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# LIMITS OF THE BOUNDARY OF RANDOM PLANAR MAPS

Loïc Richier\*

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## Abstract

We discuss asymptotics for the boundary of critical Boltzmann planar maps under the assumption that the distribution of the degree of a typical face is in the domain of attraction of a stable distribution with parameter  $\alpha \in (1, 2)$ . First, in the dense phase corresponding to  $\alpha \in (1, 3/2)$ , we prove that the scaling limit of the boundary is the random stable looptree with parameter  $(\alpha - 1/2)^{-1}$ . Second, we show the existence of a phase transition through local limits of the boundary: in the dense phase, the boundary is tree-like, while in the dilute phase corresponding to  $\alpha \in (3/2, 2)$ , it has a component homeomorphic to the half-plane. As an application, we identify the limits of loops conditioned to be large in the rigid  $O(n)$  loop model on quadrangulations, proving thereby a conjecture of Curien & Kortchemski.

## 1 Introduction

The purpose of this work is to investigate local limits, in the sense of Angel & Schramm, and scaling limits, in the Gromov-Hausdorff sense, of the boundary of bipartite Boltzmann planar maps conditioned to have a large perimeter.

Given a sequence  $\mathbf{q} = (q_1, q_2, \dots)$  of nonnegative real numbers and a planar map  $\mathbf{m}$  which is bipartite (i.e., with faces of even degree), the associated Boltzmann weight is

$$w_{\mathbf{q}}(\mathbf{m}) := \prod_{f \in \text{Faces}(\mathbf{m})} q_{\deg(f)/2}.$$

The sequence  $\mathbf{q}$  is admissible if these weights form a finite measure on the set of pointed bipartite maps (with a distinguished oriented edge and vertex). We also say that  $\mathbf{q}$  is critical if moreover the expected number of edges of the map is infinite under this measure (see Remark 2.5 for details).

One can then generate a large planar map by choosing it with a probability proportional to its weight among the set of planar maps with  $n$  faces (or vertices). The scaling limits of such large random planar maps have attracted a lot of attention. The first model to be considered was the uniform measure on  $2p$ -angulations, in which all faces have the same

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degree  $2p$ . In this case, Le Gall [41] proved the subsequential convergence towards a random metric space called the *Brownian map*, first introduced by Marckert & Mokkadem in [48] and whose distribution has been characterized later by Le Gall [43] and Miermont [51]. This result has been extended by Le Gall [43] to critical sequences  $\mathbf{q}$  such that the degree of a typical face has small exponential moments (while the first results on this model were obtained by Marckert & Miermont [47]). The result also holds for critical sequences  $\mathbf{q}$  such that the degree of a typical face has a finite variance, as shown in the recent work [49] (such a sequence  $\mathbf{q}$  is called *generic critical*). Convergence towards the Brownian map has also been established in the non-bipartite case in [50, 52]. All these results demonstrate the universality of the Brownian map, whose geometry is now well understood [45, 42].

For a different behaviour to arise, Le Gall & Miermont suggested in [44] to assume, besides criticality, that the degree of a typical face is in the domain of attraction of a stable law with parameter  $\alpha \in (1, 2)$ . The weight sequence  $\mathbf{q}$  is then called *non-generic critical* with parameter  $\alpha$ . Under slightly stronger assumptions, they proved the convergence (along a subsequence) towards a one-parameter family of random metric spaces called the *stable maps* with parameter  $\alpha$ . These maps offer a geometry which is very different from that of the Brownian map, because of large faces that remain present in the scaling limit. Their duals are studied in the recent works [16, 6], but a lot of questions remain open.

The stable maps are believed to have a phase transition at  $\alpha = 3/2$ . The regime  $\alpha \in (1, 3/2)$  is called the *dense phase* because the large faces of the map are supposed to be self-intersecting in the limit, while in the regime  $\alpha \in (3/2, 2)$ , called the *dilute phase*, they are supposed to be self-avoiding. The aim of this work is twofold: first, we identify the branching structure of the large faces in the dense phase via *scaling limits*. Then, we establish the phase transition through *local limits* of large faces.

Precisely, we consider Boltzmann distributions on bipartite maps with a boundary, meaning that the face on the right of the root edge (the root face) is interpreted as the boundary  $\partial\mathbf{m}$  of the map  $\mathbf{m}$ , and receives no weight. Any admissible weight sequence  $\mathbf{q}$  provides a probability measure  $\mathbb{P}_{\mathbf{q}}^{(k)}$  on the set of bipartite maps with perimeter  $\#\partial\mathbf{m} = 2k$ , for every  $k \geq 0$  (see Section 2.2). At large scale, this can be seen as an (unpointed) stable map rooted on a large face. Our main result is the following.

**Theorem 1.1.** *Let  $\mathbf{q}$  be a non-generic critical sequence with parameter  $\alpha \in (1, 3/2)$ . For every  $k \geq 0$ , let  $M_k$  be a random planar map with distribution  $\mathbb{P}_{\mathbf{q}}^{(k)}$ . Then, there exists a slowly varying function  $\Lambda$  such that in distribution for the Gromov-Hausdorff topology,*

$$\frac{\Lambda(k)}{(2k)^{\alpha-1/2}} \cdot \partial M_k \xrightarrow[k \rightarrow \infty]{(d)} \mathcal{L}_{\beta},$$

where  $\mathcal{L}_{\beta}$  is the random stable looptree with parameter

$$\beta := \frac{1}{\alpha - \frac{1}{2}} \in (1, 2).$$

The random stable looptrees  $(\mathcal{L}_{\beta} : \beta \in (1, 2))$  are compact metric spaces introduced by Curien & Kortchemski in [24], that can informally be seen as the random stable trees of Duquesne & Le Gall [28, 29], in which branching points are turned into topological circles. Random stable looptrees also appear as the scaling limits of discrete *looptrees* [24], which are loosely speaking collections of cycles glued along a tree structure. They have Hausdorff dimension  $\beta$  almost surely [24, Theorem 1.1].

The result of Theorem 1.1 covers the dense case, but the subcritical, dilute and generic critical regimes remain open. We believe that in the dilute and generic critical phases, the scaling limit of  $\partial M_k$  is a circle. Furthermore, in the subcritical phase, the *Continuum Random Tree* [1, 2] is expected to arise as a scaling limit. We will discuss these questions in greater detail in Section 4, and hope to investigate them in a future work.

The local limits of Boltzmann bipartite planar maps with a boundary have been studied by Curien in [21]. He proved that for any admissible weight sequence  $\mathbf{q}$ , we have the weak convergence for the local topology

$$\mathbb{P}_{\mathbf{q}}^{(k)} \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{\mathbf{q}}^{(\infty)}.$$

The probability measure  $\mathbb{P}_{\mathbf{q}}^{(\infty)}$  is supported on bipartite maps with an infinite boundary, and called the (law of the) Infinite Boltzmann Half-Planar Map with weight sequence  $\mathbf{q}$  (or  $\mathbf{q}$ -IBHPM for short). We now let  $\mathbf{M}_{\infty} = \mathbf{M}_{\infty}(\mathbf{q})$  be a planar map with distribution  $\mathbb{P}_{\mathbf{q}}^{(\infty)}$ .

We are interested in the behaviour of the boundary  $\partial \mathbf{M}_{\infty}$  of  $\mathbf{M}_{\infty}$ , depending on the weight sequence  $\mathbf{q}$ . In general,  $\partial \mathbf{M}_{\infty}$  is not simple and has self-intersections, called cut vertices (or pinch points). Then,  $\mathbf{M}_{\infty}$  can be decomposed in *irreducible components*, i.e., bipartite maps with a simple boundary attached by cut vertices of  $\partial \mathbf{M}_{\infty}$ . When  $\mathbf{M}_{\infty}$  has a unique infinite irreducible component, we call this component the *core* of  $\mathbf{M}_{\infty}$ . For technical reasons, it is more convenient to study the *scooped-out* map  $\text{Scoop}(\mathbf{M}_{\infty})$  instead of  $\partial \mathbf{M}_{\infty}$ . This map is obtained by duplicating the edges of  $\partial \mathbf{M}_{\infty}$  whose both sides belong to the root face.

It is no surprise that the boundary of  $\mathbf{M}_{\infty}$  is a local limit version of looptrees. Let us briefly sketch their construction, details being postponed to Section 5.2. Given a pair of offspring distributions  $(\rho_{\circ}, \rho_{\bullet})$ , a two-type alternated Galton-Watson tree is a random tree in which vertices at even (resp. odd) height have offspring distribution  $\rho_{\circ}$  (resp.  $\rho_{\bullet}$ ) all independently of each other. As in the monotype case, we can make sense of such trees conditioned to survive, and denote the limiting infinite tree by  $\mathbf{T}_{\infty}^{\circ, \bullet} = \mathbf{T}_{\infty}^{\circ, \bullet}(\rho_{\circ}, \rho_{\bullet})$ . When  $(\rho_{\circ}, \rho_{\bullet})$  is critical (meaning that the product of the means equals one), Stephenson established in [56] that  $\mathbf{T}_{\infty}^{\circ, \bullet}$  is a two-type version of *Kesten's tree* ([37], see also [46]). In particular,  $\mathbf{T}_{\infty}^{\circ, \bullet}$  is locally finite and has a unique *spine*. On the contrary, we will prove in Proposition 5.3, under additional assumptions, that when  $(\rho_{\circ}, \rho_{\bullet})$  is subcritical,  $\mathbf{T}_{\infty}^{\circ, \bullet}$  has a unique vertex with infinite degree (at odd height). This is an expression of the *condensation* phenomenon first observed in the monotype case by Jonsson & Stefánsson ([36], see also [39]). We now define an infinite planar map  $\mathbf{L}_{\infty} = \mathbf{L}_{\infty}(\rho_{\circ}, \rho_{\bullet})$  out of the tree structure given by  $\mathbf{T}_{\infty}^{\circ, \bullet}$ , by taking each vertex at odd height in  $\mathbf{T}_{\infty}^{\circ, \bullet}$  and connecting its neighbours by edges in cyclic order. Therefore,  $\mathbf{L}_{\infty}$  has only finite faces in the critical regime, while a (unique) infinite face arises in the subcritical regime. Note that  $\rho_{\bullet}$  dictates the size of the finite faces in  $\mathbf{L}_{\infty}$ . We can now state our local limit result.

**Theorem 1.2.** *Let  $\mathbf{q}$  be an admissible weight sequence, and  $\mathbf{M}_{\infty} = \mathbf{M}_{\infty}(\mathbf{q})$  the  $\mathbf{q}$ -IBHPM. We assume that  $\mathbf{q}$  is either subcritical, generic critical or non-generic critical with parameter  $\alpha \in (1, 2)$ . Then, there exists probability measures  $\nu_{\circ}$  (geometric) and  $\nu_{\bullet}$  such that*

$$\text{Scoop}(\mathbf{M}_{\infty}) \stackrel{(d)}{=} \mathbf{L}_{\infty}(\nu_{\circ}, \nu_{\bullet}).$$

*A phase transition is observed:*

- *If  $\mathbf{q}$  is subcritical or non-generic critical with parameter  $\alpha \in (1, 3/2]$ ,  $(\nu_{\circ}, \nu_{\bullet})$  is critical and  $\mathbf{M}_{\infty}$  has only finite irreducible components.*

- If  $\mathbf{q}$  is non-generic critical with parameter  $\alpha \in (3/2, 2)$  or generic critical,  $(\nu_\circ, \nu_\bullet)$  is subcritical and  $\mathbf{M}_\infty$  has a well defined core with an infinite simple boundary.

Moreover,  $\nu_\bullet$  has finite variance if and only if  $\mathbf{q}$  is subcritical. Otherwise,  $\nu_\bullet$  is in the domain of attraction of a stable distribution, with parameter  $(\alpha - 1/2)^{-1}$  (if  $\alpha \in (1, 3/2)$ ),  $\alpha - 1/2$  (if  $\alpha \in (3/2, 2)$ ) or  $3/2$  (if  $\mathbf{q}$  is generic critical).

In other words, in the dense phase,  $\mathbf{M}_\infty$  is tree-like, while in the dilute phase, it has an irreducible component homeomorphic to the half-plane on which finite maps are grafted (see Figure 1 for an illustration). In the subcritical and dense phases, the  $\mathbf{q}$ -IBHPM can even be recovered from the infinite looptree  $\mathbf{L}_\infty$  and a collection of independent Boltzmann bipartite maps with a simple boundary, as shown in Proposition 5.8. Note that such collections of random combinatorial structures attached to a tree also appear in the recent work [57]. In the dilute and generic critical regimes, we expect the map with an infinite simple boundary  $\text{Core}(\mathbf{M}_\infty)$  to be the local limit of Boltzmann bipartite maps constrained to have a simple boundary when the perimeter goes to infinity, as shown in the quadrangular case in [25] (see Section 5.3 for more on this). The critical parameter  $\alpha = 3/2$  plays a special role that we discuss in Section 7, under additional assumptions.

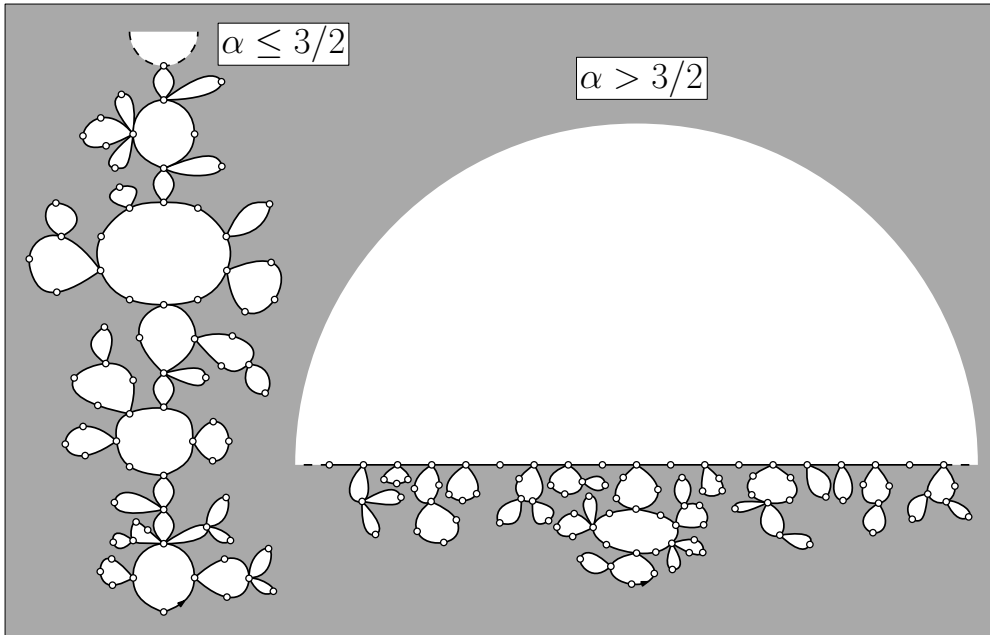


Figure 1: Schematic representation of the boundary of the  $\mathbf{q}$ -IBHPM for  $\mathbf{q}$  non-generic critical with parameter  $\alpha \in (1, 2)$ .

The study of Boltzmann distributions such that  $\mathbf{q}$  is non-generic critical with parameter  $\alpha \in (1, 2)$  is also motivated by the connection with statistical physics models on random maps. Here, we are interested in the *rigid*  $O(n)$  loop model on quadrangulations, studied by Borot, Bouttier & Guitter in [11]. We now give a brief description of this model, and refer to Section 6 for details.

A *loop-decorated* quadrangulation with a boundary  $(\mathbf{q}, \ell)$  is a planar map  $\mathbf{q}$  whose faces all are quadrangles (except the root face), together with a collection of non-crossing loops  $\ell = (\ell_1, \ell_2, \dots)$  drawn on the dual of  $\mathbf{q}$ . The configuration is called *rigid* if loops cross

quadrangles only through their opposite sides. Given  $n \in (0, 2)$  and  $g, h \geq 0$ , we define a measure on loop-decorated quadrangulations by

$$W_{(n;g,h)}((\mathbf{q}, \ell)) := g^{\#\text{Faces}(\mathbf{q}) - |\ell|} h^{|\ell|} n^{\#\ell},$$

where  $|\ell|$  is the total length of the loops and  $\#\ell$  is the number of loops. Provided this measure is finite, it induces a probability measure  $\mathbf{P}_{(n;g,h)}^{(k)}$  on loop-decorated quadrangulations with a boundary of perimeter  $2k$ , for every  $k \geq 0$ . We are particularly interested in the case  $k = 1$ , which corresponds to the rigid  $O(n)$  loop model on quadrangulations of the sphere, simply by gluing the two edges of the boundary together. See Figure 2 for an illustration.

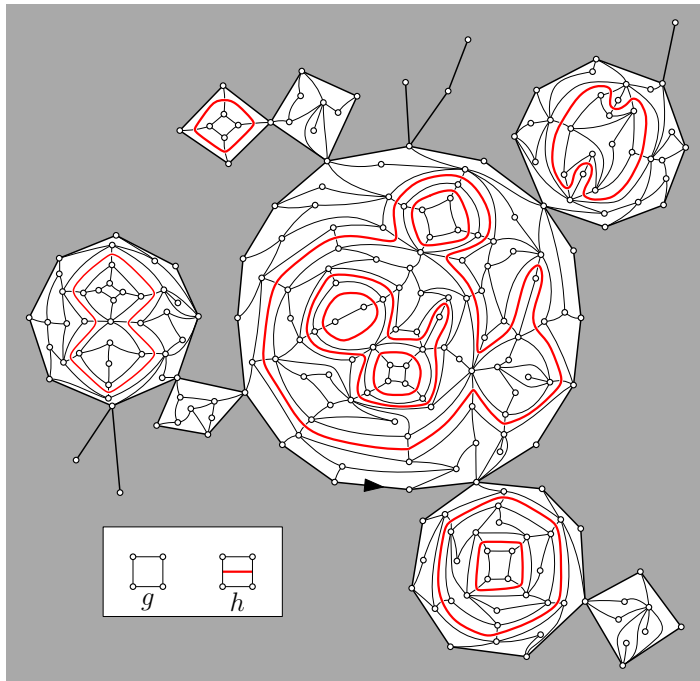


Figure 2: A rigid loop configuration  $\ell$  on a quadrangulation with a boundary  $\mathbf{q}$ .

In [11], Borot, Bouttier & Guitter introduced the *gasket* of a loop-decorated quadrangulation, obtained by pruning the interior of the outermost loops. They proved that in the rigid  $O(n)$  loop model on quadrangulations with perimeter  $2k$ , the gasket is a Boltzmann bipartite planar map with distribution  $\mathbb{P}_{\mathbf{q}}^{(k)}$ , where  $\mathbf{q} = \mathbf{q}(n; g, h)$  is the solution of a certain equation. This leads to a classification of the parameters  $(n; g, h)$  in subcritical and (non-)generic critical regimes, depending on the type of the weight sequence  $\mathbf{q}$ . It has been argued in [11] (and fully justified in [17, Appendix]) that the model admits a complete phase diagram, shown in Figure 3. In particular, for non-generic critical parameters, the gasket is a non-generic Boltzmann bipartite map with parameter  $\alpha$  satisfying

$$\alpha = \frac{3}{2} \pm \frac{1}{\pi} \arccos\left(\frac{n}{2}\right).$$

In this work, we are motivated by the study of the geometry of large loops in the rigid  $O(n)$  loop model on quadrangulations. More generally, the geometry of large interfaces in statistical physics models on random maps is of great interest. In [23], Curien and Kortchemski studied percolation on random triangulations of the sphere. They proved that the boundary of (the

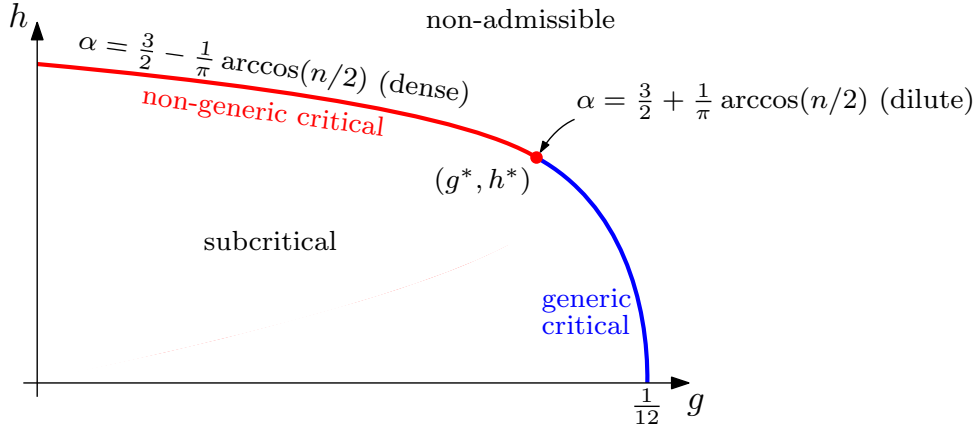


Figure 3: The phase diagram of the rigid  $O(n)$  loop model on quadrangulations. For every  $n \in (0, 2)$ , there exists a critical line  $h = h_c(n; g)$  that separates the subcritical and ill-defined parameters. The regime changes along the critical line. There is a special point  $(g^*(n), h^*(n))$  such that the parameters are non-generic critical with parameter  $\alpha < 3/2$  (dense) for  $g < g^*$ , and generic critical for  $g > g^*$ . The special point  $(g^*, h^*)$  itself is non-generic critical with parameter  $\alpha > 3/2$  (dilute).

hull of) a critical percolation cluster conditioned to be large converges after proper rescaling towards the random stable looptree with parameter  $3/2$ . They also conjecture that the whole family  $(\mathcal{L}_\beta : \beta \in (1, 2))$  appears as a scaling limit of large loops in the  $O(n)$  model on triangulations. The following application of Theorem 1.1 proves this conjecture for the rigid  $O(n)$  loop model on quadrangulations.

**Theorem 1.3.** *Let  $n \in (0, 2)$ ,  $g \in [0, g^*(n))$  and  $h := h_c(n; g)$ . For every  $k \geq 0$ , let  $(Q_k, L_k)$  be a loop-decorated quadrangulation with distribution  $\mathbf{P}_{(n;g,h)}^{(k)}$ . Then, there exists a constant  $C = C(n, g, h)$  such that in distribution for the Gromov-Hausdorff topology,*

$$\frac{C}{(2k)^{1/\beta}} \cdot \partial Q_k \xrightarrow[k \rightarrow \infty]{(d)} \mathcal{L}_\beta,$$

where  $\mathcal{L}_\beta$  is the random stable looptree with parameter

$$\beta := \left(1 - \frac{1}{\pi} \arccos\left(\frac{n}{2}\right)\right)^{-1} \in (1, 2).$$

Note that the value of  $\beta$  in Theorem 1.3 fits the prediction of [23]. We also obtain local limit results regarding large loops of the  $O(n)$  model from Theorem 1.2 and its proof.

**Theorem 1.4.** *Let  $n \in (0, 2)$  and  $g, h \geq 0$  such that  $(n; g, h)$  is admissible. For every  $k \geq 0$ , let  $(Q_k, L_k)$  be a loop-decorated quadrangulation with distribution  $\mathbf{P}_{(n;g,h)}^{(k)}$ . Then, there exists probability measures  $\nu_\circ$  (geometric) and  $\nu_\bullet$  such that in distribution for the local topology,*

$$\text{Scoop}(Q_k) \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{L}_\infty(\nu_\circ, \nu_\bullet).$$

Moreover,

- If  $(n; g, h)$  is subcritical,  $(\nu_\circ, \nu_\bullet)$  is critical and  $\nu_\bullet$  has finite variance.
- If  $h = h_c(n; g)$  and  $g < g^*(n)$  (the dense case),  $(\nu_\circ, \nu_\bullet)$  is critical and  $\nu_\bullet$  is in the domain of attraction of a stable distribution with parameter  $(1 - \frac{1}{\pi} \arccos(\frac{n}{2}))^{-1}$ .
- If  $(h, g) = (h^*(n), g^*(n))$  (the dilute case),  $(\nu_\circ, \nu_\bullet)$  is subcritical and  $\nu_\bullet$  is in the domain of attraction of a stable distribution with parameter  $1 + \frac{1}{\pi} \arccos(\frac{n}{2})$ .
- If  $h = h_c(n; g)$  and  $g > g^*(n)$  (the generic critical case),  $(\nu_\circ, \nu_\bullet)$  is subcritical and  $\nu_\bullet$  is in the domain of attraction of a stable distribution with parameter  $3/2$ .

