



HAL
open science

On stochastic modified 3D Navier-Stokes equations with anisotropic viscosity

Hakima Bessaih, Annie Millet

► **To cite this version:**

Hakima Bessaih, Annie Millet. On stochastic modified 3D Navier-Stokes equations with anisotropic viscosity. *Journal of Mathematical Analysis and Applications*, 2018, 462 (1), pp.915-956. 10.1016/j.jmaa.2017.12.053 . hal-01502048v2

HAL Id: hal-01502048

<https://hal.science/hal-01502048v2>

Submitted on 20 Nov 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON STOCHASTIC MODIFIED 3D NAVIER-STOKES EQUATIONS WITH ANISOTROPIC VISCOSITY

HAKIMA BESSAIH AND ANNIE MILLET

ABSTRACT. Navier-Stokes equations in the whole space \mathbb{R}^3 subject to an anisotropic viscosity and a random perturbation of multiplicative type is described. By adding a term of Brinkman-Forchheimer type to the model, existence and uniqueness of global weak solutions in the PDE sense are proved. These are strong solutions in the probability sense. The Brinkman-Forchheimer term provides some extra regularity in the space $L^{2\alpha+2}(\mathbb{R}^3)$, with $\alpha > 1$. As a consequence, the nonlinear term has better properties which allow to prove uniqueness. The proof of existence is performed through a control method. A Large Deviations Principle is given and proven at the end of the paper.

1. INTRODUCTION

The Navier-Stokes equations describe the time evolution of the velocity u of an incompressible fluid in a bounded or unbounded domain of \mathbb{R}^n , $n = 2, 3$ and are described by:

$$\begin{aligned}\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \quad u|_{t=0} = u_0,\end{aligned}$$

where $\nu > 0$ is the viscosity of the fluid and p denotes the pressure. If existence and uniqueness is known to hold in dimension 2, the case of dimension 3 is still only partially solved. Indeed, there exists a solution in some homogeneous Sobolev space $\dot{H}^{1/2}$ either on a small time interval or on an arbitrary time interval if the norm of the initial condition is small enough. The difficulty in dimension 3 comes from the nonlinear term $(u \cdot \nabla)u$ that requires more regularity. However, this regularity is not satisfied by the energy estimates while it is in dimension 2. In particular, the lack of this regularity is essentially the reason the uniqueness cannot be proved for weak solutions. Many regularizations have been introduced to overcome this difficulty. Here, we will discuss only two of them; a regularization by a rotating term $u \times e_3$ and a regularization by a Brinkman-Forchheimer term $|u|^{2\alpha} u$. Of course these two different regularizations give rise to different models. One is related to some rotating flows while the other is related to some porous media models. We refer to [19] and the references therein, where the following system has been investigated (in an even more general formulation)

$$\begin{aligned}\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p + a|u|^{2\alpha} u &= f, \\ \operatorname{div} u &= 0, \quad u|_{t=0} = u_0,\end{aligned}$$

2000 *Mathematics Subject Classification.* Primary 60H15, 60F10; Secondary 76D06, 76M35.

Key words and phrases. Navier-Stokes equations, anisotropic viscosity, Brinkman-Forchheimer model, nonlinear convectivity, stochastic PDEs, large deviations.

Hakima Bessaih was partially supported by NSF grant DMS 1418838.

where $a > 0$ and $\alpha > 0$ and f is an external force. Under some assumptions on the coefficient α , the authors in [19] prove the existence and uniqueness of global strong solutions.

A slightly different regularization has been investigated by Kalantarov and Zelik in [17]; more precisely they studied some versions of the following model:

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + g(u) + \nabla p &= f, \\ \operatorname{div} u &= 0, \quad u|_{t=0} = u_0, \end{aligned}$$

where $g \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ satisfies the following properties:

$$\begin{cases} g'(u)v \cdot v \geq (-K + \kappa|u|^{r-1})|v|^2, & \forall u, v \in \mathbb{R}^3, \\ |g'(u)| \leq C(1 + |u|^{r-1}), & \forall u \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where K, C, κ are some positive constants, $r \in [1, \infty)$ and $u \cdot v$ stands for the inner product in \mathbb{R}^3 . When the forcing is of random type, that is $f = \sigma(t, u)dW(t)$, M. Röckner, T. Zhang and X. Zhang tackled a stochastic version of a modification of the previous model (1.1), that they called the tamed stochastic Navier-Stokes equations, in several papers such as [23], and [24]. Let us mention that in both the deterministic and the stochastic versions of (1.1), the solutions are investigated when the regularity of initial condition is at least H^1 and the viscosity acts in all three directions.

In this paper, we are interested in the 3D Navier-Stokes equations with anisotropic viscosity that is acting only in the horizontal directions. These models have some applications in atmospheric dynamics where some informations are missing. The relevance of the anisotropic viscosity is explained through the Ekman law (see e.g. [22] or the introduction of [10]). The aim of this paper is to study an anisotropic Navier-Stokes equation in dimension 3 that is subject to some multiplicative random forcing. More precisely, we consider the following model of a modified 3D anisotropic Navier-Stokes system on a fixed time interval $[0, T]$ which can be written formally as follows:

$$\begin{aligned} \partial_t u - \nu \Delta_h u + u \cdot \nabla u + a|u|^{2\alpha} u + \nabla p &= \sigma(t, u) \dot{W} \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^3, \\ \nabla \cdot u &= 0 \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^3, \end{aligned} \quad (1.2)$$

with the initial condition u_0 independent of the driving noise W . Here the viscosity ν and the coefficient a of the nonlinear convective term are strictly positive, $\alpha > 1$, ∂_t denotes the time partial derivative, $\Delta_h := \partial_1^2 + \partial_2^2$ and ∂_i denotes the partial derivative in the direction x_i , $i = 1, 2, 3$. Thus the viscosity is only smoothing in the horizontal directions. As usual the fluid is incompressible, p denotes the pressure; the forcing term $\sigma(t, u) \dot{W}$ is a multiplicative noise driven by an infinite dimensional Brownian motion W which is white in time with spatial correlation. The convective term $a|u|^{2\alpha}u$ is of Brinkman-Forchheimer type and has a regularizing effect which can balance on one hand the vertical partial derivative of the bilinear term to prove existence, and on the other hand provide some control to obtain uniqueness. Note that the space $L^{2\alpha+2}(\mathbb{R}^3)$ appears naturally in the analysis of (1.2); it is equal to $L^4(\mathbb{R}^3)$ if $\alpha = 1$. Furthermore, the homogeneous critical Sobolev space $\dot{H}^{1/2}$ for the Navier-Stokes equation is included in L^4 . Hence it is natural to impose $\alpha > 1$.

The deterministic counterpart of (1.2), that is equation (1.2) with $\sigma = 0$, has been studied by H. Bessaih, S. Trabelsi and H. Zorgati in [5]. The authors have proved that if the initial condition $u_0 \in \tilde{H}^{0,1}$, for any $T > 0$ there exists a unique solution in $L^\infty(0, T; \tilde{H}^{0,1}) \cap$

$L^2(0, T; \tilde{H}^{1,1})$ which belongs to $C([0, T], L^2)$, for some anisotropic Sobolev spaces which will be defined in the next section (see (2.1)). We generalize this result by allowing the system to be subject to some random external force whose intensity may depend on the solution u and on its horizontal gradient $\nabla_h u$. Note that since no smoothing is provided by a viscosity in the vertical direction, in the anisotropic case, one requires that the initial condition u_0 is square integrable as well as its vertical partial derivative.

In the deterministic setting (that is $\sigma = 0$), replacing the Brinkman-Forchheimer term $a|u|^{2\alpha}u$ by the rotating term $\frac{1}{\epsilon}u \times e_3$, J.Y. Chemin, B. Desjardin, I. Gallagher and E. Grenier [9] have studied an anisotropic modified Navier Stokes equation on \mathbb{R}^3 with a vertical viscosity $\nu_v \geq 0$, which is allowed to vanish. Using some homogeneous anisotropic spaces, they have proved that if $u_0 \in H^{0,s}$ with $s > \frac{3}{2}$, there exists ϵ_0 depending only on ν and u_0 such that for $\epsilon \in (0, \epsilon_0]$,

$$\begin{aligned} \partial_t u - \nu \Delta_h u + u \cdot \nabla u + \frac{1}{\epsilon} u \times e_3 + \nabla p &= 0, \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^3, \\ \nabla \cdot u &= 0 \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^3, \quad u|_{t=0} = u_0 \end{aligned}$$

has a unique global solution in $L^\infty(0, T; H^{0,s}) \cap L^2(0, T; H^{1,s})$. The dispersive Brinkman-Forchheimer term is "larger" than the rotating term used in [9] but the regularity required on the initial condition is weaker and we allow a stochastic forcing term.

The paper is organized as follows. In section 2 we describe the functional setting of our anisotropic model and prove some technical properties of the deterministic terms. Several results were already proved in [5] and we sketch the arguments for the sake of completeness. We also describe the random forcing term and the growth and Lipschitz assumptions on the diffusion coefficient σ . In section 3 we prove that if $u_0 \in L^4(\Omega, \tilde{H}^{0,1})$ is independent of W and σ satisfies some general assumptions (in particular cases σ may contain some "small multiple" of the horizontal gradient $\nabla_h u$), equation (1.2) has a unique solution in $L^4(\Omega; L^\infty(0, T; \tilde{H}^{0,1})) \cap L^2(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2\alpha+2}(\Omega \times (0, T) \times \mathbb{R}^3)$, which is almost surely continuous from $[0, T]$ to H , where H denotes the set of square integrable divergence free functions. Examples of such coefficients σ are provided. Since we are working on the whole space \mathbb{R}^3 , and not on a bounded domain, the martingale approach used in [4], which depends on tightness properties, does not seem appropriate. We use instead the control method introduced in [20] for the 2D Navier-Stokes equation; see also [26], [15], [13] and [24], where this method has been used for the stochastic 2D Navier-Stokes equations, stochastic 2D general hydrodynamical Bénard models and the stochastic 3D tamed Navier-Stokes equations. In section 4, under stronger assumptions on σ (which may no longer depend on the horizontal gradient $\nabla_h u$), we also prove a large deviations result in $C([0, T]; H) \cap L^2(0, T; \tilde{H}^{1,0})$ when the noise intensity is multiplied by a small parameter $\sqrt{\epsilon}$ converging to 0. The proof uses the weak-convergence approach introduced by A. Budhiraja, P. Dupuis and R.S. Ellis in [16] and [6]; see also the references [26], [15], [13] and [24] where this approach, based on the equivalence of the Large Deviations and Laplace principles, is used for various stochastic 2D Hydrodynamical models and the stochastic 3D tamed Navier-Stokes equation. For the sake of completeness, some technical well-posedness result for a stochastic controlled equation and estimates which only depend on the norm stochastic control, whose proofs are similar to that of the original equation in section 3, are given in the appendix. The proof of the weak convergence and compactness arguments, which have also been used in some papers on Large Deviations Principles of stochastic hydrodynamical models, are also described in the appendix

2. THE FUNCTIONAL SETTING

2.1. Some notations. Let us describe some further notations and the functional framework we will use throughout the paper. Given a vector $x = (x_1, x_2, x_3)$ let $x_h := (x_1, x_2)$ denote the horizontal variable, which does not play the same role as the vertical variable x_3 . Due to the anisotropic feature of the model, we use anisotropic Sobolev spaces defined as follows: given $s, s' \in \mathbb{R}$ let $H^{s, s'}$ denote the set of tempered distributions $\psi \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\|\psi\|_{s, s'}^2 := \int_{\mathbb{R}^3} (1 + |(\xi_1, \xi_2)|^{2s}) (1 + |\xi_3|^{2s'}) |\mathcal{F}\psi(\xi)|^2 d\xi < \infty, \quad (2.1)$$

where \mathcal{F} denotes the Fourier transform. The set $H^{s, s'}$ endowed with the norm $\|\cdot\|_{s, s'}$ is a Hilbert space.

Set $\operatorname{div}_h u = \partial_1 u_1 + \partial_2 u_2$. Note that for $u \in (H^{1,0} \cap H^{0,1})^3$

$$\nabla \cdot u = 0 \quad \text{implies} \quad \operatorname{div}_h u = -\partial_3 u_3. \quad (2.2)$$

For exponents $p, q \in [1, \infty)$ let $\|\cdot\|_p$ denote the $L^p(\mathbb{R}^3)$ norm while $L_h^p(L_v^q)$ denotes the space $L^p(\mathbb{R}_{x_1} \mathbb{R}_{x_2}, L^q(\mathbb{R}_{x_3}))$ endowed with the norm

$$\|\phi\|_{L_h^p(L_v^q)} := \left\{ \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |\phi(x_h, x_3)|^q dx_3 \right)^{\frac{p}{q}} dx_h \right\}^{\frac{1}{p}}.$$

The space $L_v^q(L_h^p) = L^q(\mathbb{R}_{x_3}; L^p(\mathbb{R}_{x_1} \mathbb{R}_{x_2}))$ is defined in a similar way and endowed with the norm $\|\phi\|_{L_v^q(L_h^p)} := \left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |\phi(x_h, x_3)|^p dx_h \right)^{\frac{q}{p}} dx_3 \right\}^{\frac{1}{q}}$. Note that in the above definitions we may assume that p or q is ∞ changing the norm accordingly.

Let \mathcal{V} be the space of infinitely differentiable vector fields u on \mathbb{R}^3 with compact support and satisfying $\nabla \cdot u = 0$. Let us denote by H the closure of \mathcal{V} in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, that is

$$H = \{u \in L^2(\mathbb{R}^3; \mathbb{R}^3) : \nabla \cdot u = 0 \text{ in } \mathbb{R}^3\}.$$

The space H is a separable Hilbert space with the inner product inherited from L^2 , denoted in the sequel by (\cdot, \cdot) with corresponding norm $|\cdot|_{L^2}$.

To ease notations, when no confusion arises let L^p (resp. $L_v^q(L_h^p)$) also denote the set of triples of functions $u = (u_1, u_2, u_3)$ such that each component u_j belongs to L^p (resp. to $L_v^q(L_h^p)$), $j = 1, 2, 3$, that is $u \in L^p(\mathbb{R}^3; \mathbb{R}^3)$ (resp. $u \in L_v^q(L_h^p)(\mathbb{R}^3; \mathbb{R}^3)$). For non negative indices s, s' we set

$$\tilde{H}^{s, s'} := (H^{s, s'})^3 \cap H \quad \text{and again} \quad \|\cdot\|_{s, s'} \quad \text{for the corresponding norm.}$$

We denote by $(\cdot, \cdot)_{0,1}$ the scalar product in the Hilbert space $\tilde{H}^{0,1}$, that is for $u, v \in \tilde{H}^{0,1}$:

$$(u, v)_{0,1} = \sum_{j=1}^3 \int_{\mathbb{R}^3} u_j(x) v_j(x) dx + \sum_{j=1}^3 \int_{\mathbb{R}^3} \partial_3 u_j(x) \partial_3 v_j(x) dx.$$

As defined previously, we set $\Delta_h := \partial_1^2 + \partial_2^2$; integration by parts implies that given $u \in (H^{2,0})^3$ we have

$$(\Delta_h u, u) = - \sum_{j=1}^3 \int_{\mathbb{R}^3} |\nabla_h u_j|_{L^2}^2 dx, \quad \text{where} \quad \nabla_h u_j = (\partial_1 u_j, \partial_2 u_j, u_j).$$

To ease notation, we write $\nabla_h u$ to denote the triple of functions $(\nabla_h u_j, j = 1, 2, 3)$ so that $\langle \Delta_h u, u \rangle = -|\nabla_h u|_{L^2}^2$ for $u \in \tilde{H}^{1,0}$.

Note that as usual, starting with an initial condition $u_0 \in \tilde{H}^{0,1}$ and projecting equation (1.2) on the space of divergence-free fields, we get rid of the pressure and rewrite the evolution equation as follows:

$$\partial_t u - \nu A_h u + B(u, u) + a |u|^{2\alpha} u = \sigma(t, u) \dot{W} \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^3, \quad (2.3)$$

where

$$A_h u = P_{\text{div}} \Delta_h u, \quad B(u, v) = P_{\text{div}}(u \cdot \nabla v), \quad |u|^\alpha u = P_{\text{div}}(|u|^\alpha u), \quad (2.4)$$

and P_{div} denotes the projection on divergence free functions. For $u \in H^1$ such that $\nabla \cdot u = 0$, set

$$B(u) := B(u, u).$$

2.2. Some properties of the non linear terms. In this section, we describe some properties of the non linear terms $B(u) = u \cdot \nabla u$ and $|u|^{2\alpha} u$ in equation (2.3). They will be crucial to obtain apriori estimates and prove global well posedness.

First, for u, v, w in the classical (non isotropic) Sobolev space H^1 such that $\nabla \cdot u = \nabla \cdot v = \nabla \cdot w = 0$, the classical antisymmetry property is satisfied:

$$\langle (B(u, v), w) \rangle = -\langle (B(u, w), v) \rangle, \quad \text{and} \quad \langle (B(u, v), v) \rangle = 0. \quad (2.5)$$

We will prove that under proper assumptions on the initial condition u_0 and on the stochastic forcing term, the solution u to the SPDE (2.3) belongs a.s. to the set X defined by

$$X := L^\infty(0, T; \tilde{H}^{0,1}) \cap L^2(0, T; \tilde{H}^{1,1}) \cap L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3; \mathbb{R}^3) \quad (2.6)$$

and endowed with the norm

$$\|u\|_X := \sum_{j=1}^3 \left[\text{ess sup}_{t \in [0, T]} \|u_j(t)\|_{0,1} + \left(\int_0^T \|u_j(t, \cdot)\|_{1,1}^2 dt \right)^{\frac{1}{2}} + \|u_j\|_{L^{2(\alpha+1)}([0, T] \times \mathbb{R}^3)} \right].$$

For random processes, we set $\Omega_T := \Omega \times (0, T)$ endowed with the product measure $d\mathbb{P} \otimes ds$ on $\mathcal{F} \otimes \mathcal{B}(0, T)$, and

$$\mathcal{X} := L^4(\Omega; L^\infty(0, T; \tilde{H}^{0,1})) \cap L^4(\Omega; (L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3; \mathbb{R}^3)). \quad (2.7)$$

First, let us prove some integral upper estimates of the bilinear term.

Lemma 2.1. *Let $u \in L^\infty(0, T; H) \cap L^2(0, T; \tilde{H}^{1,0})$ and $v \in L^\infty(0, T; H) \cap L^2(0, T; \tilde{H}^{1,1})$. Then*

$$\int_0^T |\langle B(u(t)), v(t) \rangle| dt \leq C \left(\int_0^T \|v(t)\|_{1,1}^2 dt \right)^{\frac{1}{2}} \text{ess sup}_{t \in [0, T]} \|u(t)\|_{L^2} \left(\int_0^T \|\nabla_h u(t)\|_{L^2}^2 dt \right)^{\frac{1}{2}}, \quad (2.8)$$

$$|\langle B(u(t)) - B(v(t)), (u - v)(t) \rangle| \leq C \|v\|_{1,1} \|\nabla_h(u(t) - v(t))\|_{L^2} \|(u - v)(t)\|_{L^2}, \quad (2.9)$$

$$\int_0^T |\langle B(u(t)) - B(v(t)), (u - v)(t) \rangle| dt \leq C \left(\int_0^T \|v(t)\|_{1,1}^2 dt \right)^{\frac{1}{2}} \times \text{ess sup}_{t \in [0, T]} \|(u - v)(t)\|_{L^2} \left(\int_0^T \|\nabla_h((u - v)(t))\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \quad (2.10)$$

Proof. Let us prove some upper estimates of $\langle B(\varphi, \psi), v \rangle$ for $\varphi, \psi \in \tilde{H}^{1,0}$ and $v \in \tilde{H}^{1,1}$. Since $\nabla \cdot \varphi = \nabla \cdot \psi = \nabla \cdot v = 0$, using notations similar to that in [5] and part of the

arguments in this reference used to prove the uniqueness of the solution, the antisymmetry (2.5) of B yields

$$-\langle B(\varphi, \psi), v \rangle = \langle (B(\varphi, v), \psi) \rangle = J_1 + J_2, \quad (2.11)$$

where

$$J_1 := \sum_{k=1}^2 \sum_{l=1}^3 \int_{\mathbb{R}^3} \varphi_k(x) \partial_k v_l(x) \psi_l(x) dx, \quad J_2 := \sum_{l=1}^3 \int_{\mathbb{R}^3} \varphi_3(x) \partial_3 v_l(x) \psi_l(x) dx.$$

The Fubini theorem and Hölder's inequality applied to the Lebesgue integral with respect to dx_h imply that for almost every $t \in [0, T]$:

$$\begin{aligned} |J_1| &\leq \sum_{k=1}^2 \sum_{l=1}^3 \int_{\mathbb{R}} |\partial_k v_l(\cdot, x_3)|_{L_h^2} \|\varphi_k(\cdot, x_3)\|_{L_h^4} \|\psi_l(\cdot, x_3)\|_{L_h^4} dx_3 \\ &\leq \sum_{k=1}^2 \sum_{l=1}^3 \left(\sup_{x_3} |\partial_k v_l(\cdot, x_3)|_{L_h^2} \right) \int_{\mathbb{R}} \|\varphi_k(\cdot, x_3)\|_{L_h^4} \|\psi_l(\cdot, x_3)\|_{L_h^4} dx_3. \end{aligned}$$

The Gagliardo-Nirenberg inequality implies that for almost every $x_3 \in \mathbb{R}$ we have for $\phi = \varphi_k(\cdot, x_3)$ and $\psi = \psi_l(\cdot, x_3)$:

$$\|\phi\|_{L_h^4} \leq C |\nabla_h \phi|_{L_h^2}^{\frac{1}{2}} |\phi|_{L_h^2}^{\frac{1}{2}}. \quad (2.12)$$

On the other hand, for almost every $x_3 \in \mathbb{R}$ the Cauchy-Schwarz inequality for the Lebesgue measure on \mathbb{R}^3 implies for $k = 1, 2$ and $l = 1, 2, 3$:

$$\begin{aligned} |\partial_k v_l(\cdot, x_3)|_{L_h^2}^2 &= \int_{-\infty}^{x_3} \frac{d}{dz} |\partial_k v_l(\cdot, z)|_{L_h^2}^2 dz = 2 \int_{-\infty}^{x_3} \int_{\mathbb{R}^2} \partial_k v_l(x_h, z) \partial_z \partial_k v_l(x_h, z) dx_h dz \\ &\leq C |\nabla_h v|_{L^2} |\partial_3 \nabla_h v|_{L^2} \leq C \|v\|_{1,1}^2. \end{aligned}$$

Therefore, the Hölder inequality with respect to the Lebesgue measure dx_3 implies that

$$\begin{aligned} |J_1| &\leq C \|v\|_{1,1} \left(\int_{\mathbb{R}} |\nabla_h \varphi(\cdot, x_3)|_{L_h^2}^2 dx_3 \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} |\nabla_h \psi(\cdot, x_3)|_{L_h^2}^2 dx_3 \right)^{\frac{1}{4}} \\ &\quad \times \left(\int_{\mathbb{R}} |\varphi(\cdot, x_3)|_{L_h^2}^2 dx_3 \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} |\psi(\cdot, x_3)|_{L_h^2}^2 dx_3 \right)^{\frac{1}{4}} \\ &\leq C \|v\|_{1,1} |\nabla_h \varphi|_{L^2}^{\frac{1}{2}} |\nabla_h \psi|_{L^2}^{\frac{1}{2}} |\varphi|_{L^2}^{\frac{1}{2}} |\psi|_{L^2}^{\frac{1}{2}}. \end{aligned} \quad (2.13)$$

