphi(0,x)f^{\mathrm{in}}(x)\,\mathrm{d}x =\\
&=\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+\times\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\frac{\partial}{\partial x}\varphi(t,x)f^\varepsilon(t,x)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+\times\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)\frac{\partial}{\partial x}\varphi(t,x)f^\varepsilon(t,x)f^\varepsilon(t,x_*)x\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon)\\
&=\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\int_{\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x_*)\,\mathrm{d}x_*\int_{\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)x\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon)\\
&= I - II + O(\varepsilon)
\end{aligned}$$

We now analyze the convergence of the above terms.

For all $0 < \varepsilon < 1/2$, let $F^\varepsilon(t,x) := \int_x^{+\infty} f^\varepsilon(t,\xi)\,\mathrm{d}\xi$, then $F^\varepsilon \in C^1([0,T]; W^{1,1}(\mathbb{R}_+))$, $F^\varepsilon(t,\cdot)$ has compact support contained in E and $\frac{\partial}{\partial x}F^\varepsilon(t,x) = -f^\varepsilon(t,x)$ has still compact support contained in E.

- $F^\varepsilon(t,\cdot)$ weak* convergence in $BV(\mathbb{R}_+)$ for a.e. $t \in [0,T]$: For all fixed $t \in [0,T]$, we can see $F^\varepsilon(t,\cdot)$ as a uniformly bounded sequence of $BV(\mathbb{R}_+)$ functions, indeed

$$\|F^\varepsilon(t,\cdot)\|_{BV(\mathbb{R}_+)} = \|F^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}_+)} + \left\|\frac{\partial}{\partial x}F^\varepsilon(t,\cdot)\right\|_{L^1(\mathbb{R}_+)} \le |E|m_0 + m_0 = m_0(x_M+1).$$

  Hence, up to a subsequence that we will denote with the same index, for a.e. $t \in [0,T]$, $F^\varepsilon(t,\cdot)$ converges in $L^1(\mathbb{R}_+)$ to a function $F(t,\cdot)$ that belongs to $BV(\mathbb{R}_+)$, see Theorem 4 in Chapter 5 of [9]. Therefore, we have that $F^\varepsilon(t,\cdot)$ weakly* converges to $F(t,\cdot)$ for a.e. $t \in [0,T]$ in $BV(\mathbb{R}_+)$, see Proposition 3.15 of [1].

  In particular, this implies that $\frac{\partial}{\partial x}F^\varepsilon(t,\cdot) = -f^\varepsilon(t,\cdot)$ weakly* converges to $\frac{\partial}{\partial x}F(t,\cdot) =: -f(t,\cdot)$ for a.e. $t \in [0,T]$, where $f(t,\cdot)$ belongs to the space of Radon measures $\mathcal{M}(\mathbb{R}_+)$ for a.e. $t \in [0,T]$. Hence, for all $\varphi \in C_0(\mathbb{R}_+)$, for a.e. $t \in [0,T]$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\varphi(x)f^\varepsilon(t,x)\,\mathrm{d}x = \int_{\mathbb{R}_+}\varphi(x)\,\mathrm{d}f(t,x). \tag{17}$$

  Morover, taking a particular $\varphi \in C_0(\mathbb{R}_+)$ equal to 1 on $[0,x_M]$, we can prove that $m_0 = \lim_{\varepsilon\to 0}\int_{\mathbb{R}_+} f^\varepsilon(t,x)\,\mathrm{d}x = \int_{\mathbb{R}_+}\mathrm{d}f(t,x) = |f(t,\cdot)|(\mathbb{R}_+)$, where $|f(t,\cdot)|$ denotes the total variation of $f(t,\cdot)$ as a measure. In particular this says that $f \in L^1([0,T],\mathcal{M}(\mathbb{R}_+))$, since $\int_0^T |\int_{\mathbb{R}_+}\mathrm{d}f(t,x)|\,\mathrm{d}t = Tm_0$.
- Weak* convergence of f^ε in $\mathcal{M}([0,T]\times\mathbb{R}_+)$: Since $\|f^\varepsilon\|_{L^1([0,T]\times\mathbb{R}_+)} = Tm_0$, we have that f^ε weakly* converges to $\mu \in \mathcal{M}([0,T]\times\mathbb{R}_+)$, i.e. for all $\varphi \in C_c([0,T]\times\mathbb{R}_+)$

