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# The spherical $p$ -harmonic eigenvalue problem in non-smooth domains

Konstantinos Gkikas\*

Laurent Véron†

## Abstract

We prove the existence of  $p$ -harmonic functions under the form  $u(r, \sigma) = r^{-\beta}\omega(\sigma)$  in any cone  $C_S$  generated by a spherical domain  $S$  and vanishing on  $\partial C_S$ . We prove the uniqueness of the exponent  $\beta$  and of the normalized function  $\omega$  under a Lipschitz condition on  $S$ .

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*Key words.*  $p$ -Laplacian operator; polar sets; Harnack inequality; boundary Harnack inequality;  $p$ -Martin boundary.

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## 1 Introduction

Let  $p > 1$ ,  $S$  a domain of the unit sphere  $S^{N-1}$  of  $\mathbb{R}^N$  and  $C_S := \{(r, \sigma) : r > 0, \sigma \in S\}$  the positive cone generated by  $S$ . If one looks for  $p$ -harmonic functions in  $C_S$  under the form  $u(x) = u(r, \sigma) = r^{-\beta}\omega(\sigma)$  vanishing on  $\partial C_S \setminus \{0\}$ , then  $\omega$  satisfies the *spherical  $p$ -harmonic eigenvalue problem* on  $S$

$$\begin{aligned}
 -\operatorname{div}' \left( \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega \right) &= (p-1)\beta(\beta - \beta_0) \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega && \text{in } S \\
 \omega &= 0 && \text{in } \partial S
 \end{aligned}
 \tag{1.1}$$

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with  $\beta_0 = \frac{N-p}{p-1}$  and were  $div'$  and  $\nabla'$  denote the divergence operator and the covariant gradient on  $S^{N-1}$  endowed with the metric induced by its isometric inbedding into  $\mathbb{R}^N$ . Separable solutions play a key role for describing the boundary behaviour and the singularities of solutions of a large variety of quasilinear equations. When  $N = 2$  the equation is completely integrable and has been solved by Kroll in the regular case  $\beta < 0$  and Kichenassamy and Véron in the the singular case  $\beta > 0$ . In higher dimension, Tolksdorff [15] proved the following:

**Theorem A** *If  $S$  is a smooth spherical domain, there exist two couples  $(\beta_S, \omega_S)$  and  $(\beta'_S, \omega'_S)$  where  $\beta_S > 0$  and  $\beta'_S < 0$ ,  $\omega_S$  and  $\omega'_S$  are positive  $C^2(\bar{S})$ -functions vanishing on  $\partial S$  which solve (1.1) with  $(\beta, \omega) = (\beta_S, \omega_S)$  or  $(\beta, \omega) = (\beta'_S, \omega'_S)$ . Furthermore  $\beta_S$  and  $\beta'_S$  are unique, and  $\omega_S$  and  $\omega'_S$  are unique up to an homothety.*

A more general and transparent proof has been obtained by Porretta and Véron [13], but always in the case of a smooth spherical domain. The aim of this article is to extend Theorem A to a general spherical domain. If we consider an increasing sequence of smooth domains  $\{S_k\}$  such that  $S_k \subset \bar{S}_k \subset S_{k+1}$  and  $\cup_k S_k = S$  we prove the following:

**Theorem B** *Assume that  $S^c$  is not polar. Then the sequence of the  $\beta_{S_k} > 0$  from Theorem A is decreasing and converges to  $\beta_S > 0$ . There exists  $\omega_S \in W_0^{1,p}(S) \cap L^\infty(S)$  weak solution of (1.1) with  $\beta = \beta_S$ . Furthermore  $\beta_S > 0$  is the largest exponent  $\beta$  such that (1.1) admits a positive solution  $\omega_S \in W_0^{1,p}(S)$ .*