Using once more the Fubini theorem and Hölder's inequality with respect to dx_h we deduce that

$$\begin{aligned} |J_2| &\leq \sum_{l=1}^3 \int_{\mathbb{R}} \|\partial_3 v_l(\cdot, x_3)\|_{L_h^4} |\varphi_3(\cdot, x_3)|_{L_h^2} \|\psi_l(\cdot, x_3)\|_{L_h^4} dx_3 \\ &\leq \sum_{l=1}^3 \left(\sup_{x_3} |\varphi_3(\cdot, x_3)|_{L_h^2} \right) \int_{\mathbb{R}} \|\partial_3 v_l(\cdot, x_3)\|_{L_h^4} \|\psi_l(\cdot, x_3)\|_{L_h^4} dx_3. \end{aligned}$$

Furthermore, since $\nabla \cdot \varphi = 0$, we deduce that $\partial_3 \varphi_3(x_h, x_3) = -\operatorname{div} \varphi_h(x_h, x_3) := -[\partial_1 \varphi_1(x_h, x_3) + \partial_2 \varphi_2(x_h, x_3)]$. Therefore, the Cauchy-Schwarz inequality with respect to the Lebesgue measure on \mathbb{R}^3 yields for almost every $t \in [0, T]$ and $x_3 \in \mathbb{R}$:

$$|\varphi_3(\cdot, x_3)|_{L_h^2}^2 = 2 \int_{-\infty}^{x_3} \int_{\mathbb{R}^2} \varphi_3(x_h, z) \partial_z \varphi_3(x_h, z) dx_h dz$$

$$= -2 \int_{-\infty}^{x_3} \int_{\mathbb{R}^2} \varphi_3(x_h, z) \operatorname{div} \varphi_h(x_h, z) dx_h dz \leq 2 |\nabla_h \varphi|_{L^2} |\varphi|_{L^2}.$$

Plugging the above upper estimate, using again the Gagliardo-Nirenberg inequality (2.12) for $\phi = \partial_3 v_l(\cdot, x_3)$ and $\phi = \psi_l(\cdot, x_3)$, using the Hölder inequality with respect to the Lebesgue measure dx_h we obtain:

$$\begin{aligned} |J_2| &\leq C \sum_{l=1}^3 |\nabla_h \varphi|_{L^2}^{\frac{1}{2}} |\varphi|_{L^2}^{\frac{1}{2}} \int_{\mathbb{R}} |\nabla_h \partial_3 v_l(\cdot, x_3)|_{L_h^2}^{\frac{1}{2}} |\partial_3 v_l(\cdot, x_3)|_{L_h^2}^{\frac{1}{2}} \\ &\quad \times |\nabla_h \psi_l(t, \cdot, x_3)|_{L_h^2}^{\frac{1}{2}} |\psi_l(t, \cdot, x_3)|_{L_h^2}^{\frac{1}{2}} dx_3 \\ &\leq C |\nabla_h \varphi|_{L^2}^{\frac{1}{2}} |\varphi|_{L^2}^{\frac{1}{2}} |\nabla_h \partial_3 v|_{L^2}^{\frac{1}{2}} |\partial_3 v|_{L^2}^{\frac{1}{2}} |\nabla_h \psi|_{L^2}^{\frac{1}{2}} |\psi|_{L^2}^{\frac{1}{2}} \\ &\leq C \|v\|_{1,1} |\nabla_h \varphi|_{L^2}^{\frac{1}{2}} |\nabla_h \psi|_{L^2}^{\frac{1}{2}} |\varphi|_{L^2}^{\frac{1}{2}} |\psi|_{L^2}^{\frac{1}{2}}. \end{aligned} \quad (2.14)$$

The upper estimates (2.11), (2.13) and (2.14) imply the existence of a positive constant C such that

$$|\langle B(\varphi, \psi), v \rangle| \leq C \|v\|_{1,1} |\nabla_h \varphi|_{L^2}^{\frac{1}{2}} |\nabla_h \psi|_{L^2}^{\frac{1}{2}} |\varphi|_{L^2}^{\frac{1}{2}} |\psi|_{L^2}^{\frac{1}{2}}. \quad (2.15)$$

Let $u \in L^\infty(0, T; H) \cap L^2(0, T; \tilde{H}^{1,0})$ and $v \in L^\infty(0, T; H) \cap L^2(0, T; \tilde{H}^{1,1})$. Since for almost every $t \in [0, T]$ we have $u(t, \cdot) \in \tilde{H}^{0,1}$ and $v(t, \cdot) \in \tilde{H}^{1,1}$, using (2.15) for $\varphi = \psi = u(t)$ and Hölder's inequality with respect to the Lebesgue measure on $[0, T]$, we obtain

$$\begin{aligned} \int_0^T |\langle B(u(t)), v(t) \rangle| dt &\leq C \|v\|_{L^2(0, T; \tilde{H}^{1,1})} \left(\int_0^T |\nabla_h u(t, \cdot)|_{L^2}^2 |u(t, \cdot)|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\leq C \|v\|_{L^2(0, T; \tilde{H}^{1,1})} \operatorname{ess\,sup}_{t \in [0, T]} |u(t)|_{L^2} \left(\int_0^T |\nabla_h u(t)|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (2.16)$$

This concludes the proof of (2.8).

Expanding $B(u(t)) - B(v(t))$ and using the antisymmetry property (2.5) we deduce that

$$\langle B(u(t, \cdot)) - B(v(t, \cdot)), (u - v)(t, \cdot) \rangle = \langle B((u - v)(t, \cdot)), v(t, \cdot) \rangle, (u - v)(t, \cdot) \rangle.$$

Using once more the antisymmetry and the upper estimate (2.15) with $\varphi = \psi = (u - v)(t)$, we conclude the proof of (2.9). Integrating (2.9) on $[0, T]$ and using the Cauchy Schwarz inequality, we deduce (2.10). \square

Using Hölder's inequality with respect to the expected value in the upper estimates of Lemma 2.1, we deduce the following analog for stochastic processes.

Lemma 2.2. *Let $u \in L^4(\Omega; L^\infty(0, T; H)) \cap L^4(\Omega; L^2(0, T; \tilde{H}^{1,0}))$ and $v \in L^4(\Omega; L^\infty(0, T; H)) \cap L^4(\Omega; L^2(0, T; \tilde{H}^{1,1}))$. Then*

$$\begin{aligned} \mathbb{E} \int_0^T |\langle B(u(t)), v(t) \rangle| dt &\leq C \|v\|_{L^4(\Omega; L^2(0, T; \tilde{H}^{1,1}))} \\ &\quad \times \|u\|_{L^4(\Omega; L^\infty(0, T; H))} \left(\mathbb{E} \left[\int_0^T |\nabla_h u(t)|_{L^2}^2 dt \right]^2 \right)^{\frac{1}{4}}. \end{aligned} \quad (2.17)$$

$$\mathbb{E} \int_0^T |\langle B(u(t)) - B(v(t)), (u - v)(t) \rangle| dt \leq C \|v\|_{L^4(\Omega; L^2(0, T; \tilde{H}^{1,1}))}$$

$$\times \|u - v\|_{L^4(\Omega; L^\infty(0, T; H))} \left(\mathbb{E} \left[\int_0^T |\nabla_h(u - v)(t)|_{L^2}^2 dt \right]^2 \right)^{\frac{1}{4}}. \quad (2.18)$$

The following lemma proves upper estimates for the third partial derivatives of the bilinear term; it is essentially contained in [5]. This results shows the crucial role of the other non linear term $|u|^{2\alpha}u$ of (2.3) in the control of the partial derivative ∂_3 of the bilinear term.

Lemma 2.3. *There exists a positive constant C such that for any $\alpha \in (1, \infty)$ there exists $C_\alpha > 0$, $\epsilon_0, \epsilon_1 > 0$, $s \in [0, T]$ and $u \in X$:*

$$\begin{aligned} \left| \langle \partial_3 B(u(s)), \partial_3 u(s) \rangle \right| &\leq C \left[\epsilon_0 |\nabla_h \partial_3 u(s)|_{L^2}^2 + \frac{\epsilon_1}{4\epsilon_0} |u(s)|^\alpha |\partial_3 u(s)|_{L^2}^2 \right. \\ &\quad \left. + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} |\partial_3 u(s)|_{L^2}^2 \right]. \end{aligned} \quad (2.19)$$

Proof. We briefly sketch the proof in order to be self contained. Since $\operatorname{div}_h \partial_3 u(s) = \partial_3 \operatorname{div}_h u(s)$, the antisymmetry (2.5) yields $\langle B(u(s), \partial_3 u(s)), \partial_3 u(s) \rangle = 0$; hence for $s \in [0, T]$:

$$\langle \partial_3 B(u(s)), \partial_3 u(s) \rangle = \sum_{k,l=1}^3 \int_{\mathbb{R}^3} \partial_3 u_k(s, x) \partial_k u_l(s, x) \partial_3 u_l(s, x) dx := \bar{J}_1(s) + \bar{J}_2(s),$$

where integration by parts with respect to ∂_k , $k = 1, 2$ yields

$$\begin{aligned} \bar{J}_1(s) &= - \sum_{k=1}^2 \sum_{l=1}^3 \int_{\mathbb{R}^3} \partial_k \partial_3 u_k(s, x) u_l(s, x) \partial_3 u_l(s, x) dx \\ &\quad - \sum_{k=1}^2 \sum_{l=1}^3 \int_{\mathbb{R}^3} \partial_3 u_k(s, x) u_l(s, x) \partial_k \partial_3 u_l(s, x) dx, \\ \bar{J}_2(s) &= \sum_{l=1}^3 \int_{\mathbb{R}^3} \partial_3 u_3(s, x) (\partial_3 u_l(s, x))^2 dx = - \sum_{l=1}^3 \int_{\mathbb{R}^3} \operatorname{div}_h u_h(s, x) (\partial_3 u_l(s, x))^2 dx; \end{aligned}$$

the last identity comes from the fact that $\nabla \cdot u(s) = 0$. Since $\alpha > 1$, the Hölder and Young inequalities imply that for functions $f, g, h : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\epsilon_0 > 0$ and then $\epsilon_1 > 0$, we have for some $C_\alpha > 0$:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} f(x) g(x) h(x) dx \right| &\leq \| |f| |g|^{\frac{1}{\alpha}} \|_{L^{2\alpha}} \| |g|^{1-\frac{1}{\alpha}} \|_{L^{\frac{2\alpha}{\alpha-1}}} \| h \|_{L^2} \\ &\leq \epsilon_0 \| h \|_{L^2}^2 + \frac{\epsilon_1}{4\epsilon_0} \| |f|^\alpha |g|^2 \|_{L^2} + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} \| |g|^2 \|_{L^2}. \end{aligned} \quad (2.20)$$

Using this inequality for $f = u_l(s)$, $g = \partial_3 u_l(s)$ and $h = \partial_k \partial_3 u_k(s)$ (resp. $g = \partial_3 u_k(s)$, $h = \partial_k \partial_3 u_l(s)$) we deduce the existence of $C > 0$ such that for any $\alpha > 1$, $\epsilon_0, \epsilon_1 > 0$ and some constant $C_\alpha > 0$:

$$|\bar{J}_1(s)| \leq C \left[\epsilon_0 |\nabla_h \partial_3 u(s)|_{L^2}^2 + \frac{\epsilon_1}{4\epsilon_0} |u(s)|^\alpha |\partial_3 u(s)|_{L^2}^2 + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} |\partial_3 u(s)|_{L^2}^2 \right].$$

Integration by parts implies that $\bar{J}_2(s) = 2 \sum_{k=1}^2 \sum_{l=1}^3 \int_{\mathbb{R}^3} u_k(s, x) \partial_k \partial_3 u_l(s, x) \partial_3 u_l(s, x) dx$. Using (2.20) with $f = u_k(s)$, $g = \partial_3 u_l(s)$ and $h = \partial_k \partial_3 u_l(s)$, we deduce the existence of $C > 0$ such that for any $\alpha > 1$, $\epsilon_0, \epsilon_1 > 0$ and $C_\alpha > 0$:

$$|\bar{J}_2(s)| \leq C \left[\epsilon_0 |\nabla_h \partial_3 u(s)|_{L^2}^2 + \frac{\epsilon_1}{4\epsilon_0} |u(s)|^\alpha |\partial_3 u(s)|_{L^2}^2 + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} |\partial_3 u(s)|_{L^2}^2 \right].$$

The upper estimates of $\bar{J}_1(s)$ and $\bar{J}_2(s)$ conclude the proof. \square

For any regular enough function $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, let $F(\varphi)$ be the function defined by

$$F(\varphi) = \nu \Delta_h \varphi - B(\varphi) - a |\varphi|^{2\alpha} \varphi. \quad (2.21)$$

The following lemma proves that for $u \in X$ (resp. $u \in \mathcal{X}$), $F(u)$ belongs to the dual space of $L^2(0, T; \tilde{H}^{1,1}) \cap L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3)$ (resp. to the dual space of $L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)$).

Lemma 2.4. (i) Let $u \in X$ and $v \in L^2(0, T; \tilde{H}^{1,1}) \cap L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3; \mathbb{R}^3)$; then

$$\begin{aligned} \int_0^T |\langle F(u(t, \cdot)), v(t, \cdot) \rangle| dt &\leq C \left[\|v\|_{L^2(0, T; \tilde{H}^{1,0})} \|u\|_{L^2(0, T; \tilde{H}^{1,0})} + \|v\|_{L^{2(\alpha+1)}([0, T] \times \mathbb{R}^3)} \right. \\ &\quad \left. \times \|u\|_{L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3)}^{2\alpha+1} + \|v\|_{L^2(0, T; \tilde{H}^{1,1})} \sup_{t \in [0, T]} \|u(t)\|_{L^2} \left(\int_0^T |\nabla_h u(t)|_{L^2}^2 dt \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (2.22)$$

(ii) Let $u \in \mathcal{X}$ and $v \in L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)$. Then

$$\begin{aligned} \mathbb{E} \int_0^T |\langle F(u(t, \cdot)), v(t, \cdot) \rangle| dt &\leq C \left[\|v\|_{L^2(\Omega_T; \tilde{H}^{1,0})} \|u\|_{L^2(\Omega_T; \tilde{H}^{1,0})} + \|v\|_{L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)} \right. \\ &\quad \left. \times \|u\|_{L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)}^{2\alpha+1} + \|v\|_{L^4(\Omega; L^2(0, T; \tilde{H}^{1,1}))} \|u\|_{L^4(\Omega; L^\infty(0, T; H))} \|u\|_{L^4(\Omega; L^2(0, T; \tilde{H}^{1,0}))} \right]. \end{aligned} \quad (2.23)$$

Proof. (i) Integration by parts and the Cauchy-Schwarz inequality with respect to $dt \otimes dx$ yield

$$\begin{aligned} \left| \nu \int_0^T \langle \Delta_h u(t, \cdot), v(t, \cdot) \rangle dt \right| &= \left| -\nu \int_0^T \int_{\mathbb{R}^3} \nabla_h u(t, x) \nabla_h v(t, x) dx dt \right| \\ &\leq \nu \|u\|_{L^2(0, T; \tilde{H}^{1,0})} \|v\|_{L^2(0, T; \tilde{H}^{1,0})}. \end{aligned} \quad (2.24)$$

Note that $2\alpha+2$ and $\frac{2\alpha+2}{2\alpha+1}$ are conjugate Hölder exponents. Since $u \in L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3)$, the function $|u|^{2\alpha} u$ belongs to $L^{\frac{2(\alpha+1)}{2\alpha+1}}((0, T) \times \mathbb{R}^3)$ and

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^3} |u(t, x)|^{2\alpha} u(t, x) v(t, x) dx dt \right| &\leq \| |u|^{2\alpha} u \|_{L^{\frac{2(\alpha+1)}{2\alpha+1}}((0, T) \times \mathbb{R}^3)} \|v\|_{L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3)} \\ &\leq \|u\|_{L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3)}^{2\alpha+1} \|v\|_{L^{2(\alpha+1)}((0, T) \times \mathbb{R}^3)}. \end{aligned} \quad (2.25)$$

The inequalities (2.24), (2.8) and (2.25) conclude the proof of (2.22).

(ii) Let $u \in \mathcal{X}$ and $v \in L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)$. Then a.s. we may apply part (i) to $u(t)(\omega)$ and $v(t)(\omega)$. The Cauchy Schwarz and Hölder inequalities with respect to the expectation conclude the proof. \square

To prove uniqueness of the solution, we will need the following lemma which provides an upper estimate of $\langle F(u(t, \cdot)) - F(v(t, \cdot)), u(t, \cdot) - v(t, \cdot) \rangle$ for $u, v \in X$ and $t \in [0, T]$.

Lemma 2.5. There exists a positive constant κ depending on α , and for any $\eta \in (0, \nu)$ a positive constant C_η such that for $u, v \in \tilde{H}^{1,1} \cap L^{2(\alpha+1)}(\mathbb{R}^3)$;

$$\begin{aligned} \langle F(u) - F(v), u - v \rangle &\leq -\eta |\nabla_h(u - v)|_{L^2}^2 \\ &\quad + C_\eta \|v\|_{1,1}^2 \|u - v\|_{L^2}^2 - a \kappa (|u| + |v|)^\alpha (u - v)|_{L^2}^2. \end{aligned} \quad (2.26)$$

Proof. Integration by parts implies that

$$\nu \langle \Delta_h(u - v), u - v \rangle = -\nu |\nabla_h(u - v)|_{L^2}^2. \quad (2.27)$$

It is well-known (see [2]; see also [19] where it is used) that there exists a constant κ depending on α such that

$$\begin{aligned} \kappa |u(x) - v(x)|^2 (|u(x)| + |v(x)|)^{2\alpha} \\ \leq (|u(x)|^{2\alpha} u(x) - |v(x)|^{2\alpha} v(x)) \cdot (u(x) - v(x)), \end{aligned}$$

which clearly implies:

$$\begin{aligned} a \int_{\mathbb{R}^3} (|u(x)|^{2\alpha} u(x) - |v(x)|^{2\alpha} v(x)) \cdot (u(x) - v(x)) dx \\ \geq a \kappa \left(|u| + |v| \right)^\alpha (u - v) \Big|_{L^2}^2. \end{aligned} \quad (2.28)$$

Using Young's inequality in (2.9) we deduce that for any $\eta \in (0, \nu)$ there exists $C_\eta > 0$ such that

$$|\langle B(u) - B(v), u - v \rangle| \leq (\nu - \eta) |\nabla_h(u - v)|_{L^2}^2 + C_\eta \|v\|_{1,1}^2 |(u - v)|_{L^2}^2.$$

This upper estimate, (2.27) and (2.28) conclude the proof of (2.26). \square

2.3. The stochastic perturbation. We will consider an external random force in equation (2.3) driven by a Wiener process W and whose intensity may depend on the solution u .

More precisely, let $(e_k, k \geq 1)$ be an orthonormal basis of H whose elements belong to $H^2 := W^{2,2}(\mathbb{R}^3; \mathbb{R}^3)$ and are orthogonal in $\tilde{H}^{0,1}$. For integers $k, l \geq 1$ with $k \neq l$, we deduce that

$$(\partial_3^2 e_k, e_l) = -(\partial_3 e_k, \partial_3 e_l) = -(e_k, e_l)_{0,1} - (e_k, e_l) = 0.$$

Therefore, $\partial_3^2 e_k$ is a constant multiple of e_k . Let $\mathcal{H}_n = \text{span}(e_1, \dots, e_n)$ and let P_n (resp. \tilde{P}_n) denote the orthogonal projection from H (resp. $\tilde{H}^{0,1}$) to \mathcal{H}_n . We deduce that for $u \in \tilde{H}^{0,1}$ we have $P_n u = \tilde{P}_n u$. Indeed, for $v \in \mathcal{H}_n$, we have $\partial_3^2 v \in \mathcal{H}_n$ and for any $u \in \tilde{H}^{0,1}$:

$$(P_n u, v) = (u, v), \quad \text{and} \quad (\partial_3 P_n u, \partial_3 v) = -(P_n u, \partial_3^2 v) = -(u, \partial_3^2 v) = (\partial_3 u, \partial_3 v).$$

Hence given $u \in \tilde{H}^{0,1}$, we have $(P_n u, v)_{0,1} = (u, v)_{0,1}$ for any $v \in \mathcal{H}_n$; this proves that P_n and \tilde{P}_n coincide on $\tilde{H}^{0,1}$.

Let $(W(t), t \geq 0)$ be a $\tilde{H}^{0,1}$ -valued Wiener process with covariance operator Q on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$; that is Q is a positive operator from $\tilde{H}^{0,1}$ to itself which is trace class, and hence compact. Let $(q_k, k \geq 1)$ be the set of eigenvalues of Q with $\sum_{k \geq 1} q_k < \infty$, and let $(\psi_k, k \geq 1)$ denote the corresponding eigenfunctions (that is $Q\psi_k = q_k \psi_k$). The process W is Gaussian, has independent time increments, and for $s, t \geq 0, f, g \in \tilde{H}_{0,1}$,

$$\mathbb{E}[(W(s), f)_{0,1}] = 0 \quad \text{and} \quad \mathbb{E}[(W(s), f)_{0,1} (W(t), g)_{0,1}] = (s \wedge t) (Qf, g)_{0,1}.$$

We also have the following representation

$$W(t) = \lim_{n \rightarrow \infty} W_n(t) \quad \text{in} \quad L^2(\Omega; \tilde{H}^{0,1}) \quad \text{with} \quad W_n(t) = \sum_{k=1}^n q_k^{1/2} \beta_k(t) \psi_k, \quad (2.29)$$

where β_k are standard (scalar) mutually independent Wiener processes and ψ_k are the above eigenfunctions of Q . For details concerning this Wiener process we refer to [14].

Let $H_0 = Q^{\frac{1}{2}}\tilde{H}^{0,1}$; then H_0 is a Hilbert space with the scalar product

$$(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi)_{0,1}, \quad \forall \phi, \psi \in H_0,$$

together with the induced norm $|\cdot|_0 = \sqrt{(\cdot, \cdot)_0}$. The embedding $i : H_0 \rightarrow \tilde{H}^{0,1}$ is Hilbert-Schmidt and hence compact; moreover, $i i^* = Q$.

Let $\mathcal{L} \equiv L^{(2)}(H_0, H)$ (resp. $\tilde{\mathcal{L}} \equiv L^{(2)}(H_0, \tilde{H}^{0,1})$) be the space of linear operators $S : H_0 \mapsto H$ (resp. $S : H_0 \mapsto \tilde{H}^{0,1}$) such that $SQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from $\tilde{H}^{0,1}$ to H (resp. from $\tilde{H}^{0,1}$ to itself). Clearly, $\tilde{\mathcal{L}} \subset \mathcal{L}$. Set

$$|S|_{\mathcal{L}}^2 = \text{trace}_H([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k=1}^{\infty} \|SQ^{1/2}\phi_k\|_{L^2}^2, \quad (2.30)$$

$$|S|_{\tilde{\mathcal{L}}}^2 = \text{trace}_{\tilde{H}^{0,1}}([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k=1}^{\infty} |SQ^{1/2}\phi_k|_{0,1}^2 \quad (2.31)$$

for any orthonormal basis $\{\phi_k\}$ in $\tilde{H}^{0,1}$. Let $(\cdot, \cdot)_{\mathcal{L}}$ and $(\cdot, \cdot)_{\tilde{\mathcal{L}}}$ denote the associated scalar products.

