On The Brownian Loop Measure

Yong Han, Yuefei Wang, Michel Zinsmeister

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Yong Han * Yuefei Wang† Michel Zinsmeister‡

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Abstract

In 2003 Lawler and Werner introduced the Brownian loop measure and studied some of its properties. Cardy and Gamsa has predicted a formula for the total mass of the Brownian loop measure on the set of simple loops in the upper half plane and disconnect two given points from the boundary. In this paper we give a rigorous proof of the formula.

Keywords: Brownian loop, SLE bubble, Brownian bubble, Disconnect from boundary

1 Introduction

The conformally invariant scaling limits of a series of planar lattice models can be described by the one-parameter family of random fractal curves SLE(κ), which was introduced by Schramm. These models include site percolation on the triangular graph, loop erased random walk, Ising model, harmonic random walk, discrete Gaussian free field, FK-Ising model and uniform spanning tree. Using SLE as a tool, many problems related to the

*Institute of Mathematics, Academy of Mathemtics and Systems Sciences, Chinese Academy of Sciences and University of Chinese Academy of Sciences, Beijing 100190, China. Email: hanyong@amss.ac.cn

†Institute of Mathematics, Academy of Mathemtics and Systems Sciences and University of Chinese Academy of Sciences, Chinese Academy of Sciences, Beijing 100190, China. Email: wangyf@math.ac.cn

‡MAPMO, Université d'Orléans Orléans Cedex 2, France Email:zins@univ-orleans.fr

properties of these models have been solved, such as the arm exponents for these models. There are also some variants of SLE (conformal loop ensemble, Brownian loop measure, Brownian bubble measure) that describe the scaling limit of the random loops in these models. Therefore it is natural to use SLE to get properties of these loop measures. One of theses application is to use SLE($\frac{8}{3}$) to study the properties of the Brownian bubble measure and Brownian loop measure. In fact, by rescaling and letting the two end points tends to one common point, one can get the Brownian bubble measure(up to a additive constant).

Recently Beliaev and Viklund [1] found a formula for the probability that two given points lies to the left of the SLE($\frac{8}{3}$) curve and used it to study some connectivity functions for SLE($\frac{8}{3}$) bubbles and to reconstruct the chordal restriction measure introduced by Lawler, Werner and Schramm [2]. In this paper, we will follow their work and use the SLE($\frac{8}{3}$) bubble measure to derive the formula for the total mass of the Brownian loop that disconnects two given points from the boundary. This formula was predicted by Cardy and Gamsa [3], and the formula we get here just differ by a constant from theirs.

In the following section, we give a brief introduction to the topics that will be used in this paper, which include the definition of SLE processes, Brownian bubble measure, Brownian loop measure, SLE(κ) bubble measure and the relation between these measures. In the third section, we give the proof of the main theorem.

Theorem 1. *Denote by $\mu_{\mathbb{H}}^{loop}$ the Brownian loop measure on the upper half plane and by γ a sample of the Brownian loop. Given two points $z = x + iy, w = u + iv \in \mathbb{H}$, let $E(z, w)$ denote the event that γ disconnects both z and w from the boundary of $\mathbb{H}$. Then we have*

$$\begin{aligned}\mu_{\mathbb{H}}^{loop}[E(z,w)] = &-\frac{\pi}{5\sqrt{3}} - \frac{1}{10}\eta\, {}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\eta) - \frac{1}{10}\log(\eta(\eta-1)) \\ &+ \frac{\Gamma(\frac{2}{3})^2}{5\Gamma(\frac{4}{3})}(\eta(\eta-1))^{\frac{1}{3}}\, {}_2F_1(1,\frac{2}{3};\frac{4}{3},\eta).\end{aligned} \tag{1.1}$$

where

$$\eta = \eta(z,w) = -\frac{(x-u)^2+(y-v)^2}{4yv}, \tag{1.2}$$

and ${}_3F_2$, ${}_2F_1$ are the hypergeometric functions.

The formula (1.1) was first given by Cardy and Gamsa using conformal field theory which assumes that $O(n)$ model has the scaling limit. In fact (1.1) has a nicer form:

$$\mu^{\text{loop}}[E(z,w)] = -\frac{1}{10}[\log\sigma + (1-\sigma)\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1-\sigma)], \tag{1.3}$$

where

$$\sigma = \sigma(z,w) = \frac{|z-w|^2}{|z-\bar{w}|^2} = \frac{(x-u)^2+(y-v)^2}{(x-u)^2+(y+v)^2}, \tag{1.4}$$

and ${}_3F_2$ is the hypergeometric function.