Under a mild assumption on  $S$  it is possible to approximate it by a decreasing sequence of smooth domains  $S'_k$  such that  $S'_k \subset \bar{S}'_k \subset S'_{k-1}$  and  $\cap_k S'_k = \bar{S}$

**Theorem C** *Assume that  $S = \bar{S}$ . Then the sequence  $\beta_{S'_k} > 0$  is increasing and converges to  $\hat{\beta}_S > 0$  and there exists  $\hat{\omega}_S \in W_0^{1,p}(S) \cap L^\infty(S)$  weak solution of (1.1) with  $\beta = \hat{\beta}_S$ . Furthermore  $\hat{\beta}_S$  is the smallest exponent  $\beta$  such that (1.1) admits a positive solution  $\omega_S \in W_0^{1,p}(S)$ .*

We prove the uniqueness of the exponent  $\beta$ , under a Lipschitz assumption on  $S$ .

**Theorem D** *Assume that  $S$  is a Lipschitz domain, then  $\beta_S = \hat{\beta}_S$  and if  $\omega$  and  $\omega'$  are two positive solutions of (1.1) in  $W_0^{1,p}(S)$ , there exists a constant  $c > 0$  such that  $c^{-1}\omega' \leq \omega \leq c\omega'$ .*

The proof of Theorem C is based upon a sharp form of boundary Harnack inequality proved in [10],

$$\left| \ln \frac{\omega(\sigma_1)}{\omega'(\sigma_1)} - \ln \frac{\omega(\sigma_2)}{\omega'(\sigma_2)} \right| \leq c_1 |\sigma_1 - \sigma_2|^\alpha \quad \forall \sigma_1, \sigma_2 \in S, \quad (1.2)$$

for some  $c_1 = c_1(N, p, S) > 0$  and  $\alpha \in (0, 1)$ . Actually we have a stronger result, much more delicate to obtain.

**Theorem E** *Let  $S$  be a Lipschitz subdomain of  $S^{N-1}$ . Then two positive solutions of (1.1) in  $W_0^{1,p}(S)$  are proportional.*

The proof is based upon a non trivial adaptation of a series of deep results of Lewis and Nyström [10] concerning the  $p$ -Martin boundary of domains.

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## 2 Existence

### 2.1 Estimates

Through this article we assume that  $S^c$  is not polar, or equivalently that it has positive  $c_{1,p}^{S^{N-1}}$ -capacity.

**Lemma 2.1.** Assume  $p > 1$ . Then any solution  $\omega \in W_0^{1,p}(S)$  of (1.1) satisfies

$$\|\omega\|_{C^\gamma(S)} \leq c_1 \|\omega\|_{L^p(S)}, \quad (2.1)$$

if  $p > N - 1$  where  $\gamma = 1 - \frac{N-1}{p}$  if  $p > N - 1$  and

$$\|\omega\|_{L^\infty(S)} \leq c_1 \|\omega\|_{L^p(S)}, \quad (2.2)$$

if  $1 < p \leq N - 1$ , where  $c_1 > 0$  depends on  $p, N, \beta$ .

*Proof.* Multiplying the equation by  $\omega$  and using Hölder's inequality, we derive

$$\begin{aligned} (i) \quad & \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{2}} dS \leq (\beta(p\beta - (p-1)\beta_0))^{\frac{p}{2}} \int_S |\omega|^p dS \quad \text{if } p \geq 2, \\ (ii) \quad & \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{2}} dS \leq \beta^{p-1} (p\beta - (p-1)\beta_0) \int_S |\omega|^p dS \quad \text{if } 1 < p < 2. \end{aligned} \quad (2.3)$$

Notice that these inequalities hold for all  $p > 1$ . If  $p > N - 1$  (2.1) follows by Morrey's inequality. Here after we assume  $1 < p \leq N - 1$ . Let  $\alpha \geq 1$  and  $k > 0$ . Then  $\zeta = \min\{|\omega|, k\}^{\alpha-1} \omega$  is an admissible test function, hence