The noise intensity of the stochastic perturbation $\sigma : [0, T] \times \tilde{H}^{1,1} \rightarrow \tilde{\mathcal{L}}$ which we put in (2.3) satisfies the following classical growth and Lipschitz conditions (i) and (ii). Note that due to the anisotropic feature of our model, we have to impose growth conditions both for the $|\cdot|_{\mathcal{L}}$ and $|\cdot|_{\tilde{\mathcal{L}}}$ norms.

Condition (C): The diffusion coefficient $\sigma \in C([0, T] \times \tilde{H}^{1,1}; \tilde{\mathcal{L}})$ is a linear operator such that:

(i) **Growth condition** There exist non negative constants K_i and \tilde{K}_i such that for every $t \in [0, T]$ and $u \in \tilde{H}^{1,1}$:

$$|\sigma(t, u)|_{\mathcal{L}}^2 \leq K_0 + K_1|u|_{L^2}^2 + K_2|\nabla_h u|_{L^2}^2, \quad (2.32)$$

$$|\sigma(t, u)|_{\tilde{\mathcal{L}}}^2 \leq \tilde{K}_0 + \tilde{K}_1\|u\|_{0,1}^2 + \tilde{K}_2(|\nabla_h u|_{L^2}^2 + |\partial_3 \nabla_h u|_{L^2}^2). \quad (2.33)$$

(ii) **Lipschitz condition** There exists constants L_1 and L_2 such that:

$$|\sigma(t, u) - \sigma(t, v)|_{\mathcal{L}}^2 \leq L_1|u - v|_{L^2}^2 + L_2|\nabla_h(u - v)|_{L^2}^2, \quad t \in [0, T] \text{ and } u, v \in \tilde{H}^{1,1}.$$

Definition 2.6. An (\mathcal{F}_t) -predictable stochastic process $u(t, \omega)$ is called a weak solution in $C([0, T]; H) \cap X$ for the stochastic equation (2.3) on $[0, T]$ with initial condition u_0 if $u \in C([0, T]; H) \cap X$ a.s., where X is defined in (2.6), and u satisfies

$$\begin{aligned} (u(t), v) - (u_0, v) + \int_0^t \left[-\nu \langle u(s), \Delta_h v \rangle - \langle B(u(s), v), u(s) \rangle \right] ds \\ + a \int_0^t \int_{\mathbb{R}^3} |u(s, x)|^{2\alpha} u(s, x) v(x) dx ds = \int_0^t (\sigma(s, u(s)) dW(s), v), \quad \text{a.s.}, \end{aligned}$$

for every test function $v \in H^2(\mathbb{R}^3)$ and all $t \in [0, T]$. All terms are well defined since $u \in L^{2(\alpha+1)}([0, T] \times \mathbb{R}^3)$ for almost every $s \in [0, T]$; this implies $|u(s)|^{2\alpha} u(s) \in L^{\frac{2(\alpha+1)}{2\alpha+1}}(\mathbb{R}^3)$ which is the dual space of $L^{2(\alpha+1)}(\mathbb{R}^3)$.

Furthermore the Gagliardo-Nirenberg inequality implies $\text{Dom}(-\Delta) \subset L^p(\mathbb{R}^3)$ for any $p \in [2, \infty)$. Note that this solution is a strong one in the probabilistic meaning, that is the trajectories of u are written in terms of stochastic integrals with respect to the given Brownian motion W .

3. EXISTENCE AND UNIQUENESS OF GLOBAL SOLUTIONS

The aim of this section is to prove that equation (2.3) has a unique solution in \mathcal{X} defined in (2.7). We at first prove local well posedness of a Galerkin approximation of u and apriori estimates.

3.1. Galerkin approximation and apriori estimates. Let $(e_n, n \geq 1)$ be the orthonormal basis of H defined in section 2.3 (that is made of functions in H^2 which are also orthogonal in $\tilde{H}^{0,1}$). Recall that for every integer $n \geq 1$ we set $\mathcal{H}_n := \text{span}(e_1, \dots, e_n)$ and that the orthogonal projection P_n from H to \mathcal{H}_n restricted to $\tilde{H}^{0,1}$ coincides with the orthogonal projection from $\tilde{H}^{0,1}$ to \mathcal{H}_n .

Let Π_n denote the projection in H_0 on $Q^{1/2}(\mathcal{H}_n)$. Let $W_n(t) = \sum_{j=1}^n \sqrt{q_j} \psi_j \beta_j(t) = \Pi_n W(t)$ be defined by (2.29).

Fix $n \geq 1$ and consider the following stochastic ordinary differential equation on the n -dimensional space \mathcal{H}_n defined by $u_n(0) = P_n u_0$, and for $t \in [0, T]$ and $v \in \mathcal{H}_n$:

$$d(u_n(t), v) = \langle F(u_n(t)), v \rangle dt + (P_n \sigma(t, u_n(t)) \Pi_n dW(t), v). \quad (3.1)$$

Then for $k = 1, \dots, n$ we have for $t \in [0, T]$:

$$d(u_n(t), e_k) = \langle F(u_n(t)), e_k \rangle dt + \sum_{j=1}^n q_j^{\frac{1}{2}} (P_n \sigma(t, u_n(t)) \psi_j, e_k) d\beta_j(t).$$

Note that for $v \in \mathcal{H}_n$ the map $u \in \mathcal{H}_n \mapsto \langle F(u), v \rangle$ is locally Lipschitz. Indeed, $H^2 \subset L^{2\alpha+2}$ and there exists some constant $C(n)$ such that $\|v\|_{H^2} \leq C(n) \|v\|_{L^2}$ for $v \in \mathcal{H}_n$. Let $\varphi, \psi, v \in \mathcal{H}_n$; integration by parts implies that

$$|\langle \Delta_h \varphi - \Delta_h \psi, v \rangle| \leq \|\varphi - \psi\|_{1,0} \|v\|_{1,0} \leq C(n)^2 \|\varphi - \psi\|_{L^2} \|v\|_{L^2}.$$

In the polynomial nonlinear term, the Hölder and Gagliardo-Nirenberg inequalities imply:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (|\varphi(x)|^{2\alpha} \varphi(x) - |\psi(x)|^{2\alpha} \psi(x)) v(x) dx \right| \\ \leq C(\alpha) (\|\varphi\|_{L^{2\alpha+2}}^{2\alpha} + \|\psi\|_{L^{2\alpha+2}}^{2\alpha}) \|\varphi - \psi\|_{L^{2\alpha+2}} \|v\|_{L^{2\alpha+2}} \\ \leq C(\alpha) C(n)^{2(\alpha+1)} (\|\varphi\|_{L^2}^{2\alpha} + \|\psi\|_{L^2}^{2\alpha}) \|\varphi - \psi\|_{L^2} \|v\|_{L^2}. \end{aligned}$$

Finally, using (2.15) and integration by parts we deduce:

$$\begin{aligned} |\langle B(\varphi) - B(\psi), v \rangle| &= |-\langle B(\varphi - \psi), v \rangle, \varphi\rangle - \langle B(\psi), v \rangle, \varphi - \psi\rangle| \\ &\leq C \|\varphi - \psi\|_{1,0} (\|\varphi\|_{1,0} + \|\psi\|_{1,0}) \|v\|_{1,1} \\ &\leq CC(n)^3 \|\varphi - \psi\|_{L^2} (\|\varphi\|_{L^2} + \|\psi\|_{L^2}) \|v\|_{L^2}. \end{aligned}$$

Condition (C) implies that the map $u \in \mathcal{H}_n \mapsto (\sqrt{q_j} (\sigma(t, u) \psi_j, e_k) : 1 \leq j, k \leq n)$ satisfies the classical global linear growth and Lipschitz conditions from \mathcal{H}_n to $n \times n$ matrices uniformly in $t \in [0, T]$; indeed, the growth and Lipschitz conditions (2.32) and (C)(ii) imply:

$$\begin{aligned} |(\sigma(t, u) \sqrt{q_j} \psi_j, e_k)| &\leq |\sigma(t, u) \sqrt{q_j} \psi_j|_H |e_k|_{L^2} \leq \sqrt{K_0} + \sqrt{K_1} |u|_{L^2} + \sqrt{K_2} |\nabla_h u|_{L^2} \\ &\leq C(n) (1 + |u|_{L^2}), \\ |([\sigma(t, u) - \sigma(t, v)] \sqrt{q_j} \psi_j, e_k)| &\leq \sqrt{L_1} |u - v|_{L^2} + \sqrt{L_2} |\nabla_h(u - v)|_{L^2}^2 \leq CC(n) |u - v|_{L^2}. \end{aligned}$$

Hence by a well-known result about existence and uniqueness of solutions to stochastic differential equations (see e.g. [18]), there exists a maximal solution $u_n = \sum_{k=1}^n (u_n, e_k) e_k \in$

\mathcal{H}_n to (3.1), i.e., a stopping time $\tau_n^* \leq T$ such that (3.1) holds for $t < \tau_n^*$ and as $t \uparrow \tau_n^* < T$, $\|u_n(t)\|_{L^2} \rightarrow \infty$.

The following proposition shows that $\tau_n^* = T$ a.s., that is provides the (global) existence and uniqueness of the finite dimensional approximations u_n . It also gives apriori estimates of u_n which do not depend on n ; this will be crucial to prove well posedness of (2.3).

Proposition 3.1. *Let u_0 be a \mathcal{F}_0 measurable random variable such that $\mathbb{E}\|u_0\|_{0,1}^4 < \infty$, $T > 0$ and σ satisfy condition **(C)** with $\tilde{K}_2 < \frac{2\nu}{21}$. Then (3.1) has a unique global solution (i.e., $\tau_n^* = T$ a.s.) with a modification $u_n \in C([0, T], \mathcal{H}_n)$. Furthermore, there exists a constant $C > 0$ such that:*

$$\sup_n \mathbb{E} \left[\sup_{t \in [0, T]} \|u_n(t)\|_{0,1}^4 + \left(\int_0^T \|u_n(s)\|_{1,1}^2 ds \right)^2 + \int_0^T \|u_n(s)\|_{L^{2(\alpha+1)}}^{2(\alpha+1)} ds \right] \leq C(\mathbb{E}\|u_0\|_{0,1}^4 + 1). \quad (3.2)$$

Proof. Let $u_n(t)$ be the maximal solution to (3.1) described above. For every $N > 0$, set

$$\tau_N = \inf\{t : \|u_n(t)\|_{0,1} \geq N\} \wedge \tau_n^*.$$

Itô's formula applied to $\|\cdot\|_{0,1}$ and the antisymmetry relation (2.5) of the bilinear term yield that for $t \in [0, T]$:

$$\begin{aligned} \|u_n(t \wedge \tau_N)\|_{0,1}^2 &= \|P_n u_0\|_{0,1}^2 - 2\nu \int_0^{t \wedge \tau_N} |\nabla_h u_n(s)|_{L^2}^2 ds - 2\nu \int_0^{t \wedge \tau_N} |\nabla_h \partial_3 u_n(s)|_{L^2}^2 ds \\ &\quad - 2a \int_0^{t \wedge \tau_N} \|u_n(s)\|_{L^{2\alpha+2}}^{2\alpha+2} ds - 2a(2\alpha+1) \int_0^{t \wedge \tau_N} \int_{\mathbb{R}^3} |u_n(s, x)|^{2\alpha} |\partial_3 u_n(s, x)|^2 ds + \sum_{j=1}^3 T_j(t), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} T_1(t) &= -2 \int_0^{t \wedge \tau_N} \langle \partial_3 B(u_n(s)), \partial_3 u_n(s) \rangle ds, \\ T_2(t) &= 2 \int_0^{t \wedge \tau_N} (\sigma(s, u_n(s)) dW_n(s), u_n(s))_{0,1}, \\ T_3(t) &= \int_0^{t \wedge \tau_N} |P_n \sigma(s, u_n(s)) \Pi_n|_{\tilde{\mathcal{L}}}^2 ds. \end{aligned}$$

The growth condition (2.33) implies that

$$T_3(t) \leq \int_0^{t \wedge \tau_N} [\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{0,1}^2 + \tilde{K}_2 (|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2)] ds,$$

while (2.19) in Lemma 2.3 yields the existence of positive constants C, C_α, ϵ_0 and ϵ_1 such that

$$\begin{aligned} |T_1(t)| &\leq 2C \left[\epsilon_0 \int_0^{t \wedge \tau_N} |\nabla_h \partial_3 u_n(s)|_{L^2}^2 ds + \frac{\epsilon_1}{4\epsilon_0} \int_0^{t \wedge \tau_N} \|u_n(s)\|_{0,1}^\alpha |\partial_3 u_n(s)|_{L^2}^2 ds \right. \\ &\quad \left. + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} \int_0^{t \wedge \tau_N} |\partial_3 u_n(s)|_{L^2}^2 ds \right]. \end{aligned}$$

Finally, the Burkholder-Davies-Gundy and Young inequalities as well as (2.33) imply that for $\beta \in (0, 1)$:

$$\mathbb{E} \left(\sup_{s \leq t} \left| 2 \int_0^{s \wedge \tau_N} (\sigma(r, u_n(r)) dW_n(r), u_n(r))_{0,1} \right| \right) \leq 6\mathbb{E} \left\{ \int_0^{t \wedge \tau_N} |P_n \sigma(r, u_n(r)) \Pi_n|_{\tilde{\mathcal{L}}}^2 \|u_n(r)\|_{0,1}^2 dr \right\}^{\frac{1}{2}}$$

$$\begin{aligned} &\leq \beta \mathbb{E} \left(\sup_{s \leq \inf t \wedge \tau_N} \|u_n(s)\|_{0,1}^2 \right) \\ &\quad + \frac{9}{\beta} \mathbb{E} \int_0^{t \wedge \tau_N} [\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{0,1}^2 + \tilde{K}_2 (|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2)] ds. \end{aligned}$$

If $\tilde{K}_2 < \frac{\nu}{5}$ and $\epsilon \in (0, 2\nu - 10\tilde{K}_2)$, we may choose $\beta \in (0, 1)$ such that $2\nu - (\frac{9}{\beta} + 1)\tilde{K}_2 > \epsilon$, then $\epsilon_0 > 0$ such that $2C\epsilon_0 < \frac{\epsilon}{2}$, and finally $\epsilon_1 > 0$ such that $2a(2\alpha + 1) - \frac{\epsilon_1 C}{2\epsilon_0} > \epsilon$. For this choice of constants, the inequality $\|P_n u_0\|_{0,1} \leq \|u_0\|_{0,1}$ and the above upper estimates yield (neglecting some non negative terms in the left hand side of (3.3)):

$$\begin{aligned} (1 - \beta) \mathbb{E} \left(\sup_{s \in [0, t]} \|u_n(s \wedge \tau_N)\|_{0,1}^2 \right) &\leq \mathbb{E} \|u_0\|_{0,1}^2 + T\tilde{K}_0 \left(\frac{9}{\beta} + 1 \right) \\ &\quad + \left[\tilde{K}_1 \left(\frac{9}{\beta} + 1 \right) + \frac{2CC_\alpha}{\epsilon_0 \epsilon_1^{1/(\alpha-1)}} \right] \mathbb{E} \int_0^t \|u_n(s \wedge \tau_N)\|_{0,1}^2 ds. \end{aligned} \quad (3.4)$$

Gronwall's lemma implies that $\mathbb{E}(\sup_{s \in [0, T]} \|u_n(s \wedge \tau_N)\|_{0,1}^2) \leq C$ for some constant C which does not depend on n and N . Note that $\|\phi\|_{1,1}^2 = \|\phi\|_{0,1}^2 + |\nabla_h \phi|_{L^2}^2 + |\partial_3 \nabla_h \phi|_{L^2}^2$. We use (3.4) and the upper estimates of $T_i(t)$ for $i = 1, 2, 3$ for the same choice of constants β, ϵ_0 and ϵ_1 ; this yields

$$\mathbb{E} \left(\sup_{s \in [0, T]} \|u_n(s \wedge \tau_N)\|_{0,1}^2 \right) + \mathbb{E} \int_0^{\tau_N} (\|u_n(s)\|_{1,1}^2 + \|u_n(s)\|_{L^{2\alpha+2}}^{2\alpha+2}) ds \leq C(1 + \mathbb{E} \|u_0\|_{0,1}^2) \quad (3.5)$$

for some positive constant C depending on $\tilde{K}_i, i = 0, 1, 2, \beta, \epsilon_0$ and ϵ_1 but independent of n and N .

Apply once more the Itô formula to the square of $\|\cdot\|_{0,1}^2$. This yields

$$\begin{aligned} \|u_n(t \wedge \tau_N)\|_{0,1}^4 &= \|P_n u_0\|_{0,1}^4 - 4\nu \int_0^{t \wedge \tau_N} \|u_n(s)\|_{0,1}^2 [|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2] ds \\ &\quad - 4a \int_0^{t \wedge \tau_N} \|u_n(s)\|_{0,1}^2 \|u_n(s)\|_{L^{2\alpha+2}}^{2\alpha+2} ds \\ &\quad - 4a(2\alpha + 1) \int_0^{t \wedge \tau_N} \|u_n(s)\|_{0,1}^2 |u_n(s)|^\alpha |\partial_3 u_n(s)|_{L^2}^2 ds + \sum_{j=1}^4 \tilde{T}_j(t), \end{aligned} \quad (3.6)$$

where we let

$$\begin{aligned} \tilde{T}_1(t) &= -4 \int_0^{t \wedge \tau_N} \langle \partial_3 B(u_n(s)), \partial_3 u_n(s) \rangle \|u_n(s)\|_{0,1}^2 ds, \\ \tilde{T}_2(t) &= 4 \int_0^{t \wedge \tau_N} (P_n \sigma(s, u_n(s)) dW_n(s), u_n(s))_{0,1} \|u_n(s)\|_{0,1}^2, \\ \tilde{T}_3(t) &= 2 \int_0^{t \wedge \tau_N} |P_n \sigma(s, u_n(s)) \Pi_n|_{\tilde{\mathcal{L}}}^2 \|u_n(s)\|_{0,1}^2 ds, \\ \tilde{T}_4(t) &= 4 \int_0^{t \wedge \tau_N} |(\Pi_n \sigma(s, u_n(s)) P_n)^* u_n(s)|_0^2 \|u_n(s)\|_{0,1}^2 ds. \end{aligned}$$

The growth condition (2.33) implies that

$$\tilde{T}_3(t) + \tilde{T}_4(t) \leq 6 \int_0^{t \wedge \tau_N} [\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{0,1}^2 + \tilde{K}_2 (|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2)] \|u_n(s)\|_{0,1}^2 ds,$$

while (2.19) implies

$$|\tilde{T}_1(t)| \leq 4C \int_0^{t \wedge \tau_N} \left(\epsilon_0 |\nabla_h \partial_3 u_n(s)|_{L^2}^2 + \frac{\epsilon_1}{4\epsilon_0} \|u_n(s)\|^\alpha |\partial_3 u_n(s)|_{L^2}^2 + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} |\partial_3 u_n(s)|_{L^2}^2 \right) \times \|u_n(s)\|_{0,1}^2 ds.$$

The Burkholder-Davies-Gundy inequality, the growth condition (2.33) and Young's inequality imply that for $\beta \in (0, 1)$:

$$\begin{aligned} \mathbb{E} \left(\sup_{s \leq t} \tilde{T}_2(s) \right) &\leq 12 \mathbb{E} \left\{ \int_0^{t \wedge \tau_N} |\sigma(r, u_n(r))|_{\tilde{\mathcal{L}}}^2 \|u_n(r)\|_{0,1}^6 dr \right\}^{\frac{1}{2}} \\ &\leq \beta \mathbb{E} \left(\sup_{s \leq t \wedge \tau_N} \|u_n(s)\|_{0,1}^4 \right) \\ &\quad + \frac{36}{\beta} \mathbb{E} \int_0^{t \wedge \tau_N} [\tilde{K}_0 + \tilde{K}_1 \|u_n(s)\|_{0,1}^2 + \tilde{K}_2 (|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2)] \|u_n(s)\|_{0,1}^2 ds. \end{aligned}$$

If $\tilde{K}_2 < \frac{2\nu}{21}$ we may choose $\beta \in (0, 1)$ and $\epsilon > 0$ such that $\epsilon < 4\nu - 6(1 + 6/\beta)\tilde{K}_2$, then $\epsilon_0 > 0$ such that $4C\epsilon_0 < \frac{\epsilon}{2}$, and finally $\epsilon_1 > 0$ such that $\frac{4C\epsilon_1}{4\epsilon_0} + \epsilon < 4a(2\alpha + 1)$. For this choice of constants, neglecting some non positive integrals in the right hand side of (3.6), we deduce:

$$\begin{aligned} (1 - \beta) \mathbb{E} \left(\sup_{s \in [0, t]} \|u_n(s \wedge \tau_N)\|_{0,1}^4 \right) &+ \frac{\epsilon}{2} \mathbb{E} \int_0^{t \wedge \tau_N} \|u_n(s)\|_{0,1}^2 [|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2] ds \\ &\leq \mathbb{E} \|u_0\|_{0,1}^4 + \left(6 + \frac{36}{\beta}\right) \tilde{K}_1 \mathbb{E} \int_0^t \|u_n(s \wedge \tau_N)\|_{0,1}^4 ds + \left[6 + \frac{36}{\beta}\right] \tilde{K}_0 \mathbb{E} \int_0^t \|u_n(s \wedge \tau_N)\|_{0,1}^2 ds. \end{aligned}$$

This inequality, (3.5) and Gronwall's lemma yield $\sup_n \mathbb{E} \left(\sup_{s \in [0, T]} \|u_n(s \wedge \tau_N)\|_{0,1}^4 \right) < \infty$. We deduce the existence of a constant C , which does not depend on n and N , such that:

$$\mathbb{E} \left(\sup_{s \in [0, T]} \|u_n(s \wedge \tau_N)\|_{0,1}^4 \right) + \mathbb{E} \int_0^{\tau_N} \|u_n(s)\|_{1,1}^2 \|u_n(s)\|_{0,1}^2 ds \leq C(1 + \mathbb{E} \|u_0\|_{0,1}^4). \quad (3.7)$$

We now prove that (3.2) holds. As $N \rightarrow \infty$, the sequence of stopping times τ_N increases to τ_n^* , and on the set $\{\tau_n^* < T\}$ we have $\sup_{s \in [0, \tau_N]} \|u_n(s)\|_{0,1} \rightarrow +\infty$. Hence (3.5) proves that $P(\tau_n^* < T) = 0$ and that for almost every ω , for $N(\omega)$ large enough, $\tau_{N(\omega)}(\omega) = T$. The monotone convergence theorem used in (3.5) and (3.7), we deduce the following upper estimates for some constant which does not depend on n :

$$\mathbb{E} \left(\sup_{s \in [0, T]} \|u_n(s)\|_{0,1}^2 \right) + \mathbb{E} \int_0^T (\|u_n(s)\|_{1,1}^2 + \|u_n(s)\|_{L^{2\alpha+2}}^2) ds \leq C(1 + \mathbb{E} \|u_0\|_{0,1}^2), \quad (3.8)$$

$$\mathbb{E} \left(\sup_{s \in [0, T]} \|u_n(s)\|_{0,1}^4 \right) + \mathbb{E} \int_0^T \|u_n(s)\|_{1,1}^2 \|u_n(s)\|_{0,1}^2 ds \leq C(1 + \mathbb{E} \|u_0\|_{0,1}^4). \quad (3.9)$$