This result should be compared to the local limit of critical percolation clusters conditioned to be large in triangulations of the half-plane, studied in [54].

At first glance, Theorems 1.3 and 1.4 hold only for the boundary of loop-decorated quadrangulations. However, by the gasket decomposition, they apply to *any* loop conditioned to be large in the rigid  $O(n)$  loop model. To make it more concrete, one can choose any deterministic procedure to pick a loop in the rigid  $O(n)$  loop model on quadrangulations of the sphere (e.g. the loop that is the closest to the root edge) and condition this loop to have perimeter  $2k$ . Then, the *inner contour* of this loop is the boundary of a loop-decorated quadrangulation with distribution  $\mathbf{P}_{(n;g,h)}^{(k)}$ . See Proposition 6.1 and Remark 6.2 for more details.

Our approach is based on the decomposition of bipartite planar maps with a general boundary into a tree of bipartite planar maps with a simple boundary, which is inspired by [23] and described in Section 3. In order to deduce the results from this decomposition, we need estimates on the partition function of bipartite maps with a simple boundary. This is done in Section 2.3, by means of a simple relation between the generating functions of bipartite maps with a general (resp. simple) boundary (see Lemma 2.10).

This method is quite robust, and only needs estimates on the partition function of the model as an input. For this reason, we believe that our proofs can be adapted to more general statistical physics models on random planar maps for which Borot, Bouttier & Guitter proved results similar to those of [11]. For instance, general  $O(n)$  loop models on triangulations with *bending energy* [10] or *domain symmetry breaking* [9]. This last case covers in particular the *Potts model* and *Fortuin-Kasteleyn percolation* on general planar maps, that have been studied in [5, 55, 31, 32, 33, 19]. An interesting example is the critical Bernoulli percolation model on random triangulations, treated in [10, Section 4.2, p.23]. This corresponds to a  $O(n)$  loop model on triangulations for  $n = 1$  and a suitable choice of the parameters. The asymptotics are similar to the quadrangular case, and we get the exponent  $\beta = (1 - \arccos(1/2)/\pi)^{-1} = 3/2$ , which is consistent with the result of [23].

## 2 Boltzmann distributions

**Notation.** Throughout this work, we use the notation

$$\mathbb{N} := \{1, 2, \dots\} \quad \text{and} \quad \mathbb{Z}_+ := \mathbb{N} \cup \{0\}.$$



## 2.1 Boltzmann distributions on bipartite maps

**Maps.** A *planar map* is a proper embedding of a finite connected graph in the two-dimensional sphere  $\mathbb{S}^2$ , considered up to orientation-preserving homeomorphisms. The faces of a planar map are the connected components of the complement of the embedding, and the degree  $\deg(f)$  of a face  $f$  is the number of its incident oriented edges. The sets of vertices, edges and faces of a planar map  $\mathbf{m}$  are denoted by  $V(\mathbf{m})$ ,  $E(\mathbf{m})$  and  $F(\mathbf{m})$ . For technical reasons, the planar maps we consider are always *rooted*, which means that an oriented edge  $e_* = (e_-, e_+)$ , called the *root edge*, is distinguished. The face  $f_*$  incident on the right of the root edge is called the *root face*. A planar map *with a boundary*  $\mathbf{m}$  is a map in which we consider the root face as an *external face*, whose incident edges and vertices form the *boundary*  $\partial\mathbf{m}$  of the map. The non-root faces are then called *internal* and the degree  $\#\partial\mathbf{m}$  of the external face is the *perimeter* of the map.

In this paper, we consider *bipartite* planar maps, in which all face degrees are even. We denote by  $\mathcal{M}$  set of bipartite planar maps, and by  $\mathcal{M}_k$  be the set of bipartite planar maps with perimeter  $2k$ , for  $k \geq 0$ . By convention, the “vertex map”  $\dagger$  consisting of a single vertex is considered as the only element of  $\mathcal{M}_0$ . We will also consider *pointed* bipartite maps, which have a marked vertex  $v_*$ . A pointed bipartite map  $\mathbf{m}$  such that  $d_{\mathbf{m}}(e_+, v_*) = d_{\mathbf{m}}(e_-, v_*) + 1$  is said to be *positive*, and the corresponding set is denoted by  $\mathcal{M}_+^\bullet$  (by convention,  $\dagger \in \mathcal{M}_+^\bullet$ ). Finally, we use the notation  $M$  for the identity mapping on  $\mathcal{M}$ .

**Boltzmann distributions.** We now recall the construction of Boltzmann distributions, and first deal with positive bipartite maps. Given a *weight sequence*  $\mathbf{q} = (q_k : k \in \mathbb{N})$  of nonnegative real numbers, the *Boltzmann weight* of a bipartite planar map  $\mathbf{m}$  is defined by

$$w_{\mathbf{q}}(\mathbf{m}) := \prod_{f \in F(\mathbf{m})} q_{\deg(f)/2}. \quad (1)$$

By convention, we set  $w_{\mathbf{q}}(\dagger) = 1$ . This defines a  $\sigma$ -finite measure on  $\mathcal{M}_+^\bullet$  with total mass (or *partition function*)

$$Z_{\mathbf{q}} := w_{\mathbf{q}}(\mathcal{M}_+^\bullet) \in [1, \infty]. \quad (2)$$

Naturally, a weight sequence  $\mathbf{q}$  is said to be *admissible* if  $Z_{\mathbf{q}} < \infty$ . This is our basic assumption in the next part. Then, the Boltzmann distribution  $\mathbb{P}_{\mathbf{q}}^\bullet$  associated to  $\mathbf{q}$  is defined by

$$\mathbb{P}_{\mathbf{q}}^\bullet(\mathbf{m}) := \frac{w_{\mathbf{q}}(\mathbf{m})}{Z_{\mathbf{q}}}, \quad \mathbf{m} \in \mathcal{M}_+^\bullet.$$

Following [47], we introduce the function

$$f_{\mathbf{q}}(x) := \sum_{k=1}^{\infty} \binom{2k-1}{k-1} q_k x^{k-1}, \quad x \geq 0, \quad (3)$$

whose radius of convergence is denoted by  $R_{\mathbf{q}}$ . By [47, Proposition 1], a weight sequence  $\mathbf{q}$  is admissible iff the equation

$$f_{\mathbf{q}}(x) = 1 - \frac{1}{x}, \quad x > 0 \quad (4)$$

has a solution. In that case, the smallest such solution is  $Z_{\mathbf{q}}$  and  $Z_{\mathbf{q}}^2 f_{\mathbf{q}}'(Z_{\mathbf{q}}) \leq 1$ . In particular, we have  $Z_{\mathbf{q}} \in (1, R_{\mathbf{q}}]$ . The following definitions were introduced in [47, 11].

**Definition 2.1.** An admissible weight sequence  $\mathbf{q}$  is *critical* if  $Z_{\mathbf{q}}^2 f'_{\mathbf{q}}(Z_{\mathbf{q}}) = 1$ , and *subcritical* otherwise. A critical weight sequence is *regular critical* if  $Z_{\mathbf{q}} < R_{\mathbf{q}}$ , *generic critical* if  $f''_{\mathbf{q}}(R_{\mathbf{q}}) < \infty$ , and *non-generic critical* otherwise.

The function  $f_{\mathbf{q}}$  being of class  $C^\infty$  on  $(0, R_{\mathbf{q}})$ , a regular weight sequence is indeed generic.

The classification of weight sequences is closely related to the Bouttier-Di Francesco-Guitter bijection [12], that associates to every map  $\mathbf{m} \in \mathcal{M}_+^\bullet$  a tree  $\Phi_{\text{BDG}}(\mathbf{m})$  (together with labels on vertices at even height). The study is simplified by using additionally a bijection  $\Phi_{\text{JS}}$  due to Janson and Stefánsson [35, Section 3]. This bijection will be of independent interest in the next part, so we give a detailed presentation in Section 3.1. We are interested in the application that associates to  $\mathbf{m} \in \mathcal{M}_+^\bullet$  the tree  $\Phi(\mathbf{m}) := \Phi_{\text{JS}}(\Phi_{\text{BDG}}(\mathbf{m}))$ . By combining [47, Proposition 7] and [35, Appendix A] (see also Proposition 3.1), we get the following.

**Lemma 2.2.** *Let  $\mathbf{q}$  be an admissible weight sequence. Then, under  $\mathbb{P}_{\mathbf{q}}^\bullet$ , the plane tree  $\Phi(M)$  is a Galton-Watson tree with offspring distribution  $\mu$  defined by*

$$\mu(0) = 1 - f_{\mathbf{q}}(Z_{\mathbf{q}}) \quad \text{and} \quad \mu(k) = Z_{\mathbf{q}}^{k-1} \binom{2k-1}{k-1} q_k, \quad k \in \mathbb{N}.$$

The definition of Galton-Watson trees is postponed to Section 3.1. Recall that the offspring distribution  $\mu$  is called *critical* (resp. *subcritical*) iff it has mean  $m_\mu = 1$  (resp.  $m_\mu < 1$ ). Lemma 2.2 has a simple expression in terms of the generating function  $G_\mu$  of  $\mu$ , which reads

$$G_\mu(s) := \sum_{k=0}^{\infty} s^k \mu(k) = 1 - f_{\mathbf{q}}(Z_{\mathbf{q}}) + s f_{\mathbf{q}}(s Z_{\mathbf{q}}), \quad s \in [0, 1]. \quad (5)$$

The notions of criticality, regularity and genericity are now easily defined in terms of the offspring distribution  $\mu$ . First, we find

$$m_\mu = 1 - \frac{1 - Z_{\mathbf{q}}^2 f'_{\mathbf{q}}(Z_{\mathbf{q}})}{Z_{\mathbf{q}}}, \quad (6)$$

and assuming  $m_\mu = 1$ , the variance  $\sigma_\mu^2$  of  $\mu$  reads

$$\sigma_\mu^2 = Z_{\mathbf{q}}^2 f''_{\mathbf{q}}(Z_{\mathbf{q}}) + \frac{2}{Z_{\mathbf{q}}}. \quad (7)$$

We obtain from Definition 2.1 the following result.

**Proposition 2.3.** *An admissible weight sequence  $\mathbf{q}$  is critical iff  $\mu$  is critical. A critical weight sequence  $\mathbf{q}$  is regular critical iff  $\mu$  has small exponential moments, and generic critical (resp. non-generic critical) iff  $\mu$  has finite (resp. infinite) variance.*

In this paper, we are particularly interested in the non-generic critical case.

**Definition 2.4.** A weight sequence  $\mathbf{q}$  is *non-generic critical* with parameter  $\alpha \in (1, 2)$  if  $\mathbf{q}$  is critical and the distribution  $\mu$  is in the domain of attraction of a stable law with parameter  $\alpha$ : there exists a slowly varying function  $\ell$  on  $\mathbb{R}_+$  (eventually positive) such that

$$\mu([k, \infty)) = \frac{\ell(k)}{k^\alpha}.$$

Recall that by definition, a measurable function  $\ell$  is slowly varying (at infinity) if it satisfies  $\ell(\lambda x)/\ell(x) \rightarrow 1$  as  $x \rightarrow \infty$ , for every  $\lambda > 0$ . We emphasize that Definition 2.4 is slightly more general than that introduced in [44], which implies that the slowly varying function  $\ell$  is asymptotically constant (and is also the framework in [11, 16, 6, 21]).

**Remark 2.5.** The classification of weight sequences can be translated in terms of  $\mathbb{P}_{\mathbf{q}}^{\bullet}$  by properties of  $\Phi_{\text{BDG}}$ . First, we have  $\mathbb{E}_{\mathbf{q}}^{\bullet}(\#E(M)) = \infty$  iff  $\mathbf{q}$  is critical. Then, the probability measure  $\mu_{\bullet}(k) := \mu(k+1)/f_{\mathbf{q}}(Z_{\mathbf{q}})$  is interpreted as the law of (half) the degree of a typical face of the map under  $\mathbb{P}_{\mathbf{q}}^{\bullet}$ . Thus, a critical sequence  $\mathbf{q}$  is regular critical (resp. generic critical) iff the degree of a typical face has small exponential moments (resp. finite variance). Moreover,  $\mathbf{q}$  is non-generic critical with parameter  $\alpha \in (1, 2)$  iff the degree of a typical face is in the domain of attraction of a stable distribution with parameter  $\alpha$ .

We conclude by translating Proposition 2.3 in terms of the Laplace transform  $L_{\mu}$  of  $\mu$ . First, if  $\mathbf{q}$  is subcritical,  $\mu$  has finite mean  $m_{\mu} < 1$  and

$$L_{\mu}(t) := G_{\mu}(e^{-t}) = 1 - m_{\mu}t + o(t) \quad \text{as } t \rightarrow 0^+. \quad (8)$$

When  $\mathbf{q}$  is generic critical,  $\mu$  has mean  $m_{\mu} = 1$  and finite variance  $\sigma_{\mu}^2$  which yields

$$L_{\mu}(t) = 1 - t + \frac{\sigma_{\mu}^2 + 1}{2}t^2 + o(t^2) \quad \text{as } t \rightarrow 0^+. \quad (9)$$

The situation is different when  $\mathbf{q}$  is non-generic critical with parameter  $\alpha \in (1, 2)$ . By Karamata's Abelian theorem [8, Theorem 8.1.6], we get

$$L_{\mu}(t) = 1 - t + |\Gamma(1 - \alpha)|t^{\alpha}\ell(1/t) + o(t^{\alpha}\ell(1/t)) \quad \text{as } t \rightarrow 0^+. \quad (10)$$

## 2.2 Boltzmann distributions on maps with a boundary

We now deal with maps that have a boundary. The root face  $f_*$  is then considered as external to the map, and receives no weight. This amounts to using the Boltzmann weights

$$w_{\mathbf{q}}(\mathbf{m}) := \prod_{f \in F(\mathbf{m}) \setminus \{f_*\}} q_{\deg(f)/2}. \quad (11)$$

Let us introduce the partition functions for bipartite maps with a fixed perimeter

$$F_k := \sum_{\mathbf{m} \in \mathcal{M}_k} w_{\mathbf{q}}(\mathbf{m}), \quad k \in \mathbb{Z}_+, \quad (12)$$

where we hide the dependence in the sequence  $\mathbf{q}$  in the notation. These quantities are finite if  $\mathbf{q}$  is admissible. For  $\mathbf{q} = 0$ ,  $F_k$  is the  $k$ -th Catalan number. In particular, for any weight sequence  $\mathbf{q}$ ,  $F_k > 0$  for every  $k \geq 0$  (and  $F_0 = 1$ ). The associated Boltzmann measure on bipartite maps with fixed perimeter is defined by

$$\mathbb{P}_{\mathbf{q}}^{(k)}(\mathbf{m}) := \frac{\mathbf{1}_{\{\mathbf{m} \in \mathcal{M}_k\}} w_{\mathbf{q}}(\mathbf{m})}{F_k}, \quad \mathbf{m} \in \mathcal{M}, \quad k \in \mathbb{Z}_+. \quad (13)$$

This is the probability measure we focus on. The aim of this section is to give asymptotics for the partition function  $F_k$ . For this purpose, we define the generating function

$$F(x) := \sum_{k=0}^{\infty} F_k x^k, \quad x \geq 0, \quad (14)$$

whose radius of convergence is denoted by  $r_{\mathbf{q}}$ . We borrow ideas of [11, Section 3.1] and [21, Section 5.1], but we need to extend these results due to our more general definition of a non-generic critical weight sequence. The idea is to let the (admissible) weight sequence  $\mathbf{q}$  vary by adding a weight  $u \in [0, 1]$ . We let  $\mathbf{q}(u) := (u^{k-1}\mathbf{q}_k : k \in \mathbb{N})$ , and set  $Z_{\mathbf{q}(u)} := Z_{\mathbf{q}(u)}$ . Using the universal form of the generating function for pointed Boltzmann maps [15, Proposition 2, Section A.1] and Euler's formula, we obtain (see [21, Equation (5.2)])

$$F_k = \binom{2k}{k} \int_0^1 (uZ_{\mathbf{q}(u)})^k du, \quad k \in \mathbb{Z}_+. \quad (15)$$

In the setting of [44], the asymptotics of  $F_k$  would follow from Laplace's method, see [11, 21]. Here, we use a different technique based on Karamata's Tauberian theorem. First, the function  $X_{\mathbf{q}}(u) := uZ_{\mathbf{q}}(u)$  is increasing from  $[0, 1]$  to  $[0, Z_{\mathbf{q}}]$ . It is also continuous as a normally converging series of continuous functions and thus invertible on  $[0, 1]$ , with inverse denoted by  $Y_{\mathbf{q}}$ . Since  $Z_{\mathbf{q}}(u)$  is the smallest solution of (4) with  $\mathbf{q} = \mathbf{q}(u)$ , we have by (5)

$$Y_{\mathbf{q}}(x) = x - x f_{\mathbf{q}}(x) = 1 + x - Z_{\mathbf{q}} G_{\mu}(x/Z_{\mathbf{q}}), \quad x \in [0, Z_{\mathbf{q}}].$$

This proves that  $Y_{\mathbf{q}}$  is of class  $C^{\infty}$  on  $(0, Z_{\mathbf{q}})$ . Coming back to the integral in (15),

$$\int_0^1 (uZ_{\mathbf{q}}(u))^k du = \int_0^{Z_{\mathbf{q}}} x^k Y_{\mathbf{q}}'(x) dx = Z_{\mathbf{q}}^{k+1} \int_0^{\infty} e^{-t(k+1)} Y_{\mathbf{q}}'(Z_{\mathbf{q}} e^{-t}) dt.$$

We now introduce the increasing function

$$U(t) := \int_0^t Z_{\mathbf{q}} e^{-u} Y_{\mathbf{q}}'(Z_{\mathbf{q}} e^{-u}) du = 1 - Y_{\mathbf{q}}(Z_{\mathbf{q}} e^{-t}) = -Z_{\mathbf{q}} e^{-t} + Z_{\mathbf{q}} L_{\mu}(t), \quad t \geq 0.$$

On the one hand, the integral is expressed in terms of the Laplace transform of  $U$ :

$$\int_0^1 (uZ_{\mathbf{q}}(u))^k du = Z_{\mathbf{q}}^k \int_0^{\infty} e^{-kt} U(dt),$$

and on the other hand from (8), (9) and (10), as  $t \rightarrow 0^+$ ,

$$U(t) = \begin{cases} Z_{\mathbf{q}}(1 - m_{\mu})t + o(t) & (\mathbf{q} \text{ subcritical}) \\ Z_{\mathbf{q}}\sigma_{\mu}^2 t^2/2 + o(t^2) & (\mathbf{q} \text{ generic critical}) \\ Z_{\mathbf{q}}|\Gamma(1 - \alpha)|t^{\alpha}\ell(1/t) + o(t^{\alpha}\ell(1/t)) & (\mathbf{q} \text{ non-generic critical } \alpha) \end{cases}.$$

We can thus apply Karamata's Tauberian theorem [8, Theorem 1.7.1']. Using also Stirling's formula for the binomial coefficient in (15), we get

$$F_k \underset{k \rightarrow \infty}{\sim} \begin{cases} \frac{Z_{\mathbf{q}}(1 - m_{\mu})(4Z_{\mathbf{q}})^k}{\sqrt{\pi}k^{3/2}} & (\mathbf{q} \text{ subcritical}) \\ \frac{Z_{\mathbf{q}}\sigma_{\mu}^2(4Z_{\mathbf{q}})^k}{\sqrt{\pi}k^{5/2}} & (\mathbf{q} \text{ generic critical}) \\ \frac{Z_{\mathbf{q}}\alpha\sqrt{\pi}(4Z_{\mathbf{q}})^k\ell(k)}{\sin(\pi(\alpha - 1))k^{\alpha+1/2}} & (\mathbf{q} \text{ non-generic critical } \alpha) \end{cases}, \quad (16)$$

where we used the identity  $\Gamma(1 - \alpha)\Gamma(1 + \alpha) = \alpha\pi/\sin(\pi\alpha)$  for  $\alpha \in (1, 2)$ . The quantity  $a := \alpha + 1/2$  is of particular importance and governs the asymptotic behaviour of the partition function. Following [21], we introduce a notation for weight sequences.

**Notation.** An admissible weight sequence  $\mathbf{q}$  is said of type  $a = 3/2$  if it is subcritical, of type  $a = 5/2$  if it is generic critical and of type  $a \in (3/2, 5/2)$  if it is non-generic critical with parameter  $\alpha = a - 1/2$ .

This allows us to write (16) in a unified way. Let

$$c_{3/2} := \frac{Z_{\mathbf{q}}(1 - m_{\mu})}{\sqrt{\pi}}, \quad c_{5/2} := \frac{Z_{\mathbf{q}}\sigma_{\mu}^2}{\sqrt{\pi}} \quad \text{and} \quad c_a := \frac{Z_{\mathbf{q}}(a - 1/2)\sqrt{\pi}}{\sin(\pi(a - 3/2))}, \quad a \in (3/2, 5/2), \quad (17)$$

and set the convention that  $\ell = 1$  if  $a \in \{3/2, 5/2\}$ . Then,

$$F_k \underset{k \rightarrow \infty}{\sim} \frac{c_a(4Z_{\mathbf{q}})^k \ell(k)}{k^a}, \quad a \in [3/2, 5/2]. \quad (18)$$

We now derive from these asymptotics a singular expansion for the generating function  $F$ , whose radius of convergence is  $r_{\mathbf{q}} = (4Z_{\mathbf{q}})^{-1}$ . In particular,  $0 < r_{\mathbf{q}} \leq 1/4$  if  $\mathbf{q}$  is admissible. We also have  $1 < F(r_{\mathbf{q}}) < \infty$ , and  $F'(r_{\mathbf{q}}) < \infty$  iff  $a \in (2, 5/2]$ . For  $k \geq 0$ , let

$$\zeta(k) := \frac{F_k r_{\mathbf{q}}^k}{F(r_{\mathbf{q}})} \underset{k \rightarrow \infty}{\sim} \frac{c_a \ell(k)}{k^a F(r_{\mathbf{q}})}.$$

The function  $k \mapsto k^{-a}\zeta(k)$  is slowly varying, so that Karamata's theorem (direct half) [8, Proposition 1.5.10] yields

$$\sum_{j \geq k} \zeta(j) \underset{k \rightarrow \infty}{\sim} (a - 1)k\zeta(k) \underset{k \rightarrow \infty}{\sim} \frac{c_a \ell(k)}{k^{a-1} F(r_{\mathbf{q}})}.$$

We can now apply Karamata's Abelian theorem [8, Theorem 8.1.6] to get the asymptotics of the Laplace transform  $L_{\zeta}$  of  $\zeta$ . For  $a \in [3/2, 2)$ , we find

$$L_{\zeta}(t) = 1 - \frac{\Gamma(2 - a)c_a}{F(r_{\mathbf{q}})} t^{a-1} \ell(1/t) + o(t^{a-1} \ell(1/t)) \quad \text{as } t \rightarrow 0^+,$$

while for  $a \in (2, 5/2]$ ,

$$L_{\zeta}(t) = 1 - m_{\zeta} t + \frac{|\Gamma(2 - a)|c_a}{F(r_{\mathbf{q}})} t^{a-1} \ell(1/t) + o(t^{a-1} \ell(1/t)) \quad \text{as } t \rightarrow 0^+.$$

The function  $\ell_1$  defined for  $y \geq 0$  by  $\ell_1(y) = \ell(-(\log(1 - 1/y))^{-1})$  is slowly varying at infinity by stability properties of slowly varying functions [8, Proposition 1.3.6]. We obtain from the formula  $G_{\zeta}(s) = L_{\zeta}(-\log(s))$  that for  $a \in [3/2, 2)$ ,

$$G_{\zeta}(s) = 1 - \frac{\Gamma(2 - a)c_a}{F(r_{\mathbf{q}})} (1 - s)^{a-1} \ell_1\left(\frac{1}{1 - s}\right) (1 + o(1)) \quad \text{as } s \rightarrow 1^-,$$

and for  $a \in (2, 5/2]$ ,

$$G_{\zeta}(s) = 1 - m_{\zeta}(1 - s) + \frac{|\Gamma(2 - a)|c_a}{F(r_{\mathbf{q}})} (1 - s)^{a-1} \ell_1\left(\frac{1}{1 - s}\right) (1 + o(1)) \quad \text{as } s \rightarrow 1^-.$$

The singular expansion of  $F$  follow from the identity  $F(xr_{\mathbf{q}}) = F(r_{\mathbf{q}})G_{\zeta}(x)$ . Note that  $m_{\zeta} = r_{\mathbf{q}}F'(r_{\mathbf{q}})/F(r_{\mathbf{q}})$ , and let  $\kappa_a := c_a|\Gamma(2 - a)|$ . Recall also that  $\ell_1 = 1$  for  $a \in \{3/2, 5/2\}$ .