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}_+}\varphi(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}_+}\varphi(t,x)\,\mathrm{d}\mu(t,x)$$

  Let us prove that $\mu = f$. For all $\varphi \in C_c([0,T]\times\mathbb{R}_+)$ and for all $t \in [0,T]$ we have

$$\left|\int_{\mathbb{R}_+} f^\varepsilon(t,x)\varphi(t,x)\,\mathrm{d}x\right| \le m_0\|\varphi\|_{L^\infty([0,T]\times\mathbb{R}_+)} \in L^1([0,T])$$

  and (17) holds for a.e. $t \in [0,T]$, hence, by the dominated convergence Theorem we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}_+}\varphi(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}_+}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t.$$

  This implies that $\mu = f$.

The proven weak* convergence of f^ε to f in $\mathcal{M}([0,T]\times\mathbb{R}_+)$ gives in addiction that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}_+}\frac{\partial}{\partial t}\varphi(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}_+}\frac{\partial}{\partial t}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t,$$

since $\frac{\partial}{\partial t}\varphi \in C_c^\infty([0,T)\times\mathbb{R}_+)$.

We now observe that all the convergences proven for F^ε holds for $(F^\varepsilon)^2$. Hence, in particular, $(F^\varepsilon)^2(t,\cdot)$ converges to $F^2(t,\cdot)$ in $L^1(\mathbb{R}_+)$ for a.e. $t\in[0,T]$, $F^2(t,\cdot)$ belongs to $BV(\mathbb{R}_+)$ for a.e. $t\in[0,T]$ and $\frac{\partial}{\partial x}(F^\varepsilon)^2=-2f^\varepsilon F^\varepsilon$ weakly* converges to $\frac{\partial}{\partial x}(F)^2=-2fF$ in $\mathcal{M}([0,T]\times\mathbb{R}_+)$, with $\frac{\partial}{\partial x}(F)^2\in L^1([0,T],\mathcal{M}(\mathbb{R}_+))$.

Let us analyze I,

$$I=\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\int_{\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

For the first term

$$\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*=\int_{\mathbb{R}_+}-\frac12\frac{\partial}{\partial x}(F^\varepsilon)^2(t,x_*)x_*\,\mathrm{d}x_*.$$

Recalling that $f^\varepsilon(t,\cdot)$ has support contained in E, we can take $\Psi\in C_0(\mathbb{R}_+)$ defined as $\Psi(x):=x$ for all $x\in E$ so that by the weak* convegence of $\frac{\partial}{\partial x}(F^\varepsilon)^2=-2f^\varepsilon F^\varepsilon$ to $\frac{\partial}{\partial x}(F)^2=-2fF$ in $\mathcal{M}(\mathbb{R}_+)$ for a.e. $t\in[0,T]$ we have that

$$\int_{\mathbb{R}_+}-\frac12\frac{\partial}{\partial x}(F^\varepsilon)^2(t,x_*)\Psi(x_*)\,\mathrm{d}x_*$$

converges for a.e. $t\in[0,T]$ to

$$\int_{\mathbb{R}_+}-\frac12\frac{\partial}{\partial x}(F)^2(t,x_*)\Psi(x_*)\,\mathrm{d}x_*.$$

Moreover, we have the following uniform bound

$$\left|\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\right|\le\frac{m_0^2}{2}x_M\in L^1([0,T]).$$

Using again weak* convergence, the second part of the integral is such that, for a.e. $t\in[0,T]$

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x=\lim_{\varepsilon\to0}\int_{\mathbb{R}_+}-\frac12\frac{\partial}{\partial x}(F^\varepsilon)^2(t,x_*)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x$$
$$=\int_{\mathbb{R}_+}-\frac12\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}\frac{\partial}{\partial x}(F)^2(t,x_*)=\int_{\mathbb{R}_+}\frac{\partial}{\partial x}\varphi(t,x)\left(\int_x^{+\infty}\mathrm{d}f(t,\xi)\right)\mathrm{d}f(t,x).$$

Moreover

$$\left|\int_{\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\right|\le\frac{m_0^2}{2}\left\|\frac{\partial}{\partial x}\varphi\right\|_{L^\infty([0,T]\times\mathbb{R}_+)}\in L^1([0,T]).$$