To complete the proof and check (3.2), we finally prove that

$$\sup_n \mathbb{E} \left(\left| \int_0^T \|u_n(s)\|_{1,1}^2 ds \right|^2 \right) \leq C(1 + \mathbb{E} \|u_0\|_{0,1}^4). \quad (3.10)$$

The identity (3.3) and the upper estimates of $T_1(t)$ and $T_3(t)$ imply that for $\tilde{K}_2 < 2\nu$, $2C\epsilon_0 < \tilde{K}_2$ and ϵ_1 small enough we have for every $t \in [0, T]$ and :

$$\|u_n(t \wedge \tau_N)\|_{0,1}^2 + (2\nu - \tilde{K}_2) \int_0^{t \wedge \tau_N} (|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2) ds$$

$$\leq \|u_0\|_{0,1}^2 + \sup_{s \leq t} |T_2(s)| + J(t), \quad (3.11)$$

where for some positive constant C :

$$J(t) = \int_0^{t \wedge \tau_N} \left[\tilde{K}_1 \|u_n(s)\|_{0,1}^2 + \tilde{K}_0 + \frac{2CC_\alpha}{\epsilon_0 \epsilon_1^{1/(\alpha-1)}} |\partial_3 u_n(s)|_{L^2}^2 \right] ds \leq C \int_0^{t \wedge \tau_N} \|u_n(s)\|_{0,1}^2 ds.$$

Hence for $\tilde{K}_2 < 2\nu$, using the Doob and Cauchy Schwarz inequalities as well as (2.33), we deduce:

$$\begin{aligned} & \mathbb{E} \left[\left\{ \sup_{s \leq T} \|u_n(s \wedge \tau_N)\|_{0,1}^2 + (2\nu - \tilde{K}_2) \int_0^{\tau_N} (|\nabla_h u_n(s)|_{L^2}^2 + |\partial_3 \nabla_h u_n(s)|_{L^2}^2) ds \right\}^2 \right] \\ & \leq 3\mathbb{E}(J(T)^2) + 3\mathbb{E} \left(\sup_{s \leq T} T_2^2(s) \right) + 3\mathbb{E}(\|u_0\|_{0,1}^4) \\ & \leq 3CT\mathbb{E} \int_0^{\tau_N} \|u_n(s)\|_{0,1}^4 ds + 3C\mathbb{E} \int_0^{\tau_N} \|u_n(s)\|_{0,1}^2 |\sigma(u_n(s))\Pi_n|_{\mathcal{L}}^2 ds + 3\mathbb{E}(\|u_0\|_{0,1}^4) \\ & \leq 3C\mathbb{E} \int_0^{\tau_N} [2\tilde{K}_2 \|u_n(s)\|_{0,1}^2 \|u_n(s)\|_{1,1}^2 + (\tilde{K}_1 + T) \|u_n(s)\|_{0,1}^4 + \tilde{K}_0 \|u_n(s)\|_{0,1}^2] ds \\ & \quad + 3\mathbb{E}(\|u_0\|_{0,1}^4). \end{aligned}$$

Let $N \rightarrow \infty$ in this equation. Since $\tau_{N(\omega)}(\omega) = T$ for $N(\omega)$ large enough, the above inequality where τ_N is replaced by T (which is deduced by means of the monotone convergence theorem) coupled with (3.8) and (3.9) yield (3.10). This completes the proof. \square

3.2. Well posedness of equation (2.3). The aim of this section is to prove that if the initial condition $u_0 \in L^4(\Omega; \tilde{H}^{0,1})$, equation (2.3) has a unique (weak) solution in the space \mathcal{X} which belongs a.s. to $C([0, T]; H)$, where \mathcal{X} has been defined in (2.7).

Theorem 3.2. *Let σ satisfy condition (C) with $\tilde{K}_2 < \frac{2\nu}{21}$ and u_0 be independent of $(W(t), t \geq 0)$ such that $E(\|u_0\|_{0,1}^4) < \infty$. Then there exists a weak solution $u \in \mathcal{X}$ to (2.3) with initial condition u_0 . This solution belongs to $C([0, T]; H)$ a.s.*

Furthermore, there exists a constant $C > 0$ such that this solution satisfies the following upper estimate:

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \|u(t)\|_{0,1}^4 + \left(\int_0^T \|u(t)\|_{1,1}^2 dt \right)^2 + \int_0^T \int_{\mathbb{R}^3} |u(t, x)|^{2(\alpha+1)} dx dt \right) \leq C(1 + \mathbb{E}\|u_0\|_{0,1}^4). \quad (3.12)$$

If $L_2 < 2\nu$, then (2.3) has a pathwise unique weak solution in \mathcal{X} which belongs a.s. to $C([0, T]; H)$.

Proof. The proof is decomposed in several steps.

Recall that \mathcal{L} is defined by (2.31) and that σ satisfies (2.32).

Step 1: Weak convergence of the solution The inequalities (3.2) and (2.23) imply the existence of a subsequence of $(u_n, n \geq 1)$ (resp. of $(P_n \sigma(\cdot, u_n) \circ \Pi_n, n \geq 1)$ and of $(F(u_n), n \geq 1)$), still denoted by the same notation, of processes $u \in \mathcal{X}$ (resp. $\tilde{S} \in L^2(\Omega_T; \mathcal{L})$ and $\tilde{F} \in [L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)]^*$), and finally of a random variable $\tilde{u}(T) \in L^2(\Omega; \tilde{H}^{0,1})$, for which the following properties hold:

- (i) $u_n \rightarrow u$ weakly in $L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)$,
- (ii) u_n is weak star converging to u in $L^4(\Omega; L^\infty(0, T; \tilde{H}^{0,1}))$,
- (iii) $u_n(T) \rightarrow \tilde{u}(T)$ weakly in $L^2(\Omega; \tilde{H}^{0,1})$,

- (iv) $F(u_n) \rightarrow \tilde{F}$ weakly in $[L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)]^*$,
(v) $P_n \sigma(\cdot, u_n(\cdot)) \Pi_n \rightarrow \tilde{S}$ weakly in $L^2(\Omega_T; \mathcal{L})$.

Indeed, (i) and (ii) are straightforward consequences of Proposition 3.1, of (3.2), and of uniqueness of the limit of $\mathbb{E} \int_0^T (u_n(t), v(t)) dt$ for appropriate v . The upper estimate (2.23) proves (iv). The definition of P_n , Π_n , the growth condition (2.32) and (3.2) imply:

$$\sup_n \mathbb{E} \int_0^T |P_n \sigma(s, u_n(s)) \Pi_n|_{\mathcal{L}}^2 ds \leq \sup_n \mathbb{E} \int_0^T [K_0 + K_1 |u_n(s)|_{L^2}^2 + K_2 |\nabla_h u_n(s)|_{L^2}^2] ds < \infty.$$

This proves (v). Finally, (3.7) and the equality $\tau_N = T$ a.s. imply that $\sup_n \mathbb{E} \|u_n(T)\|_{0,1}^4 < \infty$, which proves (iii).

Furthermore, properties (i) and (ii) and (3.2) imply that

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T \|u(s)\|_{1,1}^2 ds \right)^2 + \int_0^T \|u(s)\|_{L^{2\alpha+2}}^{2\alpha+2} ds \right] &\leq C(1 + \mathbb{E} \|u_0\|_{0,1}^4), \\ \mathbb{E} \left(\sup_{s \in [0, T]} \|u(s)\|_{0,1}^4 \right) &\leq C(1 + \mathbb{E} \|u_0\|_{0,1}^4). \end{aligned}$$

Step 2: An equation for the weak limits The approach is that used in [21]. We prove that $\tilde{u}(T) = u(T)$ a.s. and that for $t \in [0, T]$:

$$u(t) = u_0 + \int_0^t \tilde{F}(s) ds + \int_0^t \tilde{S}(s) dW(s). \quad (3.13)$$

For $\delta > 0$, let $f \in H^1(-\delta, T+\delta)$ be such that $\|f\|_\infty = 1$, $f(0) = 1$ and for any integer $j \geq 1$ set $g_j(t) = f(t)e_j$, where $\{e_j\}_{j \geq 1}$ is the previous orthonormal basis of H made of elements of H^2 which are also orthogonal in $\tilde{H}^{0,1}$, such that for every $n \geq 1$, $\mathcal{H}_n = \text{span}(e_1, \dots, e_n)$.

The Itô formula implies that for any $j \geq 1$, and for $0 \leq t \leq T$:

$$(u_n(T), g_j(T)) = (u_n(0), g_j(0)) + \sum_{i=1}^3 I_{n,j}^i, \quad (3.14)$$

where

$$\begin{aligned} I_{n,j}^1 &= \int_0^T (u_n(s), e_j) f'(s) ds, & I_{n,j}^2 &= \int_0^T \langle F(u_n(s)), g_j(s) \rangle ds, \\ I_{n,j}^3 &= \int_0^T (P_n \sigma(s, u_n(s)) \Pi_n dW(s), g_j(s)). \end{aligned}$$

We study the convergence of all terms in (3.14). Since $f' \in L^2(0, T)$ and $\alpha > 1$, for every $Z \in L^{2(\alpha+1)}(\Omega) \subset L^4(\Omega)$, the map $(t, \omega) \mapsto e_j Z(\omega) f'(t)$ belongs to $L^4(\Omega; L^2(0, T; \tilde{H}^{0,1})) \subset L^{\frac{4}{3}}(\Omega; L^1(0, T; \tilde{H}^{0,1}))$. Hence, the weak-star convergence (ii) above implies that as $n \rightarrow \infty$, $I_{n,j}^1 \rightarrow \int_0^T (u(s), e_j) f'(s) ds$ weakly in $(L^{2(\alpha+1)}(\Omega))$.

Furthermore, the Gagliardo-Nirenberg inequality implies that $H^1(-\delta, T+\delta) \subset L^{2(\alpha+1)}(0, T)$ and $H^2 \subset L^{2(\alpha+1)}(\mathbb{R}^3)$. Hence $(t, \omega) \mapsto g_j(t)Z(\omega) \in L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3) \cap L^4(\Omega; L^2(0, T; \tilde{H}^{1,1}))$; therefore, (iv) implies that as $n \rightarrow \infty$, $I_{n,j}^2 \rightarrow \int_0^T \langle \tilde{F}(s), g_j(s) \rangle ds$ weakly in $(L^{2(\alpha+1)}(\Omega))$.

To prove the convergence of $I_{n,j}^3$, as in [26] (see also [13]), let \mathcal{P}_T denote the class of predictable processes in $L^2(\Omega_T, \mathcal{L})$ with the inner product

$$(G, J)_{\mathcal{P}_T} = \mathbb{E} \int_0^T (G(s), J(s))_{\mathcal{L}} ds = \mathbb{E} \int_0^T \text{trace}_H(G(s)QJ(s)^*) ds.$$

The map $\mathcal{T} : \mathcal{P}_T \rightarrow L^2(\Omega)$ defined by $\mathcal{T}(G)(t) = \int_0^t (G(s)dW(s), g_j(s))$ is linear and continuous because of the Itô isometry. Furthermore, (v) shows that for every $G \in \mathcal{P}_T$, as $n \rightarrow \infty$, $(P_n\sigma(\cdot, u_n(\cdot))\Pi_n, G)_{\mathcal{P}_T} \rightarrow (\tilde{S}(\cdot), G)_{\mathcal{P}_T}$ weakly in $L^2(\Omega)$. Hence, as $n \rightarrow \infty$, $\int_0^T (P_n\sigma(s, u_n(s))\Pi_n dW(s), g_j(s))$ converges to $\int_0^T (\tilde{S}_s dW(s), g_j(s))$.

Finally, as $n \rightarrow \infty$, $P_n u_0 = u_n(0) \rightarrow u_0$ in H . By (iii), $(u_n(T), g_j(T))$ converges to $(\tilde{u}(T), g_j(T))$ weakly in $L^2(\Omega)$. Therefore, as $n \rightarrow \infty$, (3.14) leads to:

$$\begin{aligned} (\tilde{u}(T), e_j) f(T) &= (u_0, e_j) + \int_0^T (u(s), e_j) f'(s) ds + \int_0^T \langle \tilde{F}(s), g_j(s) \rangle ds \\ &\quad + \int_0^T (\tilde{S}(s) dW(s), g_j(s)) \text{ a.s.} \end{aligned} \quad (3.15)$$

Choosing f in an appropriate way, we next prove a similar identity for any fixed $t \in [0, T]$. For $\delta > 0$, $k > \frac{1}{\delta}$, $t \in [0, T]$, let $f_k \in H^1(-\delta, T + \delta)$ be such that $\|f_k\|_\infty = 1$, $f_k = 1$ on $(-\delta, t - \frac{1}{k})$ and $f_k = 0$ on $[t, T + \delta)$. Then $f_k \rightarrow 1_{(-\delta, t)}$ in L^2 , and $f'_k \rightarrow -\delta_t$ in the sense of distributions. Hence as $k \rightarrow \infty$, (3.15) written with $f := f_k$ yields

$$0 = (u_0 - u(t), e_j) + \int_0^t \langle \tilde{F}(s), e_j \rangle ds + \int_0^t (\tilde{S}(s) dW(s), e_j)$$

for almost all $(\omega, t) \in \Omega_T$. Here, the weak continuity (after some modification) of $u(t)$ in H for almost all $\omega \in \Omega$ is deduced by using Lemma 1.4 in Chapter III in Temam [27]. Indeed, it is easy to see that (3.15) provides weak continuity with values in H^{-1} . Using the fact that the solution is also a.s. $L^\infty(0, T; H)$, Lemma 1.4 from [27] provides that the solution is a.s. in $C_w([0, T]; H)$.

Note that j is arbitrary and $\mathbb{E} \int_0^T |\tilde{S}(s)|_{\mathcal{L}}^2 ds < \infty$; hence for $0 \leq t \leq T$ and almost every ω , we deduce (3.13). Moreover $\int_0^t \tilde{F}(s) ds \in H$ a.s. Let $f = 1_{(-\delta, T+\delta)}$; using again (3.15) we obtain

$$\tilde{u}(T) = u_0 + \int_0^T \tilde{F}(s) ds + \int_0^T \tilde{S}(s) dW(s).$$

This equation and (3.13) yield that $\tilde{u}(T) = u(T)$ a.s.

Step 3: Identification of the limits In (3.13) we still have to prove that $d\mathbb{P} \otimes ds$ a.e. on Ω_T , we have:

$$\tilde{S}(s) = \sigma(s, u(s)) \text{ and } \tilde{F}(s) = F(u(s)).$$

To establish these relations we use the same idea as in [20] (see also [26]). More precisely, we introduce a discounting factor which enables us to cancel out terms where both elements in the scalar products depend on t . Let $v \in \mathcal{X}$, where \mathcal{X} has been defined in (2.7). Since σ satisfies the Lipschitz condition **(C)(ii)** with a constant $L_2 < 2\nu$, we may choose $\eta \in (0, \nu)$ such that $L_2 < 2\eta$. For this choice of η , let $C_\eta > 0$ be defined by (2.26) and for every $t \in [0, T]$, set

$$r(t) = \int_0^t \left[C_\eta \|v(s)\|_{1,1}^2 + L_1 \right] ds. \quad (3.16)$$

Then almost surely, $0 \leq r(t) < \infty$ for all $t \in [0, T]$. Moreover, we also have that

$$r \in L^1(\Omega; L^\infty(0, T)), \quad e^{-r} \in L^\infty(\Omega_T), \quad r' \in L^1(\Omega_T), \quad r' e^{-r} \in L^2(\Omega; L^1(0, T)). \quad (3.17)$$

The weak convergence in (iii) and the property $P_n u_0 \rightarrow u_0$ in H imply that

$$\mathbb{E}(|u(T)|_{L^2}^2 e^{-r(T)}) - \mathbb{E}|u_0|_{L^2}^2 \leq \liminf_n \left[\mathbb{E}(|u_n(T)|_{L^2}^2 e^{-r(T)}) - \mathbb{E}|P_n u_0|_{L^2}^2 \right]. \quad (3.18)$$

We now apply Itô's formula to $|\phi(t)|_{L^2}^2 e^{-r(t)}$ for $\phi = u$ and $\phi = u_n$. This gives the relation

$$\mathbb{E}(|\phi(T)|_{L^2}^2 e^{-r(T)}) - \mathbb{E}|\phi(0)|_{L^2}^2 = \mathbb{E} \int_0^T e^{-r(s)} d\{|\phi(s)|_{L^2}^2\} - \mathbb{E} \int_0^T r'(s) e^{-r(s)} |\phi(s)|_{L^2}^2 ds,$$

which can be justified due to (3.17) and the property $|\phi|^2 \in L^1(\Omega, L^\infty((0, T)))$ for both choices of ϕ . Using (3.13), (3.1) and letting $u = v + (u - v)$ after simplification, from (3.18) we obtain

$$\begin{aligned} \mathbb{E} \int_0^T e^{-r(s)} \left[-r'(s) \{ |u(s) - v(s)|_{L^2}^2 + 2(u(s) - v(s), v(s)) \} + 2\langle \tilde{F}(s), u(s) \rangle + |\tilde{S}(s)|_{\mathcal{L}}^2 \right] ds \\ \leq \liminf_n X_n, \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} X_n = \mathbb{E} \int_0^T e^{-r(s)} \left[-r'(s) \{ |u_n(s) - v(s)|_{L^2}^2 + 2(u_n(s) - v(s), v(s)) \} \right. \\ \left. + 2\langle F(u_n(s)), u_n(s) \rangle + |P_n \sigma(s, u_n(s)) \Pi_n|_{\mathcal{L}}^2 \right] ds. \end{aligned}$$

We write $X_n = Y_n + \sum_{i=1}^3 Z_n^i$, where Y_n need not converge but is non positive, while the sequences Z_n^i , $i = 1, 2, 3$ converge as $n \rightarrow \infty$. The upper estimate in (2.26) and the Lipschitz condition **(C)(ii)** imply that for $s \in [0, T]$ and $L_2 < 2\eta < 2\nu$:

$$\begin{aligned} 2\langle F(u_n(s)) - F(v(s)), u_n(s) - v(s) \rangle + |P_n \sigma(s, u_n(s)) \Pi_n - P_n \sigma(s, v(s)) \Pi_n|_{\mathcal{L}}^2 \\ \leq -2\eta |\nabla_h(u_n(s) - v(s))|_{L^2}^2 + C_\eta \|v(t)\|_{1,1}^2 |u_n(s) - v(s)|_{L^2}^2 + |\sigma(s, u_n(s)) - \sigma(s, v(s))|_{\mathcal{L}}^2 \\ \leq -(2\eta - L_2) |\nabla_h(u_n(s) - v(s))|_{L^2}^2 + (C_\eta \|v(s)\|_{1,1}^2 + L_1) |u_n(s) - v(s)|_{L^2}^2. \end{aligned}$$

Hence the definition of r in (3.16) implies that

$$\begin{aligned} Y_n := \mathbb{E} \int_0^T e^{-r(s)} \left[-r'(s) |u_n(s) - v(s)|_{L^2}^2 + 2\langle F(u_n(s)) - F(v(s)), u_n(s) - v(s) \rangle \right. \\ \left. + |P_n [\sigma(s, u_n(s)) - \sigma(s, v(s))] \Pi_n|_{\mathcal{L}}^2 \right] ds \leq 0. \end{aligned} \quad (3.20)$$

Furthermore, $X_n = Y_n + \sum_{j=1}^3 Z_n^j$, where

$$\begin{aligned} Z_n^1 &= \mathbb{E} \int_0^T e^{-r(s)} \left[-2r'(s) (u_n(s) - v(s), v(s)) + 2\langle F(u_n(s)), v(s) \rangle + 2\langle F(v(s)), u_n(s) \rangle \right. \\ &\quad \left. - 2\langle F(v(s)), v(s) \rangle + 2(P_n \sigma(s, u_n(s)) \Pi_n, \sigma(s, v(s)))_{\mathcal{L}} \right] ds, \\ Z_n^2 &= 2 \mathbb{E} \int_0^T e^{-r(s)} (P_n \sigma(s, u_n(s)) \Pi_n, P_n \sigma(s, v(s)) \Pi_n - \sigma(s, v(s)))_{\mathcal{L}} ds, \\ Z_n^3 &= - \mathbb{E} \int_0^T e^{-r(s)} |P_n \sigma(s, v(s)) \Pi_n|_{\mathcal{L}}^2 ds. \end{aligned}$$

We next study the convergence of Z_n^j , $j = 1, 2, 3$, and first prove that $\tilde{S}(s) = \sigma(s, u(s))$ a.e. on Ω_T . The definition of \mathcal{X} and (3.17) imply that $r'e^{-r}v \in L^2(\Omega; L^1(0, T; \tilde{H}^{0,1}))$. Hence the weak star convergence (ii) implies that as $n \rightarrow \infty$:

$$\mathbb{E} \int_0^T e^{-r(s)} r'(s) (u_n(s) - v(s), v(s)) ds \rightarrow \mathbb{E} \int_0^T e^{-r(s)} r'(s) (u(s) - v(s), v(s)) ds.$$

Since (2.23) implies that $F(v) \in (L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3))^*$, the weak convergence (i) implies that $\mathbb{E} \int_0^T e^{-r(s)} \langle F(v(s)), u_n(s) \rangle ds \rightarrow \mathbb{E} \int_0^T e^{-r(s)} \langle F(v(s)), u(s) \rangle ds$. Since $v \in L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)$, the weak convergence (iv) implies that $\mathbb{E} \int_0^T e^{-r(s)} \langle F(u_n(s)), v(s) \rangle ds \rightarrow \mathbb{E} \int_0^T e^{-r(s)} \langle \tilde{F}(s), v(s) \rangle ds$. Finally, the weak convergence (v) implies that as $n \rightarrow \infty$:

$$\mathbb{E} \int_0^T e^{-r(s)} (P_n \sigma(s, u_n(s)) \Pi_n, \sigma(s, v(s)))_{\mathcal{L}} ds \rightarrow \mathbb{E} \int_0^T e^{-r(s)} (\tilde{S}(s), \sigma(s, v(s)))_{\mathcal{L}} ds.$$

Hence as $n \rightarrow \infty$,

$$\begin{aligned} Z_n^1 \rightarrow \mathbb{E} \int_0^T e^{-r(s)} \left[-2r'(s) (u(s) - v(s), v(s)) + 2\langle \tilde{F}(s), v(s) \rangle + 2\langle F(v(s)), u(s) \rangle \right. \\ \left. - 2\langle F(v(s)), v(s) \rangle + 2(\tilde{S}(s), \sigma(s, v(s)))_{\mathcal{L}} \right] ds. \end{aligned} \quad (3.21)$$

For almost every $(\omega, t) \in \Omega_T$ and any orthonormal basis ψ_j of H_0 , $\sum_{j \geq n+1} q_j |\sigma(s, v(s)) \psi_j|_{L^2}^2$ converges to 0 as $n \rightarrow \infty$. This sequence is dominated by $|\sigma(s, v(s))|_{\mathcal{L}}^2$ which belongs to $L^1(P)$ by means of the growth condition (2.32) and the definition of \mathcal{X} . Furthermore, the inequality $|P_n \sigma(s, u_n(s)) \circ \Pi_n|_{\mathcal{L}} \leq |\sigma(s, u_n(s))|_{\mathcal{L}}$, the growth condition (2.32), (3.7) and the Cauchy-Schwarz inequality yield

$$Z_n^2 \leq \left(\mathbb{E} \int_0^T e^{-r(s)} |P_n \sigma(s, u_n(s)) \circ \Pi_n|_{\mathcal{L}}^2 ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T e^{-r(s)} \sum_{j \geq n+1} q_j |\sigma(s, v(s)) \psi_j|_{L^2}^2 ds \right)^{\frac{1}{2}}.$$

In the above right hand side, the first factor remains bounded, while as $n \rightarrow \infty$ the second one converges to 0 by the dominated convergence theorem. This yields

$$Z_n^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.22)$$

Finally, the definition of P_n , Π_n and the growth condition (2.32) imply that for a.e. $(\omega, s) \in \Omega_T$,

$$\begin{aligned} \left| \sum_{j \geq 1} q_j |P_n \sigma(s, v(s)) \Pi_n \psi_j|_{L^2}^2 - |\sigma(s, v(s))|_{\mathcal{L}}^2 \right| \leq 2 \sum_{j \geq n+1} q_j |\sigma(s, v(s)) \psi_j|_{L^2}^2 \\ + 2|(P_n - \text{Id})\sigma(s, v(s))|_{\mathcal{L}}^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Furthermore, the growth condition (2.32) implies that for every n :

$$\begin{aligned} \left| \sum_{j \geq 1} q_j |P_n \sigma(s, v(s)) \Pi_n \psi_j|_{L^2}^2 - |\sigma(s, v(s))|_{\mathcal{L}}^2 \right| \leq 2|\sigma(s, v(s))|_{\mathcal{L}}^2 \\ \leq 2[K_0 + K_1|v(s)|_{L^2}^2 + K_2|\nabla_h v(s)|_{L^2}^2] \in L^1(\Omega_T). \end{aligned}$$

Hence the dominated convergence theorem implies that

$$Z_n^3 \rightarrow -\mathbb{E} \int_0^T e^{-r(s)} |\sigma(s, v(s))|_{\mathcal{L}}^2 ds. \quad (3.23)$$

Using the inequalities (3.19)–(3.23) we obtain:

$$\begin{aligned} \mathbb{E} \int_0^T e^{-r(s)} \left[-r'(s) |u(s) - v(s)|_{L^2}^2 + 2\langle \tilde{F}(s) - F(v(s)), u(s) - v(s) \rangle \right. \\ \left. + |\tilde{S}(s) - \sigma(s, v(s))|_{\mathcal{L}}^2 \right] ds \leq 0. \end{aligned} \quad (3.24)$$

Let $v = u \in \mathcal{X}$; then we deduce that for almost every $(\omega, s) \in \Omega_T$ we have $\tilde{S}(s) = \sigma(s, u(s))$.