**Proposition 2.6.** *Let  $\mathbf{q}$  be a weight sequence of type  $a$ . For  $a \in [3/2, 2)$ ,*

$$F(x) = F(r_{\mathbf{q}}) - \kappa_a \left(1 - \frac{x}{r_{\mathbf{q}}}\right)^{a-1} \ell_1 \left(\frac{1}{1 - \frac{x}{r_{\mathbf{q}}}}\right) (1 + o(1)) \quad \text{as } x \rightarrow r_{\mathbf{q}}^-,$$

and for  $a \in (2, 5/2]$ ,

$$F(x) = F(r_{\mathbf{q}}) - r_{\mathbf{q}} F'(r_{\mathbf{q}}) \left(1 - \frac{x}{r_{\mathbf{q}}}\right) + \kappa_a \left(1 - \frac{x}{r_{\mathbf{q}}}\right)^{a-1} \ell_1 \left(\frac{1}{1 - \frac{x}{r_{\mathbf{q}}}}\right) (1 + o(1)) \quad \text{as } x \rightarrow r_{\mathbf{q}}^-.$$

**Remark 2.7.** This method fails in the special case  $a = 2$ , which is momentarily excluded. Indeed, Karamata's Abelian theorem [8, Theorem 8.1.6] requires a different assumption for integer powers [8, Equation (8.1.11c)] that we cannot prove to be satisfied in general (see [8, Proposition 1.5.8] and the comments below). This issue can be bypassed by making additional assumptions on the weight sequence, which is done in Section 7.

### 2.3 Boltzmann distributions on maps with a simple boundary

The aim of this section is to obtain asymptotics for bipartite maps that have a simple boundary, which will be of particular interest in the next part. A planar map with a *simple boundary* is a planar map whose boundary is a cycle with no self-intersection. Their set is denoted by  $\widehat{\mathcal{M}}$ . Consistently, for every  $k \geq 0$ ,  $\widehat{\mathcal{M}}_k$  is the set of bipartite maps with a simple boundary of perimeter  $2k$ . A generic element of  $\widehat{\mathcal{M}}$  is denoted by  $\widehat{\mathbf{m}}$ , and  $\dagger \in \widehat{\mathcal{M}}_0$  by convention.

The partition function for bipartite maps with a simple boundary and fixed perimeter is

$$\widehat{F}_k := \sum_{\mathbf{m} \in \widehat{\mathcal{M}}_k} w_{\mathbf{q}}(\mathbf{m}), \quad k \in \mathbb{Z}_+. \quad (19)$$

These are finite if  $\mathbf{q}$  is admissible. The associated Boltzmann measure is defined by

$$\widehat{\mathbb{P}}_{\mathbf{q}}^{(k)}(\mathbf{m}) := \frac{\mathbf{1}_{\{\mathbf{m} \in \widehat{\mathcal{M}}_k\}} w_{\mathbf{q}}(\mathbf{m})}{\widehat{F}_k}, \quad \mathbf{m} \in \mathcal{M}, \quad k \in \mathbb{Z}_+, \quad (20)$$

and the associated generating function by

$$\widehat{F}(x) := \sum_{k=0}^{\infty} \widehat{F}_k x^k \quad x \geq 0. \quad (21)$$

The radius of convergence of  $\widehat{F}$  is denoted by  $\widehat{r}_{\mathbf{q}}$ . Note that  $\widehat{F}_0 = 1$  for any weight sequence, while if  $\mathbf{q} = 0$ ,  $\widehat{F}_k = \delta_0(k) + \delta_1(k)$  (the vertex map and the map with a single oriented edge are the only bipartite maps with a simple boundary and no internal face). When we consider the Boltzmann measure  $\widehat{\mathbb{P}}_{\mathbf{q}}^{(k)}$ , we implicitly assume that  $\widehat{F}_k > 0$ .

We will prove the following analogue of Proposition 2.6 for Boltzmann maps with a simple boundary, which is the technical core of this paper. The constants  $(\widehat{c}_a : a \in \{3/2\} \cup (2, 5/2])$  and the slowly varying functions  $\widehat{\ell}_1$  (depending on  $a$ ) will be defined at the end of the section, see (27) and (28).

**Proposition 2.8.** *Let  $\mathbf{q}$  be a weight sequence of type  $a$ . For  $a = 3/2$ , as  $y \rightarrow r_{\mathbf{q}}F^2(r_{\mathbf{q}})^+$ ,*

$$\widehat{F}(y) = F(r_{\mathbf{q}}) \left( 1 - \frac{1}{2} \left( 1 - \frac{y}{r_{\mathbf{q}}F^2(r_{\mathbf{q}})} \right) + \widehat{c}_{3/2} \left( 1 - \frac{y}{r_{\mathbf{q}}F^2(r_{\mathbf{q}})} \right)^2 (1 + o(1)) \right).$$

*For  $a \in (3/2, 2) \cup (2, 5/2]$ ,  $\widehat{F}$  has radius of convergence  $\widehat{r}_{\mathbf{q}} = P(r_{\mathbf{q}})$ . Moreover, for  $a \in (3/2, 2)$ ,*

$$\widehat{F}(y) = F(r_{\mathbf{q}}) \left( 1 - \frac{1}{2} \left( 1 - \frac{y}{\widehat{r}_{\mathbf{q}}} \right) + \left( 1 - \frac{y}{\widehat{r}_{\mathbf{q}}} \right)^{\frac{1}{a-1}} \widehat{\ell}_1 \left( \frac{1}{1 - \frac{y}{\widehat{r}_{\mathbf{q}}}} \right) (1 + o(1)) \right) \quad \text{as } y \rightarrow \widehat{r}_{\mathbf{q}}^+,$$

*and for  $a \in (2, 5/2]$ ,*

$$\widehat{F}(y) = F(r_{\mathbf{q}}) \left( 1 - \frac{\widehat{c}_a}{2} \left( 1 - \frac{y}{\widehat{r}_{\mathbf{q}}} \right) + \left( 1 - \frac{y}{\widehat{r}_{\mathbf{q}}} \right)^{a-1} \widehat{\ell}_1 \left( \frac{1}{1 - \frac{y}{\widehat{r}_{\mathbf{q}}}} \right) (1 + o(1)) \right) \quad \text{as } y \rightarrow \widehat{r}_{\mathbf{q}}^+.$$

**Remark 2.9.** One may wonder if we can use the theory of singularity analysis [30, Chapter 6] to get an asymptotic expansion of the partition function  $\widehat{F}_k$ . However, it is not clear that the so-called delta-analyticity assumption is satisfied in this context. We will use instead Karamata's Tauberian theorem, which provides a weaker result. In the subcritical case, we do not know if the radius of convergence of the generating function  $\widehat{F}$  equals  $r_{\mathbf{q}}F^2(r_{\mathbf{q}})$  or not.

In the special case of quadrangulations, corresponding to the weights  $q_k = q\delta_2(k)$ , computations can be carried out explicitly using [13]. First,  $\mathbf{q}$  is admissible if  $q \leq 1/12$  and critical if  $q = 1/12$ . The generating function  $F$  satisfies from [13, Equations (3.4), (3.11) and (3.15)]

$$r_{\mathbf{q}} = \frac{1}{4R(q)} \quad \text{and} \quad F(r_{\mathbf{q}}) = 2(1 - qR^2(q)), \quad \text{where} \quad R(q) := \frac{1 - \sqrt{1 - 12q}}{6q}, \quad 0 < q \leq \frac{1}{12}.$$

Furthermore, the generating function  $\widehat{F}$  has from [13, Equation (5.16)] radius of convergence given by

$$\widehat{r}_{\mathbf{q}} = \frac{4}{27qR^3(q)}.$$

We conclude that  $\widehat{r}_{\mathbf{q}} > r_{\mathbf{q}}F^2(r_{\mathbf{q}})$  for subcritical quadrangulations. Moreover, [13, Equation (5.16)] also provides an equivalent of the partition function for quadrangulations:

$$\widehat{F}_k \underset{k \rightarrow \infty}{\sim} \frac{2\sqrt{3}\widehat{r}_{\mathbf{q}}^{-k}}{27\sqrt{\pi}k^{5/2}} \quad (\mathbf{q} \text{ critical}) \quad \text{and} \quad \widehat{F}_k \underset{k \rightarrow \infty}{\sim} \frac{\sqrt{3}\widehat{r}_{\mathbf{q}}^{-k}}{27\sqrt{\pi}k^{3/2}} \left( \frac{1}{qR^2(q)} - 3 \right) \quad (\mathbf{q} \text{ subcritical}).$$

Our approach relies on a simple relation between the generating functions  $F$  and  $\widehat{F}$ , which was first observed in [14] (see also [13] for quadrangulations). This relation is itself based on the following decomposition of bipartite maps.

Following [25, Section 2.2], every  $\mathbf{m} \in \mathcal{M}$  can be decomposed into a collection of irreducible components, that is bipartite maps with a simple boundary attached by the cut vertices of  $\partial\mathbf{m}$ . Let  $\widehat{\mathbf{m}} \in \widehat{\mathcal{M}}$  be the irreducible component of  $\mathbf{m}$  containing the root edge. Then,  $\widehat{\mathbf{m}} \setminus V(\partial\widehat{\mathbf{m}})$  disconnects  $\mathbf{m}$  into  $\#\partial\widehat{\mathbf{m}}$  connected components. We root each of them at the oriented edge of  $\partial\mathbf{m}$  which is the closest to the root edge of  $\mathbf{m}$  (and such that the root face lies on its right). This provides a unique decomposition of  $\mathbf{m}$  into  $\widehat{\mathbf{m}} \in \widehat{\mathcal{M}}$  and a collection  $(\mathbf{m}_i : 1 \leq i \leq \#\partial\widehat{\mathbf{m}})$  of elements of  $\mathcal{M}$  attached to the vertices of  $\partial\widehat{\mathbf{m}}$ . See Figure 4 for an illustration.

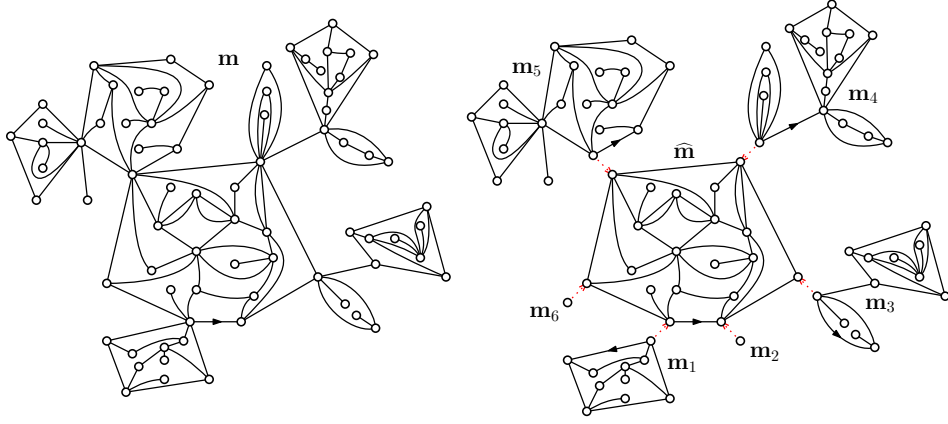


Figure 4: The decomposition of a bipartite map  $\mathbf{m}$ .

**Lemma 2.10.** *For every  $x \geq 0$ , we have*

$$F(x) = \widehat{F}(xF^2(x)).$$

*In particular, the radius of convergence of  $\widehat{F}$  satisfies  $\widehat{r}_q \geq r_q F^2(r_q)$ .*

*Proof.* Let  $x \geq 0$ . By the above decomposition, we have

$$\begin{aligned} F(x) &= \sum_{\mathbf{m} \in \mathcal{M}} x^{\#\partial \mathbf{m}/2} w_q(\mathbf{m}) = \sum_{\widehat{\mathbf{m}} \in \widehat{\mathcal{M}}} x^{\#\partial \widehat{\mathbf{m}}/2} w_q(\widehat{\mathbf{m}}) \prod_{i=1}^{\#\partial \widehat{\mathbf{m}}} x^{\#\partial \mathbf{m}_i/2} w_q(\mathbf{m}_i) \\ &= \sum_{\widehat{\mathbf{m}} \in \widehat{\mathcal{M}}} x^{\#\partial \widehat{\mathbf{m}}/2} w_q(\widehat{\mathbf{m}}) \left( \sum_{\mathbf{m} \in \mathcal{M}} x^{\#\partial \mathbf{m}/2} w_q(\mathbf{m}) \right)^{\#\partial \widehat{\mathbf{m}}} \\ &= \sum_{\widehat{\mathbf{m}} \in \widehat{\mathcal{M}}} (xF^2(x))^{\#\partial \widehat{\mathbf{m}}/2} w_q(\widehat{\mathbf{m}}) = \widehat{F}(xF^2(x)). \end{aligned}$$

We have that  $F(x) < \infty$  if  $x < r_q$ . Additionally, the function  $x \mapsto xF^2(x)$  is continuous increasing on  $[0, r_q)$ , so that  $\widehat{F}(y) < \infty$  if  $y < r_q F^2(r_q)$ . We deduce that  $\widehat{r}_q \geq r_q F^2(r_q)$ .  $\square$

We now use this relation to derive a singular expansion of  $\widehat{F}$  from Proposition 2.6. Let us introduce the function

$$P(x) := xF^2(x), \quad x \geq 0.$$

The function  $P$  is continuous and increasing from  $[0, r_q]$  onto  $[0, r_q F^2(r_q)]$ , with inverse  $P^{-1}$ . The results of Proposition 2.6 readily transfer to  $P$ . For  $a \in [3/2, 2)$ ,

$$P(x) = P(r_q) - \kappa'_a \left(1 - \frac{x}{r_q}\right)^{a-1} \ell_1 \left(\frac{1}{1 - \frac{x}{r_q}}\right) (1 + o(1)) \quad \text{as } x \rightarrow r_q^-, \quad (22)$$

and for  $a \in (2, 5/2]$ ,

$$P(x) = P(r_q) - C_q \left(1 - \frac{x}{r_q}\right) + \kappa'_a \left(1 - \frac{x}{r_q}\right)^{a-1} \ell_1 \left(\frac{1}{1 - \frac{x}{r_q}}\right) (1 + o(1)) \quad \text{as } x \rightarrow r_q^-, \quad (23)$$



where  $C_q := r_q F(r_q)(F(r_q) + 2r_q F'(r_q)) > 0$  and  $\kappa'_a := 2r_q F(r_q)\kappa_a$ . We now invert this expansion to get that of  $P^{-1}$ , and treat the cases  $a \in [3/2, 2)$  and  $a \in (2, 5/2]$  separately. Recall that a measurable function  $f$  is regularly varying (at infinity) with index  $\gamma \in \mathbb{R}$  if it satisfies  $f(\lambda x)/f(x) \rightarrow \lambda^\gamma$  as  $x \rightarrow \infty$ , for every  $\lambda > 0$ .

**Lemma 2.11.** *Let  $f$  be a continuous decreasing regularly varying function with index  $-\gamma < 0$ .*

*Then,  $f$  is invertible and the function  $y \mapsto f^{-1}(1/y)$  is regularly varying with index  $1/\gamma$ .*

*Proof.* Observe that  $x \mapsto 1/f(x)$  is regularly varying with index  $\gamma > 0$ . From [8, Theorem 1.5.12], there exists  $g$  regularly varying with index  $1/\gamma$  such that

$$\frac{1}{f(g(x))} \sim x \quad \text{as } x \rightarrow \infty.$$

One version of  $g$  is the inverse  $y \mapsto f^{-1}(1/y)$  of  $x \mapsto 1/f(x)$ , defined for  $y$  large enough since  $f$  vanishes at infinity. Thus,  $y \mapsto f^{-1}(1/y)$  is regularly varying with index  $1/\gamma$ .  $\square$

Let  $a \in [3/2, 2)$ . From (22), we know that

$$R(x) := P(r_q) - P(r_q(1 - 1/x)) \sim \kappa'_a x^{1-a} \ell_1(x) \quad \text{as } x \rightarrow \infty,$$

thus  $R$  is regularly varying with index  $-(a - 1) < 0$ . Moreover,  $R$  is continuous decreasing on  $[1, \infty)$  with inverse  $R^{-1}$  defined by

$$R^{-1}(y) = \left(1 - \frac{1}{r_q} P^{-1}(P(r_q) - y)\right)^{-1}, \quad y \in (0, P(r_q)].$$

By Lemma 2.11,  $y \mapsto R^{-1}(1/y)$  is regularly varying with index  $1/(a - 1)$ , so that [8, Theorem 1.4.1] ensures the existence of a positive slowly varying function  $\bar{\ell}_1$  such that

$$R^{-1}(1/y) = y^{\frac{1}{a-1}} \bar{\ell}_1(y), \quad y \in [1/P(r_q), \infty).$$

As a consequence,

$$P^{-1}(y) = r_q - r_q (P(r_q) - y)^{\frac{1}{a-1}} \left(\bar{\ell}_1\left(\frac{1}{P(r_q) - y}\right)\right)^{-1}, \quad y \in [0, P(r_q)]. \quad (24)$$

When  $a = 3/2$ ,  $\ell_1 = 1$  so that computation can be made more explicit. Indeed, we find

$$R(x) \sim \frac{\kappa'_{3/2}}{\sqrt{x}} \quad \text{as } x \rightarrow \infty.$$

Then, the function  $Q(x) := R((\kappa'_{3/2}/x)^2)$  is continuous increasing from  $(0, \kappa'_{3/2}]$  onto  $(0, P(r_q)]$  with inverse  $Q^{-1}$ . Additionally,  $Q$  is right-differentiable at 0 with  $Q'(0^+) = 1$ , so that  $Q^{-1}$  is right-differentiable at 0 and  $(Q^{-1})'(0^+) = 1$ . Thus,  $Q^{-1}(y) \sim y$  as  $y \rightarrow 0^+$  and we get

$$R^{-1}(y) = \left(\frac{\kappa'_{3/2}}{Q^{-1}(y)}\right)^2 \sim \left(\frac{\kappa'_{3/2}}{y}\right)^2 \quad \text{as } y \rightarrow 0^+.$$

As a conclusion,

$$P^{-1}(y) = r_q - \frac{r_q}{(\kappa'_{3/2})^2} (P(r_q) - y)^2 (1 + o(1)) \quad \text{as } y \rightarrow P(r_q)^-. \quad (25)$$

We are now interested in the case where  $a \in (2, 5/2]$ . From (23), we have

$$R(x) := C_q^{-1} [P(r_q) - P(r_q(1-x))] = x - \frac{\kappa'_a}{C_q} x^{a-1} \ell_1(1/x) (1 + o(1)) \quad \text{as } x \rightarrow 0^+.$$

The function  $R$  is continuous increasing on  $[0, 1]$ , with inverse  $R^{-1}$  defined by

$$R^{-1}(y) = 1 - \frac{1}{r_q} P^{-1}(P(r_q) - C_q y), \quad y \in [0, C_q^{-1} P(r_q)].$$

It also satisfies  $R'(0^+) = 1$ ,  $(R^{-1})'(0^+) = 1$  and thus  $R^{-1}(y) \sim y$  as  $y \rightarrow 0^+$ . In particular,  $y \mapsto R^{-1}(1/y)$  is regularly varying with index  $-1$  and by [8, Proposition 1.5.7], the function  $\bar{\ell}_1(y) := \ell_1(1/R^{-1}(1/y))$  is slowly varying. We get

$$R^{-1}(y) - y \sim \frac{\kappa'_a}{C_q} (R^{-1}(y))^{a-1} \ell_1(1/R^{-1}(y)) \sim \frac{\kappa'_a}{C_q} y^{a-1} \bar{\ell}_1(1/y) \quad \text{as } y \rightarrow 0^+,$$

and as a conclusion

$$P^{-1}(y) = r_q - \frac{r_q}{C_q} (P(r_q) - y) - \frac{\kappa'_a}{C_q^a} (P(r_q) - y)^{a-1} \bar{\ell}_1 \left( \frac{C_q}{P(r_q) - y} \right) (1 + o(1)) \quad \text{as } y \rightarrow P(r_q)^-. \quad (26)$$

We can now introduce the constants involved in the statement of Proposition 2.8,

$$\widehat{c}_{3/2} := \frac{P(r_q)^2}{2(\kappa'_{3/2})^2} - \frac{1}{8} \quad \text{and} \quad \widehat{c}_a = 1 - \frac{P(r_q)}{C_q} \in (0, 1) \quad \text{for } a \in (2, 5/2]. \quad (27)$$

and the functions  $\widehat{\ell}_1$  defined by

$$\widehat{\ell}_1(y) := \frac{P(r_q)^{\frac{1}{a-1}}}{2\bar{\ell}_1\left(\frac{y}{P(r_q)}\right)}, \quad a \in (3/2, 2) \quad \text{and} \quad \widehat{\ell}_1(y) := \frac{\kappa'_a P(r_q)^{a-1} \bar{\ell}_1\left(\frac{C_q y}{P(r_q)}\right)}{2C_q^a}, \quad a \in (2, 5/2]. \quad (28)$$

These functions are positive slowly varying, from [8, Proposition 1.3.6]. Note that for  $a = 5/2$ , we have  $\ell_1 = 1$  so that  $\widehat{\ell}_1$  is constant.

*Proof of Proposition 2.8.* By Lemma 2.10, we have that  $\widehat{F}(r_q F^2(r_q)) = F(r_q)$ , as well as

$$\widehat{F}(y) = \sqrt{\frac{y}{P^{-1}(y)}}, \quad 0 < y \leq P(r_q).$$

We obtain asymptotic expansions for  $\widehat{F}$  around  $P(r_q)$  using (24), (25), and (26). These expansions are singular for  $a \neq 3/2$ , and thus  $\widehat{F}$  is not of class  $C^\infty$  at  $P(r_q)$ . Together with Lemma 2.10, this proves that the radius of convergence of  $\widehat{F}$  is  $\widehat{r}_q = P(r_q)$  in these cases.  $\square$

## 3 Structure of the boundary of Boltzmann maps

### 3.1 Random trees and the Janson-Stefánsson bijection

We focus on the branching structure of the boundary of Boltzmann bipartite maps. We start with generalities on plane trees.

**Trees.** A (finite) plane tree  $\mathbf{t}$  [40, 53] is a finite subset of the sequences of positive integers

$$\mathcal{U} := \bigcup_{n \in \mathbb{Z}_+} \mathbb{N}^n$$

satisfying the following properties. First,  $\emptyset \in \mathbf{t}$  and is called the *root vertex*. Then, for every  $u = (u_1, \dots, u_k) \in \mathbf{t}$ ,  $\widehat{u} := (u_1, \dots, u_{k-1}) \in \mathbf{t}$  (and is called the *parent* of  $u$  in  $\mathbf{t}$ ). Finally, for every  $u = (u_1, \dots, u_k) \in \mathbf{t}$ , there exists  $k_u = k_u(\mathbf{t}) \in \mathbb{Z}_+$  (the number of children of  $u$  in  $\mathbf{t}$ ) such that  $uj := (u_1, \dots, u_k, j) \in \mathbf{t}$  iff  $1 \leq j \leq k_u$ . The height  $|u|$  of a vertex  $u = (u_1, \dots, u_k) \in \mathbf{t}$  is  $|u| = k$ . The vertices at even height are called white, and those at odd height are called black. We let  $\mathbf{t}_\circ$  and  $\mathbf{t}_\bullet$  be the corresponding subsets of vertices of  $\mathbf{t}$ . The total number of vertices of a tree  $\mathbf{t}$  is denoted by  $|\mathbf{t}|$ . The set of finite plane trees is denoted by  $\mathcal{T}_f$ . We use the notation  $T$  for the identity mapping on  $\mathcal{T}_f$ .