Hence,

$$\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\int_{\mathbb{R}_+}\left(\int_x^{+\infty}f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x$$

converges for a.e. $t\in[0,T]$ and is bounded by $\frac{m_0^4}{4}x_M\left\|\frac{\partial}{\partial x}\varphi\right\|_{L^\infty([0,T]\times\mathbb{R}_+)}$, which belongs to $L^1([0,T])$. By the dominated convergence theorem, we have that I converges to

$$\frac{\eta}{m_0^2}\int_0^T\int_{\mathbb{R}_+}\left(\int_{x_*}^{+\infty}\mathrm{d}f(t,\xi)\right)x_*\,\mathrm{d}f(t,x_*)\int_{\mathbb{R}_+}\left(\int_x^{+\infty}\mathrm{d}f(t,\xi)\right)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t.$$

Let us analyze II. We have

$$II = \frac{\eta}{m_0^2}\int_0^T \int_{\mathbb{R}_+} \left(\int_{x_*}^{+\infty} f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right) f^\varepsilon(t,x_*)\,\mathrm{d}x_* \int_{\mathbb{R}_+} \left(\int_x^{+\infty} f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right) f^\varepsilon(t,x) x\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Now

$$\int_{\mathbb{R}_+} \left(\int_{x_*}^{+\infty} f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right) f^\varepsilon(t,x_*)\,\mathrm{d}x_* = \frac{m_0^2}{2} = \int_{\mathbb{R}_+} \left(\int_{x_*}^{+\infty} \mathrm{d}f(t,\xi)\right)\mathrm{d}f(t,x_*).$$

Moreover $F^\varepsilon f^\varepsilon$ weakly* converges to Ff in $\mathcal{M}([0,T]\times\mathbb{R}_+)$. Hence, since $x\frac{\partial}{\partial x}\varphi \in C_c^\infty([0,T)\times\mathbb{R}_+)$, we have

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\mathbb{R}_+} \left(\int_x^{+\infty} f^\varepsilon(t,\xi)\,\mathrm{d}\xi\right) f^\varepsilon(t,x) x\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}_+} \left(\int_x^{+\infty} \mathrm{d}f(t,\xi)\right) x\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t$$

so that II converges.

Hence, letting $\varepsilon \to 0$ we have

$$-\int_0^T \int_{\mathbb{R}_+} \frac{\partial}{\partial t}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t - \int_{\mathbb{R}_+}\varphi(0,x) f^{\mathrm{in}}(x)\,\mathrm{d}x =$$

$$= \frac{\eta}{m_0^2}\int_0^T \int_{\mathbb{R}_+\times\mathbb{R}_+} \left(\int_x^{+\infty} \mathrm{d}f(t,\xi)\right)\left(\int_{x_*}^{+\infty} \mathrm{d}f(t,\xi)\right)(x_*-x)\,\mathrm{d}f(t,x_*)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t.$$

Therefore $f \in L^1([0,T];\mathcal{M}(\mathbb{R}_+))$ is a distributional solution of

$$\frac{\partial}{\partial t}f(t,x) = \frac{\eta}{m_0^2}\frac{\partial}{\partial x}\left[\int_{\mathbb{R}_+} \left(\int_x^{+\infty} \mathrm{d}f(t,\xi)\right)\left(\int_{x_*}^{+\infty} \mathrm{d}f(t,\xi)\right)(x-x_*) f(t,x)\,\mathrm{d}f(t,x_*)\right],$$

with initial condition $f(0,x) = f^{\mathrm{in}}(x)$ for all $x \in \mathbb{R}_+$. □

6. Numerical simulations

This section collects some numerical results on the model with linear interaction rule, i.e. when the payoff rules of the game are those described in Table 1.

The quantitative results of this section have been obtained by using a particle method. The unknown function f has been discretized by means of a sum of Dirac masses, centered in $x_k(t)$, $1 \le k \le N$, representing a set composed by $N \in \mathbb{N}$ macro-particles that evolve in time.

More precisely, we have approximated the distribution function in this way:

$$f = \sum_{k=1}^N \omega_k\,\delta(x - x_k(t)),$$

where ω_k is the weight of the k-th particle.