Using another choice of v , we next prove that $\tilde{F}(s) = F(u(s))$ a.e. on Ω_T . Let $\lambda \in \mathbb{R}$ and $\tilde{v} \in \mathcal{X}$ and set $v_\lambda = u + \lambda \tilde{v} \in \mathcal{X}$. Then if r_λ is defined in terms on v_λ using (3.16), the inequality (3.24) yields

$$\begin{aligned} \lambda^2 \mathbb{E} \int_0^T e^{-r_\lambda(s)} r'_\lambda(s) |\tilde{v}(s)|_{L^2}^2 ds + 2\lambda \mathbb{E} \int_0^T e^{-r_\lambda(s)} \langle \tilde{F}(s) - F(u(s)), \tilde{v}(s) \rangle ds \\ + 2\lambda \mathbb{E} \int_0^T e^{-r_\lambda(s)} \langle F(u(s)) - F(v_\lambda(s)), \tilde{v}(s) \rangle ds \leq 0. \end{aligned} \quad (3.25)$$

The upper estimate (2.26) and Hölder's inequality imply that for $\eta \in (0, \nu)$ and $\lambda \in (0, 1]$,

$$|\langle F(v_\lambda(s)) - F(u(s)), \tilde{v}(s) \rangle| = \frac{1}{|\lambda|} |\langle F(v_\lambda(s)) - F(u(s)), v_\lambda(s) - u(s) \rangle| \leq |\lambda| \phi(t),$$

where by Hölder's inequality we have

$$\begin{aligned} \phi(t) = \eta \|\tilde{v}(t)\|_{1,1}^2 + 2C_\eta (\|u(s)\|_{1,1}^2 + \|\tilde{v}\|_{1,1}^2) \|\tilde{v}(s)\|_{L^2}^2 \\ + C a \kappa (\|u(s)\|_{L^{2(\alpha+1)}}^{2\alpha} + \|\tilde{v}(s)\|_{L^{2(\alpha+1)}}^{2\alpha}) \|\tilde{v}(s)\|_{L^{2(\alpha+1)}}^2. \end{aligned}$$

Using once more Hölder's inequality, we deduce that

$$\begin{aligned} \mathbb{E} \int_0^T \phi(t) dt \leq C \mathbb{E} \left[\int_0^T \|\tilde{v}(s)\|_{1,1}^2 ds + \left[\left(\int_0^T \|u(s)\|_{1,1}^2 ds \right)^2 \right]^{\frac{1}{2}} + \left[\int_0^T \|\tilde{v}(s)\|_{1,1}^2 ds \right]^{\frac{1}{2}} \right] \\ \times \left(\sup_{s \in [0, T]} \|\tilde{v}(s)\|_{L^2}^4 \right)^{\frac{1}{2}} + C \left[\|u\|_{L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)}^{2\alpha} + \|\tilde{v}\|_{L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)}^{2\alpha} \right] \|\tilde{v}\|_{L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3)}^2 < \infty. \end{aligned}$$

Since $r_\lambda(s) \geq 0$, the dominated convergence theorem implies that

$$\mathbb{E} \int_0^T e^{-r_\lambda(s)} \langle F(u(s)) - F(v_\lambda(s)), \tilde{v}(s) \rangle ds \rightarrow 0 \text{ as } \lambda \rightarrow 0.$$

Furthermore, since $\tilde{F}(s) - F(u(s)) \in (L^4(\Omega; L^2(0, T; \tilde{H}^{1,1})) \cap L^{2(\alpha+1)}(\Omega_T \times \mathbb{R}^3))^*$, using once more the dominated convergence theorem we deduce that as $\lambda \rightarrow 0$:

$$\mathbb{E} \int_0^T e^{-r_\lambda(s)} \langle \tilde{F}(s) - F(u(s)), \tilde{v}(s) \rangle ds \rightarrow \mathbb{E} \int_0^T e^{-r_0(s)} \langle \tilde{F}(s) - F(u(s)), \tilde{v}(s) \rangle ds.$$

Dividing (3.25) by λ and letting $\lambda \rightarrow 0^+$ and $\lambda \rightarrow 0^-$, we deduce that for every $\tilde{v} \in \mathcal{X}$,

$$\mathbb{E} \int_0^T e^{-r_0(s)} \langle \tilde{F}(s) - F(u(s)), \tilde{v}(s) \rangle ds = 0.$$

This implies that $\tilde{F}(s) = F(u(s))$ a.e. on Ω_T .

Step 4: Continuity of the solution We next prove that $u \in C([0, T]; H)$ a.s. The proof, based on some regularization of the solution, is similar to that in [13]; however, the functional setting is different which requires some changes. Set $A = P_{\text{div}} \Delta$; then $e^{-\delta A}$ maps $(\tilde{H}^{1,1})^* \subset H^{-2}$ to H for any $\delta > 0$. Furthermore, the Gagliardo Nirenberg inequality implies that $H^2(\mathbb{R}^3) \subset L^{2(\alpha+1)}(\mathbb{R}^3)$, so that the semi-group $e^{-\delta A}$ also maps $L^{\frac{2(\alpha+1)}{2\alpha+1}}(\mathbb{R}^3) =$

$(L^{2(\alpha+1)}(\mathbb{R}^3))^* \subset H^{-2}$ to H . Since $|u(s)|^{2\alpha}u(s) \in L^{\frac{2(\alpha+1)}{2\alpha+1}}(\mathbb{R}^3)$ for almost every (ω, t) , we deduce that $e^{-\delta A} \int_0^t F(u(s)) ds$ belongs to $C([0, T], H)$. Finally, (2.32) in the growth condition **C(i)** implies $\mathbb{E} \int_0^T |e^{-\delta A} \sigma(s, u(s))|_{L_Q}^2 ds < +\infty$. Thus $\int_0^t e^{-\delta A} \sigma(s, u(s)) dW(s)$ belongs to $C([0, T], H)$ a.s. (see e.g. [14], Theorem 4.12). Therefore, it is sufficient to prove that a.s. $e^{-\delta A}u$ converges to u uniformly on the time interval $[0, T]$, that is

$$\lim_{\delta \rightarrow 0} \mathbb{E} \left\{ \sup_{0 \leq t \leq T} |u(t) - e^{-\delta A}u(t)|_{L^2}^2 \right\} = 0. \quad (3.26)$$

Let $G_\delta = Id - e^{-\delta A}$ and apply Itô's formula to $|G_\delta u(t)|_{L^2}^2$. This yields

$$\begin{aligned} |G_\delta u(t)|_{L^2}^2 &= |G_\delta u_0|_{L^2}^2 - 2\nu \int_0^t \|G_\delta u(s)\|_{\tilde{H}^{1,0}}^2 ds + 2I(t) + \int_0^t |G_\delta \sigma(u(s))|_{\mathcal{L}}^2 ds \\ &\quad - 2 \int_0^t \langle B(u(s)) + 2a|u(s)|^{2\alpha}u(s), G_\delta^2 u(s) \rangle ds, \end{aligned} \quad (3.27)$$

where $I(t) = \int_0^t (G_\delta \sigma(u(s)) dW(s), G_\delta u(s))$. By the Burkholder-Davies-Gundy and Schwarz inequalities we have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |I(t)| &\leq C \mathbb{E} \left(\int_0^T |G_\delta u(s)|^2 |G_\delta \sigma(s, u(s))|_{\mathcal{L}}^2 ds \right)^{1/2} \\ &\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T} |G_\delta u(t)|_{L^2}^2 + \frac{C^2}{2} \mathbb{E} \int_0^T |G_\delta \sigma(s, u(s))|_{\mathcal{L}}^2 ds. \end{aligned}$$

Hence for some constant C , (3.27) yields

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} |G_\delta u(t)|_{L^2}^2 &\leq 2|G_\delta u_0|_{L^2}^2 + C \mathbb{E} \int_0^T |G_\delta \sigma(s, u(s))|_{\mathcal{L}}^2 ds \\ &\quad + 4 \mathbb{E} \int_0^T |\langle B(u(s)) + 2a|u(s)|^{2\alpha}u(s), G_\delta^2 u(s) \rangle| ds. \end{aligned}$$

Since for every $u \in H$, $|G_\delta u|_{L^2} \rightarrow 0$ as $\delta \rightarrow 0$ and $\sup_{\delta > 0} |G_\delta|_{L(H,H)} \leq 2$, we deduce that if $\{\varphi_k\}$ denotes an orthonormal basis in H , then $|G_\delta \sigma(s, u_h(s)) Q^{1/2} \varphi_k|_{L^2}^2 \rightarrow 0$ for every k and almost every $(\omega, t) \in \Omega \times [0, T]$. Since

$$\sup_{\delta > 0} |G_\delta \sigma(s, u(s))|_{\mathcal{L}}^2 \leq \sum_k \sup_{\delta > 0} |G_\delta \sigma(s, u(s)) Q^{1/2} \varphi_k|_{L^2}^2 \leq C |\sigma(s, u(s))|_{\mathcal{L}}^2 \in L^1(\Omega \times [0, T]),$$

the Lebesgue dominated convergence theorem implies $\mathbb{E} \int_0^T |G_\delta \sigma(s, u(s))|_{\mathcal{L}}^2 ds \rightarrow 0$. Given $u \in \tilde{H}^{1,1} \subset H^1$ we have $\|G_\delta^2 u\|_{H^1} \rightarrow 0$ as $\delta \rightarrow 0$; furthermore, $\sup_{\delta > 0} |G_\delta|_{L(\tilde{H}^{1,1}, \tilde{H}^{1,1})} \leq 2$. Hence $\langle B(u(s)), G_\delta^2 u(s) \rangle \rightarrow 0$ for almost every $(\omega, s) \in \Omega_T$. Furthermore, $e^{-\delta A}$ is a bounded operator of $L^{2(\alpha+1)}$ (see e.g. the Appendix of [14]). Hence $\langle |u(s)|^{2\alpha}u(s), G_\delta^2 u(s) \rangle \rightarrow 0$ for almost every $(\omega, s) \in \Omega_T$. Therefore, as above, the Lebesgue dominated convergence theorem concludes the proof of (3.26).

Step 5: Pathwise uniqueness of the solution We finally prove that if L_2 is small enough, there exists a unique process in \mathcal{X} and a.s. in $C([0, T]; H)$ which is a weak solution to (2.3). Let $u, v \in \mathcal{X}$ be solutions to (2.3) and belong a.s. to $C([0, T]; H)$. For every N set

$$\tau_N = \inf\{s \geq 0 : |u(s)|_{L^2} \vee |v(s)|_{L^2} \geq N\} \wedge T.$$

Since $|u(\cdot)|_{L^2}$ and $|v(\cdot)|_{L^2}$ are a.s. bounded on $[0, T]$ by the definition of \mathcal{X} , we deduce that a.s. $\tau_N \rightarrow T$ as $N \rightarrow \infty$. Set $U = u - v$; since $L_2 < 2\nu$, we may choose $\eta \in (0, \nu)$ such that $L_2 < 2\eta < 2\nu$. Let C_η be a constant defined in (2.26); as in the argument of Step 4, despite of the lack of regularity of u , we may apply Itô's formula to the square of the H norm and deduce

$$e^{-2C_\eta \int_0^{t \wedge \tau_N} \|v(r)\|_{1,1}^2 dr} |U(t \wedge \tau_N)|_{L^2}^2 = 2M(t \wedge \tau_N) + \int_0^{t \wedge \tau_N} \psi(s) ds,$$

where

$$\begin{aligned} M(\tau) &= \int_0^\tau e^{-2C_\eta \int_0^s \|v(r)\|_{1,1}^2 dr} (U(s), [\sigma(s, u(s)) - \sigma(s, v(s))] dW(s)), \\ \psi(s) &= e^{-2C_\eta \int_0^s \|v(r)\|_{1,1}^2 dr} [-2C_\eta \|v(s)\|_{1,1}^2 |U(s)|_{L^2}^2 + 2\langle F(u(s)) - F(v(s)), U(s) \rangle \\ &\quad + |\sigma(s, u(s)) - \sigma(s, v(s))|_{\mathcal{L}}^2]. \end{aligned}$$

We at first check that the process M is a square integrable martingale. Indeed, the Cauchy-Schwarz and the Young inequalities, the Lipschitz condition **(C)(ii)** and the definition of \mathcal{X} imply that

$$\begin{aligned} \mathbb{E} \int_0^T e^{-4C_\eta \int_0^s \|v(r)\|_{1,1}^2 dr} |U(s)|_{L^2}^2 |\sigma(s, u(s)) - \sigma(s, v(s))|_{\mathcal{L}}^2 ds \\ \leq \mathbb{E} \int_0^T |U(s)|_{L^2}^2 [L_1 |U(s)|_{L^2}^2 + L_2 |\nabla_h U(s)|_{L^2}^2] ds \\ \leq C \mathbb{E} \left(\sup_{t \in [0, T]} |U(s)|_{L^2}^4 \right) + C \mathbb{E} \left(\left| \int_0^T |\nabla_h U(s)|_{L^2}^2 ds \right|^2 \right) < \infty. \end{aligned}$$

Furthermore, the upper estimate (2.26) and the Lipschitz condition **(C)(ii)** imply that for $L_2 < 2\eta < 2\nu$, we have

$$|\psi(s)| \leq (L_2 - 2\eta) |\nabla_h U(s)|_{L^2}^2 + L_1 |U(s)|_{L^2}^2 \leq L_1 |U(s)|_{L^2}^2.$$

Hence taking expected values, we deduce that for any $t \in [0, T]$:

$$\mathbb{E} \left(e^{-2C_\eta \int_0^{t \wedge \tau_N} \|v(r)\|_{1,1}^2 dr} |U(t \wedge \tau_N)|_{L^2}^2 \right) \leq \int_0^t \mathbb{E} \left(e^{-2C_\eta \int_0^{s \wedge \tau_N} \|v(r)\|_{1,1}^2 dr} |U(s \wedge \tau_N)|_{L^2}^2 \right) ds.$$

The Gronwall lemma implies that for every $t \in [0, T]$, we have $U(t \wedge \tau_N) = 0$ a.s. Since U a.s. belongs to $C([0, T]; H)$, this completes the proof as $N \rightarrow \infty$. \square

3.3. Examples. Here, we provide two examples of coefficients σ which satisfy condition **(C)**

Let $\{\psi_k, k \geq 1\}$ denote an orthonormal basis of $H_0 = Q^{\frac{1}{2}} \tilde{H}^{0,1}$ and for $t \in [0, T]$, $u \in \tilde{H}^{1,1}$ and $\psi \in H_0$; set

$$\sigma(t, u)\psi(x) := \sum_{k=1}^{\infty} (\psi, \psi_k)_0 \sigma_k(t, x, u(x), \nabla_h u(x)),$$

where $\sigma_k : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^6 \rightarrow \mathbb{R}^3$ are measurable functions with appropriate regularity and $\nabla_h = (\partial_1 u, \partial_2 u)$.

Example 1: For $t \in [0, T]$, $x \in \mathbb{R}^3$, $y \in \mathbb{R}^3$ and $z = (\zeta, \tilde{\zeta})$ for $\zeta, \tilde{\zeta} \in \mathbb{R}^3$ set

$$\sigma_k(t, x, y, z) = \sigma_{k,0}(t, x) + \sigma_{k,1}(t, x)y + \sigma_{k,2}(t, x)\zeta + \tilde{\sigma}_{k,2}(t, x)\tilde{\zeta},$$

where $\sigma_{k,0}(t, \cdot) \in \tilde{H}^{0,1}$, $\sigma_{k,1}(t, \cdot)$, $\sigma_{k,2}(t, \cdot)$, $\tilde{\sigma}_{k,2}(t, \cdot)$, $\partial_3 \sigma_{k,0}(t, \cdot)$; $\partial_3 \sigma_{k,2}(t, \cdot)$ and $\partial_3 \tilde{\sigma}_{k,2}(t, \cdot)$ belong to $L^\infty(\mathbb{R}^3)$. Suppose furthermore that:

$$\sup_{t \in [0, T]} \sum_{k \geq 1} \left[\|\sigma_{k,0}(t, \cdot)\|_{0,1}^2 + \|\sigma_{k,1}(t, \cdot)\|_{L^\infty}^2 + \|\sigma_{k,2}(t, \cdot)\|_{L^\infty}^2 + \|\tilde{\sigma}_{k,2}(t, \cdot)\|_{L^\infty}^2 \right] < \infty,$$

$$\sup_{t \in [0, T]} \sum_{k \geq 1} \left[\|\partial_3 \sigma_{k,1}(t, x)\|_{L^\infty}^2 + \|\partial_3 \sigma_{k,2}(t, x)\|_{L^\infty}^2 + \|\partial_3 \tilde{\sigma}_{k,2}(t, x)\|_{L^\infty}^2 \right] < \infty.$$

Then condition (2.32) holds with $K_0 = 3 \sup_t \sum_k \|\sigma_{k,0}(t, \cdot)\|_{L^2}^2$, $K_1 = 3 \sup_t \sum_k \|\sigma_{k,1}(t, \cdot)\|_{L^\infty}^2$ and $K_2 = 3 \sup_t \sum_k (\|\sigma_{k,2}(t, \cdot)\|_{L^\infty}^2 + \|\tilde{\sigma}_{k,2}(t, \cdot)\|_{L^\infty}^2)$. The Lipschitz condition **(C)(ii)** holds with $L_1 = \frac{2}{3}K_1$ and $L_2 = \frac{2}{3}K_2$.

Taking the partial derivative with respect to x_3 , we deduce that (2.33) holds with

$$\tilde{K}_0 = 5 \sup_t \sum_k \|\sigma(t, \cdot)\|_{0,1}^2,$$

$$\tilde{K}_1 = K_1 + 5 \sup_t \sum_k \left(\|\sigma_{k,1}(t, \cdot)\|_{L^\infty}^2 + \|\partial_3 \sigma_{k,1}(t, \cdot)\|_{L^\infty}^2 \right)$$

and finally

$$\tilde{K}_2 = K_2 + 5 \sup_t \sum_k \left(\|\sigma_{k,2}(t, \cdot)\|_{L^\infty}^2 + \|\tilde{\sigma}_{k,2}(t, \cdot)\|_{L^\infty}^2 + \|\partial_3 \sigma_{k,2}(t, \cdot)\|_{L^\infty}^2 + \|\partial_3 \tilde{\sigma}_{k,2}(t, \cdot)\|_{L^\infty}^2 \right).$$

Example 2 The following example has some more general Lipschitz structure.