Remark 2. By the conformal invariance of Brownian loop measure (see [4]), for any simply connected domain $D \subset \mathbb{C}$ with $z, w \in D$, we can get the total mass of the Brownian loop in D that disconnect both z and w from ∂D by the conformal map from D to $\mathbb{H}$. In particular, if $D = \mathbb{D}$, we choose the conformal map $\phi(z) = i\frac{1+z}{1-z}$ from $\mathbb{D}$ onto $\mathbb{H}$. Then the total mass of the Brownian loop measure in $\mathbb{D}$ that disconnects $z, w \in \mathbb{D}$ from $\partial\mathbb{D}$ is

$$-\frac{1}{10}[\log\tilde{\sigma} + (1-\tilde{\sigma})\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1-\tilde{\sigma})],$$

where $\tilde{\sigma} = \tilde{\sigma}(z,w) = \frac{|z-w|^2}{|1-z\bar{w}|^2}$.

2 Background

In this section very brief introductions on the chordal SLE, Brownian loop measure and SLE bubble measure are given.

2.1 Chordal SLE process

The chordal SLE process from 0 to ∞ in $\mathbb{H}$ is a random family of conformal maps $(g_t : t \geq 0)$ that satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad 0 \leq t < \tau(z), \quad g_0(z) = z. \tag{2.1}$$

where B is a standard Brownian motion and $\tau(z)$ is the blow-up time for the differential equation (2.1). The SLE(κ) process is generated by a continuous curve γ, (see [5] and [6]). That is, the following limits exists:

$$\gamma(t) = \lim_{y\downarrow 0} g_t^{-1}(iy + \sqrt{\kappa}B_t).$$

The curve γ is a simple curve if and only if $\kappa \in (0, 4]$, (see[5]). The SLE(κ) curve satisfies the scaling invariance property (see [5]), i.e for any $r > 0$, $r\gamma$ has the same distribution as γ (with a time rescaling). So given any triple set (D, a, b), where D is a simply connected domain with two given boundary points a and b, we can define the SLE process on D from a to b by the conformal map from $\mathbb{H}$ to D that sends 0 to a and ∞ to b.

The following lemma will be useful in defining the SLE-bubble measure.

Lemma 3 (see [7]). *Given $z = x + iy \in \mathbb{H}$, suppose $\kappa \in (0, 4]$ and γ is the* SLE(κ) *curve from 0 to ∞ in $\mathbb{H}$. Then the probability that γ passes the left of z is*

$$p(z) = C \int_{-\infty}^{\frac{x}{y}} (1 + t^2)^{-\frac{4}{\kappa}} dt, \tag{2.2}$$

where $C = C(\kappa)$ is the constant that make the total integral above equal to 1.

2.2 Brownian loop measure and bubble measure

In this section, we will introduce several measures on the space of continuous curves in the plane. To keep the present article short, we will not provide the detailed discussions but instead refer the reader to the fifth chapter of Lawler's book [8] and [4].

Let $\mu(z; t)$ be the law of a complex Brownian motion $(B_s : 0 \leq s \leq t)$ starting from z. And $\mu(z; t)$ can be written as

$$\mu(z; t) = \int_{\mathbb{C}} \mu(z, w; t) dw$$

where the above integral can be regarded as the integral of measure-valued functions. Here $\mu(z; t)$ and $\mu(z, w; t)$ are regarded as measures on the space of curves in the plane. Using the density function of the complex Brownian motion, we can see that the total mass of $\mu(z, w; t)$ is $\frac{1}{2\pi t} \exp\{-\frac{1}{2t}|z - w|^2\}$.

Let $\mu(z, w)$ be the measure defined by $\mu(z, w) = \int_0^\infty \mu(z, w; t)$. This is a $\sigma-$finite infinite measure. If $D \subset \mathbb{C}$ is a simply connected domain with nice boundary and $z, w \in D$, we can define $\mu_D(z, w)$ be the restriction of $\mu(z, w)$ on the space of curves that lie inside D. If $z \neq w$, the total measure of $\mu_D(z, w)$ is $\pi G_D(z, w)$, where $G_D(z, w)$ is the Green function on D.

If D is a simply connected domain with nice boundary, let B be a complex Brownian motion starting from $z \in D$ and τ_D the exit time. Denote

$\mu_D(z, \partial D)$ the law of $(B_t : 0 \le t \le \tau_D)$, we can write

$$\mu_D(z, \partial D) = \int_{\partial D} \mu_D(z, w) dw.$$

Here $\mu_D(z, w)$ can be regarded as a measure on the space of curves in D from z to $w \in \partial D$, and the total mass of $\mu_D(z, w)$ is the the Poisson kernel $H_D(z, w)$. For $z \in D, w \in \partial D$, $\mu_D(z, w)$ can also be equivalently defined by the limits

$$\mu_D(z, w) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mu_D(z, w + \epsilon \mathbf{n}_w),$$

where $\mathbf{n}_w$ is the inner normal at w.