Once the number N of numerical particles has been chosen, the problem has been initialized by approximating the initial condition f^{in} with

$$f^{\mathrm{in}}(x,v) = \sum_{k=1}^N \omega_k\,\delta(x - x_k^0),$$

and then the time evolution of the system has been obtained by deducing the time evolution of the macro-particles trough the binary exchange rules (9).

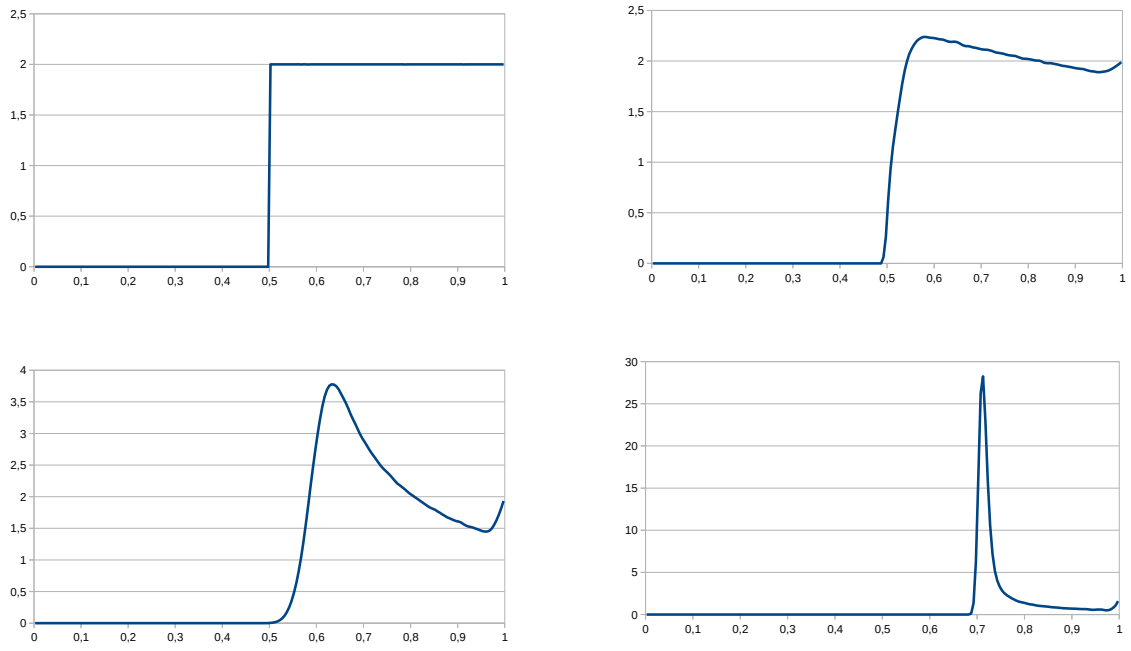

FIGURE 2. Evolution history of the population density with initial condition $f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2 \leq x \leq 1}$, at $t = 0$, $t = 0.1$, $t = 0.65$ and $t = 5$.

All the weights of the particles are identical and their magnitude has been chosen for reproducing the mass of the initial condition, which has been normalized by convenience:

$$\omega_k = \frac{1}{N} \|f^{\text{in}}\|_{L^1(\mathbb{R}_+)} = \frac{1}{N},$$

for all $1 \leq k \leq N$.

The main phenomenon which governs the time evolution of the system is the game defined in Table 1, which allows a net gain to the poorer player only. However, the wealth of the competitor being unknown, the agents choose their strategies according to their relative wealth with respect to the population. Denote the wealth of the i-th player by x_i. Obviously, the probability that the i-th player is richer than than the j-th player is given by

$$\mathbb{P}(x_i > x_j, (i,j)) = \frac{1}{N} \sum_{k=0}^{N} \mathcal{H}(x_i - x_k), \tag{18}$$

where $\mathcal{H}$ is the Heaviside function. Therefore, a generic player i chooses the strategy 1 with a probability that is given by $\mathbb{P}(x_i > x_j, (i,j))$.