For $t \in [0, T]$, $x \in \mathbb{R}^3$, $y, y' \in \mathbb{R}^3$ and $z, z' \in \mathbb{R}^6$ set

$$|\sigma_k(t, x, y, z) - \sigma_k(t, x, y', z')| \leq C_{k,1}(t, x)|y - y'| + C_{k,2}(t, x)|z - z'|,$$

$$|\partial_{x_3} \sigma_k(t, x, y, z)| \leq \tilde{C}_{k,0}(t, x) + \tilde{C}_{k,1}(t, x)|y| + \tilde{C}_{k,2}(t, x)|z|,$$

where $\sigma_k(t, \cdot, 0, 0)$ and $\tilde{C}_{k,0}$ belong to $L^2(\mathbb{R}^3)$, while $C_{k,1}(t, \cdot)$, $C_{k,2}(t, \cdot)$, $\tilde{C}_{k,1}(t, \cdot)$ and $\tilde{C}_{k,2}(t, \cdot)$ belong to $[L^\infty(\mathbb{R}^3)]^3$. Moreover, we suppose that

$$\sup_{t \in [0, T]} \sum_{k \geq 1} \sup_{(x, y, z) \in \mathbb{R}^{12}} |\nabla_y \sigma_k(t, x, y, z)|^2 = \tilde{C}_3 < \infty,$$

$$\sup_{t \in [0, T]} \sum_{k \geq 1} \sup_{(x, y, z) \in \mathbb{R}^{12}} |\nabla_z \sigma_k(t, x, y, z)|^2 = \tilde{C}_4 < \infty,$$

and

$$\sup_{t \in [0, T]} \sum_{k \geq 1} \left(\|\sigma_k(t, \cdot, 0, 0)\|_{L^2}^2 + \|\tilde{C}_{k,0}(t, \cdot)\|_{L^2}^2 \right) < \infty$$

$$\sup_{t \in [0, T]} \sum_{k \geq 1} \left(\|C_{k,1}(t, \cdot)\|_{L^\infty}^2 + \|C_{k,2}(t, \cdot)\|_{L^\infty}^2 + \|\tilde{C}_{k,1}(t, \cdot)\|_{L^\infty}^2 + \|\tilde{C}_{k,2}(t, \cdot)\|_{L^\infty}^2 \right) < \infty.$$

The growth condition (2.32) holds with:

$$K_0 = 3 \sup_t \sum_k \|\sigma_k(t, \cdot, 0, 0)\|_{L^2}^2, \quad K_1 = 3 \sup_t \sum_k \|C_{k,1}(t, \cdot)\|_{L^\infty}^2, \quad K_2 = 3 \sup_t \sum_k \|C_{k,2}(t, \cdot)\|_{L^\infty}^2.$$

The Lipschitz condition **(C)(ii)** holds with $L_1 = \frac{2}{3}K_1$ and $L_2 = \frac{2}{3}K_2$. Taking partial derivatives with respect to x_3 yields that the growth condition (2.33) is satisfied with:

$$\tilde{K}_0 = K_0 + 5 \sup_t \sum_k \|\tilde{C}_{k,0}(t, \cdot)\|_{L^2}^2,$$

$$\begin{aligned}\tilde{K}_1 &= K_1 + 5\tilde{C}_3 + \sup_t \sum_k (3\|C_{k,1}(t, \cdot)\|_{L^\infty}^2 + 5\|\tilde{C}_{k,1}(t, \cdot)\|_{L^\infty}^2), \\ \tilde{K}_2 &= K_2 + 5\tilde{C}_4 + \sup_t \sum_k (3\|C_{k,2}(t, \cdot)\|_{L^\infty}^2 + 5\|\tilde{C}_{k,2}(t, \cdot)\|_{L^\infty}^2).\end{aligned}$$

4. LARGE DEVIATIONS

Recall that the set of processes \mathcal{X} has been defined in (2.7). For $\epsilon > 0$, let $u^\epsilon \in \mathcal{X}$ such that $u^\epsilon \in C([0, T]; H)$ a.s. denote the solution of (2.3) where the noise intensity is multiplied by a small parameter $\epsilon > 0$, that is

$$u^\epsilon(t) = u_0 + \int_0^t [\nu A_h u^\epsilon(s) - B(u^\epsilon(s)) - a|u^\epsilon(s)|^{2\alpha} u^\epsilon(s)] ds + \sqrt{\epsilon} \int_0^t \sigma(s, u^\epsilon(s)) dW(s). \quad (4.1)$$

For any constants K_i , \tilde{K}_i and \tilde{L}_i in Condition **(C)**, for ϵ small enough there is a unique solution to (4.1) which is denoted $u^\epsilon = \mathcal{G}^\epsilon(\sqrt{\epsilon}W)$ for some Borel-measurable function $\mathcal{G}^\epsilon : C([0, T]; \tilde{H}^{0,1}) \rightarrow X$.

In this section we prove that u^ϵ satisfies a large deviations principle in the space $Y := C([0, T]; H) \cap L^2(0, T; \tilde{H}^{1,0})$. We use the weak convergence approach introduced in [6] and [7]. We at first prove apriori estimate for stochastic control equations deduced from (2.3) by shifting W by some random element. To describe a set of admissible random shifts, we introduce the class \mathcal{A} as the set of H_0 -valued (\mathcal{F}_t) -predictable stochastic processes ϕ such that $\int_0^T |\phi(s)|_0^2 ds < \infty$, a.s. Let

$$S_M = \left\{ \phi \in L^2(0, T; H_0) : \int_0^T |\phi(s)|_0^2 ds \leq M \right\}.$$

The set S_M endowed with the following weak topology is a Polish space (complete separable metric space) [7]: $d_1(\phi, \psi) = \sum_{i=1}^\infty \frac{1}{2^i} \left| \int_0^T (\phi(s) - \psi(s), \tilde{e}_i(s))_0 ds \right|$, where $\{\tilde{e}_i(s)\}_{i=1}^\infty$ is an orthonormal basis for $L^2(0, T; H_0)$. Define

$$\mathcal{A}_M = \{ \phi \in \mathcal{A} : \phi(\omega) \in S_M, \text{ a.s.} \}. \quad (4.2)$$

Let $\mathcal{B}(Y)$ denote the Borel σ -field of the Polish space Y endowed with the metric associated with the norm

$$\|u\|_Y = \sup_{t \in [0, T]} |u(t)|_{L^2} + \left(\int_0^T \|u(t)\|_{1,0}^2 ds \right)^{\frac{1}{2}}. \quad (4.3)$$

We recall some classical definitions; by convention the infimum over an empty set is $+\infty$.

Definition 4.1. *The random family (u^ϵ) is said to satisfy a large deviation principle on Y with the good rate function I if the following conditions hold:*

***I is a good rate function.** The function $I : Y \rightarrow [0, \infty]$ is such that for each $M \in [0, \infty[$ the level set $\{\phi \in Y : I(\phi) \leq M\}$ is a compact subset of Y .*

For $A \in \mathcal{B}(Y)$, set $I(A) = \inf_{u \in A} I(u)$.

Large deviation upper bound. *For each closed subset F of Y :*

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(u^\epsilon \in F) \leq -I(F).$$

Large deviation lower bound. *For each open subset G of Y :*

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(u^\epsilon \in G) \geq -I(G).$$

For all $\phi \in L^2(0, T; H_0)$, we will prove that there exists a unique solution let $u_\phi^0 \in Y$ of the deterministic control equation (4.4) with initial condition $u_\phi^0(0) = u_0 \in L^4(\Omega, \tilde{H}^{0,1})$:

$$du_\phi^0(t) + [-\nu A_h u_\phi^0(t) + B(u_\phi^0(t)) + a|u_\phi^0(t)|^{2\alpha} u_\phi^0(t)]dt = \sigma(t, u_\phi^0(t))\phi(t)dt. \quad (4.4)$$

Let $\mathcal{C}_0 = \{\int_0^\cdot \phi(s)ds : \phi \in L^2(0, T; H_0)\} \subset C([0, T], H_0)$. Define $\mathcal{G}^0 : C([0, T], H_0) \rightarrow Y$ by $\mathcal{G}^0(\Phi) = u_\phi$ for $\Phi = \int_0^\cdot \phi(s)ds \in \mathcal{C}_0$ and $\mathcal{G}^0(\Phi) = 0$ otherwise.

Since the argument below requires some information about the difference of the solution at two different times, we need an additional assumption about the regularity of the map $\sigma(\cdot, u)$. Furthermore, for technical reasons, we will suppose that condition **(C)** holds with stronger growth and Lipschitz conditions, which forbid any gradient. This is summarized in the following:

Condition (C')

(i) (*Stronger growth and Lipschitz conditions*): The coefficient σ satisfies condition **(C)** with the constants $K_2 = \tilde{K}_2 = L_2 = 0$.

(ii) (*Time Hölder regularity of σ*): There exist constants $\gamma > 0$ and $C \geq 0$ such that for $t_1, t_2 \in [0, T]$ and $u \in \tilde{H}^{1,0}$:

$$|\sigma(t_1, u) - \sigma(t_2, u)|_{\mathcal{L}} \leq C (1 + \|u\|_{1,0}) |t_1 - t_2|^\gamma.$$

The following theorem is the main result of this section.

Theorem 4.2. *Suppose that condition (C') is satisfied and that $u_0 \in \tilde{\mathcal{H}}^{0,1}$. Then the solution (u^ε) to (4.1) satisfies the large deviation principle in $Y = C([0, T]; H) \cap L^2(0, T; \tilde{H}^{1,0})$, with the good rate function*

$$I_\xi(u) = \inf_{\{\phi \in L^2(0, T; H_0) : u = \mathcal{G}^0(\int_0^\cdot \phi(s)ds)\}} \left\{ \frac{1}{2} \int_0^T |\phi(s)|_0^2 ds \right\}. \quad (4.5)$$

The proof relies on properties of a stochastic control equation. Let $M > 0$, $\phi \in \mathcal{A}_M$ and $u_0 \in L^4(\Omega; \tilde{H}^{0,1})$. Suppose that σ satisfies condition **(C')(i)** and consider the following non linear SPDE with initial condition $u_\phi(0) = u_0$:

$$\begin{aligned} dt u_\phi(t) + [-\nu A_h \Delta_h u_\phi(t) + B(u_\phi(t)) + a|u_\phi(t)|^{2\alpha} u_\phi(t)] dt \\ = \sigma(t, u_\phi(t)) dW(t) + \sigma(t, u_\phi(t)) \phi(t) dt. \end{aligned} \quad (4.6)$$

The following theorem shows that Theorem 3.2 holds in this setting. Its proof, which is similar to that of Theorem 3.2 (see also Theorem 2.4 in [13]), is given in the appendix. Note that the result would still be valid with "small enough" K_2 , \tilde{K}_2 and L_2 . However, some further arguments needed to prove the Large Deviations Principle require these coefficients to vanish.

Theorem 4.3. *Let σ satisfy condition (C')(i) Then for every $M > 0$ and $T > 0$ and any \mathcal{F}_0 -measurable u_0 such that $\mathbb{E}\|u_0\|_{0,1}^4 < \infty$ and any $\phi \in \mathcal{A}_M$, there exists a unique weak solution u_ϕ in \mathcal{X} of the equation (4.6) with initial data $u_\phi(0) = u_0 \in L^4(\Omega; \tilde{H}^{0,1})$. Furthermore, $u_\phi \in C(0, T; H)$ a.s. and there exists a constant $C := C(K_0, K_1, \tilde{K}_0, \tilde{K}_1, T, M)$ such that for $\phi \in \mathcal{A}_M$,*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \|u_\phi(t)\|_{0,1}^4 + \left(\int_0^T \|u_\phi(t)\|_{1,1}^2 dt \right)^2 + \int_0^T \|u_\phi(t)\|_{L^{2\alpha+2}}^{2\alpha+2} dt \right) \leq C (1 + E\|u_0\|_{0,1}^4). \quad (4.7)$$

We next consider stochastic control evolution equations deduced from (4.1) by a random shift by a function $\phi \in \mathcal{A}_M$, that is the solution u_ϕ^ε to the evolution equation:

$$\begin{aligned} u_\phi^\varepsilon(t) &= u_0 + \int_0^t [\nu A_h u_\phi^\varepsilon(s) - B(u_\phi^\varepsilon(s)) - a|u_\phi^\varepsilon(s)|^{2\alpha} u_\phi^\varepsilon(s) + \sigma(s, u_\phi^\varepsilon(s))\phi(s)] ds \\ &\quad + \sqrt{\varepsilon} \int_0^t \sigma(s, u_\phi^\varepsilon(s)) dW(s). \end{aligned} \quad (4.8)$$

Let $\varepsilon_0 > 0$, $(\phi_\varepsilon, 0 < \varepsilon \leq \varepsilon_0)$ be a family of random elements taking values in the set \mathcal{A}_M given by (4.2). Let $u_{\phi_\varepsilon}^\varepsilon$, be the solution of the corresponding stochastic control equation with initial condition $u_{\phi_\varepsilon}^\varepsilon(0) = u_0 \in \tilde{H}^{0,1}$:

$$\begin{aligned} d_t u_{\phi_\varepsilon}^\varepsilon(t) &+ [-\nu A_h u_{\phi_\varepsilon}^\varepsilon(t) + B(u_{\phi_\varepsilon}^\varepsilon(t)) + a|u_{\phi_\varepsilon}^\varepsilon(t)|^{2\alpha} u_{\phi_\varepsilon}^\varepsilon(t)] dt \\ &= \sigma(t, u_{\phi_\varepsilon}^\varepsilon(t)) [\phi_\varepsilon(t) dt + \sqrt{\varepsilon} dW(t)]. \end{aligned} \quad (4.9)$$

Note that for $W^\varepsilon = W + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \phi_\varepsilon(s) ds$ we have $u_{\phi_\varepsilon}^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon} W^\varepsilon)$.

The following proposition establishes the weak convergence of the family (u_{ϕ_ε}) as $\varepsilon \rightarrow 0$. Its proof, which is similar to that of Proposition 4.3 in [15] (see also Proposition 3.4 in [13]), is given in the appendix.

Proposition 4.4. *Suppose that condition (C') is satisfied. Let u_0 be \mathcal{F}_0 -measurable such that $E\|u_0\|_{0,1}^4 < +\infty$, and let ϕ_ε converge to ϕ in distribution as random elements taking values in \mathcal{A}_M , where this set is defined by (4.2) and endowed with the weak topology of the space $L_2(0, T; H_0)$. Then as $\varepsilon \rightarrow 0$, the solution $u_{\phi_\varepsilon}^\varepsilon$ of (4.9) converges in distribution to the solution u_ϕ^0 of (4.4) in $Y = C([0, T]; H) \cap L^2(0, T; \tilde{H}^{1,0})$ endowed with the norm (4.3). That is, as $\varepsilon \rightarrow 0$, $\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}\left(W + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \phi_\varepsilon(s) ds\right)\right)$ converges in distribution to $\mathcal{G}^0\left(\int_0^\cdot \phi(s) ds\right)$ in Y .*

The following compactness result is the second ingredient which allows to transfer the LDP from $\sqrt{\varepsilon}W$ to u^ε . Its proof is similar to that of Proposition 4.4 and easier; it will be sketched in the appendix.

Proposition 4.5. *Suppose that condition (C') holds. Fix $M > 0$, $u_0 \in \tilde{H}^{0,1}$ and let $K(M) = \{u_\phi^0 \in X : \phi \in S_M\}$, where u_ϕ^0 is the unique solution of the deterministic control equation (4.4), and let $Y = C([0, T]; H) \cap L^2(0, T; \tilde{H}^{1,0})$. Then $K(M)$ is a compact subset of Y .*

Using the above results, we can complete the proof of the Large Deviations Principle for our stochastic Brinkman-Forchheimer 3D Navier-Stokes equations.

Proof of Theorem 4.2: Propositions 4.5 and 4.4 imply that the family $\{u^\varepsilon\}$ satisfies the Laplace principle, which is equivalent to the large deviation principle, in Y with the good rate function defined by (4.5); see Theorem 4.4 in [6] or Theorem 5 in [7]. This concludes the proof of Theorem 4.2. \square

5. APPENDIX

The computations in this section are similar to the ones established for the stochastic equation (2.3). Equation (4.4) is a particular case of equation (4.6) and the proof of the well posedness of (4.6) follows the steps used to prove that of (2.3). However, for the sake of completeness, we show some of the estimates that are performed for (4.6) to show how the extra term $\sigma(t, u_\phi(t))\phi(t)$ with respect to (2.3) can be dealt with.

5.1. A priori estimates for the stochastic control equation. In this section we will only show how to obtain the estimates given in Theorem 4.3. The argument is similar to that of Theorem 3.2 (see also Theorem 2.4 in [13]). We briefly sketch it only pointing out the changes to be made to deal with the random shift ϕ .

We at first consider an analog of (3.1). For $t \in [0, T]$, $\phi \in \mathcal{A}_M$, $v \in \mathcal{H}_n$ and $u_{n,\phi}(0) = P_n u_0$, let $u_{n,\phi}$ be defined on \mathcal{H}_n as follows:

$$d(u_{n,\phi}(t), v) = \langle F(u_{n,\phi}(t), v) dt + (P_n \sigma(t, u_{n,\phi}(t))) dW_n(t), v \rangle + (P_n \sigma(t, u_{n,\phi}(t))) \Pi_n \phi(t), v \rangle dt. \quad (5.1)$$

We check that an analog of (3.2) can be obtained for these processes with a constant C which only depends on M (but not on ϕ and n). We let $\tau_N = \inf\{t : \|u_{n,\phi}(t)\|_{0,1} \geq N\} \wedge T$.

We apply the Itô formula to $\|\cdot\|_{0,1}^2$ and the process $u_{n,\phi}$. This yields an equation similar to (3.3) where u_n is replaced by $u_{n,\phi}$, and where we add the term $T_4(t)$ in the right hand side, with

$$T_4(t) = 2 \int_0^{t \wedge \tau_N} (\sigma(s, u_{n,\phi}(s)) \phi, u_{n,\phi}(s)) ds.$$

The growth condition (2.33) with $\tilde{K}_2 = 0$, the Cauchy-Schwarz inequality, and the inequality $|y| \leq 1 + y^2$ imply

$$\begin{aligned} |T_4(t)| &\leq 2 \int_0^{t \wedge \tau_N} \left[\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1} |u_{n,\phi}(s)|_{L^2} \right] |\phi(s)|_0 \|u_{n,\phi}(s)\|_{0,1} ds \\ &\leq 2\sqrt{\tilde{K}_0} MT + 2\left(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\right) \int_0^{t \wedge \tau_N} |\phi(s)|_0 \|u_{n,\phi}(s)\|_{0,1}^2 ds. \end{aligned}$$

Fix $\epsilon > 0$; as in the proof of Proposition 3.1, choose $\epsilon_0 > 0$ small enough to ensure $2C\epsilon_0 < 2\nu - \epsilon$, where C is the constant in the right hand side of (2.22), and then $\epsilon_1 > 0$ small enough to ensure $\frac{\epsilon_1}{4\epsilon_0} < 2a(2\alpha + 1) - \epsilon$. Set

$$\begin{aligned} X(t) &= \sup_{s \leq t \wedge \tau_N} \|u_{n,\phi}(s)\|_{0,1}^2 + 2a \int_0^{t \wedge \tau_N} \|u_{n,\phi}(s)\|_{L^{2\alpha+2}}^{2\alpha+2} ds, \\ Y(t) &= \int_0^{t \wedge \tau_N} \left[(|\nabla_h u_{n,\phi}(s)|_{L^2}^2 + |\partial_3 \nabla_h u_{n,\phi}(s)|_{L^2}^2) + \|u_{n,\phi}(s)\|_{L^{2\alpha+2}}^{2\alpha+2} \right] ds. \end{aligned}$$

For this choice of constants, we deduce that

$$X(t) + \epsilon Y(t) \leq Z + \int_0^t \varphi(r) X(r) dr + I(t),$$

where $\varphi(r) = \tilde{K}_1 + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} + 2(\sqrt{\tilde{K}_1} + \sqrt{\tilde{K}_0}) |\phi(r)|_0$ and

$$Z = \|u_0\|_{0,1}^2 + \tilde{K}_0 T + 2\sqrt{\tilde{K}_0} TM, \quad I(t) = 2 \sup_{s \in [0, T]} \left| \int_0^{s \wedge \tau_N} (\sigma(r, u_\phi(r)) dW(r), u_\phi(r))_{0,1} \right|.$$

The Burkholder-Davies-Gundy inequality, the growth condition (2.33) with $\tilde{K}_2 = 0$ and arguments similar to those in the proof of Proposition 3.1 imply that for $\beta \in (0, 1)$, $\gamma = \frac{9}{\tilde{K}_1}$, $\tilde{C} = \frac{9}{\beta} \tilde{K}_0 T$ we have

$$\mathbb{E}(I(t)) \leq \beta \mathbb{E}(X(t)) + \gamma \int_0^t \mathbb{E}(X(s)) ds + \tilde{C}$$

Then $\int_0^T \varphi(s) ds \leq \tilde{K}_1 T + C_\alpha \epsilon_0^{-1} \epsilon_1^{-\frac{1}{\alpha-1}} T + 2\left(\sqrt{\tilde{K}_1} + \sqrt{\tilde{K}_0}\right) \sqrt{MT} := C(1)$.

Since ϕ is random, we need an extension of Gronwall's lemma (see [15], Lemma 3.9 for the proof of a more general result).

Lemma 5.1. *Let X, Y, I and φ be non-negative processes and Z be a non-negative integrable random variable. Assume that I is non-decreasing and there exist non-negative constants C, κ, β, γ with the following properties*

$$\int_0^T \varphi(s) ds \leq C \text{ a.s.}, \quad 2\beta e^C \leq 1, \quad (5.2)$$

and such that for $0 \leq t \leq T$,

$$\begin{aligned} X(t) + \kappa Y(t) &\leq Z + \int_0^t \varphi(r) X(r) dr + I(t), \text{ a.s.}, \\ \mathbb{E}(I(t)) &\leq \beta \mathbb{E}(X(t)) + \gamma \int_0^t \mathbb{E}(X(s)) ds + \tilde{C}, \end{aligned}$$

where $\tilde{C} > 0$ is a constant. If $X \in L^\infty([0, T] \times \Omega)$, then we have

$$\mathbb{E}[X(t) + \kappa Y(t)] \leq 2 \exp(C + 2t\gamma e^C) (\mathbb{E}(Z) + \tilde{C}), \quad t \in [0, T]. \quad (5.3)$$

Lemma 5.1 implies that for all $t \in [0, T]$ we have $\mathbb{E}(X(t) + \epsilon Y(t)) \leq 2 \exp(C(1) + 2t\gamma e^{C(1)}) [\mathbb{E}Z + \tilde{C}]$.

Hence there exists a constant C , which only depends on M, T and the constants \tilde{K}_i , $i = 0, 1$ in Condition **(C)**, such that for every $\phi \in \mathcal{A}_M$

$$\mathbb{E} \left(\sup_{s \in [0, T]} \|u_{n, \phi}(s \wedge \tau_N)\|_{0,1}^2 + \int_0^{\tau_N} (\|u_{n, \phi}(s)\|_{1,1}^2 + \|u_{n, \phi}(s)\|_{L^{2\alpha+2}}^{2\alpha+2}) ds \right) \leq C(1 + \mathbb{E}\|u_0\|_{0,1}^2). \quad (5.4)$$

We then apply once more the Itô formula to the square of $\|u_{n, \phi}\|_{0,1}^2$. This yields an upper estimate similar to (3.6) with $u_{n, \phi}$ instead of u_n , and where we add $\tilde{T}_5(t)$ in the right hand side, with

$$\tilde{T}_5(t) = 4 \int_0^{t \wedge \tau_N} (\sigma(s, u_{n, \phi}(s)) \phi(s), u_{n, \phi}(s))_{0,1} \|u_{n, \phi}(s)\|_{0,1}^2 ds.$$

Using the Cauchy-Schwarz inequality and the growth condition (2.33) with $\tilde{K}_2 = 0$, we deduce that

$$|\tilde{T}_5(t)| \leq 4 \int_0^{t \wedge \tau_N} \left(\sqrt{\tilde{K}_1} + \sqrt{\tilde{K}_0} \right) \|u_{n, \phi}(s)\|_{0,1}^4 |\phi(s)|_0 ds + 4\sqrt{\tilde{K}_0 TM}.$$

Let

$$\bar{X}(t) = \sup_{s \in [0, t]} \|u_{n, \phi}(s \wedge \tau_N)\|_{0,1}^4, \quad \bar{Y}(t) = \int_0^{t \wedge \tau_N} \|u_{n, \phi}(s)\|_{0,1}^2 (|\nabla_h u_{n, \phi}(s)|_{L^2}^2 + \partial_3 \nabla_h u_{n, \phi}(s)|_{L^2}^2) ds.$$

Then choosing again ϵ_0 and ϵ_1 small enough, we deduce that for some $\epsilon > 0$,

$$\bar{X}(t) + \epsilon \bar{Y}(t) \leq \bar{Z} + \bar{I}(t) + \int_0^t \bar{\varphi}(s) \bar{X}(s) ds,$$

where $\bar{\varphi}(s) = 6\tilde{K}_1 + 4(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1})|\phi(s)|_0$, $\bar{I}(t) = \sup_{s \in [0, t]} \tilde{T}_2(s)$ for $\tilde{T}_2(s)$ defined in (3.6) and $\bar{Z} = \sqrt{4\tilde{K}_0 TM} + 6\tilde{K}_0 \int_0^{\tau_N} \|u_{n, \phi}(s)\|_{0,1}^2 ds$. Then $\int_0^T \bar{\varphi}(s) ds \leq C(2) := 6\tilde{K}_1 T + 4(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1})\sqrt{TM}$. For $\bar{\beta} \in (0, 1)$ and $\bar{\gamma} = \frac{36}{\bar{\beta}} \tilde{K}_1$, we have $\mathbb{E}\bar{I}(t) \leq \bar{\beta} \mathbb{E}\bar{X}(t) + \bar{\gamma} \int_0^t \mathbb{E}\bar{X}(s) ds + \bar{C}'$

where $\bar{C}' = \frac{36}{\beta} \mathbb{E} \int_0^{\tau_N} \|u_{n,\phi}(s)\|_{0,1}^2 ds < \infty$ by (5.4). Using once more Lemma 5.1 we deduce the existence of a constant C depending on M, T and the constants \tilde{K}_i in (2.33) such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_{n,\phi}(s \wedge \tau_N)\|_{0,1}^4 + \int_0^{\tau_N} \|u_{n,\phi}(s)\|_{0,1}^2 \|u_{n,\phi}(s)\|_{1,1}^2 ds \right] \leq C(1 + \mathbb{E}\|u_0\|_{0,1}^4). \quad (5.5)$$

holds for any $\phi \in \mathcal{A}_M$.