And similarly for $z, w \in \partial D$, we can also define

$$\mu_D(z, w) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon^2} \mu_D(z + \epsilon \mathbf{n}_z, w + \epsilon \mathbf{n}_w)$$

It can be showed that the above limits exist in the sense of Prohorov convergence(see Chapter 5 of [8]).

Given $z \in \partial D$, the **Brownian bubble measure** $\mu_D^{\text{bub}}(z)$ is defined as the limit

$$\mu_D^{\text{bub}}(z) := \lim_{w \in \partial D, w \to z} \pi \mu_D(z, w).$$

The **Brownian loop measure** is defined as following:

$$\mu_{\mathbb{C}}^{loop} := \int_{\mathbb{C}} \frac{1}{t_\gamma} \mu(z, z) dz = \int_{\mathbb{C}} \int_0^\infty \frac{1}{t_\gamma} \mu(z, z; t) dt dz.$$

Since $\mu(z, z)$ is a measure defined on loops with z as a marked point(called a root), the Brownian loop measure should be understood as the above integral of measures by forgetting the root. For any domain D, let μ_D^{loop} be the restriction of the Brownian loop measure on the space of loops inside D.

For any $a \in \mathbb{R}$, define $\mathbb{H}_a := \{x + iy \in \mathbb{C} : y > a\}$, according to the lowest point of the Brownian loop, the Brownian loop can be decomposed into the following integral of Brownian bubbles(see [4]):

$$\mu_{\mathbb{C}}^{\text{loop}} = \frac{1}{\pi} \int_{\mathbb{C}} \mu_{\mathbb{H}_y}^{\text{bub}}(x + iy) dx dy. \tag{2.3}$$

(2.3) will be very important in our computation.

2.3 SLE bubble measure

In this section we will define the SLE bubble measure and give the relation between SLE($\frac{8}{3}$) bubble and Brownian bubble measure.

Suppose $\kappa \in (0,4], \epsilon > 0$ and γ^ϵ is the SLE(κ) curve from 0 to ϵ in the upper half plane. Let μ^ϵ denote the law of γ^ϵ.

Lemma 4. *The limit of the following limit exists:*

$$\mu^{bub}_{SLE(\kappa)}(0) = \lim_{\epsilon \to 0} \epsilon^{1-\frac{8}{\kappa}} \mu^\epsilon. \tag{2.4}$$

We call $\mu^{bub}_{SLE(\kappa)}(0)$ *the* SLE(κ)***-bubble measure***.

Proof. We only need to show that the limit restricted to some generated algebras that consist of finite mass exists. Here we choose the measurable sets $\{\gamma : \gamma \text{ disconnects } z \text{ from } \infty\}$ for fixed $z \in \mathbb{H}$. By the definition of the SLE(κ) from 0 to ϵ, we choose the auto-conformal map $F_\epsilon(z) = \frac{\epsilon z}{z+1}$ that sends ∞ to ϵ and fixes 0. We have

$$\mathbf{P}[\gamma \text{ disconnects } z \text{ from } \infty] = p(F_\epsilon^{-1}(z)),$$

where $p(z)$ is defined in (2.2). Therefore

$$\lim_{\epsilon \to 0} \epsilon^{1-\frac{8}{\kappa}} p(F_\epsilon^{-1}(z)) = \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})(\frac{8}{\kappa}-1)} (\frac{x^2+y^2}{y})^{1-\frac{8}{\kappa}} \tag{2.5}$$

So for fix $z \in \mathbb{H}$, if we denote $\mu^\epsilon(z)$ the restriction of μ^ϵ restricted to the curves that disconnect z from ∞, then by the above equation, we know that the limit

$$\mu^{\text{bub}}_{\text{SLE}(\kappa)}(0,z) := \lim_{\epsilon \to 0} \epsilon^{1-\frac{8}{\kappa}} \mu^\epsilon(z)$$

exists and therefore we can define $\mu^{\text{bub}}_{\text{SLE}(\kappa)}(0)$ as the limit of $\mu^{\text{bub}}_{\text{SLE}(\kappa)}(0,z)$ as z tends to zero. □

If $\kappa = \frac{8}{3}$, from (2.5), we get that the total mass of the SLE($\frac{8}{3}$)-bubble that disconnects a given point $z = x + iy \in \mathbb{H}$ is $\frac{1}{4}(\frac{y}{x^2+y^2})^2 = \frac{1}{4}(\text{Im}\,\frac{1}{z})^2$ which corresponds to the part (a) of proposition 3.1 in [1]. In fact, [1] also gives the measure of the SLE($\frac{8}{3}$)-bubble that disconnects two points $z, w \in \mathbb{H}$ from ∞ which we will state as the following lemma.