Given a probability measure  $\rho$  on  $\mathbb{Z}_+$ , a *Galton-Watson tree* with offspring distribution  $\rho$  is a random tree in which every vertex has a number of children distributed according to  $\rho$ , all independently of each other. The tree is called *critical* (resp. *subcritical*) if the mean  $m_\rho$  of  $\rho$  is equal to 1 (resp. less than 1). In these cases, its distribution  $\mathbf{GW}_\rho$  is characterized by

$$\mathbf{GW}_\rho(\mathbf{t}) = \prod_{u \in \mathbf{t}} \rho(k_u), \quad \forall \mathbf{t} \in \mathcal{T}_f. \quad (29)$$

We will also deal with (alternated) *two-type Galton-Watson trees* with offspring distribution  $(\rho_\circ, \rho_\bullet)$ , in which every vertex at even (resp. odd) generation has a number of children distributed according to  $\rho_\circ$  (resp.  $\rho_\bullet$ ), all independently of each other. Such a tree is *critical* (resp. *subcritical*) if  $m_{\rho_\circ} m_{\rho_\bullet} = 1$  (resp.  $m_{\rho_\circ} m_{\rho_\bullet} < 1$ ). Then, its distribution  $\mathbf{GW}_{\rho_\circ, \rho_\bullet}$  is characterized by

$$\mathbf{GW}_{\rho_\circ, \rho_\bullet}(\mathbf{t}) = \prod_{u \in \mathbf{t}_\circ} \rho_\circ(k_u) \prod_{u \in \mathbf{t}_\bullet} \rho_\bullet(k_u), \quad \forall \mathbf{t} \in \mathcal{T}_f. \quad (30)$$

**The Janson-Stefánsson bijection.** We now describe the Janson-Stefánsson bijection  $\Phi_{\text{JS}}$  introduced in [35, Section 3]. First,  $\Phi_{\text{JS}}(\{\emptyset\}) = \{\emptyset\}$ . For  $\mathbf{t} \neq \{\emptyset\}$ ,  $\Phi_{\text{JS}}(\mathbf{t})$  has the same vertices as  $\mathbf{t}$  but different edges defined as follows. For every  $u \in \mathbf{t}_\circ$ , set the convention that  $u0 = \widehat{u}$  (if  $u \neq \emptyset$ ) and  $u(k_u + 1) = u$ . Then, for every  $j \in \{0, 1, \dots, k_u\}$ , add the edge  $(uj, u(j+1))$  to  $\Phi_{\text{JS}}(\mathbf{t})$ . If  $u$  is a leaf, this amounts to adding an edge between  $u$  and its parent. We obtain a tree  $\Phi_{\text{JS}}(\mathbf{t})$  embedded in the plane. The vertex 1 of  $\mathbf{t}$  is the root vertex of  $\Phi_{\text{JS}}(\mathbf{t})$ , and its first children in  $\Phi_{\text{JS}}(\mathbf{t})$  is chosen according to the lexicographical order of  $\mathbf{t}$ . For further notice, we give a brief description of the inverse application  $\Phi_{\text{JS}}^{-1}$ . For  $\mathbf{t} \neq \{\emptyset\}$ ,  $\Phi_{\text{JS}}^{-1}(\mathbf{t})$  has the same vertices as  $\mathbf{t}$ , and its edges can be defined as follows. For every leaf  $u \in \mathbf{t}$ , let  $(u_1, u_2, \dots)$  be the sequence of vertices after  $u$  in the contour order of  $\mathbf{t}$ , and let  $\ell(u)$  be the largest index such that  $u_1, \dots, u_{\ell(u)}$  all are ancestors of  $u$  in  $\mathbf{t}$ . Then, add an edge between  $u$  and  $u_k$  in  $\Phi_{\text{JS}}^{-1}(\mathbf{t})$  for every  $k \in \{1, \dots, \ell(u)\}$ . We obtain a tree  $\Phi_{\text{JS}}^{-1}(\mathbf{t})$  embedded in the plane, and choose the last leaf  $u'$  of  $\mathbf{t}$  in contour order as the root vertex, and  $u'_{\ell(u)}$  as its first child.

The application  $\Phi_{\text{JS}}$  is a bijection from  $\mathcal{T}_f$  onto itself. It has the property that every vertex of  $\mathbf{t}_\circ$  is mapped to a leaf of  $\Phi_{\text{JS}}(\mathbf{t})$ , and every vertex of  $\mathbf{t}_\bullet$  with degree  $k$  is mapped to a vertex of  $\Phi_{\text{JS}}(\mathbf{t})$  with degree  $k+1$ . See Figure 5 for an illustration.

This bijection greatly simplifies the study of alternated two-type Galton-Watson trees, because of the following result of [35] (see also [23, Proposition 3.6]).

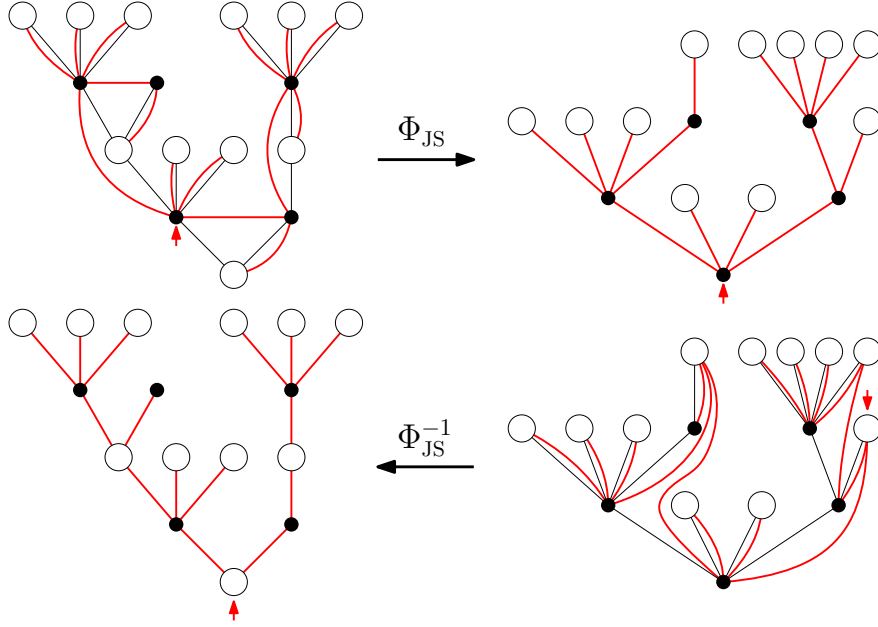


Figure 5: The Janson-Stefánsson bijection and its inverse application.

**Proposition 3.1.** [35, Appendix A] *Let  $\rho_\circ$  and  $\rho_\bullet$  be probability measures on  $\mathbb{Z}_+$  such that  $m_{\rho_\circ} m_{\rho_\bullet} \leq 1$  and  $\rho_\circ$  has geometric distribution with parameter  $1-p \in (0, 1)$ :  $\rho_\circ(k) = (1-p)p^k$  for  $k \geq 0$ . Then, the image of  $\text{GW}_{\rho_\circ, \rho_\bullet}$  under  $\Phi_{\text{JS}}$  is  $\text{GW}_\rho$ , where*

$$\rho(0) = 1 - p \quad \text{and} \quad \rho(k) = p \cdot \rho_\bullet(k - 1), \quad k \in \mathbb{N}.$$

Note that  $m_\rho - p = (1-p)m_{\rho_\circ}m_{\rho_\bullet}$ , so that  $(\rho_\circ, \rho_\bullet)$  is critical iff  $\rho$  itself is critical.

### 3.2 Random looptrees and scooped-out maps

We now introduce random looptrees and their *tree of components* to represent the boundary of a planar map as a tree. This idea goes back to [23], while random looptrees were first introduced in [24]. The following presentation is inspired by [23, Section 2.3].

**Random looptrees.** A looptree is a planar map whose edges are incident to two distinct faces, one of them being the root face (such a map is also called edge-outerplanar). We denote the set of finite looptrees by  $\mathcal{L}_f$ . Informally, a looptree is a collection of simple cycles glued along a tree structure. Consistently, there is a way to build looptrees from trees and conversely.

We associate to every plane tree  $\mathbf{t} \in \mathcal{T}_f$  a looptree  $\text{Loop}(\mathbf{t})$  as follows. For every (black) vertex  $u \in \mathbf{t}_\bullet$ , connect all the incident (white) vertices of  $u$  in cyclic order. Then,  $\text{Loop}(\mathbf{t})$  is the planar map obtained when discarding the black vertices and the edges of  $\mathbf{t}$ . The root edge of  $\text{Loop}(\mathbf{t})$  connects the origin of  $\mathbf{t}$  to the last child of its first offspring in  $\mathbf{t}$ . The inverse application associates to a looptree  $\mathbf{l} \in \mathcal{L}_f$  a plane tree  $\text{Tree}(\mathbf{l})$ , called the tree of components, as follows. We add an extra vertex in every internal face of  $\mathbf{l}$ , which we connect by an edge to all the vertices of this face. The plane tree  $\text{Tree}(\mathbf{l})$  is then obtained by discarding the edges of  $\mathbf{l}$ . The root edge of  $\text{Tree}(\mathbf{l})$  connects the origin of  $\mathbf{l}$  to the vertex lying inside the internal face incident to the root edge of  $\mathbf{l}$ . This is illustrated in Figure 6.

**Remark 3.2.** In a looptree  $\mathbf{l}$ , every internal face is rooted at the oriented edge whose origin is the closest vertex to the origin of  $\mathbf{l}$ , and such that the root face of  $\mathbf{l}$  lies on its right. The gluing of a planar map with a simple boundary of perimeter  $k$  into an internal face of degree  $k$  is then determined by the convention that the root edges of the map and the face match.

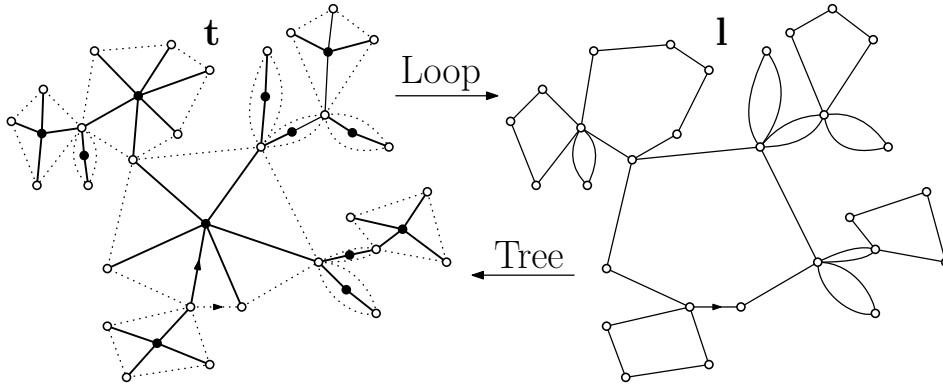


Figure 6: A looptree  $\mathbf{l}$  and the associated tree of components  $\mathbf{t}$ .

This definition of looptree slightly differs from that of [24, 23], that we now recall. Given a plane tree  $\mathbf{t} \in \mathcal{T}_f$ , the looptree  $\mathbf{Loop}(\mathbf{t})$  (or  $\mathbf{Loop}'(\mathbf{t})$  in [24]) is build from  $\mathbf{t}$  as follows. For every  $u, v \in \mathbf{t}$ , there is an edge between  $u$  and  $v$  iff one of the following conditions is fulfilled:  $u$  and  $v$  are consecutive siblings in  $\mathbf{t}$ , or  $v$  is either the first or the last child of  $u$  in  $\mathbf{t}$ . We will also need  $\overline{\mathbf{Loop}}(\mathbf{t})$ , which is obtained from  $\mathbf{Loop}(\mathbf{t})$  by contracting the edges linking a vertex of  $\mathbf{t}$  and its last child in  $\mathbf{t}$ . These objects are rooted at the oriented edge between the origin of  $\mathbf{t}$  and its last child in  $\mathbf{t}$  (resp. penultimate for  $\overline{\mathbf{Loop}}$ ). See Figure 7 for an example. We use the bold print  $\mathbf{Loop}$  to distinguish this construction from Loop. Note that  $\mathbf{Loop}(\mathbf{t})$  is a looptree and can be obtained as the image of a plane tree by Loop, but the converse does not hold:  $\mathbf{Loop}$  does not allow several loops to be glued at the same vertex.

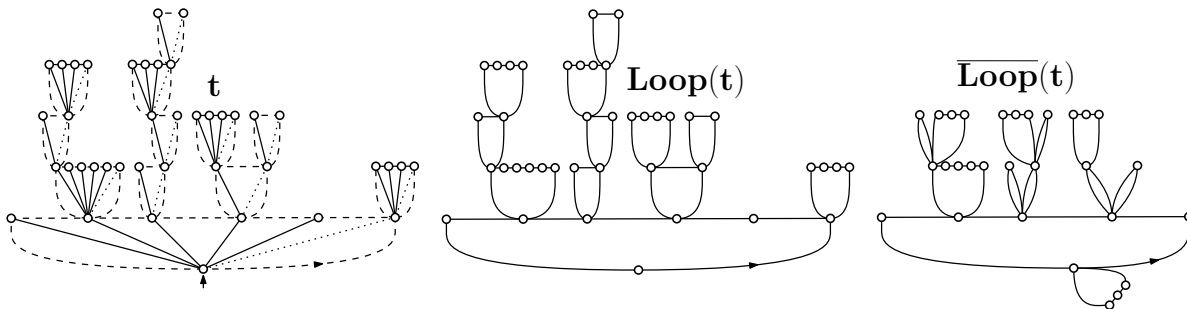


Figure 7: A tree  $\mathbf{t}$  and the looptrees  $\mathbf{Loop}(\mathbf{t})$  and  $\overline{\mathbf{Loop}}(\mathbf{t})$ .

**The scooped-out map.** The *scooped-out* map of a planar map  $\mathbf{m}$  was defined in [23] as the looptree  $\mathbf{Scoop}(\mathbf{m})$  obtained from the boundary  $\partial\mathbf{m}$  by duplicating the edges whose both sides belong to the root face. We call tree of components of  $\mathbf{m}$  the tree of components of  $\mathbf{Scoop}(\mathbf{m})$ , denoted by  $\mathbf{Tree}(\mathbf{m}) := \mathbf{Tree}(\mathbf{Scoop}(\mathbf{m}))$ .

Any planar map  $\mathbf{m}$  is recovered from  $\mathbf{Scoop}(\mathbf{m})$  by gluing into the internal faces of  $\mathbf{Scoop}(\mathbf{m})$  the proper maps with a simple boundary (using Remark 3.2). These maps are

the irreducible components of  $\mathbf{m}$ , rooted at the oriented edge of  $\partial\mathbf{m}$  which is the closest to the root edge of  $\mathbf{m}$  (and such that the root face lies on its right). Otherwise said, to every black vertex  $u$  of  $\mathbf{t} := \mathbf{Tree}(\mathbf{m})$  corresponds a cycle of  $\text{Scoop}(\mathbf{m})$ , and thus a planar map with a simple boundary of perimeter  $\deg(u)$ . This construction provides a bijection

$$\Phi_{\text{TC}} : \mathbf{m} \mapsto (\mathbf{Tree}(\mathbf{m}), (\widehat{\mathbf{m}}_u : u \in \mathbf{Tree}(\mathbf{m})_\bullet))$$

that associates to a bipartite map  $\mathbf{m} \in \mathcal{M}$  the plane tree  $\mathbf{t} = \mathbf{Tree}(\mathbf{m})$ , whose vertices at odd height have even degree, and a collection  $(\widehat{\mathbf{m}}_u : u \in \mathbf{t}_\bullet)$  of bipartite maps with a simple boundary of respective perimeter  $\deg(u)$ . See Figure 8 for an example. The following relations between the perimeter of a map and the size of its tree of components will be useful:

$$|\mathbf{t}| = \#\partial\mathbf{m} + 1 \quad \text{and} \quad \sum_{u \in \mathbf{t}_\bullet} \deg(u) = \#\partial\mathbf{m} \quad (\mathbf{t} = \mathbf{Tree}(\mathbf{m})). \quad (31)$$

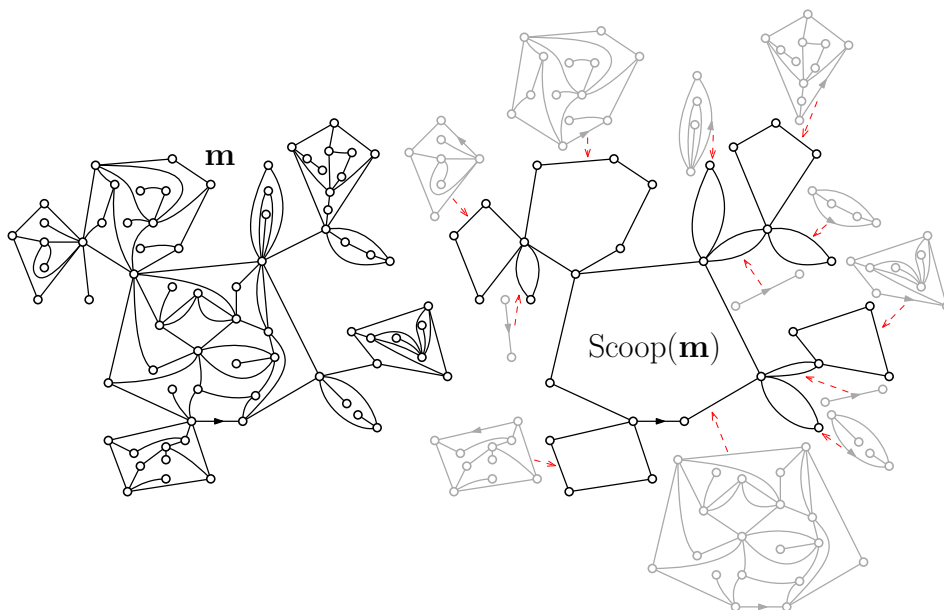


Figure 8: A planar map  $\mathbf{m}$  and the associated scooped-out map  $\text{Scoop}(\mathbf{m})$ .

### 3.3 Distribution of the tree of components

The purpose of this section is to provide the distribution of the tree of components of a bipartite map under the probability measure  $\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}$  defined by

$$\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}(\mathbf{m}) := \frac{r_{\mathbf{q}}^{\#\partial\mathbf{m}/2} w_{\mathbf{q}}(\mathbf{m})}{F(r_{\mathbf{q}})}, \quad \mathbf{m} \in \mathcal{M}. \quad (32)$$

It is related to  $\mathbb{P}_{\mathbf{q}}^{(k)}$  by conditioning with respect to the perimeter of the map: for every  $k \geq 0$ ,

$$\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}(\mathbf{m} \mid \mathcal{M}_k) = \mathbb{P}_{\mathbf{q}}^{(k)}(\mathbf{m}), \quad \mathbf{m} \in \mathcal{M}. \quad (33)$$

The main result of this section generalizes [4, Proposition 6] that deals with quadrangulations.

**Proposition 3.3.** *Let  $\mathbf{q}$  be a weight sequence of type  $a \in [3/2, 5/2]$ . Under  $\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}$ ,  $\mathbf{Tree}(M)$  is a two-type Galton-Watson tree with offspring distribution  $(\nu_{\circ}, \nu_{\bullet})$  defined by*

$$\nu_{\circ}(k) = \frac{1}{F(r_{\mathbf{q}})} \left(1 - \frac{1}{F(r_{\mathbf{q}})}\right)^k \quad \text{and} \quad \nu_{\bullet}(2k+1) = \frac{1}{F(r_{\mathbf{q}}) - 1} (r_{\mathbf{q}} F^2(r_{\mathbf{q}}))^{k+1} \widehat{F}_{k+1}, \quad k \in \mathbb{Z}_+.$$

(With  $\nu_{\bullet}(k) = 0$  for  $k$  even.) Moreover, conditionally on  $\mathbf{Tree}(M)$ , the bipartite maps with a simple boundary  $(\widehat{M}_u : u \in \mathbf{Tree}(M)_{\bullet})$  associated to  $M$  by  $\Phi_{\text{TC}}$  are independent Boltzmann bipartite maps with a simple boundary, having respective distribution  $\widehat{\mathbb{P}}_{\mathbf{q}}^{(\deg(u)/2)}$ .

**Remark 3.4.** The probability measure  $\nu_{\bullet}$  is supported by odd integers, so the internal faces of  $\text{Scoop}(M)$  have even degree. This is consistent with the fact that  $\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}$  is supported by bipartite maps. Note that  $M$  may have edges whose both sides are incident to the external face: this corresponds to a vertex of degree 2 of  $\mathbf{Tree}(M)$  associated to the bipartite map with a simple boundary made of a single oriented edge.

*Proof.* Let us check that  $\nu_{\circ}$  and  $\nu_{\bullet}$  are probability measures. This is clear for  $\nu_{\circ}$  which is a geometric distribution. For  $\nu_{\bullet}$ , we get recalling that  $1 < F(r_{\mathbf{q}}) = \widehat{F}(r_{\mathbf{q}} F^2(r_{\mathbf{q}}))$

$$\sum_{k \in \mathbb{Z}_+} \nu_{\bullet}(k) = \frac{1}{F(r_{\mathbf{q}}) - 1} \left( \widehat{F}(r_{\mathbf{q}} F^2(r_{\mathbf{q}})) - 1 \right) = 1.$$

Let  $\mathbf{m} \in \mathcal{M}$ , and recall that  $\Phi_{\text{TC}}$  associates to  $\mathbf{m}$  its tree of components  $\mathbf{t} = \mathbf{Tree}(\mathbf{m})$  and a collection  $(\widehat{\mathbf{m}}_u : u \in \mathbf{t}_{\bullet})$  of bipartite maps with a simple boundary and perimeter  $\deg(u)$ . Moreover, vertices of  $\mathbf{t}_{\bullet}$  have even degree. Using (32) and (31), we have

$$\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}(\mathbf{m}) = \frac{r_{\mathbf{q}}^{\#\partial \mathbf{m}/2} w_{\mathbf{q}}(\mathbf{m})}{F(r_{\mathbf{q}})} = \frac{1}{F(r_{\mathbf{q}})} \prod_{u \in \mathbf{t}_{\bullet}} r_{\mathbf{q}}^{\deg(u)/2} w_{\mathbf{q}}(\widehat{\mathbf{m}}_u).$$

Then, for every  $c > 0$

$$1 = \prod_{u \in \mathbf{t}_{\circ}} c^{k_u} \left(\frac{1}{c}\right)^{|\mathbf{t}_{\bullet}|} \quad \text{and} \quad \frac{1}{c} = \prod_{u \in \mathbf{t}_{\bullet}} c^{k_u} \left(\frac{1}{c}\right)^{|\mathbf{t}_{\circ}|}.$$

Applying the first identity with  $c = 1 - 1/F(r_{\mathbf{q}})$  and the second one with  $c = F(r_{\mathbf{q}})$  yields

$$\begin{aligned} \mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}(\mathbf{m}) &= \prod_{u \in \mathbf{t}_{\circ}} \frac{1}{F(r_{\mathbf{q}})} \left(1 - \frac{1}{F(r_{\mathbf{q}})}\right)^{k_u} \prod_{u \in \mathbf{t}_{\bullet}} \frac{1}{F(r_{\mathbf{q}}) - 1} (r_{\mathbf{q}} F^2(r_{\mathbf{q}}))^{(k_u+1)/2} w_{\mathbf{q}}(\widehat{\mathbf{m}}_u) \\ &= \prod_{u \in \mathbf{t}_{\circ}} \nu_{\circ}(k_u) \prod_{u \in \mathbf{t}_{\bullet}} \nu_{\bullet}(k_u) w_{\mathbf{q}}(\widehat{\mathbf{m}}_u) \frac{1}{\widehat{F}_{(k_u+1)/2}}. \end{aligned}$$

By convention, both sides equal zero if there exists  $u \in \mathbf{t}_{\bullet}$  such that  $\widehat{F}_{(k_u+1)/2} = 0$ . Finally,

$$\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}(\mathbf{Tree}(M) = \mathbf{t}, \widehat{M}_u = \widehat{\mathbf{m}}_u : u \in \mathbf{t}_{\bullet}) = \mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}(M = \mathbf{m}) = \text{GW}_{\nu_{\circ}, \nu_{\bullet}}(\mathbf{t}) \prod_{u \in \mathbf{t}_{\bullet}} \widehat{\mathbb{P}}_{\mathbf{q}}^{(\deg(u)/2)}(\widehat{\mathbf{m}}_u),$$

which is the expected result.  $\square$

By Proposition 3.1, we obtain the following.