1- If  $p \geq 2$ ,

$$\begin{aligned} & \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega, \nabla' \zeta \rangle dS = (p-1)\beta(\beta - \beta_0) \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega \zeta dS \\ & \leq c_2 \int_S |\nabla' \omega|^{p-2} \omega^2 \min\{|\omega|, k\}^{\alpha-1} dS + c_2 \beta^p \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \\ & \leq c_2 \left( \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \right)^{\frac{p-2}{p}} \left( \int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \right)^{\frac{2}{p}} \\ & + c_2 \beta^p \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \end{aligned} \quad (2.4)$$

where  $c_2 = c_2(N, p, \beta) > 0$ . Since

$$\int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega, \nabla' \zeta \rangle dS \geq c_3(p) \int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS,$$

it implies that there exists  $c_4 = c_4(N, p, \beta)$  such that

$$\int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \leq c_4 \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \quad (2.5)$$

which yields

$$\int_S |\nabla' j(\omega)|^p dS \leq c_4 \int_S |j(\omega)|^p dS, \quad (2.6)$$

where  $j(\omega) = \min\{|\omega|, k\}^{\frac{\alpha-1}{p}} \omega$ .

2- If  $1 < p < 2$ , then

$$\begin{aligned} & \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega, \nabla' \zeta \rangle dS = \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega|^2 \min\{|\omega|, k\}^{\alpha-1} dS \\ & + (\alpha - 1) \int_{S \cap \{|\omega| < k\}} \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega|^2 |\omega|^{\alpha-1} dS. \end{aligned} \quad (2.7)$$

Since

$$\begin{aligned} & \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega|^2 \min\{|\omega|, k\}^{\alpha-1} dS = \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{2}} \min\{|\omega|, k\}^{\alpha-1} dS \\ & \quad - \beta^2 \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \min\{|\omega|, k\}^{\alpha-1} \omega^2 dS \\ & \geq \int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS - \beta^2 \int_S \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \min\{|\omega|, k\}^{\alpha-1} \omega^2 dS, \end{aligned}$$

we derive

$$\int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \leq \beta^{p-1} (p\beta - (p-1)\beta_0) \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \quad (2.8)$$

which leads to (2.6). Letting  $k \rightarrow \infty$  we infer by Fatou's lemma,

$$\int_S \left| \nabla' |\omega|^{\frac{\alpha-1}{p}+1} \right|^p dS \leq c_4 \int_S |\omega|^{\alpha-1+p} dS. \quad (2.9)$$

If  $p < N - 1$  we derive from Sobolev inequality and putting  $q = \alpha - 1 + p$  and  $s = \frac{N-1}{N-1-p} > 1$

$$\left( \int_S |\omega|^{sq} dS \right)^{\frac{1}{s}} \leq c_5 \int_S |\omega|^q dS, \quad (2.10)$$

and  $c_5 > 0$  depends on  $N, p$  and  $\beta$ . Iterating this estimate by Moser's method we derive (2.10).

If  $p = N - 1$  we have for  $1 \leq m < p - 1$  and  $m^* = \frac{m(N-1)}{N-1-m}$

$$c_6 \left( \int_S |\omega|^{(\frac{\alpha-1}{p}+1)m^*} dS \right)^{\frac{pm}{m^*}} \leq \left( \int_S \left| \nabla' |\omega|^{\frac{\alpha-1}{p}+1} \right|^m dS \right)^{\frac{p}{m}} \leq |S|^{\frac{p}{m}-1} c_4 \int_S |\omega|^{\alpha-1+p} dS,$$

and  $c_6 = c_6(N, p)$ , hence

$$\left( \int_S |\omega|^{tq} dS \right)^{\frac{1}{t}} \leq c_5 \int_S |\omega|^q dS, \quad (2.11)$$

with  $t = \frac{m(N-1)}{p(N-1-m)} = \frac{m}{N-1-m}$ . The proof follows again by Moser's iterative scheme.  $\square$