Lemma 5 (see [1]). *Let $E(z,w)$ be the event that two points $z,w \in \mathbb{H}$ are disconnected from ∞ by a* SLE($\frac{8}{3}$) *curve from 0 to ϵ, then*

$$\mu^{\epsilon}[E(z,w)] = \frac{1}{4}\mathrm{Im}\,(\frac{1}{z})\mathrm{Im}\,(\frac{1}{w})G(\sigma(z,w))\epsilon^2 + O(\epsilon^3). \tag{2.6}$$

where σ is defined as (1.4) *and*

$$G(t) = 1 - t\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; 1-t). \tag{2.7}$$

Here ${}_2F_1$ is the hypergeometric function.

Notice that when $\kappa = \frac{8}{3}$, it holds that $1 - \frac{8}{\kappa} = -2$. so we have

$$\mu^{\mathrm{bub}}_{\mathrm{SLE}}(0)[E(z,w)] = \frac{1}{4}\mathrm{Im}\,(\frac{1}{z})\mathrm{Im}\,(\frac{1}{w}) = \frac{1}{4}\frac{yv}{(x^2+y^2)(u^2+v^2)}G(\sigma(z,w)). \tag{2.8}$$

SLE($\frac{8}{3}$)-bubble measure is closely related to Brownian bubble, in fact they only differ by a constant. Given a loop γ such that $\gamma(0) = \gamma(t_\gamma) = 0$ and $\gamma(0,t_\gamma) \subset \mathbb{H}$, call the complement of the unbounded connected component of $\mathbb{H} \setminus \gamma$ the hull enclosed by γ. In the following we will let γ denote both the loop and hull enclosed by it.

Lemma 6 (see [2]). *As measures on the hulls enclosed by loops, the following holds:*

$$\mu^{bub}_{\mathbb{H}}(0) = \frac{8}{5}\mu^{bub}_{SLE(\frac{8}{3})}(0).$$

Proof. By the construction of the Brownian bubble measure at 0 (see the Chapter 5 of Lawler's book [8]), it is the unique measure on loops(or hulls enclosed by loops) in $\mathbb{H}$ rooted at 0 such that the total mass that the sample intersect $|z| = r$ is $\frac{1}{r^2}$ for any $r > 0$. Since

$$\big\{\{\gamma : \gamma \cap \{|z| = r\} = \emptyset\} : r > 0\big\}$$

is a algebra that generate the $\sigma-$algebra of the space of loops, we only need to show that the total mass of the SLE($\frac{8}{3}$)-bubble sample intersecting $|z| = r$ is $\frac{5}{8r^2}$. Define $F_\epsilon(z) = \frac{z}{\epsilon - z}$, the imagine of the circle $|z| = r$ under F_ϵ is a circle with center $c_0 = -\frac{r^2}{r^2-\epsilon^2}$ and radius $\rho = \frac{\epsilon r}{r^2-\epsilon^2}$. Define the conformal

map $\phi_\epsilon(z) = z - c_0 + \frac{\rho^2}{z-c_0}$ which maps $\mathbb{H} \setminus B(c_0, \rho)$ onto $\mathbb{H}$ with the derivative at ∞ equaling to 1. By the conformal restriction property of SLE($\frac{8}{3}$), we have

$$\mu^\epsilon[\gamma \cap |z| = r = \emptyset] = \mu^\infty[\gamma \cap B(c_0, \rho) = \emptyset] = \phi'_\epsilon(0)^{\frac{5}{8}}.$$

Therefore we can check that

$$\mu^{\text{bub}}_{\text{SLE}(\kappa)}(0)[\gamma \cap \{|z| = r\} \neq \emptyset] = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2}(1 - \phi'_\epsilon(0))^{\frac{5}{8}} = \frac{5}{8r^2}.$$

□

3 Proof of the main theorem

In this section we will give a detailed proof of our main theorem.

3.1 Proof of the nicer form

Given two points $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0 \in \mathbb{H}$. By the symmetry property of the the Brownian loop measure, we may assume without loss of generality $y_0 \leq v_0$, $u_0 \geq x_0$. By (2.3),

$$\begin{aligned}\mu^{\text{loop}}_{\mathbb{H}}[E(z_0, w_0)] &= \frac{1}{\pi} \int_{\mathbb{H}} \mu^{\text{bub}}_{\mathbb{H}_y}(x + iy)[E(z_0, w_0)] dxdy \\ &= \frac{1}{\pi} \int_0^{y_0} \int_{\mathbb{R}} \mu^{\text{bub}}_{\mathbb{H}_y}(x + iy)[E(z_0, w_0)] dxdy.\end{aligned}$$

Here $E(z_0, w_0)$ denotes the event that the Brownian loop sample in $\mathbb{H}$ disconnects both z_0 and w_0 from the boundary of $\mathbb{H}$.