**Corollary 3.5.** *Let  $\mathbf{q}$  be a weight sequence of type  $a \in [3/2, 5/2]$ . Under  $\mathbb{P}_{\mathbf{q}, r_{\mathbf{q}}}$ ,  $\Phi_{\text{JS}}(\mathbf{Tree}(M))$  is a Galton-Watson tree with offspring distribution  $\nu$  defined by*

$$\nu(2k) = \frac{1}{F(r_{\mathbf{q}})} (r_{\mathbf{q}} F^2(r_{\mathbf{q}}))^k \widehat{F}_k, \quad k \in \mathbb{Z}_+ \quad (\text{and } \nu(k) = 0 \text{ for } k \text{ odd}).$$

As a consequence, the generating function of  $\nu$  reads

$$G_{\nu}(s) = \frac{1}{F(r_{\mathbf{q}})} \sum_{k=0}^{\infty} s^{2k} (r_{\mathbf{q}} F^2(r_{\mathbf{q}}))^k \widehat{F}_k = \frac{1}{F(r_{\mathbf{q}})} \widehat{F}(r_{\mathbf{q}} F^2(r_{\mathbf{q}}) s^2), \quad s \in [0, 1]. \quad (34)$$

From Lemma 2.10, we easily deduce the following formula for the mean of  $\nu$

$$m_{\nu} = G'_{\nu}(1) = \frac{1}{F(r_{\mathbf{q}})} 2r_{\mathbf{q}} F^2(r_{\mathbf{q}}) \widehat{F}'(r_{\mathbf{q}} F^2(r_{\mathbf{q}})) = \frac{1}{1 + \frac{F(r_{\mathbf{q}})}{2r_{\mathbf{q}} F'(r_{\mathbf{q}})}}. \quad (35)$$

Similarly, the generating function of  $\nu_{\bullet}$  satisfies  $G_{\nu_{\bullet}}(0) = 0$  and

$$G_{\nu_{\bullet}}(s) = \frac{1}{F(r_{\mathbf{q}}) - 1} \cdot \frac{1}{s} \left( \widehat{F}(r_{\mathbf{q}} F^2(r_{\mathbf{q}}) s^2) - 1 \right), \quad s \in (0, 1]. \quad (36)$$

We also have  $m_{\nu_{\bullet}} = F(r_{\mathbf{q}}) - 1$ , so that by Proposition 3.1 and (35),

$$m_{\nu_{\bullet}} = \frac{m_{\nu}}{1 - \frac{1}{F(r_{\mathbf{q}})}} - 1 = \frac{1}{1 + \frac{F(r_{\mathbf{q}})}{2r_{\mathbf{q}} F'(r_{\mathbf{q}})}} \cdot \frac{F(r_{\mathbf{q}})}{F(r_{\mathbf{q}}) - 1} - 1. \quad (37)$$

The next result is a consequence of (18).

**Lemma 3.6.** *The offspring distribution  $\nu$  and the pair of offspring distributions  $(\nu_{\circ}, \nu_{\bullet})$  are critical if  $a \in [3/2, 2]$  and subcritical if  $a \in (2, 5/2]$ .*

We now describe the tail of the measures  $\nu$  and  $\nu_{\bullet}$ . The following is obtained by Proposition 2.8, (34) and (36). For  $a = 3/2$ , as  $t \rightarrow 0^+$

$$L_{\nu}(t) = 1 - t + (1 + 4\widehat{c}_{3/2}) t^2 + o(t^2), \quad (38)$$

$$L_{\nu_{\bullet}}(t) = 1 - \left( \frac{1}{F(r_{\mathbf{q}}) - 1} \right) t + \left( \frac{1}{2} + 4\widehat{c}_{3/2} \frac{F(r_{\mathbf{q}})}{F(r_{\mathbf{q}}) - 1} \right) t^2 + o(t^2). \quad (39)$$

For  $a \in (3/2, 2)$ , as  $t \rightarrow 0^+$

$$L_{\nu}(t) = 1 - t + 2^{\frac{1}{a-1}} t^{\frac{1}{a-1}} \widehat{\ell}(1/t) + o\left(t^{\frac{1}{a-1}} \widehat{\ell}(1/t)\right), \quad (40)$$

$$L_{\nu_{\bullet}}(t) = 1 - \left( \frac{1}{F(r_{\mathbf{q}}) - 1} \right) t + \frac{F(r_{\mathbf{q}})}{F(r_{\mathbf{q}}) - 1} 2^{\frac{1}{a-1}} t^{\frac{1}{a-1}} \widehat{\ell}(1/t) + o\left(t^{\frac{1}{a-1}} \widehat{\ell}(1/t)\right). \quad (41)$$

Finally, for  $a \in (2, 5/2]$ , as  $t \rightarrow 0^+$ ,

$$L_{\nu}(t) = 1 - \widehat{c}_a t + 2^{a-1} t^{a-1} \widehat{\ell}(1/t) + o\left(t^{a-1} \widehat{\ell}(1/t)\right), \quad (42)$$

$$L_{\nu_{\bullet}}(t) = 1 - \left( 1 - \widehat{c}_a \frac{F(r_{\mathbf{q}})}{F(r_{\mathbf{q}}) - 1} \right) t + \frac{F(r_{\mathbf{q}})}{F(r_{\mathbf{q}}) - 1} 2^{a-1} t^{a-1} \widehat{\ell}(1/t) + o\left(t^{a-1} \widehat{\ell}(1/t)\right). \quad (43)$$



The function  $\widehat{\ell}(x) = \widehat{\ell}_1((1 - \exp(-2/x))^{-1})$  is slowly varying by [8, Proposition 1.5.7]. Note that we recover formulas (35) and (37) from the definitions of  $\widehat{c}_a$  (27) and  $C_q$  (23).

For  $a = 3/2$ , (38) and (27) entail that  $\nu$  and  $\nu_\bullet$  have finite variance equal to

$$\sigma_\nu^2 = \left( \frac{2P(r_q)}{\kappa'_{3/2}} \right)^2 = \left( \frac{F(r_q)}{Z_q(1 - m_\mu)} \right)^2 \quad \text{and} \quad \sigma_{\nu_\bullet}^2 = \frac{F(r_q)}{F(r_q) - 1} \left( \left( \frac{F(r_q)}{Z_q(1 - m_\mu)} \right)^2 - 1 \right). \quad (44)$$

If we assume additionally that  $\widehat{r}_q > r_q F^2(r_q)$ , there exists  $s > 1$  such that  $G_\nu(s) < \infty$  and  $G_{\nu_\bullet}(s) < \infty$ , so that  $\nu$  and  $\nu_\bullet$  have small exponential moments.

For  $a \in (3/2, 2)$ , Karamata's Tauberian theorem [8, Theorem 8.1.6], (40) and (41) give

$$\nu([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{2^{\frac{1}{a-1}}}{|\Gamma(\frac{a-2}{a-1})|} \cdot \frac{\widehat{\ell}(k)}{k^{\frac{1}{a-1}}} \quad \text{and} \quad \nu_\bullet([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{F(r_q)}{F(r_q) - 1} \cdot \frac{2^{\frac{1}{a-1}}}{|\Gamma(\frac{a-2}{a-1})|} \cdot \frac{\widehat{\ell}(k)}{k^{\frac{1}{a-1}}}. \quad (45)$$

Finally, when  $a \in (2, 5/2]$ , the same version of Karamata's Tauberian theorem gives

$$\nu([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{2^{a-1}}{|\Gamma(2-a)|} \cdot \frac{\widehat{\ell}(k)}{k^{a-1}} \quad \text{and} \quad \nu_\bullet([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{F(r_q)}{F(r_q) - 1} \cdot \frac{2^{a-1}}{|\Gamma(2-a)|} \cdot \frac{\widehat{\ell}(k)}{k^{a-1}}. \quad (46)$$

**Proposition 3.7.** *For  $a = 3/2$ ,  $\nu$  and  $\nu_\bullet$  have finite variance (and small exponential moments iff  $\widehat{r}_q > r_q F^2(r_q)$ ). For  $a \in (3/2, 2)$ ,  $\nu$  and  $\nu_\bullet$  are in the domain of attraction of a stable distribution with parameter  $1/(a-1) \in (1, 2)$  and for  $a \in (2, 5/2]$ ,  $\nu$  and  $\nu_\bullet$  are in the domain of attraction of a stable distribution with parameter  $a-1 \in (1, 3/2]$ .*

**Remark 3.8.** Again, the value  $a = 2$  has to be excluded, even when an analogue of Proposition 2.6 holds. In this case, the expansion of the Laplace transform of  $\nu$  is expected to have a singularity of integer order, for which Karamata's Tauberian theorem [8, Theorem 8.1.6] provides a weaker result. This issue can be circumvented by using De Haan theory [8, Chapter 3], as we will see in Section 7 (under additional assumptions on the weight sequence).

The results of Proposition 3.3 and Corollary 3.5 transfer to  $\mathbb{P}_q^{(k)}$ . For every  $n \geq 1$ , we denote by  $\mathbf{GW}_\rho^{(n)}$  (resp.  $\mathbf{GW}_{\rho_\circ, \rho_\bullet}^{(n)}$ ) the law of a Galton-Watson tree with offspring distribution  $\rho$  (resp.  $(\rho_\circ, \rho_\bullet)$ ) conditioned to have  $n$  vertices, provided this makes sense.

**Corollary 3.9.** *Let  $q$  be a weight sequence of type  $a \in [3/2, 5/2]$ . For every  $k \geq 0$ , under  $\mathbb{P}_q^{(k)}$ ,  $\mathbf{Tree}(M)$  has distribution  $\mathbf{GW}_{\nu_\circ, \nu_\bullet}^{(2k+1)}$ , and  $\Phi_{\text{JS}}(\mathbf{Tree}(M))$  has distribution  $\mathbf{GW}_\nu^{(2k+1)}$ . Moreover, conditionally on  $\mathbf{Tree}(M)$ , the maps  $(\widehat{M}_u : u \in \mathbf{Tree}(M)_\bullet)$  associated to  $M$  by  $\Phi_{\text{TC}}$  are independent with respective distribution  $\widehat{\mathbb{P}}_q^{(\deg(u)/2)}$ .*

*Proof.* Recall from (31) that for every  $\mathbf{m} \in \mathcal{M}$ , the size of  $\mathbf{t} = \Phi_{\text{JS}}(\mathbf{Tree}(\mathbf{m}))$  (or equivalently of  $\mathbf{Tree}(\mathbf{m})$ ) equals  $\#\partial\mathbf{m} + 1$ . Then, by Proposition 3.3, for every  $k \geq 1$ ,

$$\mathbf{GW}_\rho(\{|\mathbf{t}| = 2k + 1\}) = \mathbf{GW}_{\rho_\circ, \rho_\bullet}(\{|\mathbf{t}| = 2k + 1\}) = \mathbb{P}_{q, r_q}(\mathcal{M}_k) = \frac{r_q^k F_k}{F(r_q)},$$

which is positive by the results of Section 2.2. We conclude by applying Proposition 3.3.  $\square$

## 4 Scaling limits of the boundary of Boltzmann maps

This section deals with the scaling limits of the boundary of Boltzmann bipartite maps, in the Gromov-Hausdorff sense. We refer to [18] for a complete definition of this topology. We start with a preliminary result directly adapted of [23, Lemma 4.3].

**Lemma 4.1.** [23] *For every  $\mathbf{m} \in \mathcal{M}$ , we have  $\text{Scoop}(\mathbf{m}) = \overline{\mathbf{Loop}}(\Phi_{\text{JS}}(\mathbf{Tree}(\mathbf{m})))$ .*

**Scaling limits for the boundary: the dense regime.** We first focus on the dense phase  $a \in (3/2, 2)$  and prove Theorem 1.1. The proof parallels that of [23, Theorem 1.2]. It is based on the fact that the random  $\beta$ -stable looptree  $\mathcal{L}_\beta$  is the scaling limit of looptrees associated to critical Galton-Watson trees conditioned to survive, when the offspring distribution is in the domain of attraction of a stable law with parameter  $\beta$  [24, Theorem 4.1].

*Proof of Theorem 1.1.* For every  $k \geq 0$ , let  $M_k$  be a random planar map with distribution  $\mathbb{P}_q^{(k)}$  and set  $T_k := \Phi_{\text{JS}}(\mathbf{Tree}(M_k))$ . By definition of  $\overline{\mathbf{Loop}}$ , we have

$$d_{\text{GH}}(\mathbf{Loop}(T_k), \overline{\mathbf{Loop}}(T_k)) \leq 2H(T_k), \quad (47)$$

where  $H(T_k)$  is the overall height of  $T_k$ . Indeed, the longest path of vertices of  $T_k$  that are identified in  $\overline{\mathbf{Loop}}(T_k)$  has length at most  $H(T_k)$ . By the scaling limits results for conditioned Galton-Watson trees ([27, Theorem 3.1], [38, Theorem 3]) we have that

$$\frac{H(T_k)}{k^{a-1}} \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{in probability.} \quad (48)$$

The results of [27, 38] together with (48) ensure that the invariance principle of [24, Theorem 4.1] applies: there exists a slowly varying function  $\Lambda$  such that

$$\frac{\Lambda(k)}{(2k)^{a-1}} \cdot \mathbf{Loop}(T_k) \xrightarrow[k \rightarrow \infty]{(d)} \mathcal{L}_{\frac{1}{a-1}},$$

in the Gromov-Hausdorff sense. Applying (47), (48) and Lemma 4.1, we deduce that

$$\frac{\Lambda(k)}{(2k)^{a-1}} \cdot \text{Scoop}(M_k) \xrightarrow[k \rightarrow \infty]{(d)} \mathcal{L}_{\frac{1}{a-1}},$$

in the Gromov-Hausdorff sense. This concludes the proof since for any planar map  $\mathbf{m}$ ,  $\partial\mathbf{m}$  and  $\text{Scoop}(\mathbf{m})$  define the same metric space.  $\square$

**Remark 4.2.** As a byproduct, we get a scaling limit result for the tree  $T_k := \Phi_{\text{JS}}(\mathbf{Tree}(M_k))$ : there exists a slowly varying function  $\Lambda'$  such that in the Gromov-Hausdorff sense

$$\frac{\Lambda'(k)}{(2k)^{2-a}} \cdot T_k \xrightarrow[k \rightarrow \infty]{(d)} \mathcal{T}_{\frac{1}{a-1}},$$

where  $\mathcal{T}_\beta$  is the stable tree with parameter  $\beta$  [28, 29].

**Scaling limits for the boundary: the subcritical regime.** In the subcritical case, we need to make assumptions on the partition function  $\widehat{F}_k$  to obtain a convergence result. Let  $\mathbf{q}$  be a subcritical weight sequence and assume that  $\widehat{r}_q > r_q F^2(r_q)$ . For every  $k \geq 0$ , let  $M_k$

be a random planar map with distribution  $\mathbb{P}_{\mathbf{q}}^{(k)}$ . Then, there exists a constant  $K_{\mathbf{q}}$  such that in distribution for the Gromov-Hausdorff topology

$$\frac{K_{\mathbf{q}}}{\sqrt{2k}} \cdot \partial M_k \xrightarrow[k \rightarrow \infty]{(d)} \mathcal{T}_{\mathbf{e}}, \quad (49)$$

where  $\mathcal{T}_{\mathbf{e}}$  is the Continuum Random Tree [1, 2]. The proof relies on the convergence result of looptrees [22, Theorem 14], for which we need the offspring distribution  $\nu$  to have small exponential moments, i.e. that  $\widehat{r}_{\mathbf{q}} > r_{\mathbf{q}} F^2(r_{\mathbf{q}})$ . As mentioned in Remark 2.9, we do not know if this is satisfied for any subcritical sequence  $\mathbf{q}$  (but only for subcritical Boltzmann quadrangulations). However, we believe that [22, Theorem 14] holds under a finite variance assumption, and hope to investigate this in a future work.

**Scaling limits for the boundary: the generic and dilute regimes.** In the generic and dilute regimes, we also need extra assumptions on the partition function  $\widehat{F}_k$ . Let  $\mathbf{q}$  be a weight sequence of type  $a = (2, 5/2)$ , and assume that there exists a slowly varying function  $\ell_d$  such that

$$\widehat{F}_k \underset{k \rightarrow \infty}{\sim} \frac{\ell_d(k)}{k^a} \widehat{r}_{\mathbf{q}}^{-k}. \quad (50)$$

For every  $k \geq 0$ , let  $M_k$  be a random planar map with distribution  $\mathbb{P}_{\mathbf{q}}^{(k)}$ . Then, there exists a constant  $K_{\mathbf{q}}$  such that in distribution for the Gromov-Hausdorff topology

$$\frac{K_{\mathbf{q}}}{2k} \cdot \partial M_k \xrightarrow[k \rightarrow \infty]{(d)} \mathbb{S}_1, \quad (51)$$

where  $\mathbb{S}_1$  stands for the unit circle. The proof can be adapted from [23, Theorem 1.2], which is itself based on the results of [36, 39] about condensation in non-generic trees. As mentioned in Remark 2.9, we do not have an equivalent of the partition function  $\widehat{F}_k$  in general, which is needed to apply the results of [36, 39]. However, we believe that (51) holds independently of (50) and also hope to investigate this in a future work. By Remark 2.9, (50) is satisfied for critical Boltzmann quadrangulations. In this case, [7, Theorem 8] also proves that

$$\sqrt{\frac{3}{2k}} \cdot M_k \xrightarrow[k \rightarrow \infty]{(d)} \text{FBD}_1,$$

in the Gromov-Hausdorff sense. The random compact metric space  $\text{FBD}_1$  is the Free Brownian Disk with perimeter 1, a.s. homeomorphic to the unit disc [7] (which is consistent with (51)).

## 5 Local limits of the boundary of Boltzmann maps

In this section, we are interested in local limits of Boltzmann planar maps and their boundary.

### 5.1 Local limits of Galton-Watson trees

**The local topology.** The *local topology* on the set  $\mathcal{M}$  is induced by the local distance

$$d_{\text{loc}}(\mathbf{m}, \mathbf{m}') := (1 + \sup \{R \geq 0 : \mathbf{B}_R(\mathbf{m}) \sim \mathbf{B}_R(\mathbf{m}')\})^{-1}, \quad \mathbf{m}, \mathbf{m}' \in \mathcal{M}. \quad (52)$$

Here,  $\mathbf{B}_R(\mathbf{m})$  is the ball of radius  $R$  in  $\mathbf{m}$  for the graph distance, centered at the origin vertex. More precisely,  $\mathbf{B}_0(\mathbf{m})$  is the origin of  $\mathbf{m}$ , and for every  $R \in \mathbb{N}$ ,  $\mathbf{B}_R(\mathbf{m})$  contains all the vertices of  $\mathbf{m}$  at distance less than  $R$  from the origin, and all the edges whose endpoints are in this set. Equipped with  $d_{\text{loc}}$ ,  $\mathcal{M}$  is a metric space whose completion is denoted by  $\mathcal{M}'$ . The elements of  $\mathcal{M}_\infty := \mathcal{M}' \setminus \mathcal{M}$  can be considered as infinite planar maps, i.e. the proper embedding of an infinite and locally finite graph into a non-compact surface, dissecting the latter into a collection of simply connected domains (see [26, Appendix] for more on this). The set  $\mathcal{T}_{\text{loc}}$  of locally finite trees is the completion of  $\mathcal{T}_f$  for  $d_{\text{loc}}$ . It is also obtained by extending the definition of a finite plane tree to possibly infinite trees, but whose vertices all have finite degree (i.e.  $k_u(\mathbf{t}) < \infty$  for every  $u \in \mathbf{t}$ ).

In order to take account of convergence towards plane trees with vertices of infinite degree, a weaker form of local convergence has been introduced in [36] (see also [34, Section 6] for a detailed presentation). The idea is to replace the ball  $\mathbf{B}_R(\mathbf{t})$  in (52) by the sub-tree  $\mathbf{B}_R^{\leftarrow}(\mathbf{t})$ , called the *left ball* of radius  $R$  of  $\mathbf{t}$ . Formally, the root vertex belongs to  $\mathbf{B}_R^{\leftarrow}(\mathbf{t})$ , and a vertex  $u = \widehat{u}k \in \mathbf{t}$  belongs to  $\mathbf{B}_R^{\leftarrow}(\mathbf{t})$  iff  $\widehat{u} \in \mathbf{t}$ ,  $k \leq R$  and  $|u| \leq R$ .

For our purposes, a slightly stronger form of convergence is needed. Let us introduce a notation. For every  $\mathbf{t} \in \mathcal{T}_f$  and every  $u \in \mathbf{t}$ , we denote by  $(-u1, -u2, \dots, -uk_u) = (uk_u, u(k_u - 1), \dots, u1)$  the children of  $u$  in counterclockwise order. For every  $\mathbf{t} \in \mathcal{T}_f$  and every  $R \geq 0$ , the *left-right ball* of radius  $R$  in  $\mathbf{t}$  is the sub-tree  $\mathbf{B}_R^{\leftrightarrow}(\mathbf{t})$  defined as follows. First,  $\emptyset \in \mathbf{B}_R^{\leftrightarrow}(\mathbf{t})$ . Then, a vertex  $u \in \mathbf{t}$  belongs to  $\mathbf{B}_R^{\leftrightarrow}(\mathbf{t})$  iff  $\widehat{u} \in \mathbf{B}_R^{\leftrightarrow}(\mathbf{t})$ ,  $|u| \leq 2R$  and  $u \in \{\widehat{u}1, \dots, \widehat{u}R\} \cup \{-\widehat{u}1, \dots, -\widehat{u}R\}$  (i.e.  $u$  is among the  $R$  first or last children of its parent).

We call *local-\* topology* the topology on  $\mathcal{T}_f$  induced by the distance

$$d_{\text{loc}}^*(\mathbf{t}, \mathbf{t}') := (1 + \sup \{R \geq 0 : \mathbf{B}_R^{\leftrightarrow}(\mathbf{t}) = \mathbf{B}_R^{\leftrightarrow}(\mathbf{t}')\})^{-1}, \quad \mathbf{t}, \mathbf{t}' \in \mathcal{T}_f.$$

The set  $\mathcal{T}$  of general plane trees is the completion of  $\mathcal{T}_f$  for  $d_{\text{loc}}^*$ . An element of  $\mathcal{T}$  can also be seen as a tree embedded in the plane. In restriction to  $\mathcal{T}_{\text{loc}}$ , the local and local-\* topologies coincide.

**Local limits of conditioned Galton-Watson trees.** We next recall results concerning local limits of Galton-Watson trees conditioned to survive.

*The critical case.* The critical setting was first investigated by Kesten in [37] (see also [46]) for monotone trees and extended by Stephenson in [56, Theorem 3.1] to multi-type trees. Let  $(\rho_\circ, \rho_\bullet)$  be a critical pair of offspring distributions, and recall that for every probability measure  $\rho$  on  $\mathbb{Z}_+$  with mean  $m_\rho \in (0, \infty)$ , the size-biased distribution  $\bar{\rho}$  is defined by

$$\bar{\rho}(k) := \frac{k\rho(k)}{m}, \quad k \in \mathbb{Z}_+.$$

The infinite random tree  $\mathbf{T}_\infty^{\circ, \bullet} = \mathbf{T}_\infty^{\circ, \bullet}(\rho_\circ, \rho_\bullet)$  is defined as follows in [56]. It has a.s. a unique infinite spine, i.e. a distinguished sequence of vertices  $(u_0 = \emptyset, u_1, \dots)$  such that  $\widehat{u}_k = u_{k-1}$  for every  $k \geq 1$ . White (resp. black) vertices of the spine have offspring distribution  $\bar{\rho}_\circ$  (resp.  $\bar{\rho}_\bullet$ ), and a unique child in the spine chosen uniformly at random among their offspring. Outside of the spine, white (resp. black) vertices have offspring distribution  $\rho_\circ$  (resp.  $\rho_\bullet$ ), and all the number of offspring are independent. The tree  $\mathbf{T}_\infty^{\circ, \bullet}$  is illustrated in Figure 10, and its distribution is denoted by  $\text{GW}_{\rho_\circ, \rho_\bullet}^{(\infty)}$ .