This estimate being established, we follow the steps in the proof of Theorem 3.2 and prove that the weak limit u_ϕ of a proper subsequence of the sequence $(u_{n,\phi}, n \geq 1)$ is a solution of the evolution equation (4.6). In order to conclude the proof of Theorem 4.3, it remains only to prove the almost sure continuity of the process u_ϕ .

Let $W^\phi(t) = W(t) + \int_0^t \phi(s) ds$; the Girsanov theorem implies that W^ϕ is a Brownian motion under the probability $\tilde{\mathbb{P}}$ with density $\exp(-\int_0^t \phi(s) dW(s) - \frac{1}{2} \int_0^t |\phi(s)|_0^2 ds)$ with respect to \mathbb{P} on \mathcal{F}_t . Under $\tilde{\mathbb{P}}$ the process u_ϕ is the unique solution to the evolution equation (2.3) in \mathcal{X} and belongs $\tilde{\mathbb{P}}$ a.s. to $C([0, T] : H)$. Since the probabilities $\tilde{\mathbb{P}}$ and \mathbb{P} are equivalent and this completes the proof of Theorem 4.3.

5.2. Weak convergence of the stochastic control equations (Proposition 4.4).

We at first prove the following technical lemma, which studies time increments of the solution to the stochastic control problem (4.8). To state the lemma mentioned above, we need the following notations. For every integer n , let $\psi_n : [0, T] \rightarrow [0, T]$ denote a measurable map such that for every $s \in [0, T]$, $s \leq \psi_n(s) \leq (s + c2^{-n}) \wedge T$ for some positive constant c . Given $N > 0$, $\phi \in \mathcal{A}_M$, and for $t \in [0, T]$, let

$$G_N(t) = \left\{ \omega : \left(\sup_{0 \leq s \leq t} |u_\phi^\varepsilon(s)(\omega)|_{L^2}^2 \right) \vee \left(\int_0^t \|u_\phi^\varepsilon(s)(\omega)\|_{1,1}^2 ds \right) \vee \left(\int_0^t \|u_\phi^\varepsilon(s)\|_{L^{2\alpha+2}}^{2\alpha+2} ds \right) \leq N \right\}.$$

Lemma 5.2. *Let $\varepsilon_0, M, N > 0$, σ satisfy condition **(C')**(i). Let $u_0 \in L^4(\Omega; \tilde{H}^{0,1})$ be \mathcal{F}_0 -measurable, and let $u_\phi^\varepsilon(t)$ be solution of (4.8). Then there exists a positive constant C (depending on $K_i, \tilde{K}_i, i = 0, 1, L_1, T, M, N, \varepsilon_0$) such that for any $\phi \in \mathcal{A}_M$, $\varepsilon \in [0, \varepsilon_0]$:*

$$I_n(\phi, \varepsilon) := \mathbb{E} \left[1_{G_N(T)} \int_0^T \left\{ |u_\phi^\varepsilon(s) - u_\phi^\varepsilon(\psi_n(s))|_{L^2}^2 + \int_s^{\psi_n(s)} (|\nabla_h u_\phi^\varepsilon(r)|_{L^2}^2 dr + \|u_\phi^\varepsilon(r)\|_{L^{2\alpha+2}}^{2\alpha+2} dr) ds \right\} \right] \leq C 2^{-\frac{n}{2}}. \quad (5.6)$$

Proof. The proof is close to that of Lemma 3.3 in [13]. Let $\phi \in \mathcal{A}_M$, $\varepsilon \geq 0$; for any $s \in [0, T]$, Itô's formula yields $|u_\phi^\varepsilon(\psi_n(s)) - u_\phi^\varepsilon(s)|_{L^2}^2 = \sum_{i=1}^6 I_{n,i}$, where

$$\begin{aligned} I_{n,1} &= 2\sqrt{\varepsilon} \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} (\sigma(r, u_\phi^\varepsilon(r)) dW(r), u_\phi^\varepsilon(r) - u_\phi^\varepsilon(s)) \right), \\ I_{n,2} &= \varepsilon \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} |\sigma(r, u_\phi^\varepsilon(r))|_{\mathcal{L}}^2 dr \right), \\ I_{n,3} &= 2 \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} (\sigma(r, u_\phi^\varepsilon(r)) \phi(r), u_\phi^\varepsilon(r) - u_\phi^\varepsilon(s)) dr \right), \\ I_{n,4} &= 2\nu \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} \langle \Delta_h u_\phi^\varepsilon(r), u_\phi^\varepsilon(r) - u_\phi^\varepsilon(s) \rangle dr \right), \\ I_{n,5} &= -2 \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} \langle B(u_\phi^\varepsilon(r)), u_\phi^\varepsilon(r) - u_\phi^\varepsilon(s) \rangle dr \right), \end{aligned}$$

$$I_{n,6} = -2a \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} \int_{\mathbb{R}^3} |u_\phi^\varepsilon(r, x)|^{2\alpha} u_\phi^\varepsilon(r, x) (u_\phi^\varepsilon(r, x) - u_\phi^\varepsilon(s, x)) dx dr \right).$$

Clearly $G_N(T) \subset G_N(r)$ for $r \in [0, T]$. In particular this means that $|u_\phi^\varepsilon(r)|_{L^2}^2 + |u_\phi^\varepsilon(s)|_{L^2}^2 \leq N$ on $G_N(r)$ for $0 \leq s \leq r \leq T$. We use this observation in the considerations below. The Burkholder-Davis-Gundy inequality and the growth condition (2.32) yield for $\varepsilon \in [0, \varepsilon_0]$:

$$\begin{aligned} |I_{n,1}| &\leq 6\sqrt{\varepsilon} \int_0^T ds \mathbb{E} \left(\int_s^{\psi_n(s)} |\sigma(r, u_\phi^\varepsilon(r))|_{\mathcal{L}}^2 1_{G_N(r)} |u_\phi^\varepsilon(r) - u_\phi^\varepsilon(s)|^2 dr \right)^{\frac{1}{2}} \\ &\leq 6\sqrt{2\varepsilon_0 N} \int_0^T ds \mathbb{E} \left(\int_s^{\psi_n(s)} [K_0 + K_1 |u_\phi^\varepsilon(r)|_{L^2}^2] dr \right)^{\frac{1}{2}}. \end{aligned}$$

Schwarz's inequality and Fubini's theorem as well as (4.7), which holds uniformly in $\varepsilon \in [0, \varepsilon_0]$ for fixed $\varepsilon_0 > 0$ (since the constants K_i and L_1 are multiplied by at most ε_0), imply

$$|I_{n,1}| \leq 6\sqrt{2\varepsilon_0 NT} \left[\mathbb{E} \int_0^T (K_0 + K_1 |u_\phi^\varepsilon(r)|_{L^2}^2) \left(\int_{(r-c2^{-n})\vee 0}^r ds \right) dr \right]^{\frac{1}{2}} \leq C_1 2^{-\frac{n}{2}} \quad (5.7)$$

for some constant C_1 depending only on $K_i, i = 0, 1, M, \varepsilon_0, N$ and T . The growth condition (2.32) and Fubini's theorem imply that for $\varepsilon \in [0, \varepsilon_0]$:

$$|I_{n,2}| \leq \varepsilon_0 \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} (K_0 + K_1 |u_\phi^\varepsilon(r)|_{L^2}^2) dr \right) \leq C_2 2^{-n} \quad (5.8)$$

for some constant C_2 depending on the same parameters as C_1 . The Cauchy-Schwarz inequality, Fubini's theorem, the growth condition (2.32) and the definition of \mathcal{A}_M yield

$$\begin{aligned} |I_{n,3}| &\leq 2 \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} (K_0 + K_1 |u_\phi^\varepsilon(r)|_{L^2}^2)^{\frac{1}{2}} |\phi(r)|_0 |u_\phi^\varepsilon(r) - u_\phi^\varepsilon(s)|_{L^2} dr \right) \\ &\leq 4\sqrt{N} \mathbb{E} \int_0^T 1_{G_N(T)} |\phi(r)|_0 (K_0 + K_1 N)^{\frac{1}{2}} \left(\int_{(r-c2^{-n})\vee 0}^r ds \right) dr \leq C_3 2^{-n}, \quad (5.9) \end{aligned}$$

for some constant C_3 depending on the same parameters as C_1 . Using the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned} |I_{n,4}| &= 2 \left| \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} dr \left[-|\nabla_h u_\phi^\varepsilon(r)|_{L^2}^2 + |\nabla_h u_\phi^\varepsilon(r)|_{L^2} |\nabla_h u_\phi^\varepsilon(s)|_{L^2} \right] \right) \right| \\ &\leq \frac{1}{2} \mathbb{E} \left(1_{G_N(T)} \int_0^T ds |\nabla_h u_\phi^\varepsilon(s)|_{L^2}^2 \int_s^{\psi_n(s)} dr \right) \leq C N 2^{-n}. \quad (5.10) \end{aligned}$$

The antisymmetry relation (2.5), the inequality (2.15), the Cauchy-Schwarz inequality and Fubini's theorem and inequality yield:

$$\begin{aligned} |I_{n,5}| &\leq 2 \mathbb{E} \left(1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} dr |\langle B(u_\phi^\varepsilon(r)), u_\phi^\varepsilon(s) \rangle| \right) \\ &\leq CN \mathbb{E} \left[1_{G_N(T)} \left(\int_0^T \|u_\phi^\varepsilon(s)\|_{1,1}^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\int_s^{\psi_n(s)} |\nabla_h u_\phi^\varepsilon(r)|_{L^2} dr \right) ds \right)^{\frac{1}{2}} \right] \\ &\leq C(T) N^{\frac{3}{2}} 2^{-\frac{n}{2}} \mathbb{E} \left[1_{G_N(T)} \left\{ \int_0^T dr \left(\int_{(r-c2^{-n})\vee 0}^r ds \right) |\nabla_h u_\phi^\varepsilon(r)|_{L^2}^2 \right\}^{\frac{1}{2}} \right] \leq C_5 2^{-n} \quad (5.11) \end{aligned}$$

for some constant C_5 which depends on T and N .

Finally, Fubini's theorem and Hölder's inequality imply:

$$\begin{aligned}
|I_{n,6}| &\leq 2a \mathbb{E} \left[1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} dr \int_{\mathbb{R}^3} (|u_\phi^\varepsilon(r)|^{2\alpha+2} + |u_\phi^\varepsilon(r)|^{2\alpha+1} |u_\phi^\varepsilon(s)|) dx \right] \\
&\leq 2a \mathbb{E} \left[1_{G_N(T)} \int_0^T ds \int_s^{\psi_n(s)} \|u_\phi^\varepsilon(s)\|_{L^{2\alpha+2}} \|u_\phi^\varepsilon(r)\|_{L^{2\alpha+2}}^{2\alpha+1} dr \right] \\
&\quad + 2a \mathbb{E} \left[1_{G_N(T)} \int_0^T \|u_\phi^\varepsilon(r)\|_{L^{2\alpha+2}}^{2\alpha+2} \left(\int_{(r-c2^{-n}) \vee 0}^r ds \right) dr \right] \\
&\quad + 2ac2^{-n}N \\
&\leq 2a(c2^{-n})^{\frac{2\alpha+1}{2\alpha+2}} \mathbb{E} \left[1_{G_N(T)} \left(\int_0^T \|u_\phi^\varepsilon(r)\|_{L^{2\alpha+2}}^{2\alpha+2} \right)^{\frac{2\alpha+1}{2\alpha+2}} \left(\int_0^T ds \|u_\phi^\varepsilon(s)\|_{L^{2\alpha+2}}^{2\alpha+2} \int_s^{\psi_n(s)} dr \right)^{\frac{1}{2\alpha+2}} \right] \\
&\quad + 2ac2^{-n}N \leq C_6 2^{-n} \tag{5.12}
\end{aligned}$$

for some constant C_6 depending on T and N . Collecting the upper estimates from (5.7)-(5.12), we conclude the proof of (5.6). \square

In the setting of large deviations, we will use Lemma 5.2 with the following choice of the function ψ_n . For any integer n define a step function $s \mapsto \bar{s}_n$ on $[0, T]$ by the formula

$$\bar{s}_n = t_{k+1} \equiv (k+1)T2^{-n} \quad \text{for } s \in [kT2^{-n}, (k+1)T2^{-n}]. \tag{5.13}$$

Then the map $\psi_n(s) = \bar{s}_n$ clearly satisfies the previous requirements with $c = T$.

Proof of Proposition 4.4

Now we return to the setting of this proposition and recall that for random elements $(\phi_\varepsilon, 0 < \varepsilon \leq \varepsilon_0)$ taking values in the set \mathcal{A}_M , we let $u_{\phi_\varepsilon}^\varepsilon$ denote the solution to (4.9) with initial condition $u_{\phi_\varepsilon}^\varepsilon(0) = u_0 \in \tilde{H}^{0,1}$.

Since \mathcal{A}_M is a Polish space (complete separable metric space), by the Skorokhod representation theorem, we can construct processes $(\tilde{\phi}_\varepsilon, \tilde{\phi}, \tilde{W}^\varepsilon)$ such that the joint distribution of $(\tilde{\phi}_\varepsilon, \tilde{W}^\varepsilon)$ is the same as that of $(\phi_\varepsilon, W^\varepsilon)$, the distribution of $\tilde{\phi}$ coincides with that of ϕ , and $\tilde{\phi}_\varepsilon \rightarrow \tilde{\phi}$, a.s., in the (weak) topology of S_M . Hence a.s. for every $t \in [0, T]$, $\int_0^t \tilde{\phi}_\varepsilon(s) ds - \int_0^t \tilde{\phi}(s) ds \rightarrow 0$ weakly in H_0 . To lighten notations, we will write $(\tilde{\phi}_\varepsilon, \tilde{\phi}, \tilde{W}^\varepsilon) = (\phi_\varepsilon, \phi, W)$. Let $U_\varepsilon = u_{\phi_\varepsilon}^\varepsilon - u_\phi^0$; then $U_\varepsilon(0) = 0$ and

$$\begin{aligned}
dU_\varepsilon(t) &= [F(u_{\phi_\varepsilon}^\varepsilon(t)) - F(u_\phi^0(t)) + \sigma(t, u_{\phi_\varepsilon}^\varepsilon(t))\phi_\varepsilon(t) - \sigma(t, u_\phi^0(t))\phi(t)] dt \\
&\quad + \sqrt{\varepsilon} \sigma(t, u_{\phi_\varepsilon}^\varepsilon(t)) dW(t). \tag{5.14}
\end{aligned}$$

Let $\eta \in (0, \nu)$ and C_η be defined in (2.26); Itô's formula, the upper estimate (2.26), the growth condition (2.32) and the Lipschitz condition **(C')**(i) imply for $t \in [0, T]$:

$$\begin{aligned}
|U_\varepsilon(t)|_{L^2}^2 &+ 2\eta \int_0^t |\nabla_h U_\varepsilon(s)|_{L^2}^2 ds + 2a\kappa \int_0^t (|u_{\phi_\varepsilon}^\varepsilon(s)| + |u_\phi^0(s)|)^\alpha |U_\varepsilon(s)|_{L^2}^2 ds \\
&\leq \sum_{i=1}^3 T_i(t, \varepsilon) + 2 \int_0^t (C_\eta \|u_\phi^0(s)\|_{1,1}^2 + \sqrt{L_1} |\phi_\varepsilon(s)|_0) |U_\varepsilon(s)|_{L^2}^2 ds, \tag{5.15}
\end{aligned}$$

where

$$T_1(t, \varepsilon) = 2\sqrt{\varepsilon} \int_0^t (U_\varepsilon(s), \sigma(s, u_{\phi_\varepsilon}^\varepsilon(s)) dW(s)),$$

$$\begin{aligned} T_2(t, \varepsilon) &= \varepsilon \int_0^t (K_0 + K_1 |u_{\phi_\varepsilon}^\varepsilon(s)|_{L^2}^2) ds, \\ T_3(t, \varepsilon) &= 2 \int_0^t \left(\sigma(s, u_\phi^0(s)) (\phi_\varepsilon(s) - \phi(s)), U_\varepsilon(s) \right) ds. \end{aligned}$$

We want to show that as $\varepsilon \rightarrow 0$, $\|U_\varepsilon\|_Y \rightarrow 0$ in probability, which implies that $u_{h_\varepsilon}^\varepsilon \rightarrow u_h$ in distribution in Y . Fix $N > 0$ and for $t \in [0, T]$ let

$$\begin{aligned} G_N(t) &= \left\{ \sup_{0 \leq s \leq t} |u_\phi^0(s)|_{L^2}^2 \leq N \right\} \cap \left\{ \int_0^t (\|u_\phi^0(s)\|_{1,1}^2 + \|u_\phi^0(s)\|_{L^{2\alpha+2}}^{2\alpha+2}) ds \leq N \right\}, \\ G_{N,\varepsilon}(t) &= G_N(t) \cap \left\{ \sup_{0 \leq s \leq t} |u_{\phi_\varepsilon}^\varepsilon(s)|_{L^2}^2 \leq N \right\} \cap \left\{ \int_0^t (\|u_{\phi_\varepsilon}^\varepsilon(s)\|_{1,1}^2 + \|u_{\phi_\varepsilon}^\varepsilon(s)\|_{L^{2\alpha+2}}^{2\alpha+2}) ds \leq N \right\}. \end{aligned}$$

The proof consists in two steps.

Step 1: For any $\varepsilon_0 \in]0, 1]$, we have $\sup_{0 < \varepsilon \leq \varepsilon_0} \sup_{\phi, \phi_\varepsilon \in \mathcal{A}_M} \mathbb{P}(G_{N,\varepsilon}(T)^c) \rightarrow 0$ as $N \rightarrow \infty$.

Indeed, for $\varepsilon \in]0, \varepsilon_0]$, $\phi, \phi_\varepsilon \in \mathcal{A}_M$, the Markov inequality and the a priori estimate (4.7), which holds uniformly in $\varepsilon \in]0, \varepsilon_0]$, imply

$$\begin{aligned} \mathbb{P}(G_{N,\varepsilon}(T)^c) &\leq \mathbb{P}\left(\sup_{0 \leq s \leq T} |u_\phi^0(s)|_{L^2}^2 > N \right) + \mathbb{P}\left(\int_0^T (\|u_\phi^0(s)\|_{1,1}^2 + \|u_\phi^0(s)\|_{L^{2\alpha+2}}^{2\alpha+2}) ds > N \right) \\ &\quad + \mathbb{P}\left(\sup_{0 \leq s \leq T} |u_{\phi_\varepsilon}^\varepsilon(s)|_{L^2}^2 > N \right) + \mathbb{P}\left(\int_0^T (\|u_{\phi_\varepsilon}^\varepsilon(s)\|_{1,1}^2 + \|u_{\phi_\varepsilon}^\varepsilon(s)\|_{L^{2\alpha+2}}^{2\alpha+2}) ds > N \right) \\ &\leq C (1 + E \|u_0\|_{0,1}^4) N^{-1}, \end{aligned} \tag{5.16}$$

for some constant C depending on T and M .