By the translation invariance of the Brownian bubble measure, we have

$$\mu^{\text{bub}}_{\mathbb{H}_y}(x + iy)[E(z_0, w_0)] = \mu^{\text{bub}}_{\mathbb{H}}(0)[E(z_0 - z, w_0 - z)].$$

By Lemma 6 we have $\mu^{\text{bub}}_{\mathbb{H}}(0) = \frac{8}{5}\mu^{\text{bub}}_{\text{SLE}(\kappa)}(0)$. Therefore by (2.8),

$$\begin{aligned}\mu^{\text{loop}}_{\mathbb{H}}[E(z_0, w_0)] &= \frac{8}{5\pi} \int_0^{y_0} \int_{\mathbb{R}} \mu^{\text{bub}}_{\text{SLE}(\kappa)}(0)[E(z_0 - x - iy, w_0 - x - iy)] dxdy \\ &= \frac{8}{5\pi} \int_0^{y_0} \int_{\mathbb{R}} \frac{1}{4} \text{Im}\left(\frac{1}{z_0 - x - iy}\right) \text{Im}\left(\frac{1}{w_0 - x - iy}\right) \\ &\qquad G(\sigma(z_0 - x - iy, w_0 - x - iy)) dxdy. \quad (3.1)\end{aligned}$$

So in order to prove the theorem, we only need to compute above integral. Define two functions as follows:

$$f(x,y) := \frac{(y_0-y)(v_0-y)}{[(x_0-x)^2+(y_0-y)^2]\times[(u_0-x)^2+(v_0-y)^2]}. \tag{3.2}$$

$$g(y) := \frac{(x_0-u_0)^2+(y_0-v_0)^2}{(x_0-u_0)^2+(y_0+v_0-2y)^2} \\ {}_2F_1(1,\frac{4}{3};\frac{5}{3};\frac{4(y_0-y)(v_0-y)}{(x_0-u_0)^2+(y_0+v_0-2y)^2}). \tag{3.3}$$

Lemma 7. *Take the notations as above, for fixed $y>0$,*

$$\int_{\mathbb{R}} f(x,y)dx = \frac{2(y_0-y)+v_0-y_0}{(x_0-u_0)^2+\big(2(y_0-y)+v_0-y_0\big)^2}\pi. \tag{3.4}$$

Proof. For fixed $y>0$, denote

$$a=y_0-y,\quad b=v_0-y,\quad c=u_0-x_0,\quad d=v_0-y_0.$$

Then we have

$$f(x,y)=\frac{ab}{[(x_0-x)^2+a^2][(u_0-x)^2+b^2]}.$$

By standard calculus,

$$\begin{aligned}
&\int_{\mathbb{R}} f(x,y)dx = \int_{\mathbb{R}} \frac{ab}{[(x_0-x)^2+a^2][(u_0-x)^2+b^2]}dx\\
=&ab\int_{\mathbb{R}}\frac{1}{[x^2+a^2][(x+c)^2+b^2]}dx\\
=&\frac{ab\pi}{ab\big(a^4-2a^2(b^2-c^2)+(b^2+c^2)^2\big)}\Big(b(b^2+c^2-a^2)\arctan[\frac{x}{a}]\\
&+a\big[(a^2+c^2-b^2)\arctan[\frac{c+x}{b}]+bc\log\frac{b^2+(c+x)^2}{a^2+x^2}\big]\Big)|_{-\infty}^{\infty}\\
=&\pi\frac{b(b^2+c^2-a^2)+a(a^2+c^2-b^2)}{a^4-2a^2(b^2-c^2)+(b^2+c^2)^2}.
\end{aligned}$$

Replace b by $a+d$, we get

$$\int_{\mathbb{R}} f(x,y)dx = \frac{(2a+d)\pi}{c^2+(2a+d)^2},$$

which is what we want. □

By (3.1), we have

$$\mu_{\mathbb{H}}^{\text{loop}}[E(z_0,w_0)] = \frac{8}{5\pi}\int_0^{y_0}\int_{\mathbb{R}} \frac{1}{4}f(x,y)(1-g(y))dxdy$$

$$=\frac{8}{5\pi}\int_0^{y_0}\frac{\pi}{4}\frac{2a+d}{c^2+(2a+d)^2}[1-g(y)]dy = \frac{2}{5}(A-B), \tag{3.5}$$

where

$$A = A(z_0,w_0) = \int_0^{y_0}\frac{2(y_0-y)+v_0-y_0}{(x_0-u_0)^2+\big(2(y_0-y)+v_0-y_0\big)^2}dy, \tag{3.6}$$

and

$$B = B(z_0,w_0) = \int_0^{y_0}\frac{2(y_0-y)+v_0-y_0}{(x_0-u_0)^2+\big(2(y_0-y)+v_0-y_0\big)^2}g(y)dy. \tag{3.7}$$

Lemma 8.

$$A = \frac{1}{4}\log\frac{1}{\sigma},$$

where σ is defined as (1.4).