**Proposition 5.1.** [56, Theorem 3.1] *Let  $(\rho_\circ, \rho_\bullet)$  be a critical pair of offspring distributions. For every  $k \geq 1$ , assume that  $\text{GW}_{\rho_\circ, \rho_\bullet}(\{|\mathbf{t}| = k\}) > 0$  and let  $T_k^{\circ, \bullet}$  be a tree with distribution*

$\text{GW}_{\rho_\circ, \rho_\bullet}^{(k)}$ . Then, we have in distribution for the local topology

$$T_k^{\circ, \bullet} \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{T}_\infty^{\circ, \bullet}(\rho_\circ, \rho_\bullet).$$

The condition  $\text{GW}_{\rho_\circ, \rho_\bullet}(\{|\mathbf{t}| = k\}) > 0$  for every  $k \geq 1$  can be relaxed, provided we consider subsequences along which it is satisfied. A similar result holds for critical monotype Galton-Watson trees conditioned to survive. Then, the limiting tree is called *Kesten's tree*.

*The subcritical case.* We first deal with subcritical monotype trees, first considered in [36], and studied in full generality in [34, Theorem 7.1]. Let  $\rho$  be a subcritical offspring distribution (such that  $\rho(0) \in (0, 1)$ ). The infinite random tree  $\mathbf{T}_\infty = \mathbf{T}_\infty(\rho)$  is defined as follows in [36, 34]. It has a.s. a unique finite spine of random size  $L$ , such that

$$P(L = k) = (1 - m_\rho)m_\rho^{k-1}, \quad k \in \mathbb{N}.$$

The last vertex of the spine has infinite degree. The  $L - 1$  first vertices of the spine have offspring distribution  $\bar{\rho}$ , and a unique child in the spine chosen uniformly among the offspring. Outside of the spine, vertices have offspring distribution  $\rho$ , and all the number of offsprings are independent. This defines a random element of  $\mathcal{T}$ .

**Proposition 5.2.** [34, Theorem 7.1] *Let  $\rho$  be a subcritical offspring distribution with no exponential moment (and  $\rho(0) \in (0, 1)$ ). For every  $k \geq 1$ , assume that  $\text{GW}_\rho(\{|\mathbf{t}| = k\}) > 0$  and let  $T_k$  be a tree with distribution  $\text{GW}_\rho^{(k)}$ . Then, in distribution for the local-\* topology,*

$$T_k \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{T}_\infty(\rho).$$

*Proof.* The proof follows from [34, Theorem 7.1]. However, the notion of convergence in this result is equivalent to the convergence of left-balls of any radii (see [34, Lemma 6.3]), which is weaker than our statement. Then, observe that for every  $\mathbf{t} \in \mathcal{T}_f$ ,  $k \geq 0$  and  $R \geq 0$  we have

$$\text{GW}_\rho^{(k)}(\mathbf{B}_R^{\leftrightarrow}(T) = \mathbf{t}) = \text{GW}_\rho^{(k)}(\mathbf{B}_{2R}^{\leftarrow}(T) = \mathbf{t}).$$

Indeed,  $\text{GW}_\rho^{(k)}$  is invariant under the operation consisting in exchanging the descendants of  $(u(R + 1), \dots, u(2R))$  and  $(-u1, \dots, -uR)$  for every  $u \in T$  such that  $k_u(T) > 2R$  (which exchanges  $\mathbf{B}_R^{\leftrightarrow}(T)$  and  $\mathbf{B}_{2R}^{\leftarrow}(T)$ ). This concludes the argument.  $\square$

We extend Proposition 5.2 to a two-type Galton-Watson tree. Let  $(\rho_\circ, \rho_\bullet)$  be a subcritical pair of offspring distributions. We build a two-type version  $\mathbf{T}_\infty^{\circ, \bullet} = \mathbf{T}_\infty^{\circ, \bullet}(\rho_\circ, \rho_\bullet)$  of  $\mathbf{T}_\infty$  as follows. It has a.s. a unique spine, with random number of vertices  $2L'$  satisfying

$$P(L' = k) = (1 - m_{\rho_\circ}m_{\rho_\bullet})(m_{\rho_\circ}m_{\rho_\bullet})^{k-1}, \quad k \in \mathbb{N}.$$

The topmost (black) vertex of the spine has infinite degree. The  $2L' - 1$  first vertices of the spine have offspring distribution  $\bar{\rho}_\circ$  (if white) and  $\bar{\rho}_\bullet$  (if black), with a unique child in the spine chosen uniformly among the offspring. Outside of the spine, white (resp. black) vertices have offspring distribution  $\rho_\circ$  (resp.  $\rho_\bullet$ ), and all the number of offsprings are independent. We keep the notation  $\text{GW}_{\rho_\circ, \rho_\bullet}^{(\infty)}$  for the distribution of  $\mathbf{T}_\infty^{\circ, \bullet}$ . See Figure 10 for an illustration.

**Proposition 5.3.** *Let  $(\rho_\circ, \rho_\bullet)$  be a subcritical pair of offspring distributions, such that  $\rho_\circ$  has geometric distribution with parameter  $1 - p \in (0, 1)$ :  $\rho_\circ(k) = (1 - p)p^k$  for  $k \geq 0$ , and such that  $\rho_\bullet$  has no exponential moment. For every  $k \geq 1$ , assume that  $\text{GW}_{\rho_\circ, \rho_\bullet}(\{|\mathbf{t}| = k\}) > 0$  and let  $T_k^{\circ, \bullet}$  be a tree with distribution  $\text{GW}_{\rho_\circ, \rho_\bullet}^{(k)}$ . Then, in distribution for the local-\* topology,*

$$T_k^{\circ, \bullet} \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{T}_\infty^{\circ, \bullet}(\rho_\circ, \rho_\bullet).$$

The condition  $\text{GW}_{\rho_\circ, \rho_\bullet}(\{|\mathbf{t}| = k\}) > 0$  for every  $k \geq 1$  can be relaxed by considering subsequences. For every  $\mathbf{t} \in \mathcal{T}_f$  and every  $u \in \mathbf{t}$ , we denote by  $[\emptyset, u]$  (resp.  $[\emptyset, u)$ ) the ancestral line of  $u$  in  $\mathbf{t}$ ,  $u$  included (resp. excluded).

*Proof.* For every  $k \geq 1$ , let  $T_k := \Phi_{\text{JS}}(T_k^{\circ, \bullet})$ . By Proposition 3.1,  $T_k$  has distribution  $\text{GW}_\rho^{(k)}$ , where  $\rho$  satisfies

$$\rho(0) = 1 - p \quad \text{and} \quad \rho(k) = p \cdot \rho_\bullet(k - 1), \quad k \in \mathbb{N}.$$

In particular,  $\rho$  satisfies the hypothesis of Proposition 5.2. For every  $N \geq 1$ , let  $u_N = u_N(T_k)$  be the first vertex of  $\mathbf{B}_N^{\leftrightarrow}(T_k)$  in contour order having  $2N$  offspring (or the root vertex if such a vertex does not exist). For every  $R \geq 0$ , we also let  $T_k \langle u_N, R \rangle$  be the collection of subtrees of  $T_k$  containing all the children of  $u_N$  different from  $\{\pm u_N 1, \dots, \pm u_N R\}$ , as well as their descendants. Finally, set  $T_k[N, R] := \mathbf{B}_N^{\leftrightarrow}(T_k) \setminus T_k \langle u_N, R \rangle$ , and extend these definitions to  $\mathbf{T}_\infty = \mathbf{T}_\infty(\rho)$ . We denote by  $u_\infty$  the a.s. unique vertex with infinite degree of  $\mathbf{T}_\infty$ , and let  $\mathbf{T}_\infty[R]$  be the subtree of  $\mathbf{T}_\infty$  in which children of  $u_\infty$  different from  $\{u_\infty 1, \dots, u_\infty R\}$  and their descendants are discarded. This definition immediately extends to  $\mathbf{T}_\infty^{\circ, \bullet}$ .

Fix  $R \geq 0$ . By Proposition 5.2 and the definition of  $\mathbf{T}_\infty$ , we have in the local sense

$$T_k[N, R + 1] \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{T}_\infty[N, R + 1], \quad \text{and} \quad \mathbf{T}_\infty[N, R + 1] \xrightarrow[N \rightarrow \infty]{(d)} \mathbf{T}_\infty[2(R + 1)]. \quad (53)$$

In particular, the event (measurable with respect to  $\mathbf{B}_N^{\leftrightarrow}(T_k)$ )

$$\mathcal{E}(R, N, k) := \left\{ \sup_{u \in T_k[N, R + 1]} (|u| \vee k_u) < N \right\}$$

has probability tending to one when  $k$  and then  $N$  go to infinity. On the event  $\mathcal{E}(R, N, k)$ , one has  $T_k \setminus T_k[N, R + 1] \subseteq T_k \langle u_N, R + 1 \rangle$ , which in turn enforces

$$\mathbf{B}_R^{\leftrightarrow}(T_k^{\circ, \bullet}) = \mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(T_k)) \subseteq \Phi_{\text{JS}}^{-1}(T_k[N, R + 1]). \quad (54)$$

Indeed, under this assumption, the images of vertices of  $T_k \setminus T_k[N, R + 1]$  in  $\Phi_{\text{JS}}^{-1}(T_k)$  are descendants of the children of  $u'_N := \Phi_{\text{JS}}^{-1}(u_N)$  that are not in  $\{\pm u'_N 1, \dots, \pm u'_N R\}$ . (See Section 3.1 for details on the inverse Janson-Stefánsson bijection, and Figure 9 for an illustration.)

Let  $d \geq 0$ , and keep the notation  $u_\infty$  for the pointed vertex of degree  $d$  in  $\mathbf{T}_\infty[d]$  and  $\mathbf{T}_\infty^{\circ, \bullet}[d]$ . We denote by  $\text{GW}_\rho^{[d]}$  the distribution of  $(\mathbf{T}_\infty[d], u_\infty)$ , and by  $\text{GW}_{\rho_\circ, \rho_\bullet}^{[d]}$  the distribution of  $(\mathbf{T}_\infty^{\circ, \bullet}[d], u_\infty)$ . Then, the image of  $\text{GW}_{\rho_\circ, \rho_\bullet}^{[d]}$  under  $\Phi_{\text{JS}}$  reads

$$\Phi_{\text{JS}} \left( \text{GW}_{\rho_\circ, \rho_\bullet}^{[d]} \right) = \text{GW}_\rho^{[d+1]}. \quad (55)$$

We temporarily admit (55) and conclude the proof. Let  $A$  be a Borel set for the local-\* topology. We have by (54) that for every  $k \geq 1$  and  $N \geq 1$

$$\left| P(\mathbf{B}_R^{\leftrightarrow}(T_k^{\circ, \bullet}) \in A) - P(\mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(T_k[N, R + 1])) \in A) \right| \leq 2P(\mathcal{E}(R, N, k)^c).$$

Next, for every  $N \geq 1$ , (53) entails

$$\left| P(\mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(T_k[N, R+1])) \in A) - P(\mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(\mathbf{T}_\infty[N, R+1])) \in A) \right| \xrightarrow[k \rightarrow \infty]{} 0.$$

Then, by (53) again and the fact that  $\mathbf{T}_\infty[2(R+1)]$  is a.s. finite,

$$\left| P(\mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(\mathbf{T}_\infty[N, R+1])) \in A) - P(\mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(\mathbf{T}_\infty[2(R+1)])) \in A) \right| \xrightarrow[N \rightarrow \infty]{} 0.$$

Finally, for every  $R \geq 0$ ,  $\mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet}) = \mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet}[2R+1])$  by definition so that by (55),

$$P(\mathbf{B}_R^{\leftrightarrow}(\Phi_{\text{JS}}^{-1}(\mathbf{T}_\infty[2(R+1)])) \in A) = P(\mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet}[2R+1]) \in A) = P(\mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet}) \in A).$$

As a conclusion, by letting  $k$  and then  $N$  go to infinity, we have

$$\lim_{k \rightarrow \infty} |P(\mathbf{B}_R^{\leftrightarrow}(T_k^{\circ, \bullet}) \in A) - P(\mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet}) \in A)| = 0,$$

which ends the proof.

Let us now prove assertion (55). Let  $(\mathbf{t}, u^*)$  be a pointed plane tree such that  $k_{u^*}(\mathbf{t}) = d+1$ . By definition,  $(\mathbf{t}', v^*) := \Phi_{\text{JS}}^{-1}(\mathbf{t}, u^*)$  is a pointed plane tree satisfying  $k_{v^*}(\mathbf{t}') = d$ , and  $v^* \in \mathbf{t}'_\bullet$ . Then, we have by definition of  $\rho_\circ$  and the identity  $\sum_{u \in \mathbf{t}'_\circ} k_u(\mathbf{t}') = |\mathbf{t}'_\bullet|$ ,

$$\begin{aligned} \text{GW}_{\rho_\circ, \rho_\bullet}^{[d]}(\Phi_{\text{JS}}^{-1}((\mathbf{t}, u^*))) &= \text{GW}_{\rho_\circ, \rho_\bullet}^{[d]}((\mathbf{t}', v^*)) = \frac{1 - m_{\rho_\circ} m_{\rho_\bullet}}{m_{\rho_\circ}} \prod_{u \in \mathbf{t}'_\circ} (1-p) p^{k_u(\mathbf{t}')} \prod_{\substack{u \in \mathbf{t}'_\bullet \\ u \neq v^*}} \rho_\bullet(k_u(\mathbf{t}')) \\ &= \frac{p(1 - m_{\rho_\circ} m_{\rho_\bullet})}{m_{\rho_\circ}} \prod_{u \in \mathbf{t}'_\circ} (1-p) \prod_{\substack{u \in \mathbf{t}'_\bullet \\ u \neq v^*}} p \cdot \rho_\bullet(k_u(\mathbf{t}')). \end{aligned}$$

Vertices of  $\mathbf{t}'_\circ$  are mapped to leaves of  $\mathbf{t}$  by  $\Phi_{\text{JS}}$ , while vertices of  $\mathbf{t}'_\bullet \setminus \{v^*\}$  with degree  $k$  are mapped to vertices of  $\mathbf{t}$  with degree  $k+1$ . By the formula  $m_\rho - p = (1-p)m_{\rho_\circ}m_{\rho_\bullet}$ , we get

$$\begin{aligned} \text{GW}_{\rho_\circ, \rho_\bullet}^{[d]}(\Phi_{\text{JS}}^{-1}((\mathbf{t}, u^*))) &= (1 - m_\rho) \prod_{\substack{u \in \mathbf{t} \\ k_u(\mathbf{t})=0}} (1-p) \prod_{\substack{u \in \mathbf{t} \setminus \{u^*\} \\ k_u(\mathbf{t})>0}} p \cdot \rho_\bullet(k_u(\mathbf{t}) - 1) \\ &= (1 - m_\rho) \prod_{u \in \mathbf{t} \setminus \{u^*\}} \rho(k_u(\mathbf{t})), \end{aligned}$$

which is  $\text{GW}_\rho^{[d+1]}((\mathbf{t}, u^*))$ , as expected.  $\square$

We conclude with a property of  $\mathbf{T}_\infty^{\circ, \bullet}$  under the assumptions of Proposition 5.3. Let  $u_\infty$  be the a.s. unique vertex with infinite degree of  $\mathbf{T}_\infty^{\circ, \bullet}$ , and  $\widehat{u}_\infty$  its parent. There exists  $j \in \{1, \dots, k_{\widehat{u}_\infty}\}$  such that  $u_\infty = \widehat{u}_\infty j$ . We define the vertex  $u_\infty^{\leftarrow}$  as  $\widehat{u}_\infty(j-1)$  if  $j > 1$ , and  $\widehat{u}_\infty$  itself if  $j = 1$ . The vertex  $u_\infty$  and its incident edges disconnect  $\mathbf{T}_\infty^{\circ, \bullet}$  in infinitely many connected components that we denote by  $(\mathbf{T}_i : i \in \mathbb{Z})$ . For every  $i \neq 0$ ,  $\mathbf{T}_i$  is the connected component containing  $u_\infty i$ , rooted at the oriented edge going from  $u_\infty i$  to its first child in  $\mathbf{T}_\infty^{\circ, \bullet}$ . Finally,  $\mathbf{T}_0$  is the connected component containing the root vertex of  $\mathbf{T}_\infty^{\circ, \bullet}$ , and has the same root edge as  $\mathbf{T}_\infty^{\circ, \bullet}$ . Note that  $u_\infty^{\leftarrow}$  is a vertex of  $\mathbf{T}_0$ .

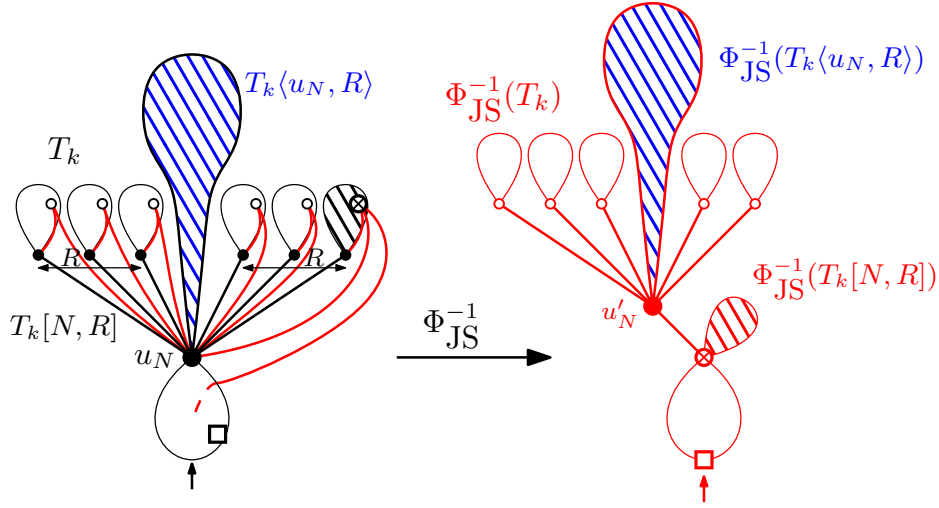


Figure 9: The image of  $T_k$  by  $\Phi_{\text{JS}}^{-1}$ , on the event  $\mathcal{E}(R, N, k)$ . The boxed vertex is the last leaf of  $T_k$  in contour order, while the crossed vertex is the last leaf among the descendants of  $u_N$ .

**Lemma 5.4.** *The plane trees  $(\mathbf{T}_i : i \in \mathbb{Z})$  are independent. For every  $i \neq 0$ ,  $\mathbf{T}_i$  has distribution  $\text{GW}_{\rho_\circ, \rho_\bullet}$ , while  $\mathbf{T}_0$  has the size-biased distribution  $\overline{\text{GW}}_{\rho_\circ, \rho_\bullet}$  defined by*

$$\overline{\text{GW}}_{\rho_\circ, \rho_\bullet}(\mathbf{t}) = \frac{|\mathbf{t}| \text{GW}_{\rho_\circ, \rho_\bullet}(\mathbf{t})}{\text{GW}_{\rho_\circ, \rho_\bullet}(|T|)}, \quad \mathbf{t} \in \mathcal{T}_f.$$

Moreover, conditionally on  $\mathbf{T}_0$ ,  $u_\infty^\leftarrow$  has uniform distribution on  $\mathbf{T}_0$ .

*Proof.* We focus on  $\mathbf{T}_0$ . Let  $(\mathbf{t}, u^*)$  be a pointed plane tree, and let  $u^\circ$  be either the parent of  $u^*$  in  $\mathbf{t}$  if  $u^* \in \mathbf{t}_\bullet$ , or  $u^*$  itself otherwise. Then,  $(\mathbf{T}_0, u_\infty^\leftarrow) = (\mathbf{t}, u^*)$  enforces  $\widehat{u}_\infty = u^\circ$ . Since  $k_{\widehat{u}_\infty}(\mathbf{T}_0) = k_{\widehat{u}_\infty}(\mathbf{T}_\infty^{\circ, \bullet}) - 1$  a.s. and by definition of  $\rho_\circ$ , we obtain

$$\begin{aligned} P((\mathbf{T}_0, u_\infty^\leftarrow) = (\mathbf{t}, u^*)) &= \prod_{\substack{u \in \mathbf{t}_\circ \\ u \in [\emptyset, u^\circ]}} \frac{\bar{\rho}_\circ(k_u(\mathbf{t}))}{k_u(\mathbf{t})} \prod_{\substack{u \in \mathbf{t}_\bullet \\ u \in [\emptyset, u^\circ]}} \frac{\bar{\rho}_\bullet(k_u(\mathbf{t}))}{k_u(\mathbf{t})} \prod_{\substack{u \in \mathbf{t}_\circ \\ u \notin [\emptyset, u^\circ]}} \rho_\circ(k_u(\mathbf{t})) \prod_{\substack{u \in \mathbf{t}_\bullet \\ u \notin [\emptyset, u^\circ]}} \rho_\bullet(k_u(\mathbf{t})) \\ &\quad \times \bar{\rho}_\circ(k_{u^\circ}(\mathbf{t}) + 1) \frac{1}{k_{u^\circ}(\mathbf{t}) + 1} (1 - m_{\rho_\circ} m_{\rho_\bullet}) (m_{\rho_\circ} m_{\rho_\bullet})^{\frac{|u^\circ|}{2}} \\ &= \frac{p(1 - m_{\rho_\circ} m_{\rho_\bullet})}{m_{\rho_\circ}} \prod_{u \in \mathbf{t}_\circ} \rho_\circ(k_u(\mathbf{t})) \prod_{u \in \mathbf{t}_\bullet} \rho_\bullet(k_u(\mathbf{t})) = (1 - m_\rho) \text{GW}_{\rho_\circ, \rho_\bullet}(\mathbf{t}). \end{aligned}$$

We conclude by Proposition 3.1, which gives  $\text{GW}_{\rho_\circ, \rho_\bullet}(|T|) = \text{GW}_\rho(|T|) = (1 - m_\rho)^{-1}$ .  $\square$

## 5.2 Random infinite looptrees.

We now define infinite looptrees out of the infinite random trees  $\mathbf{T}_\infty^{\circ, \bullet} = \mathbf{T}_\infty^{\circ, \bullet}(\rho_\circ, \rho_\bullet)$ .

*The critical case.* When  $(\rho_\circ, \rho_\bullet)$  is critical,  $\mathbf{T}_\infty^{\circ, \bullet}$  is a.s. locally finite. We extend the mapping Loop to any locally finite plane tree  $\mathbf{t} \in \mathcal{T}_{\text{loc}}$  by defining  $\text{Loop}(\mathbf{t})$  as the consistent sequence of maps  $\{\text{Loop}(\mathbf{B}_{2R}(\mathbf{t})) : R \geq 0\}$ . This mapping is continuous on  $\mathcal{T}_{\text{loc}}$  for the local topology. When  $\mathbf{t}$  is infinite and one-ended (i.e., with a unique infinite spine),  $\text{Loop}(\mathbf{t})$  is an infinite



looptree, that is an edge-outerplanar map whose root face is the unique infinite face. Then,  $\mathbf{L}_\infty = \mathbf{L}_\infty(\rho_\circ, \rho_\bullet)$  is the random infinite looptree

$$\mathbf{L}_\infty := \text{Loop}(\mathbf{T}_\infty^{\circ, \bullet}).$$

We call  $\mathbf{T}_\infty^{\circ, \bullet}$  the tree of components of  $\mathbf{L}_\infty$ , and denote it by  $\text{Tree}(\mathbf{L}_\infty)$ . The looptree  $\mathbf{L}_\infty$  is illustrated in Figure 10.