The method is based on the modeling hypotheses which are at the basis of the equations themselves: it is hence robust by its own nature and it can also be easily generalized to more complicated games, possibly with high dimensionality. It is moreover very simple to implement

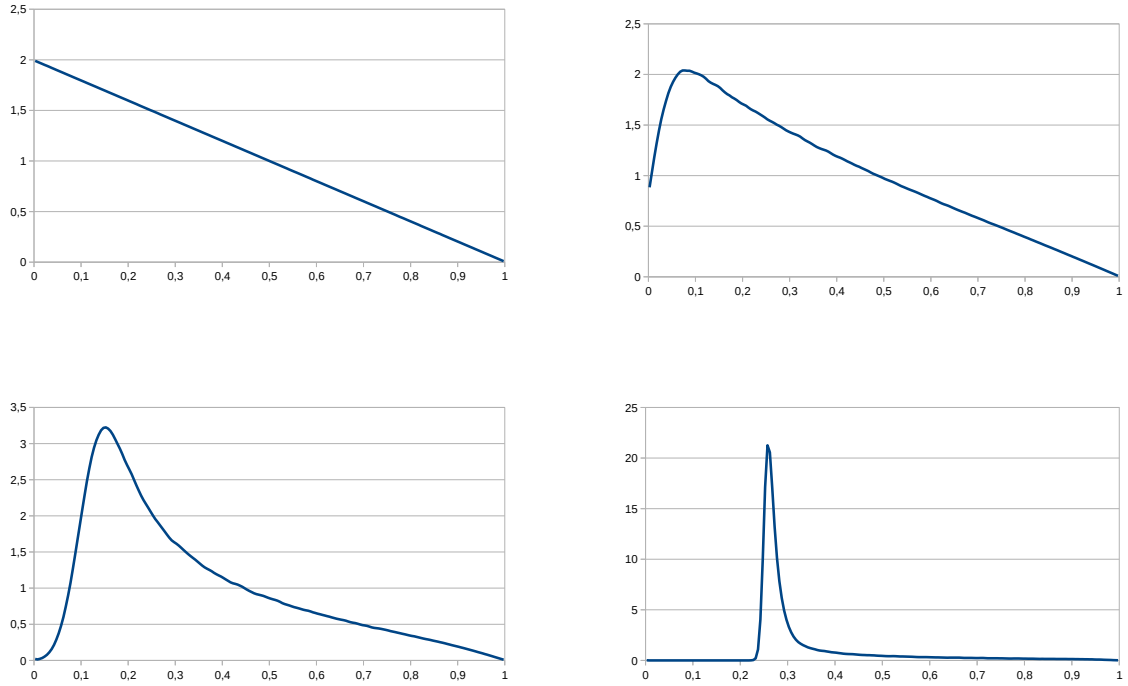

FIGURE 3. Evolution history of the population density with initial condition $f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0\leq x\leq 1}$, at $t = 0$, $t = 0.1$, $t = 0.65$ and $t = 5$.

and the evolution law of the game is treated in an exact way, provided that its analytical formulation is known.

Unfortunately, the number of particles that is needed in order to achieve a reasonable accuracy is very large, and it has a great influence on the performances. In our tests we used $N = 2 \times 10^6$ numerical particles, and a time step $\Delta t = 10^{-2}$. The domain of definition of the problem is the interval $\Omega = [0, 1]$, which has been subdivided in 200 sub-intervals of length $\Delta x = 5 \times 10^{-3}$.

The simulations, written in `C`, heavily needed sequences of random numbers. In our tests, we used the standard `rand()` pseudo-random number generator, which has provided a satisfactory approximation of the uniform distribution.

The numerical reconstruction of the density profile has been obtained by projecting the particles on the mesh by convoluting the Dirac mass which represents the particle and an hat function with mass $1/N$ and base width $4\Delta x$.

For our simulations, we have chosen two initial conditions:

$$f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2\leq x\leq 1}, \qquad f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0\leq x\leq 1}.$$

The results of the simulations with initial condition $f_1^{\text{in}}(x) = 2\times\mathbf{1}_{1/2\leq x\leq 1}$ are plotted in Figure 2. The profiles are ordered from the left to the right and from the top to the bottom and describe the system at various time instants: $t = 0$, $t = 0.1$, $t = 0.65$ and $t = 5$.