Proof. By (3.6) we have

$$A = \int_0^{y_0}\frac{2(y_0-y)+d}{c^2+(2(y_0-y)+d)^2}dy = \int_0^{y_0}\frac{2y+d}{c^2+(2y+d)^2}dy$$

$$=\frac{1}{2}\int_{d/c}^{\frac{2y_0+d}{c}}\frac{y}{1+y^2}dy = \frac{1}{4}\log\frac{c^2+(2y_0+d)^2}{c^2+d^2}.$$

In the second equation we used the change of variable $y\to y_0-y$ and in the last equation we used the change of variable $y\to\frac{2y+d}{c}$. Notice that

$$\frac{c^2+(2y_0+d)^2}{c^2+d^2} = \frac{(u_0-x_0)^2+(y_0+v_0)^2}{(u_0-x_0)^2+(v_0-y_0)^2} = \frac{1}{\sigma}.$$

□

Lemma 9.

$$B = \frac{1}{4}(1-\sigma)\,{}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; 1-\sigma),$$

where σ is defined as (1.4).

Proof. By (3.7) and the definition of $g(y)$, we have

$$\begin{aligned}
B =& \int_0^{y_0} \frac{2(y_0-y)+d}{c^2+(2(y_0-y)+d)^2} \cdot \frac{c^2+d^2}{c^2+(2(y_0-y)+d)^2} \cdot \\
& {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{4(y_0-y)((y_0-y)+d)}{c^2+(2(y_0-y)+d)^2}) dy \\
=& \int_0^{y_0} \frac{2y+d}{c^2+(2y+d)^2} \cdot \frac{c^2+d^2}{c^2+(2y+d)^2} \cdot {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{4y(y+d)}{c^2+(2y+d)^2}) dy \\
=& \int_{\frac{d}{c}}^{\frac{2y_0+d}{c}} \frac{cy}{c^2+c^2y^2} \cdot \frac{c^2+d^2}{c^2+c^2y^2} \cdot {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{c^2y^2-d^2}{c^2+c^2y^2}) \cdot \frac{c}{2} dy \\
=& \frac{1}{2}\frac{c^2+d^2}{c^2} \int_{\frac{d}{c}}^{\frac{2y_0+d}{c}} \frac{y}{(1+y^2)^2} \cdot {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{c^2y^2-d^2}{c^2+c^2y^2}) dy \\
=& \frac{1}{4} \int_0^{\frac{4y_0(y_0+d)}{c^2+(2y_0+d)^2}} {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; y) dy \\
=& \frac{1}{4}\frac{4y_0(y_0+d)}{c^2+(2y_0+d)^2} \cdot {}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; \frac{4y_0(y_0+d)}{c^2+(2y_0+d)^2}) \\
=& \frac{1}{4}(1-\sigma)\,{}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; 1-\sigma).
\end{aligned}$$

Here the second equation used the change of variable $y \to y_0 - y$, the third equation used the change of variable $y \to \frac{2y+d}{c}$, the fifth equation used the change of variable $\frac{c^2y^2-d^2}{c^2+c^2y^2} \to y$ and the sixth equation used the equation about hypergeometric functions below:

$$\int_0^x {}_2F_1(a, b; c, y) dy = x\,{}_3F_2(a, b, 1; c, 2, x).$$

□

Now by (3.5) and Lemma 8 and Lemma 9 we get (1.3).

3.2 Proof of Cardy's formula

In this section we will prove the equivalence of formula (1.1) and (1.3). First we will recall some identities for the hypergeometric functions which will be used in our proof. We will assume that our hypergeometric functions are all well defined. And they satisfies the following identities (see Chapter 8 of [9]):

$$ {}_2F_1(a,b;c;x) = (1-x)^{-b}\,{}_2F_1(c-a,b;c;\frac{x}{x-1}). \tag{3.8}$$

$$\begin{aligned} {}_2F_1(a,b;c;x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b+1-c;1-x) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c+1-a-b;1-x). \end{aligned} \tag{3.9}$$

$$ {}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x). \tag{3.10}$$

Notice that $\eta = \frac{\sigma}{\sigma-1}$ and $\sigma \in (0,1)$. We define a function ϕ on $[0,1]$ as following.

$$\begin{aligned} \phi(t) = \frac{2\pi}{\sqrt{3}} &+ \frac{t}{t-1}\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\frac{t}{t-1}) - (1-t)\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1-t) \\ &- 2\log(1-t) - 2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}\sqrt[3]{\frac{t}{(t-1)^2}}\,{}_2F_1(1,\frac{2}{3};\frac{4}{3};\frac{t}{t-1}). \end{aligned} \tag{3.11}$$