*The subcritical case.* When  $(\rho_\circ, \rho_\bullet)$  is subcritical,  $\rho_\circ$  geometric and  $\rho_\bullet$  has no exponential moment,  $\mathbf{T}_\infty^{\circ, \bullet}$  has a.s. a unique vertex  $u_\infty$  with infinite degree. Since  $u_\infty$  has odd height, the sequence  $(\mathbf{B}_r(\text{Loop}(\mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet}))) : R \geq 0)$  is eventually stationary, for every  $r \geq 0$ . Consequently, we define  $\mathbf{L}_\infty = \mathbf{L}_\infty(\rho_\circ, \rho_\bullet)$  as the local limit

$$\mathbf{L}_\infty := \lim_{R \rightarrow \infty} \text{Loop}(\mathbf{B}_R^{\leftrightarrow}(\mathbf{T}_\infty^{\circ, \bullet})). \quad (56)$$

Although  $\mathbf{L}_\infty$  is not a looptree in the aforementioned sense, we keep the notation  $\mathbf{L}_\infty = \text{Loop}(\mathbf{T}_\infty^{\circ, \bullet})$  and  $\mathbf{T}_\infty^{\circ, \bullet} = \text{Tree}(\mathbf{L}_\infty)$ . By Lemma 5.4,  $\mathbf{L}_\infty$  can be obtained as follows. We associate to  $u_\infty^{\leftarrow}$  the oriented edge  $e_\infty^{\leftarrow}$  of  $\mathbf{T}_0$  that links either  $\hat{u}_\infty$  to  $u_\infty^{\leftarrow}$  if  $u_\infty^{\leftarrow}$  has odd height, or  $\hat{u}_\infty$  to its parent otherwise. For every  $i \in \mathbb{Z}$ , we define the looptree  $\mathbf{L}_i := \text{Loop}(\mathbf{T}_i)$  (with root edge  $e_i$ ). Following the rooting convention of Section 3.2, to  $e_\infty^{\leftarrow}$  is associated an oriented edge  $e_0^{\leftarrow}$  of  $\mathbf{L}_0$ . We now consider the graph of  $\mathbb{Z}$  embedded in the plane. For every  $i \neq 0$ , we embed  $\mathbf{L}_i$  in the lower half-plane such that the vertex  $i$  of  $\mathbb{Z}$  matches the origin vertex of  $\mathbf{L}_i$ , and the edges  $(i-1, i)$  and  $e_i$  are consecutive in counterclockwise order around  $i$ . We apply the same construction to  $\mathbf{L}_0$ , but use  $e_0^{\leftarrow}$  instead of  $e_0$ . The resulting planar map, rooted at  $e_0$ , is  $\mathbf{L}_\infty = \text{Loop}(\mathbf{T}_\infty^{\circ, \bullet})$ . See Figure 10 for an illustration.

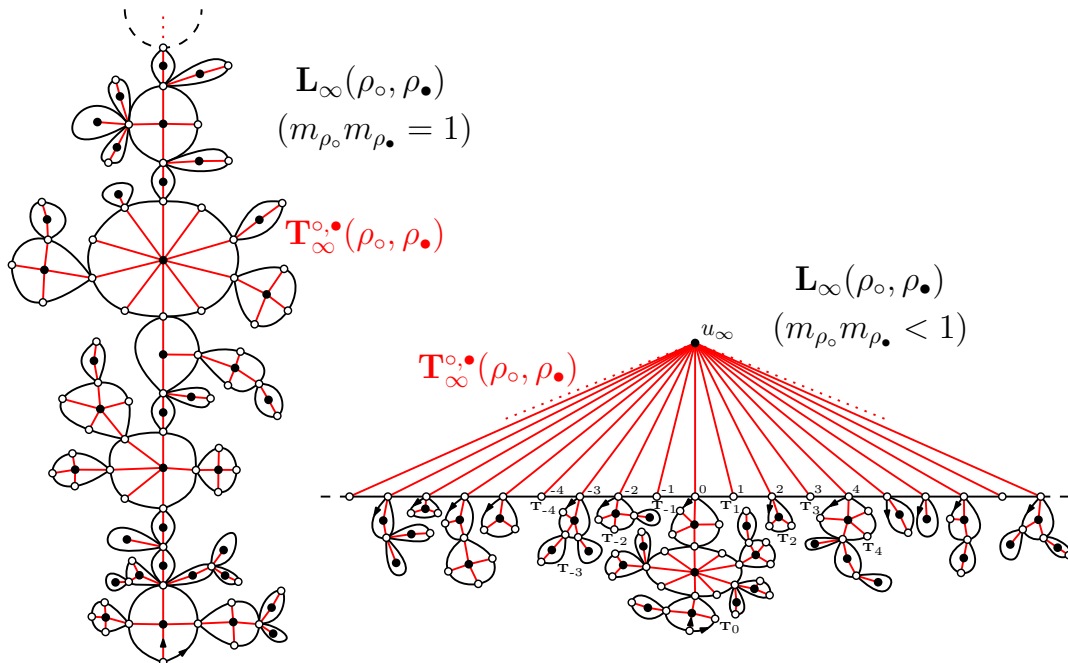


Figure 10: The infinite looptree  $\mathbf{L}_\infty$  and its tree of components  $\mathbf{T}_\infty^{\circ, \bullet}$ .

Using the continuity of the mapping  $\text{Loop}$  and Proposition 5.1 in the critical case, and (56) and Proposition 5.2 in the subcritical case we get the following result.

**Lemma 5.5.** *Let  $(\rho_\circ, \rho_\bullet)$  be a (sub)critical pair of offspring distributions such that  $\rho_\circ$  has a geometric distribution and  $\rho_\bullet$  has no exponential moment. Assume that for every  $k \geq 1$ ,  $\text{GW}_{\rho_\circ, \rho_\bullet}(\{|t| = k\}) > 0$  and let  $T_k^{\circ, \bullet}$  be a tree with distribution  $\text{GW}_{\rho_\circ, \rho_\bullet}^{(k)}$ . Then, in distribution for the local topology*

$$L_k := \text{Loop}(T_k^{\circ, \bullet}) \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{L}_\infty(\rho_\circ, \rho_\bullet).$$

The internal faces of  $\mathbf{L}_\infty = \text{Loop}(\mathbf{T}_\infty^{\circ, \bullet})$  are all finite in the critical case, while there is a unique infinite internal face in the subcritical case. In both cases, the internal faces of  $\mathbf{L}_\infty$  are in bijection with  $\text{Tree}(\mathbf{L}_\infty)_\bullet = (\mathbf{T}_\infty^{\circ, \bullet})_\bullet$ , so that the degree of the face and the degree of the vertex match. Following Remark 3.2, we define a fill-in mapping that associates to  $\mathbf{T}_\infty^{\circ, \bullet}$  and a collection  $(\widehat{M}_u : u \in (\mathbf{T}_\infty^{\circ, \bullet})_\bullet)$  of bipartite maps with a simple boundary of perimeter  $\deg(u)$  the map

$$M_\infty = \Phi_{\text{TC}}^{-1} \left( \mathbf{T}_\infty^{\circ, \bullet}, \left( \widehat{M}_u : u \in (\mathbf{T}_\infty^{\circ, \bullet})_\bullet \right) \right),$$

obtained from  $\mathbf{L}_\infty$  by gluing the map  $\widehat{M}_u$  in the face of  $\mathbf{L}_\infty$  associated to  $u$ , for every  $u \in (\mathbf{T}_\infty^{\circ, \bullet})_\bullet$ . We keep the notation  $\Phi_{\text{TC}}^{-1}$  by consistency, although we consider infinite trees.

### 5.3 Local limits of Boltzmann planar maps with a boundary

The local limits of Boltzmann bipartite maps with a boundary have been studied in [21].

**Proposition 5.6.** [21, Theorem 7] *Let  $\mathbf{q}$  be an admissible weight sequence. Then, we have the weak convergence for the local topology*

$$\mathbb{P}_{\mathbf{q}}^{(k)} \xrightarrow[k \rightarrow \infty]{} \mathbb{P}_{\mathbf{q}}^{(\infty)}.$$

The probability measure  $\mathbb{P}_{\mathbf{q}}^{(\infty)}$  is supported on infinite planar maps with a.s. one end and a unique face of infinite degree, which is the root face.

We let  $\mathbf{M}_\infty = \mathbf{M}_\infty(\mathbf{q})$  be a planar map with distribution  $\mathbb{P}_{\mathbf{q}}^{(\infty)}$ , called the Infinite Boltzmann Half-Planar Map with weight sequence  $\mathbf{q}$  ( $\mathbf{q}$ -IBHPM in short). In the quadrangular case,  $\mathbf{M}_\infty$  is a Uniform Infinite Half-Planar Quadrangulation with skewness  $\text{UIHPQ}_p$  considered in [4] (which includes the standard UIHPQ with a general boundary of [25]).

By definition, the boundary  $\partial \mathbf{m}$  of an infinite map  $\mathbf{m} \in \mathcal{M}_\infty$  is the map made by vertices and edges incident to its root face. Therefore, the definition of the scooped-out map extends to  $\mathcal{M}_\infty$ . We are interested in the continuity of Scoop with respect to the local topology.

**Lemma 5.7.** *Let  $(\mathbf{m}_k : k \in \mathbb{N} \cup \{\infty\})$  be a sequence of planar maps such that  $\mathbf{m}_\infty$  has one end and in the local sense*

$$\mathbf{m}_k \xrightarrow[k \rightarrow \infty]{} \mathbf{m}_\infty.$$

Then, in the local sense,

$$\text{Scoop}(\mathbf{m}_k) \xrightarrow[k \rightarrow \infty]{} \text{Scoop}(\mathbf{m}_\infty).$$

*Proof.* First, if  $(\#\partial \mathbf{m}_k : k \geq 1)$  is bounded, there exists  $R \geq 0$  such that for every  $k \geq 1$ ,  $\partial \mathbf{m}_k \subseteq \mathbf{B}_R(\mathbf{m}_k)$  and the result follows. Thus, we can assume that  $\#\partial \mathbf{m}_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

For every  $k \in \mathbb{N} \cup \{\infty\}$ , let  $p(k) := \#\partial \mathbf{m}_k / 2$  and denote by  $(v_k(0), v_k(1), \dots, v_k(p(k)))$  the sequence of vertices associated with the corners of the root face of  $\mathbf{m}_k$ , starting at the

origin, in right contour order. We use the notation  $(v_k(0), v_k(-1), \dots, v_k(-p(k)))$  for the left contour order, so that  $v_k(p(k)) = v_k(-p(k))$ .

Let  $r \geq 0$ . We now prove that there exists  $R \geq 0$  and  $K \geq 1$  such that for every  $k \geq K$ ,

$$\mathbf{V}(\mathbf{B}_r(\mathbf{m}_k)) \cap \{v_k(l) : |l| > R\} = \emptyset. \quad (57)$$

We proceed by contradiction. Because of the local convergence assumption, the sequence  $(\#\mathbf{V}(\mathbf{B}_r(\mathbf{m}_k)) : k \geq 0)$  is bounded. Moreover, for every  $v \in \mathbf{V}(\mathbf{B}_r(\mathbf{m}_k))$  we have

$$\#\{-p(k) \leq l \leq p(k) : v_k(l) = v\} \leq \deg_{\mathbf{m}_k}(v) \leq \sup_{u \in \mathbf{V}(\mathbf{B}_r(\mathbf{m}_k))} \deg_{\mathbf{m}_k}(u),$$

which is also bounded. Therefore, there exists  $M \geq 0$  such that for every  $k \geq 0$ ,

$$\#\{-p(k) \leq l \leq p(k) : v_k(l) \in \mathbf{V}(\mathbf{B}_r(\mathbf{m}_k))\} \leq M.$$

Let  $N \geq 0$ . By assumption, there exists infinitely many  $k$  such that  $p(k) > 2M(N+2)$  and

$$\mathbf{V}(\mathbf{B}_r(\mathbf{m}_k)) \cap \{v_{\mathbf{m}_k}(l) : |l| > M(N+2)\} \neq \emptyset.$$

As a consequence, in the cycle  $(-p(k), \dots, p(k))$ , there exists two distinct sequences of consecutive indices  $(i, \dots, i+x)$  and  $(j, \dots, j+y)$  such that  $x, y \geq N+2$  and

$$\mathbf{V}(\mathbf{B}_r(\mathbf{m}_k)) \cap \{v_k(l) : i \leq l \leq i+x\} = \{v_k(i), v_k(i+x)\},$$

and similarly for  $(j, \dots, j+y)$ . In particular, the sets of vertices  $E_1 := \{v_k(i+1), \dots, v_k(i+x-1)\}$  and  $E_2 := \{v_k(j+1), \dots, v_k(j+y-1)\}$  are disjoint. Indeed, a vertex  $v \in E_1 \cap E_2$  would disconnect  $\text{Scoop}(\mathbf{m}_k)$  in two submaps each containing a vertex at distance less than  $r$  from the origin, which is in contradiction with  $v \notin \mathbf{B}_r(\mathbf{m}_k)$ . Now, for every  $-p(k) \leq i < p(k)$ ,  $(v_k(i), v_k(i+1))$  is an edge of  $\text{Scoop}(\mathbf{m}_k)$ . Therefore, the sets of edges  $\{(v_k(l), v_k(l+1)) : i < l \leq i+N+1\}$  and  $\{(v_k(l), v_k(l+1)) : j < l \leq j+N+1\}$  are disjoint sets of  $N$  half-edges contained in  $\mathbf{B}_{r+N}(\mathbf{m}_k) \setminus \mathbf{B}_r(\mathbf{m}_k)$ . This holds for infinitely many  $k \geq 1$ , thus for  $\mathbf{m}_\infty$  by local convergence. Since  $\mathbf{m}_\infty$  has one end and  $N$  is arbitrary, this is a contradiction.

Let us choose  $R$  and  $K$  such that assertion (57) holds for every  $k \geq K$ . By local convergence, (57) holds for  $\mathbf{m}_\infty$  as well. For every  $k \geq K$ , let  $\langle v_k(-R), \dots, v_k(R) \rangle$  be the sub-map induced by the  $R$  first half-edges of  $\text{Scoop}(\mathbf{m}_k)$  in left and right contour order. We denote by  $H$  the measurable function such that  $\langle v_k(-R), \dots, v_k(R) \rangle = H(\mathbf{m}_k) = H(\mathbf{B}_R(\mathbf{m}_k))$ . By (57) and local convergence, we have for every  $k \geq K$

$$\mathbf{B}_r(\text{Scoop}(\mathbf{m}_k)) = \mathbf{B}_r(H(\mathbf{B}_R(\mathbf{m}_k))) \xrightarrow[k \rightarrow \infty]{} \mathbf{B}_r(H(\mathbf{B}_R(\mathbf{m}_\infty))) = \mathbf{B}_r(\text{Scoop}(\mathbf{m}_\infty)),$$

which concludes the proof.  $\square$

When a planar map  $\mathbf{m}$  has a unique infinite irreducible component, it is called the *core* of  $\mathbf{m}$  and denoted by  $\text{Core}(\mathbf{m})$ . Then,  $\mathbf{m}$  is recovered from  $\text{Core}(\mathbf{m})$  by gluing finite bipartite maps (with a general boundary) on vertices of the boundary of  $\text{Core}(\mathbf{m})$ . Note that the boundary of  $\text{Core}(\mathbf{m})$  may be finite or infinite. We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* For every  $k \geq 1$ , let  $M_k$  be a planar map with distribution  $\mathbb{P}_q^{(k)}$ . By Corollary 3.9,  $T_k^{\circ, \bullet} := \mathbf{Tree}(M_k)$  is a two-type Galton-Watson tree with offspring distribution  $(\nu_\circ, \nu_\bullet)$  conditioned to have  $2k+1$  vertices. By Proposition 5.6 and Lemma 5.7, we have

$$\text{Scoop}(M_k) \xrightarrow[k \rightarrow \infty]{(d)} \text{Scoop}(\mathbf{M}_\infty).$$

On the other hand, by Lemma 5.5,

$$\text{Scoop}(M_k) = \text{Loop}(T_k^{\circ, \bullet}) \xrightarrow[k \rightarrow \infty]{(d)} \mathbf{L}_\infty(\nu_\circ, \nu_\bullet),$$

in distribution for the local topology. Lemma 3.6 and Proposition 3.7 conclude the first part of the proof. For  $a \in [3/2, 2]$ ,  $\text{Scoop}(\mathbf{M}_\infty)$  has only finite internal faces, which are the boundaries of the irreducible components of  $\mathbf{M}_\infty$ . Since  $\text{Scoop}(\mathbf{M}_\infty)$  and  $\mathbf{M}_\infty$  are one-ended, these irreducible components are necessarily finite. For  $a \in (2, 5/2]$ ,  $\text{Scoop}(\mathbf{M}_\infty)$  has a unique infinite internal face, which is the boundary of an infinite irreducible component. Since  $\mathbf{M}_\infty$  is one-ended, the other irreducible components are finite, and  $\mathbf{M}_\infty$  has a well defined core.  $\square$

**Local limits: the subcritical and dense regimes.** When  $\mathbf{q}$  is of type  $a \in [3/2, 2]$ ,  $\mathbf{M}_\infty(\mathbf{q})$  can be entirely described by the looptree  $\mathbf{L}_\infty(\nu_\circ, \nu_\bullet)$  and a collection of independent Boltzmann maps. This generalizes [4, Theorem 4] which deals with subcritical quadrangulations.

**Proposition 5.8.** *Let  $\mathbf{q}$  be a weight sequence of type  $a \in [3/2, 2]$ , and  $\mathbf{T}_\infty^{\circ, \bullet} = \mathbf{T}_\infty^{\circ, \bullet}(\nu_\circ, \nu_\bullet)$ . Conditionally on  $\mathbf{T}_\infty^{\circ, \bullet}$ , let  $(\widehat{M}_u : u \in (\mathbf{T}_\infty^{\circ, \bullet})_\bullet)$  be a collection of independent bipartite maps with a simple boundary and distribution  $\widehat{\mathbb{P}}_{\mathbf{q}}^{(\deg(u)/2)}$ . Then, the infinite bipartite map*

$$M_\infty = \Phi_{\text{TC}}^{-1} \left( \mathbf{T}_\infty^{\circ, \bullet}, \left( \widehat{M}_u : u \in (\mathbf{T}_\infty^{\circ, \bullet})_\bullet \right) \right)$$

has distribution  $\mathbb{P}_{\mathbf{q}}^{(\infty)}$ , the law of the  $\mathbf{q}$ -IBHPM.

*Proof.* The proof closely follows that of [4, Theorem 4]. For every  $\mathbf{t} \in \mathcal{T}_{\text{loc}}$  and every  $R \geq 1$ , let  $\text{Cut}_R(\mathbf{t})$  be the subtree of  $\mathbf{t}$  made of vertices  $u \in \mathbf{t}$  such that  $|u| \leq 2R$ . Consistently, if  $\mathbf{m} = \Phi_{\text{TC}}^{-1}(\mathbf{t}, (\widehat{\mathbf{m}}_u : u \in \mathbf{t}_\bullet))$ ,  $\text{Cut}_R(\mathbf{m})$  is the bipartite map  $\Phi_{\text{TC}}^{-1}(\text{Cut}_R(\mathbf{t}), (\widehat{\mathbf{m}}_u : u \in \text{Cut}_R(\mathbf{t})_\bullet))$ .

Let  $R \geq 1$  and for every  $k \geq 0$ , let  $M_k$  be a bipartite map with distribution  $\mathbb{P}_{\mathbf{q}}^{(k)}$ . Let  $\mathbf{m} \in \mathcal{M}$  and  $(\mathbf{t}, (\widehat{\mathbf{m}}_u : u \in \mathbf{t}_\bullet)) = \Phi_{\text{TC}}(\mathbf{m})$ . By Proposition 3.3 and 5.1, we have

$$\begin{aligned} \mathbb{P}_{\mathbf{q}}^{(k)}(\text{Cut}_R(M) = \mathbf{m}) &= \text{GW}_{\nu_\circ, \nu_\bullet}^{(2k+1)}(\text{Cut}_R(T) = \mathbf{t}) \prod_{u \in \mathbf{t}_\bullet} \widehat{\mathbb{P}}_{\mathbf{q}}^{(\deg(u)/2)}(\widehat{\mathbf{m}}_u) \\ &\xrightarrow[k \rightarrow \infty]{} \text{GW}_{\nu_\circ, \nu_\bullet}^{(\infty)}(\text{Cut}_R(T) = \mathbf{t}) \prod_{u \in \mathbf{t}_\bullet} \widehat{\mathbb{P}}_{\mathbf{q}}^{(\deg(u)/2)}(\widehat{\mathbf{m}}_u) = P(\text{Cut}_R(M_\infty) = \mathbf{m}). \end{aligned}$$

This concludes since  $\mathbf{B}_R(\mathbf{m}) = \mathbf{B}_R(\text{Cut}_R(\mathbf{m}))$  if  $\mathbf{m} = \Phi_{\text{TC}}^{-1}(\mathbf{t}, (\widehat{\mathbf{m}}_u : u \in \mathbf{t}_\bullet))$  with  $\mathbf{t} \in \mathcal{T}_{\text{loc}}$ .  $\square$

**Remark 5.9.** The tree-like structure of  $\mathbf{M}_\infty$  when  $a \in [3/2, 2]$  makes statistical mechanics models on it easier to study. In particular, the simple random walk on  $\mathbf{M}_\infty$  is a.s. recurrent (see [4, Corollary 2] for a proof) and the critical thresholds for Bernoulli site, bond and face percolation on  $\mathbf{M}_\infty$  equal one a.s..

**Local limits: the dilute and generic regimes.** When  $\mathbf{q}$  is of type  $a \in (2, 5/2]$ ,  $\mathbf{M}_\infty = \mathbf{M}_\infty(\mathbf{q})$  cannot be fully described using finite bipartite maps. By the construction of Section 5.2,  $\text{Scoop}(\mathbf{M}_\infty)$  is obtained from the infinite simple boundary of  $\text{Core}(\mathbf{M}_\infty)$  by attaching to its vertices independent looptrees  $(\mathbf{L}_i : i \in \mathbb{Z})$  whose trees of components have distribution  $\text{GW}_{\nu_\circ, \nu_\bullet}$ , except for that looptree containing the root edge of  $\mathbf{M}_\infty$ , whose tree of components has distribution  $\overline{\text{GW}}_{\nu_\circ, \nu_\bullet}$ . We believe that the finite irreducible components of  $\mathbf{M}_\infty$  are independent Boltzmann bipartite maps with a simple boundary (conditionally on  $\partial \mathbf{M}_\infty$ ).

Moreover, given Propositions 3.3 and 5.3, we conjecture that there exists a distribution  $\widehat{\mathbb{P}}_{\mathbf{q}}^{(\infty)}$  supported on bipartite planar maps with an infinite simple boundary such that  $\widehat{\mathbb{P}}_{\mathbf{q}}^{(k)} \Rightarrow \widehat{\mathbb{P}}_{\mathbf{q}}^{(\infty)}$  as  $k \rightarrow \infty$ , and that  $\text{Core}(\mathbf{M}_{\infty})$  has distribution  $\widehat{\mathbb{P}}_{\mathbf{q}}^{(\infty)}$ . This would provide a complete description of the  $\mathbf{q}$ -IBHPM, and has been achieved in the special case of the UIHPQ in [25, Proposition 6]. However, our techniques are not sufficient to prove these assertions.

## 6 Application to the rigid $O(n)$ loop model on quadrangulations

We now give applications to the rigid  $O(n)$  loop model on quadrangulations, building on [11].