We note that the support of the distribution function does not vary with respect to time. However, the density numerically tends to concentrate around the only value which is compatible

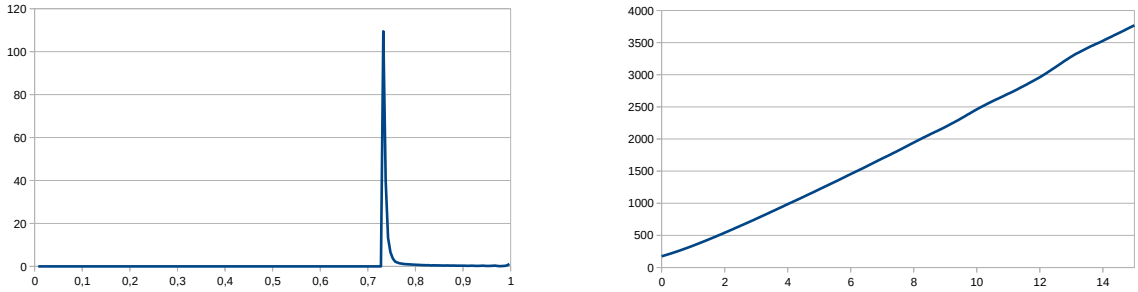

FIGURE 4. Profile of the population density at $t = 15$ (left) and time history of $m_2^{-4/3}$ (right) with initial condition $f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2 \le x \le 1}$.

with the conservation of the first moment, i.e. around $x = 0.75$ (a longer simulation, up to $t = 15$ is in agreement with this expected behavior). We observe moreover that the fraction of the population around the upper bound of the interval, i.e. around $x = 1$ practically does not evolve in time. Indeed the richest agents never play the option 0 (since this event is governed by the probability of finding someone richer). As a consequence, a boundary layer appears in $x = 1$. It is not caused by any boundary conditions (which are absent in our problem), but it is rather a consequence of the rationality of the agents, which tend to maximize their earnings: the richest agents use a conservative strategy, which suggests them to play 1 exclusively.

In Figure 3 we have collected some results of the simulations with initial condition $f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0 \le x \le 1}$. Also in this case, the profiles are ordered from the left to the right and from the top to the bottom and describe the system at the same time instants as in Figure 2: $t = 0$, $t = 0.1$, $t = 0.65$ and $t = 5$.

We numerically observed that the support of the distribution function does not evolve in time. In this case, $f_1^{\text{in}}(1) = 0$, and hence no boundary layer exists around $x = 1$. In this case, the time evolution of the system induces a concentration of the density, which is peaked around $x = 0.33$.

In both cases, the convergence speed to the asymptotic profile (which is expected to be a Dirac mass satisfying the conservation of mass and first order moment) is very slow. The numerical simulations in long time seem to suggest a time decay of order $t^{-3/4}$, as shown in Figures 4 and 5, which represent the profile of the solution at $t = 15$ and the time evolution of $m_2^{-4/3}$, with initial condition $f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2 \le x \le 1}$ and $f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0 \le x \le 1}$ respectively, where

$$m_2(t) := \int_{\mathbb{R}_+} \left| x - \frac{m_1}{m_0} \right|^2 f(t,x)\,\mathrm{d}x.$$

We aim to study the long-time behavior of the model and of its quasi-invariant approximation at the analytical level in the near future.

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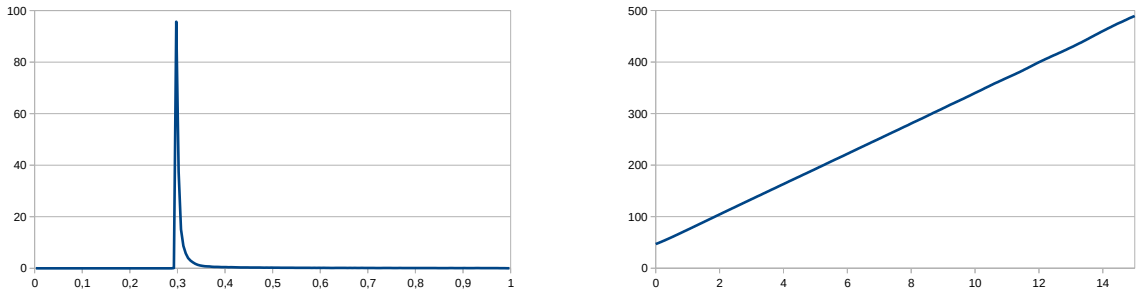

FIGURE 5. Profile of the population density at $t = 15$ (left) and time history of $m_2^{-4/3}$ (right) with initial condition $f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0\leq x\leq 1}$.

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