To prove that (1.1) and (1.3) are equivalent, we only need to show that $\phi(t) \equiv 0$. Notice that $\phi(0) = \frac{2\pi}{\sqrt{3}} - {}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1) = 0$, it is left to show that $\phi'(t) \equiv 0$. Take the following notations.

$$I(t) := \frac{t}{t-1}\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\frac{t}{t-1}) - 2\log(1-t),$$

$$J(t) := -(1-t)\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1-t),$$

$$K(t) := -2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}\sqrt[3]{\frac{t}{(t-1)^2}}\,{}_2F_1(1,\frac{2}{3};\frac{4}{3};\frac{t}{t-1}).$$

Define $f(x) = x\,{}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; x)$. It is easy to check that

$$f'(x) = {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; x).$$

So

$$\frac{dI(t)}{dt} = \frac{2}{1-t} + f'(\frac{t}{t-1})\frac{-1}{(1-t)^2} = \frac{2}{1-t} - \frac{1}{(1-t)^2}\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{t}{t-1})$$

$$= \frac{2}{1-t} - \frac{1}{1-t}\,{}_2F_1(1, \frac{1}{3}; \frac{5}{3}; t). \tag{3.12}$$

The last equation follows from (3.8) by assigning $a = \frac{1}{3}, b = 1, c = \frac{5}{3}$. Similarly we get

$$\frac{dJ(t)}{dt} = f'(1-t) = {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; 1-t).$$

Using (3.9) with $a = 1, b = \frac{4}{3}, c = \frac{5}{3}$, we have

$${}_2F_1(1, \frac{4}{3}; \frac{5}{3}; 1-t) = -\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; t) + \frac{2}{3}\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}t^{-\frac{2}{3}}\,{}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{3}; t)$$

By letting $a = \frac{1}{3}, b = \frac{2}{3}, c = \frac{1}{3}$ in (3.10), the following holds

$${}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{3}; t) = (1-t)^{-\frac{2}{3}}\,{}_2F_1(0, -\frac{1}{3}; \frac{1}{3}, x) = (1-t)^{-\frac{2}{3}}.$$

Therefore

$$\frac{dJ(t)}{dt} = -\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; t) + \frac{2}{3}\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}(t(1-t))^{-\frac{2}{3}}. \tag{3.13}$$

Lastly we deal with the derivative of $K(t)$ with respect to t. By letting $a = \frac{1}{3}, b = \frac{2}{3}, c = \frac{4}{3}$ in (3.8), we can get

$${}_2F_1(1, \frac{2}{3}; \frac{4}{3}; \frac{t}{t-1}) = (1-t)^{\frac{2}{3}}\,{}_2F_1(1, \frac{2}{3}; \frac{4}{3}; t).$$

Consequently,

$$K(t) = -2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}t^{\frac{1}{3}}\,{}_2F_1(1, \frac{2}{3}; \frac{4}{3}; t).$$

And

$$\frac{dK(t)}{dt} = -2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}\big[\frac{1}{3}t^{-\frac{2}{3}}\,{}_2F_1(1,\frac{2}{3};\frac{4}{3};t) + t^{\frac{1}{3}}\frac{(1-t)^{-\frac{2}{3}} - {}_2F_1(1,\frac{2}{3};\frac{4}{3};t)}{3t}\big]$$

$$= -\frac{2}{3}\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}t^{-\frac{2}{3}}(1-t)^{-\frac{2}{3}}. \tag{3.14}$$

Combining (3.12),(3.13) and (3.14), we have

$$\phi'(t) = \frac{dI(t)}{dt} + \frac{dJ(t)}{dt} + \frac{dK(t)}{t} = \frac{2}{1-t} - \frac{1}{1-t}\,{}_2F_1(1,\frac{1}{3};\frac{5}{3};t) - {}_2F_1(1,\frac{4}{3};\frac{5}{3};t).$$

Lemma 10.

$$2 - {}_2F_1(1,\frac{1}{3};\frac{5}{3};t) - (1-t)\,{}_2F_1(1,\frac{4}{3};\frac{5}{3};t) = 0.$$

Proof. By definition we have

$${}_2F_1(1,\frac{4}{3};\frac{5}{3};t) = 1 + \sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{4}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{5}{3})}t^n.$$

Therefore

$$t\,{}_2F_1(1,\frac{4}{3};\frac{5}{3};t) = \sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{2}{3})}t^n.$$

Similarly

$${}_2F_1(1,\frac{1}{3};\frac{5}{3};t) = 1 + \sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})\Gamma(n+\frac{5}{3})}t^n.$$

By using the relation $\Gamma(x+1) = x\Gamma(x)$, we can see that the coefficient of t^n in the sum is

$$\frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{2}{3})} - \frac{\Gamma(n+\frac{4}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{5}{3})} - \frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})\Gamma(n+\frac{5}{3})} = 0.$$

□

From Lemma 10, we have $\phi'(t) \equiv 0$, and therefore $\phi \equiv \phi(0) = 0$. This completes the proof of the equivalence between (1.1) and (1.3).