**The rigid  $O(n)$  loop model on quadrangulations.** We describe the setup of [11] (see also [20]). A *quadrangulation with a boundary* is a planar map with a boundary whose internal faces all have degree 4. Given a quadrangulation with a boundary  $\mathbf{q}$ , a *loop configuration* on  $\mathbf{q}$  is a collection  $\ell = (\ell_k : k \in \mathbb{N})$  of disjoint closed simple paths in the dual of  $\mathbf{q}$  that do not visit the root face  $f_*$ . The loop configuration is known as *rigid* if moreover every loop crosses a quadrangle through opposite sides. See Figure 2 for an illustration. The pair  $(\mathbf{q}, \ell)$  is then called a (rigid) *loop-decorated quadrangulation with a boundary*. The set of all such pairs  $(\mathbf{q}, \ell)$  is denoted by  $\mathcal{O}$  (resp.  $\mathcal{O}_k$  if additionally  $\mathbf{q}$  has perimeter  $2k$ ). The set  $\mathcal{O}_1$  is in bijection with loop-decorated quadrangulations of the sphere. For every  $(\mathbf{q}, \ell) \in \mathcal{O}$ , we denote by  $\#\ell$  the number of loops in  $\ell$ , by  $|\ell|$  the total length (i.e. the total number of edges) of the loops of  $\ell$ , and by  $|\ell|$  the number of edges, or perimeter, of a loop  $\ell \in \ell$ .

For every  $n \in (0, 2)$  and every  $g, h \geq 0$ , we define the measure  $W_{(n;g,h)}$  on  $\mathcal{O}$  by

$$W_{(n;g,h)}((\mathbf{q}, \ell)) := g^{\#\mathbf{q} - |\ell|} h^{|\ell|} n^{\#\ell}, \quad (\mathbf{q}, \ell) \in \mathcal{O}. \quad (58)$$

In other words, we put a weight  $g$  per empty quadrangle of  $\mathbf{q}$ , a weight  $h$  per quadrangle crossed by a loop, and a weight  $n$  per loop. We also define the partition function

$$F_k^{\circ} := \sum_{(\mathbf{q}, \ell) \in \mathcal{O}_k} W_{(n;g,h)}((\mathbf{q}, \ell)), \quad k \in \mathbb{Z}_+. \quad (59)$$

When this partition function is finite (which does not depend on  $k$ ), we say that  $(n; g, h)$  is admissible and define the  $O(n)$  probability measure on  $\mathcal{O}_k$  with parameters  $(n; g, h)$  by

$$\mathbf{P}_{(n;g,h)}^{(k)}((\mathbf{q}, \ell)) := \frac{W_{(n;g,h)}((\mathbf{q}, \ell))}{F_k^{\circ}}, \quad (\mathbf{q}, \ell) \in \mathcal{O}_k, \quad k \in \mathbb{Z}_+. \quad (60)$$

$\mathbf{P}_{(n;g,h)} := \mathbf{P}_{(n;g,h)}^{(1)}$  is the  $O(n)$  distribution on loop-decorated quadrangulations of the sphere.

**The gasket decomposition.** The work [11] is based on the *gasket decomposition* of loop-decorated quadrangulations with a boundary, that we now recall (see also [20]). First, for every  $(\mathbf{q}, \ell) \in \mathcal{O}$  and every  $\ell \in \ell$ , the interior and exterior of  $\ell$  are well defined thanks to the root edge of  $\mathbf{q}$ . Then, the *inner* (resp. *outer*) contour of  $\ell$  is formed by the edges of  $\mathbf{q}$  that are incident to faces of  $\mathbf{q}$  crossed by  $\ell$ , and that belong to the interior (resp. exterior) of  $\ell$ .

The gasket decomposition of  $(\mathbf{q}, \ell) \in \mathcal{O}_k$  consists in discarding the outer-most loops  $(\ell_i : i \in \mathcal{I})$  of  $\ell$  (i.e. that are not contained in the interior of another loop) as well as the

edges crossed by these loops. This disconnects  $(\mathbf{q}, \ell)$  in  $\#\mathcal{I} + 1$  connected components, as shown in Figure 11. The gasket is the connected component  $\text{Gasket}(\mathbf{q}, \ell)$  containing the root edge of  $\mathbf{q}$ . It is the element of  $\mathcal{M}_k$  formed by the edges of  $\mathbf{q}$  that are exterior to all loops. The faces of  $\text{Gasket}(\mathbf{q}, \ell)$  are either quadrangles of  $\mathbf{q}$ , or *holes* corresponding to the loops  $(\ell_i : i \in \mathcal{I})$  (with degree  $|\ell_i|$ ). The other connected components are the interiors of the outer-most loops, which are loop-decorated quadrangulations  $((\mathbf{q}_i, \ell_i) : i \in \mathcal{I})$  with perimeter  $|\ell_i|$ . By convention, the root edge of  $\mathbf{q}_i$  lies on the leftmost shortest path from the root edge of  $\mathbf{q}$  to  $\mathbf{q}_i$  (with the convention that  $\ell_i$  lies on its right).

Given  $\mathbf{m} \in \mathcal{M}_k$  and  $(\mathbf{q}, \ell) \in \mathcal{O}_k$  such that  $\text{Gasket}(\mathbf{q}, \ell) = \mathbf{m}$ ,  $(\mathbf{q}, \ell)$  is recovered by gluing into each face of  $\text{Gasket}(\mathbf{q}, \ell)$  of degree  $2p$  the proper elements of  $\mathcal{O}_p$ , with in-between a *ring* or *necklace* of  $2p$  quadrangles crossed by a loop. When  $p = 2$ , we can also glue an empty quadrangle. The following result is proved in [11], see in particular [11, Equation 2.3].

**Proposition 6.1.** [11] *Let  $(n; g, h)$  be admissible,  $k \geq 0$  and  $(Q, L)$  a loop-decorated quadrangulation with distribution  $\mathbf{P}_{(n;g,h)}^{(k)}$ . Denote by  $(\ell_i : i \in \mathcal{I})$  the outer-most loops of  $(Q, L)$ , and by  $((Q_i, L_i) : i \in \mathcal{I})$  the associated loop-decorated quadrangulations. Then,  $\text{Gasket}(Q, L)$  has distribution  $\mathbb{P}_{\mathbf{q}}^{(k)}$ , where the weight sequence  $\mathbf{q} = \mathbf{q}(n; g, h) = (q_k : k \in \mathbb{N})$  satisfies*

$$q_k = g\delta_2(k) + nh^{2k}F_k^\circ(n; g, h). \quad (61)$$

Moreover, conditionally on  $(|\ell_i| : i \in \mathcal{I})$ ,  $((Q_i, L_i) : i \in \mathcal{I})$  are independent loop-decorated quadrangulations with distribution  $\mathbf{P}_{(n;g,h)}^{(|\ell_i|)}$ .

**Remark 6.2.** We are interested in limits of large loops in the rigid  $O(n)$  model on quadrangulations. Due to the rigidity constraint on loops, we can substitute loops (that are paths in the dual map) for their inner contours (in the primal map). Proposition 6.1 ensures that for every  $k \geq 0$ , in the rigid  $O(n)$  loop model on  $\mathcal{O}_k$ , every loop of perimeter  $2p$  is distributed as the boundary of a Boltzmann bipartite map with law  $\mathbb{P}_{\mathbf{q}}^{(p)}$  for a suitable value of  $\mathbf{q}$ . Therefore, the study of large loops reduces to the study of the boundary of a large Boltzmann bipartite map. For instance, consider the rigid  $O(n)$  loop model on  $\mathcal{O}_1$  with parameters  $(n; g, h)$ , and pick a loop using a deterministic criterion (e.g. the loop that is the closest to the root edge). Now, condition this loop to have perimeter  $2p$  (which is an event of positive probability). Then, its inner contour is the boundary of a map with law  $\mathbb{P}_{\mathbf{q}}^{(p)}$ , for  $\mathbf{q}$  satisfying (61).

**The phase diagram.** In [11], the parameters of the  $O(n)$  model have been classified according to the distribution of the gasket. A triplet  $(n; g, h)$  is called *subcritical*, *generic critical* or *non-generic critical with parameter  $\alpha$*  if the weight sequence  $\mathbf{q}$  associated to  $(n; g, h)$  by (61) is subcritical, generic critical or non-generic critical with parameter  $\alpha$ . This results in the exact phase diagram of Figure 3 (see also [11, Figure 12]). As mentioned in [20], the work [11] must be completed by [17, Appendix] to get this diagram. Let  $n \in (0, 2)$ , and set

$$b := \frac{1}{\pi} \arccos\left(\frac{n}{2}\right).$$

Then, we have a critical line  $h = h_c(n; g)$  that separates the region where the model is subcritical ( $a = 3/2$ ) and the pairs  $(g, h)$  such that  $(n; g, h)$  is non-admissible. The critical line has two parts. First, an arc of parabola that links  $(g = 0, h = 2b^2/(2-n))$  to the special point  $(g^*, h^*) = (g^*(n), h^*(n))$ , with explicit equation [11, Equation (6.18)]. Then, a second part

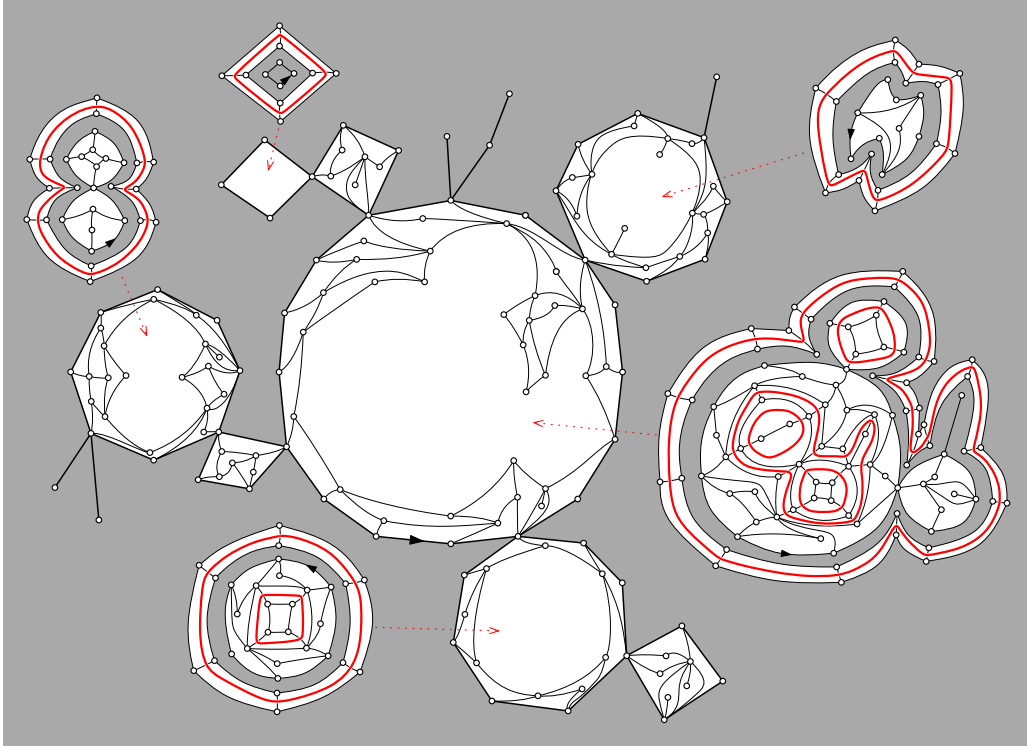


Figure 11: The gasket decomposition of the loop-decorated quadrangulation  $(\mathbf{q}, \ell)$  of Figure 2. Note that holes may be non-simple faces of  $\text{Gasket}(\mathbf{q}, \ell)$ .

that links  $(g^*, h^*)$  to  $(g = 1/12, h = 0)$  (corresponding to pure quadrangulations) with a more intricate parametrization [11, Equation (6.14)]. The regime changes along the critical line: for  $g < g^*$ , the parameters are non-generic critical with parameter  $a = 2 - b \in (3/2, 2)$ , which corresponds to the dense phase, while for  $g > g^*$ , the parameters are generic critical ( $a = 5/2$ ). Finally the special point  $(g^*, h^*)$  is non-generic critical with parameter  $a = 2 + b \in (2, 5/2)$ , which corresponds to the dilute phase.

The situation is simpler in the non-generic critical cases. In [11] (as in [44, 16]), the definition of a non-generic critical weight sequence  $\mathbf{q}$  is less general than our, and implies that there exists a constant  $\chi_{\mathbf{q}}$  such that

$$F_k \underset{k \rightarrow \infty}{\sim} \chi_{\mathbf{q}} \frac{(4Z_{\mathbf{q}})^k}{k^a},$$

see [11, Equation (3.15)], [16, Equation (6)] or [21, Equation (5.8)]. In particular, the slowly varying function  $\ell$  defined in Definition 2.4 is equivalent to a constant. Using this in the computations of Sections 2.2, 2.3 and 4, we finally obtain that the slowly varying function  $\Lambda$  of Theorem 1.2 can be replaced by a constant  $C = C(\mathbf{q}) = C(n, g, h)$ . Then, Theorems 1.3 and 1.4 follow from Proposition 6.1 and the phase diagram, by applying Theorems 1.1 and 1.2. By Remark 6.2, these results extend to any loop conditioned to be large in the rigid  $O(n)$  loop model on quadrangulations (possibly with a boundary).

## 7 The non-generic critical case with parameter $\alpha = 3/2$

We have seen in Remarks 2.7 and 3.8 that the critical parameter  $\alpha = 3/2$  ( $a = 2$ ) plays a special role. In particular, Karamata's Tauberian theorem does not yield an equivalent for the tail of the probability measure  $\nu$ . We now provide such an estimate by calling on De Haan theory [8, Chapter 3] and using a special weight sequence introduced in [3].

**The special weight sequence.** We recall the definition of the special weight sequence of [3]. Here, we draw on [16, Section 5] and define the weight sequence  $\mathbf{q}^* = (q_k^* : k \in \mathbb{N})$  by

$$q_k^* := \frac{1}{4} 6^{1-k} \frac{\Gamma(k - 3/2)}{\Gamma(k + 5/2)} \mathbf{1}_{k \geq 2} \quad k \in \mathbb{N}.$$

Then,  $\mathbf{q}^*$  is admissible, critical, and of type  $a = 2$ . There exists a continuous family of such sequences covering all the values of  $a \in (3/2, 5/2]$  (the case  $a = 5/2$  corresponding to critical quadrangulations). This weight sequence is convenient because we obtain an explicit formula for the partition function  $F_k$  by combining [16, Lemma 14] and [16, Equation (7)]:

$$F_k = \frac{3}{4} \frac{6^k}{(k + 3/2)(k + 1/2)}, \quad k \in \mathbb{Z}_+.$$

Consequently,  $r_{\mathbf{q}} = 1/6$  and we have the explicit formula

$$F(x) = \frac{1}{4x} - \frac{3}{4(6x)^{3/2}} (1 - 6x) \log \left( \frac{1 + \sqrt{6x}}{1 - \sqrt{6x}} \right), \quad (62)$$

from which we deduce the asymptotic expansions as  $x \rightarrow r_{\mathbf{q}}^-$

$$F(x) = \frac{3}{2} + \frac{3}{4} \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) \log \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) + \frac{3}{2} (1 - \log(2)) \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) (1 + o(1)), \quad (63)$$

$$F'(x) = -\frac{9}{2} (3 - 2 \log(2)) - \frac{9}{2} \log \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) + o(1). \quad (64)$$

**The generating function of bipartite maps with a simple boundary.** We now focus on estimates for the generating function  $\widehat{F}$ . Unlike the previous cases, an asymptotic expansion of  $\widehat{F}$  itself is not sufficient; we rather need an expansion of its derivative. As in Section 2.3, the function  $P(x) = xF^2(x)$  is continuous increasing from  $[0, r_{\mathbf{q}}]$  onto  $[0, P(r_{\mathbf{q}})]$  with inverse denoted by  $P^{-1}$ , and  $P(r_{\mathbf{q}}) = 3/8$ . Moreover, we have as  $x \rightarrow r_{\mathbf{q}}^-$

$$P(x) = P(r_{\mathbf{q}}) + P(r_{\mathbf{q}}) \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) \log \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) + P(r_{\mathbf{q}}) (2 \log(2) - 1) \left( 1 - \frac{x}{r_{\mathbf{q}}} \right) (1 + o(1)). \quad (65)$$

In what follows, we put  $c^* = 2 \log(2) - 1$ . We define the function

$$R(x) := \frac{1}{P(r_{\mathbf{q}})} (P(r_{\mathbf{q}}) - P(r_{\mathbf{q}}(1 - x))), \quad x \in [0, 1],$$

which is continuous increasing onto  $[0, 1]$ , with inverse  $R^{-1}$  defined by

$$R^{-1}(y) = 1 - \frac{1}{r_{\mathbf{q}}} P^{-1}(P(r_{\mathbf{q}})(1 - y)), \quad y \in [0, 1]. \quad (66)$$



The asymptotic expansion of  $R$  reads

$$R(x) = -x \log(x) - c^* x + o(x) \quad \text{as } x \rightarrow 1^-. \quad (67)$$

We now need the Lambert  $W$  function, defined as the (multivalued) inverse function of  $x \mapsto xe^x$ . Here, we use the lower branch  $W_{-1}$ , continuous decreasing from  $[-e^{-1}, 0)$  onto  $(-\infty, -1]$ , which satisfies the identities

$$W_{-1}(-x) = \log\left(\frac{-x}{W_{-1}(-x)}\right) \quad \text{and} \quad W_{-1}(x \log(x)) \log(x), \quad x \in (0, e^{-1}]. \quad (68)$$

We also have the asymptotic expansion

$$W_{-1}(-x) = \log(x) - \log(-\log(x)) + o(1) \quad \text{as } x \rightarrow 0^+. \quad (69)$$

The Lambert  $W$  function has a principal branch  $W_0$ , but the lower branch is more suitable to our needs. We introduce the function

$$Q(x) := R\left(\frac{-x}{W_{-1}(-x)}\right), \quad x \in (0, e^{-1}],$$

which is continuous increasing from  $(0, e^{-1}]$  onto  $(0, R(e^{-1})]$ . By (68), its inverse function  $Q^{-1}$  satisfies

$$Q^{-1}(y) = -R^{-1}(y) \log(R^{-1}(y)) \quad \text{and} \quad R^{-1}(y) = \frac{-Q^{-1}(y)}{W_{-1}(-Q^{-1}(y))}, \quad y \in (0, R(e^{-1})]. \quad (70)$$

Using (67), (68) and (69) we get

$$Q(x) = x - c^* \frac{x}{\log(x)} + o\left(\frac{x}{\log(x)}\right) \quad \text{as } x \rightarrow 0^+. \quad (71)$$

Then,  $Q'(0^+) = 1$ ,  $(Q^{-1})'(0^+) = 1$  and  $Q^{-1}(y) \sim y$  as  $y \rightarrow 0^+$ . Back to (71), we have

$$Q^{-1}(y) = y - c^* \frac{y}{\log(y)} + o\left(\frac{y}{\log(y)}\right) \quad \text{as } y \rightarrow 0^+. \quad (72)$$

Together with (70) and (69), this yields

$$R^{-1}(y) = -\frac{y}{\log(y)} - \frac{y \log(-\log(y))}{\log^2(y)} - c^* \frac{y}{\log^2(y)} + o\left(\frac{y}{\log^2(y)}\right) \quad \text{as } y \rightarrow 0^+. \quad (73)$$

Finally, by (66) we obtain

$$\begin{aligned} P^{-1}(y) &= r_q + r_q \left(1 - \frac{y}{P(r_q)}\right) \frac{1}{\log\left(1 - \frac{y}{P(r_q)}\right)} + r_q \left(1 - \frac{y}{P(r_q)}\right) \frac{\log\left(-\log\left(1 - \frac{y}{P(r_q)}\right)\right)}{\log^2\left(1 - \frac{y}{P(r_q)}\right)} \\ &\quad + r_q c^* \left(1 - \frac{y}{P(r_q)}\right) \frac{1}{\log^2\left(1 - \frac{y}{P(r_q)}\right)} (1 + o(1)) \quad \text{as } y \rightarrow P(r_q)^-. \end{aligned} \quad (74)$$

This proves that  $\widehat{r}_{\mathbf{q}} = P(r_{\mathbf{q}})$  by Lemma 2.10. The next step is to derive the asymptotic expansion of  $\widehat{F}'$ . By differentiating both sides in the equation of Lemma 2.10 we find

$$\widehat{F}'(y) = \frac{1}{y} \left( \frac{1}{P^{-1}(y)F'(P^{-1}(y))} + \frac{2}{F(P^{-1}(y))} \right)^{-1}, \quad y \in (0, P(r_{\mathbf{q}})). \quad (75)$$

Using (63), (64) and (74) we obtain the wanted expansion: as  $y \rightarrow P(r_{\mathbf{q}})^-$ ,

$$\widehat{F}'(y) = 2 + \frac{2}{\log\left(1 - \frac{y}{P(r_{\mathbf{q}})}\right)} + \frac{2 \log\left(-\log\left(1 - \frac{y}{P(r_{\mathbf{q}})}\right)\right)}{\log^2\left(1 - \frac{y}{P(r_{\mathbf{q}})}\right)} - \frac{2(3 - 2 \log(2))}{\log^2\left(1 - \frac{y}{P(r_{\mathbf{q}})}\right)}(1 + o(1)). \quad (76)$$

**The tree of components.** We are now interested in properties of the tail of the probability measures  $\nu$  and  $\nu_{\bullet}$  of Section 3.3. To do so, we need estimates on the derivative of the Laplace transform  $L_{\nu}$ . Recalling the form of the generating function of  $\nu$  from (34), we get

$$L'_{\nu}(t) = -\frac{2P(r_{\mathbf{q}})}{F(r_{\mathbf{q}})} e^{-2t} \widehat{F}'(P(r_{\mathbf{q}})e^{-2t}), \quad t > 0. \quad (77)$$

By (76), we obtain

$$-L'_{\nu}(t) = 1 + \frac{1}{\log(2t)} + \frac{\log(-\log(2t))}{\log^2(2t)} - \frac{3 - 2 \log(2)}{\log^2(2t)} + o\left(\frac{1}{\log^2(t)}\right), \quad \text{as } t \rightarrow 0^+. \quad (78)$$

Since  $\nu$  is critical, the Laplace transform  $L_{\bar{\nu}}$  of the size-biased measure  $\bar{\nu}$  equals  $-L'_{\nu}$ . As a consequence,

$$\frac{L_{\bar{\nu}}\left(\frac{1}{\lambda x}\right) - L_{\bar{\nu}}\left(\frac{1}{x}\right)}{\log^2(x)} \xrightarrow{x \rightarrow \infty} \log(\lambda), \quad \forall \lambda > 0. \quad (79)$$

Let us introduce a notation for the tail of the probability measure  $\bar{\nu}$ , say

$$T(x) := \sum_{k \geq x} k \nu(k), \quad x \in \mathbb{R}.$$

By De Haan's Tauberian theorem [8, Theorem 3.9.1], (79) is equivalent to

$$\frac{T(\lambda x) - T(x)}{\log^2(x)} \xrightarrow{x \rightarrow \infty} \log(\lambda), \quad \forall \lambda > 0. \quad (80)$$

The function  $T$  is said to be in the class  $\Pi_{\log^2}$  with index 1. By an integration by parts (see also the last line in the proof of [8, Theorem 8.1.6]), we have for every  $x > 0$

$$x \nu((x, \infty)) = T(x) - x \int_x^{\infty} \frac{T(t)}{t^2} dt. \quad (81)$$

Finally, by De Haan's Theorem [8, Theorem 3.7.3], (80) and (81) we obtain the following.

**Proposition 7.1.** *Let  $\mathbf{q}^*$  be the special weight sequence of type  $a = 2$ . Then, we have*

$$\nu([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{1}{k \log^2(k)} \quad \text{and} \quad \nu_{\bullet}([k, \infty)) \underset{k \rightarrow \infty}{\sim} \frac{3}{k \log^2(k)}.$$

*In particular,  $\nu$  and  $\nu_{\bullet}$  are in the domain of attraction of a Cauchy distribution (stable with parameter  $a - 1 = 1$ ).*

**Remark 7.2.** We have seen in Theorem 1.2 that when  $a = 2$ , the local limits of the boundary of Boltzmann bipartite maps behave as in the dense phase. However, Proposition 7.1 suggests that  $\nu_\bullet$  has a very heavy tail, meaning that the local limit of the boundary has very large loops. We believe that the scaling limits of the boundary behave as in the dilute phase when  $a = 2$ , meaning that we expect the limit to be a circle, but the normalizing sequence to be negligible compared to the perimeter  $2k$  of the map (typically of order  $k/\log(k)$ ).

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