Step 2: Fix $N > 0$, $\phi, \phi_\varepsilon \in \mathcal{A}_M$ such that as $\varepsilon \rightarrow 0$, $\phi_\varepsilon \rightarrow \phi$ a.s. in the weak topology of $L^2(0, T; H_0)$; then one has as $\varepsilon \rightarrow 0$:

$$\mathbb{E}\left[1_{G_{N,\varepsilon}(T)} \left(\sup_{0 \leq t \leq T} |U_\varepsilon(t)|_{L^2}^2 + \int_0^T |\nabla_h U_\varepsilon(t)|_{L^2}^2 dt \right) \right] \rightarrow 0. \tag{5.17}$$

Indeed, (5.15) and Gronwall's lemma imply that on $G_{N,\varepsilon}(T)$,

$$\sup_{0 \leq t \leq T} |U_\varepsilon(t)|_{L^2}^2 \leq \left[\sup_{0 \leq t \leq T} (T_1(t, \varepsilon) + T_3(t, \varepsilon)) + \varepsilon C_* \right] \exp\left(2C_\eta N + 2\sqrt{L_1 M T} \right),$$

where $C_* = T(K_0 + K_1 N)$. Using again (5.15) we deduce that for some constant $\tilde{C} = C(T, M, N)$, one has for every $\varepsilon \in [0, \varepsilon_0]$:

$$\mathbb{E}(1_{G_{N,\varepsilon}(T)} \|U_\varepsilon\|_Y^2) \leq \tilde{C} \left(\varepsilon + \mathbb{E}\left[1_{G_{N,\varepsilon}(T)} \sup_{0 \leq t \leq T} (T_1(t, \varepsilon) + T_3(t, \varepsilon)) \right] \right). \tag{5.18}$$

Since the sets $G_{N,\varepsilon}(\cdot)$ decrease, $\mathbb{E}(1_{G_{N,\varepsilon}(T)} \sup_{0 \leq t \leq T} |T_1(t, \varepsilon)|) \leq \mathbb{E}(\lambda_\varepsilon)$, where

$$\lambda_\varepsilon := 2\sqrt{\varepsilon} \sup_{0 \leq t \leq T} \left| \int_0^t 1_{G_{N,\varepsilon}(s)} \left(U_\varepsilon(s), \sigma(s, u_{\phi_\varepsilon}^\varepsilon(s)) dW(s) \right) \right|.$$

The scalar-valued random variables λ_ε converge to 0 in L^1 as $\varepsilon \rightarrow 0$. Indeed, by the Burkholder-Davis-Gundy inequality, (2.32) and the definition of $G_{N,\varepsilon}(s)$, we have

$$\mathbb{E}(\lambda_\varepsilon) \leq 6\sqrt{\varepsilon} \mathbb{E}\left\{ \int_0^T 1_{G_{N,\varepsilon}(s)} |U_\varepsilon(s)|_{L^2}^2 |\sigma(s, u_{\phi_\varepsilon}^\varepsilon(s))|_{\mathcal{L}}^2 ds \right\}^{\frac{1}{2}}$$

$$\leq 6\sqrt{\varepsilon} \mathbb{E} \left[\left\{ 4N \int_0^T 1_{G_{N,\varepsilon}(s)} (K_0 + K_1 |u_{\phi_\varepsilon}^\varepsilon(s)|_{L^2}^2) ds \right\}^{\frac{1}{2}} \right] \leq C(T, N) \sqrt{\varepsilon}. \quad (5.19)$$

In further estimates we use Lemma 5.2 with $\psi_n = \bar{s}_n$, where \bar{s}_n is defined in (5.13). For any $n, N \geq 1$, if we set $t_k = kT2^{-n}$ for $0 \leq k \leq 2^n$, we obviously have:

$$\mathbb{E} \left(1_{G_{N,\varepsilon}(T)} \sup_{0 \leq t \leq T} |T_3(t, \varepsilon)| \right) \leq 2 \sum_{i=1}^4 \tilde{T}_i(N, n, \varepsilon) + 2 \mathbb{E}(\bar{T}_5(N, n, \varepsilon)), \quad (5.20)$$

where

$$\begin{aligned} \tilde{T}_1(N, n, \varepsilon) &= \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \sup_{0 \leq t \leq T} \left| \int_0^t \left(\sigma(s, u_\phi^0(s)) (\phi_\varepsilon(s) - \phi(s)), [U_\varepsilon(s) - U_\varepsilon(\bar{s}_n)] \right) ds \right| \right], \\ \tilde{T}_2(N, n, \varepsilon) &= \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \right. \\ &\quad \times \left. \sup_{0 \leq t \leq T} \left| \int_0^t \left([\sigma(s, u_\phi^0(s)) - \sigma(\bar{s}_n, u_\phi^0(s))] (\phi_\varepsilon(s) - \phi(s)), U_\varepsilon(\bar{s}_n) \right) ds \right| \right], \\ \tilde{T}_3(N, n, \varepsilon) &= \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \right. \\ &\quad \times \left. \sup_{0 \leq t \leq T} \left| \int_0^t \left([\sigma(\bar{s}_n, u_\phi^0(s)) - \sigma(\bar{s}_n, u_\phi^0(\bar{s}_n))] (\phi_\varepsilon(s) - \phi(s)), U_\varepsilon(\bar{s}_n) \right) ds \right| \right], \\ \tilde{T}_4(N, n, \varepsilon) &= \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \sup_{1 \leq k \leq 2^n} \sup_{t_{k-1} \leq t \leq t_k} \left| \left(\sigma(t_k, u_\phi^0(t_k)) \int_{t_{k-1}}^t (\phi_\varepsilon(s) - \phi(s)) ds, U_\varepsilon(t_k) \right) \right| \right], \\ \bar{T}_5(N, n, \varepsilon) &= 1_{G_{N,\varepsilon}(T)} \sum_{k=1}^{2^n} \left| \left(\sigma(t_k, u_\phi^0(t_k)) \int_{t_{k-1}}^{t_k} (\phi_\varepsilon(s) - \phi(s)) ds, U_\varepsilon(t_k) \right) \right|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, the growth condition (2.32) and Lemma 5.2 with $\psi_n = \bar{s}_n$, we deduce that for some constant $\bar{C}_1 := C(T, M, N)$ and any $\varepsilon \in]0, \varepsilon_0]$:

$$\begin{aligned} \tilde{T}_1(N, n, \varepsilon) &\leq \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \int_0^T (K_0 + K_1 |u_\phi^0(s)|_{L^2}^2)^{\frac{1}{2}} |\phi_\varepsilon(s) - \phi(s)|_0 |U_\varepsilon(s) - U_\varepsilon(\bar{s}_n)|_{L^2} ds \right] \\ &\leq \left(\mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \int_0^T \{ |u_{\phi_\varepsilon}^\varepsilon(s) - u_{\phi_\varepsilon}^\varepsilon(\bar{s}_n)|_{L^2}^2 + |u_\phi^0(s) - u_\phi^0(\bar{s}_n)|_{L^2}^2 \} ds \right] \right)^{\frac{1}{2}} \\ &\quad \times \sqrt{2(K_0 + K_1 N)} \left(\mathbb{E} \int_0^T |\phi_\varepsilon(s) - \phi(s)|_0^2 ds \right)^{\frac{1}{2}} \leq \bar{C}_1 2^{-\frac{n}{4}}. \end{aligned} \quad (5.21)$$

A similar computation based on the Lipschitz condition **(C)(ii)** and Lemma 5.2 yields for some constant $\bar{C}_3 := C(T, M, N)$ and any $\varepsilon \in]0, \varepsilon_0]$

$$\begin{aligned} \tilde{T}_3(N, n, \varepsilon) &\leq \sqrt{2NL_1} \left(\mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \int_0^T |u_\phi^0(s) - u_\phi^0(\bar{s}_n)|_{L^2}^2 ds \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T |\phi_\varepsilon(s) - \phi(s)|_0^2 ds \right)^{\frac{1}{2}} \\ &\leq \bar{C}_3 2^{-\frac{n}{4}}. \end{aligned} \quad (5.22)$$

The time Hölder regularity **(C')(ii)** on $\sigma(\cdot, u)$ and the Cauchy-Schwarz inequality imply:

$$\tilde{T}_2(N, n, \varepsilon) \leq C \sqrt{N} 2^{-n\gamma} \mathbb{E} \left(1_{G_{N,\varepsilon}(T)} \int_0^T (1 + \|u_\phi^0(s)\|_{1,0}) |\phi_\varepsilon(s) - \phi(s)|_0 ds \right) \leq \bar{C}_2 2^{-n\gamma} \quad (5.23)$$

for some constant $\bar{C}_2 = C(T, M, N)$. Using the Cauchy-Schwarz inequality and the growth condition (2.32), we deduce for $\bar{C}_4 = C(T, N, M)$ and any $\varepsilon \in]0, \varepsilon_0]$

$$\begin{aligned} \tilde{T}_4(N, n, \varepsilon) &\leq \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \sup_{1 \leq k \leq 2^n} (K_0 + K_1 |u_\phi^0(t_k)|_{L^2}^2)^{\frac{1}{2}} \int_{t_{k-1}}^{t_k} |\phi_\varepsilon(s) - \phi(s)|_0 ds |U_\varepsilon(t_k)|_{L^2} \right] \\ &\leq 2\sqrt{N(K_0 + K_1 N)} \mathbb{E} \left(\sup_{1 \leq k \leq 2^n} \int_{t_{k-1}}^{t_k} |\phi_\varepsilon(s) - \phi(s)|_0 ds \right) \leq 4\bar{C}_4 2^{-\frac{n}{2}}. \end{aligned} \quad (5.24)$$

Finally, note that the weak convergence of ϕ_ε to ϕ implies that for any $a, b \in [0, T]$, $a < b$, as $\varepsilon \rightarrow 0$ the integral $\int_a^b \phi_\varepsilon(s) ds$ converges to $\int_a^b \phi(s) ds$ in the weak topology of H_0 . Therefore, since for the operator $\sigma(t_k, u_\phi^0(t_k))$ is compact from H_0 to H , we deduce that for every k ,

$$\left| \sigma(t_k, u_\phi^0(t_k)) \left(\int_{t_{k-1}}^{t_k} \phi_\varepsilon(s) ds - \int_{t_{k-1}}^{t_k} \phi(s) ds \right) \right|_{L^2} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Hence a.s., for fixed n as $\varepsilon \rightarrow 0$, $\bar{T}_5(N, n, \varepsilon, \omega) \rightarrow 0$. Furthermore, $\bar{T}_5(N, n, \varepsilon, \omega) \leq C(K_0, K_1, N, M)$ and hence the dominated convergence theorem proves that for any fixed n, N , $\mathbb{E}(\bar{T}_5(N, n, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Thus, (5.20)–(5.24) imply that for any fixed $N \geq 1$ and any integer $n \geq 1$

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \sup_{0 \leq t \leq T} |T_3(t, \varepsilon)| \right] \leq C_{N,T,M} 2^{-n(\gamma \wedge \frac{1}{4})}.$$

Since n is arbitrary, this yields for any integer $N \geq 1$:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[1_{G_{N,\varepsilon}(T)} \sup_{0 \leq t \leq T} |T_3(t, \varepsilon)| \right] = 0.$$

Therefore from (5.18) and (5.19) we obtain (5.17). By the Markov inequality

$$\mathbb{P}(\|U_\varepsilon\|_Y > \delta) \leq \mathbb{P}(G_{N,\varepsilon}(T)^c) + \frac{1}{\delta^2} \mathbb{E} \left(1_{G_{N,\varepsilon}(T)} \|U_\varepsilon\|_Y^2 \right) \text{ for any } \delta > 0.$$

Finally, (5.16) and (5.17) yield that for any integer $N \geq 1$,

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\|U_\varepsilon\|_Y > \delta) \leq C(T, M) N^{-1},$$

for some constant $C(T, M)$ which does not depend on N . This implies $\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\|U_\varepsilon\|_Y > \delta) = 0$ for any $\delta > 0$, which concludes the proof of Proposition 4.4. \square

5.3. Proof of the compactness of the set of controlled equations (Proposition 4.5). Recall that we want to prove that the set $K(M) = \{u_\phi^0 \in X : \phi \in S_M\}$ is a compact subset of Y . Let $\{u_n^0\}$ be a sequence in $K(M)$, corresponding to solutions of (4.4) with controls $\{\phi_n\}$ in S_M :

$$du_n^0(t) = F(u_n^0(t))dt + \sigma(t, u_n^0(t))\phi_n(t)dt, \quad u_n^0(0) = u_0 \in \mathcal{H}^{0,1}.$$

Since S_M is a bounded closed subset in the Hilbert space $L^2(0, T; H_0)$, it is weakly compact. So there exists a subsequence of $\{\phi_n\}$, still denoted as $\{\phi_n\}$, which converges weakly to a limit ϕ in $L^2(0, T; H_0)$. Note that in fact $\phi \in S_M$ as S_M is closed. We now show that the corresponding subsequence of solutions, still denoted as $\{u_n^0\}$, converges in Y to u_ϕ^0 which is the solution of the following “limit” equation

$$du_\phi^0(t) = F(u_\phi^0(t))dt + \sigma(t, u_\phi^0(t))\phi(t)dt, \quad u(0) = u_0.$$

This will complete the proof of the compactness of $K(M)$. To ease notation we will often drop the time parameters s, t, \dots in the equations and integrals.

Let $U_n = u_n^0 - u_\phi^0$; using (2.26) with $\eta \in (0, \nu)$, Condition **(C)** and Young's inequality, we deduce for $t \in [0, T]$:

$$\begin{aligned}
& |U_n(t)|_{L^2}^2 + 2\eta \int_0^t |\nabla_h U_n(s)|_{L^2}^2 ds \leq 2C_\eta \int_0^t \|u_\phi^0(s)\|_{1,1}^2 |U_n(s)|_{L^2}^2 ds \\
& + 2 \int_0^t \left\{ \left([\sigma(s, u_n^0(s)) - \sigma(s, u_\phi^0(s))] \phi_n(s), U_n(s) \right) \right. \\
& \quad \left. + (\sigma(s, u_\phi^0(s))(\phi_n(s) - \phi(s)), U_n(s)) \right\} ds \\
& \leq 2 \int_0^t |U_n(s)|^2 (C_\eta \|u_\phi^0(s)\|_{1,1}^2 + \sqrt{L_1} |\phi_n(s)|_0) ds \\
& + 2 \int_0^t (\sigma(s, u_\phi^0(s)) [\phi_n(s) - \phi(s)], U_n(s)) ds. \tag{5.25}
\end{aligned}$$

The inequality (4.7) implies that there exists a finite positive constant \bar{C} such that

$$\sup_n \left[\sup_{0 \leq t \leq T} (|u(t)|_{L^2}^2 + |u_n(t)|_{L^2}^2) + \int_0^T (\|u_\phi^0(s)\|_{1,1}^2 + \|u_n^0(s)\|_{1,1}^2) ds \right] = \bar{C}. \tag{5.26}$$

Thus Gronwall's lemma implies that

$$\sup_{t \leq T} |U_n(t)|^2 + 2\eta \int_0^T |\nabla_h U_n(t)|_{L^2}^2 dt \leq \exp \left(2(C_\eta \bar{C} + \sqrt{L_1 M T}) \right) \sum_{i=1}^5 I_{n,N}^i, \tag{5.27}$$

where, as in the proof of Proposition 4.4, we have for $t_k = kT2^{-N}$:

$$\begin{aligned}
I_{n,N}^1 &= \int_0^T |(\sigma(s, u_\phi^0(s)) [\phi_n(s) - \phi(s)], U_n(s) - U_n(\bar{s}_N))| ds, \\
I_{n,N}^2 &= \int_0^T \left| \left([\sigma(s, u_\phi^0(s)) - \sigma(\bar{s}_N, u_\phi^0(s))] [\phi_n(s) - \phi(s)], U_n(\bar{s}_N) \right) \right| ds, \\
I_{n,N}^3 &= \int_0^T \left| \left([\sigma(\bar{s}_N, u_\phi^0(s)) - \sigma(\bar{s}_N, u_\phi^0(\bar{s}_N))] [\phi_n(s) - \phi(s)], U_n(\bar{s}_N) \right) \right| ds, \\
I_{n,N}^4 &= \sup_{1 \leq k \leq 2^N} \sup_{t_{k-1} \leq t \leq t_k} \left| \left(\sigma(t_k, u_\phi^0(t_k)) \int_t^{t_k} (\phi_\varepsilon(s) - \phi(s)) ds, U_n(t_k) \right) \right|, \\
I_{n,N}^5 &= \sum_{k=1}^{2^N} \left(\sigma(t_k, u_\phi^0(t_k)) \int_{t_{k-1}}^{t_k} [\phi_n(s) - \phi(s)] ds, U_n(t_k) \right).
\end{aligned}$$

The Cauchy-Schwarz inequality, condition **(C')** **(i)** and Lemma 5.2 imply that for some constants C_i , which depend on M and T , but do not depend on n and N :

$$\begin{aligned}
I_{n,N}^1 &\leq (K_0 + K_1 \bar{C})^{\frac{1}{2}} \left(\int_0^T (|u_n^0(s) - u_n^0(\bar{s}_N)|_{L^2}^2 + |u_\phi^0(s) - u_\phi^0(\bar{s}_N)|_{L^2}^2) ds \right)^{\frac{1}{2}} \\
&\times \left(\int_0^T |\phi_n(s) - \phi(s)|_0^2 ds \right)^{\frac{1}{2}} \leq C_1 2^{-\frac{N}{4}}, \tag{5.28}
\end{aligned}$$

$$I_{n,N}^3 \leq 2\sqrt{L_1 \bar{C}} \left(\int_0^T |u_\phi^0(s) - u_\phi^0(\bar{s}_N)|_{L^2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^T |\phi_n(s) - \phi(s)|_0^2 ds \right)^{\frac{1}{2}} \leq C_3 2^{-\frac{N}{4}}, \tag{5.29}$$

$$I_{n,N}^4 \leq (K_0 + K_1 \bar{C})^{\frac{1}{2}} 2\sqrt{\bar{C}} \sup_{k=1, \dots, 2^N} \left(\int_{t_{k-1}}^{t_k} |\phi_n(s) - \phi(s)|_0^2 ds \right)^{\frac{1}{2}} \leq C_4 2^{-\frac{N}{2}}. \quad (5.30)$$

Furthermore, condition **(C')**(ii) implies that

$$\begin{aligned} I_{n,N}^2 &\leq C 2^{-N\gamma} \sup_{0 \leq t \leq T} (|u_\phi^0(t)|_{L^2} + |u_n^0(t)|_{L^2}) \int_0^T (1 + \|u_\phi^0(s)\|_{1,0})(|\phi(s)|_0 + |\phi_n(s)|_0) ds \\ &\leq C_2 2^{-N\gamma}. \end{aligned} \quad (5.31)$$

For fixed N and $k = 1, \dots, 2^N$, as $n \rightarrow \infty$, the weak convergence of ϕ_n to ϕ implies that of $\int_{t_{k-1}}^{t_k} (\phi_n(s) - \phi(s)) ds$ to 0 weakly in H_0 . Since $\sigma(t_k, u_\phi^0(t_k))$ is a compact operator, we deduce that for fixed k the sequence $\sigma(t_k, u_\phi^0(t_k)) \int_{t_{k-1}}^{t_k} (\phi_n(s) - \phi(s)) ds$ converges to 0 strongly in H as $n \rightarrow \infty$. Since $\sup_{n,k} |U_n(t_k)| \leq 2\sqrt{\bar{C}}$, we have $\lim_n I_{n,N}^5 = 0$. Thus (5.27)–(5.31) yield for every integer $N \geq 1$

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{t \leq T} |U_n(t)|_{L^2}^2 + \int_0^T \|U_n(t)\|_{1,0}^2 dt \right\} \leq C 2^{-N(\gamma \wedge \frac{1}{4})}.$$

Since N is arbitrary, we deduce that $\|U_n\|_Y \rightarrow 0$ as $n \rightarrow \infty$. This shows that every sequence in $K(M)$ has a convergent subsequence. Hence $K(M)$ is a sequentially relatively compact subset of Y . Finally, let $\{u_n^0\}$ be a sequence of elements of $K(M)$ which converges to v in Y . The above argument shows that there exists a subsequence $\{u_{n_k}^0, k \geq 1\}$ which converges to some element $u_\phi \in K(M)$ for the same topology of Y . Hence $v = u_\phi^0$, $K(M)$ is a closed subset of Y , and this completes the proof of Proposition 4.5. \square

Acknowledgements. This research was started in the fall 2016 while Annie Millet visited the University of Wyoming. She would like to thank this University for the hospitality and the very pleasant working conditions. Hakima Bessaih is partially supported by NSF grant DMS-1418838.

The authors wish to thank anonymous referees for helpful comments and careful reading.

REFERENCES

- [1] H. Bahouri, J.-Y. Chemin & R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren des Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343, Springer, Heidelberg (2011).
- [2] J. W. Barret & W. B. Liu, Finite element approximation of the parabolic p-Laplacian, *SIAM J. Numer. Anal.*, **31** (1994), 413–428.
- [3] R. Bennacer, A. Tobbal & H. Beji, Convection naturelle thermostable dans une cavité poreuse anisotrope: Formulation de Darcy-Brinkman, *Rev. Energ. Ren.*, **5** (2002), 1–21.
- [4] H. Bessaih, A. Millet, Large deviations and the zero viscosity limit for 2D stochastic Navier-Stokes equations with free boundary. *SIAM J. Math. Anal.*, **44** (2012), no. 3, 1861–1893.
- [5] H. Bessaih, S. Trabelsi & H. Zorgati, Existence and uniqueness of global solutions for the modified anisotropic ED Navier-Stokes equations. *M2AN*, **50** (2016), 1817–1823.
- [6] A. Budhiraja & P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, *Prob. and Math. Stat.* **20** (2000), 39–61.
- [7] A. Budhiraja, P. Dupuis & V. Maroulas, Large deviations for infinite dimensional stochastic dynamical systems. *Ann. Prob.* **36** (2008), 1390–1420.
- [8] X. Cai & Q. Jui, Weak and strong solutions for the incompressible Navier-Stokes equations with dumping, *J. of Math. Ana. and App.*, **343(2)** (2008), 799–809.
- [9] J.-Y. Chemin, B. Desjardin, I. Gallagher & E. Grenier, Fluids with anisotropic viscosity, *M2AN* **34** (2000), 315–335.

- [10] J.-Y. Chemin, B. Desjardin, I. Gallagher & E. Grenier, *Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier-Stokes Equations*, Oxford Lecture Series in Mathematics and its Applications **32** (2006).
- [11] M. Capinsky & D. Gatarek, Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension, *J. Funct. Anal.* **126** (1994) 26–35.
- [12] P. Constantin & C. Foias, *Navier-Stokes Equations*, U. of Chicago Press, Chicago, 1988.
- [13] I. Chueshov & A. Millet, Stochastic 2D hydrodynamical type systems: Well posedness and large deviations. *Applied Math. Optim.* **61(3)** (2010) , 379–420.
- [14] G. Da Prato & J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.
- [15] J. Duan & A. Millet, Large deviations for the Boussinesq equations under random influences, *Stochastic Processes and their Applications* **119-6** (2009), 2052–2081.
- [16] P. Dupuis & R.S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley-Interscience, New York, 1997.
- [17] V. Kalantarov, S. Zelik, Smooth attractors for the Brinkman-Forchheimer equations with fast growing nonlinearities, *Commun. Pure Appl. Anal.* **11-5** (2012), 2037–2054.
- [18] H. Kunita, *Stochastic flows and stochastic differential equations*, Cambridge University Press, Cambridge-New York, 1990.
- [19] P. A. Markowich, E. S. Titii & S. Trabelsi, Continuous data assimilation for the three-dimensional Brinkman Forchheimer-extended Darcy model, *Nonlinearity*, **29** (2016), 129–1328.
- [20] J.L. Menaldi & S.S. Sritharan, Stochastic 2-D Navier-Stokes equation, *Appl. Math. Optim.* **46** (2002), 31–53.
- [21] E. Pardoux, Equations aux dérivées partielles stochastiques non linéaires monotones; Etude de solutions fortes de type Itô. Thèse, Univ. Paris Sud, 1975.
- [22] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer Verlag, New York 1987.
- [23] M. Röckner, T. Zhang & X. Zhang, Large deviations for the tamed stochastic Navier-Stokes equations, *Appl. Math. Optim.* **61 (2)** (2010), 267–285.
- [24] M. Röckner & T. Zhang, Stochastic 3D tamed equations: Existence, uniqueness and small time large deviation principle, *J. Differential Equations* **252** (2012), 717–744.
- [25] M. Röckner & X. Zhang, Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and ergodicity, *Probab. Theory Related Fields* **145 (1-2)** (2009), 211-267.
- [26] S. S. Sritharan & P. Sundar, Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise, *Stoch. Proc. and Appl.* **116** (2006), 1636–1659.
- [27] R. Temam: *Navier-Stokes equations. Theory and numerical analysis. Studies in Mathematics and its Applications 2*, North-Holland Publishing Co., Amsterdam-New York, 1979.

UNIVERSITY OF WYOMING, DEPARTMENT OF MATHEMATICS, DEPT. 3036, 1000 EAST UNIVERSITY AVENUE, LARAMIE WY 82071, UNITED STATES
E-mail address: `bessaih@uwyo.edu`

SAMM, EA 4543, UNIVERSITÉ PARIS 1 PANTHÉON SORBONNE, 90 RUE DE TOLBIAC, 75634 PARIS CEDEX FRANCE *and* LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, UNIVERSITÉS PARIS 6-PARIS 7
E-mail address: `amillet@univ-paris1.fr`