4 The other cases

Given $z, w \in \mathbb{H}$, and γ the sample of the Brownian loop in the upper half plane. According to the property of Brownian path, almost surely, $z, w \notin \gamma$. So except the case that γ disconnects both z and w from the boundary, there are three other cases:

(1) γ disconnects z from the boundary but does not disconnect w from the boundary;

(2) γ disconnects w from the boundary but does not disconnect z from the boundary;

(3) γ does neither disconnects z from the boundary nor disconnects w from the boundary.

We will show that the total measure of above three cases are infinite. In fact, using the same method as [1], we can show the following lemma.

Lemma 11. *Suppose that γ is the sample of the* SLE($\frac{8}{3}$) *from* 0 *to* ϵ *and denote above three cases by $E_1(z,w), E_2(z,w)$ and $E_3(z,w)$ respectively. Then*

$$\mathbf{P}[E_1(z,w)] = \frac{1}{4}\epsilon^2\big((\mathrm{Im}\,\frac{1}{z})^2 - \mathrm{Im}\,\frac{1}{z}\mathrm{Im}\,\frac{1}{w}G(\sigma)\big) + O(\epsilon^3), \tag{4.1}$$

$$\mathbf{P}[E_2(z,w)] = \frac{1}{4}\epsilon^2\big[(\mathrm{Im}\,\frac{1}{w})^2 - \mathrm{Im}\,\frac{1}{z}\mathrm{Im}\,\frac{1}{w}G(\sigma)\big] + O(\epsilon^3), \tag{4.2}$$

$$\mathbf{P}[E_3(z,w)] = 1 - \frac{1}{4}\epsilon^2\big[(\mathrm{Im}\,\frac{1}{w})^2 + (\mathrm{Im}\,\frac{1}{z})^2 - \mathrm{Im}\,\frac{1}{z}\mathrm{Im}\,\frac{1}{w}G(\sigma)\big] + O(\epsilon^3). \tag{4.3}$$

The proof of this lemma is the same as in [1]. We only need to prove that for SLE($\frac{8}{3}$) γ from 0 to ∞, the following holds.

$$\begin{aligned}
&\mathbf{P}[\gamma \quad \text{passes the left of } z \text{ and the right of } w] \\
&= \tfrac{1}{4}(1-\tfrac{x}{|z|})(1+\tfrac{u}{|w|})(1-\tfrac{y}{|z|-x}\tfrac{v}{|w|+u}G(\sigma)). \\
&\mathbf{P}[\gamma \quad \text{passes the left of } w \text{ and the right of } z] \\
&= \tfrac{1}{4}(1+\tfrac{x}{|z|})(1-\tfrac{u}{|w|})(1-\tfrac{y}{|z|+x}\tfrac{v}{|w|-u}G(\sigma)). \\
&\mathbf{P}[\gamma \quad \text{passes the right of both } z \text{ and } w] \\
&= \tfrac{1}{4}(1-\tfrac{x}{|z|})(1-\tfrac{u}{|w|})(1+\tfrac{y}{|z|-x}\tfrac{v}{|w|-u}G(\sigma)).
\end{aligned}$$

where $G(\sigma)$ is the same as (2.7). Then using the conformal map $F_\epsilon(z) = \frac{\epsilon z}{1+z}$ to convert the SLE($\frac{8}{3}$) from 0 to ∞ into the SLE($\frac{8}{3}$) from 0 to ϵ. Combing

above lemma and the definition of the Brownian bubble measure and lemma 6, we can get

$$\mu_{\mathbb{H}}^{\text{bub}}(0)(E_1(z,w)) = \frac{1}{10}[(\frac{y}{x^2+y^2})^2 - \frac{y}{x^2+y^2}\frac{v}{u^2+v^2}G(\sigma(z,w))].$$

$$\mu_{\mathbb{H}}^{\text{bub}}(0)(E_2(z,w)) = \frac{1}{10}[(\frac{v}{u^2+v^2})^2 - \frac{y}{x^2+y^2}\frac{v}{u^2+v^2}G(\sigma(z,w))].$$

$$\mu_{\mathbb{H}}^{\text{bub}}(0)(E_3(z,w)) = \infty.$$

By relation (2.3) and calculating the integral on the upper half plane, we can see that the total mass of the Brownian loop measure on these three sets are infinite. In fact, we can see intuitively that these three cases all contain the loops with arbitrary small diameter, while the event $E(z,w)$ in the main theorem exclude these small loops.

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