**Proposition 2.2.** *Let  $S_1$  and  $S_2$  be two subdomains of  $S^{N-1}$  such that  $S_1 \subset \overline{S_1} \subset S_2$  and  $S_2$  not polar. Let  $\beta_j > 0, j=1,2$ , such that there exist positive solutions  $\omega_j \in W_0^{1,p}(S_j)$  solutions of*

$$\begin{aligned} -\operatorname{div}' \left( \left( \beta_j^2 \omega_j^2 + |\nabla' \omega_j|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_j \right) &= (p-1)\beta_j(\beta_j - \beta_0) \left( \beta_j^2 \omega_j^2 + |\nabla' \omega_j|^2 \right)^{\frac{p-2}{2}} \omega_j && \text{in } S_j \\ \omega_j &= 0 && \text{in } \partial S_j. \end{aligned} \quad (2.12)$$

Then  $\beta_1 \geq \beta_2$ .

*Proof.* Set  $u_j(r, \sigma) = r^{-\beta_j} \omega_j(\sigma)$ ,  $C_{S_j} = (0, \infty) \times S_j$  and assume  $\beta_1 < \beta_2$ . By Harnack inequality  $\omega_2 \geq c > 0$  on  $S_1$ , thus

$$u_2(r, \sigma) \geq cr^{-\beta_2} \quad \text{a.e. in } C_{S_1}.$$

For  $\epsilon > 0$  there exist  $r_\epsilon > 0$  such that

$$\epsilon u_2(x) \geq u_1(x) \quad \forall x \in C_{S_1} \cap \overline{B_{r_\epsilon}}.$$

Let  $\delta > 0$ , there exists  $R_\delta > 0$  such that

$$u_1(x) \leq \delta \quad \forall x \in C_{S_1} \cap B_{R_\delta}^c.$$

Hence  $\zeta = (u_1 - \epsilon u_2 - \delta)_+ \in W_0^{1,p}(Q_{S_1}^{r_\epsilon, R_\delta})$ , where  $Q_{S_1}^{r_\epsilon, R_\delta} = \{x \in C_{S_1} : r_\epsilon < |x| < R_\delta\}$ . This implies

$$\begin{aligned} 0 &= \int_{Q_{S_1}^{r_\epsilon, R_\delta}} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla(\epsilon u_1)|^{p-2} \nabla(\epsilon u_1), \nabla \zeta \rangle dx \\ &= \int_{Q_{S_1}^{r_\epsilon, R_\delta} \cap \{u_1 - \epsilon u_2 \geq \delta\}} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla(\epsilon u_1)|^{p-2} \nabla(\epsilon u_1), \nabla(u_1 - u_2) \rangle dx. \end{aligned}$$

Therefore  $\nabla(u_1 - \epsilon u_2 - \delta)_+ = 0$  a.e. in  $Q_{S_1}^{r_\epsilon, R_\delta}$ , which leads to  $u_1 - \epsilon u_2 \leq \delta$  in the same set. Letting  $\delta \rightarrow 0$  yields  $R_\delta \rightarrow \infty$ , thus we obtain  $u_1 \leq \epsilon u_2$  in  $C_{S_1} \cap \overline{B}_{r_\epsilon}^c$  hence  $u_1 \leq 0$  in  $C_{S_1}$ , contradiction.  $\square$

## 2.2 Approximations from inside

*Proof of Theorem B.* Let  $\{S_k\}$  be an increasing sequence of smooth domains such that  $S_k \subset \overline{S}_k \subset S_{k+1}$ . We denote by  $\{(\beta_{S_k}, \omega_k)\}$  the corresponding sequence of solutions of (1.1) with  $\beta = \beta_{S_k}$  and  $\omega = \omega_k$ . The sequence  $\{\beta_{S_k}\}$  is uniquely determined by [15], it admits a limit  $\beta := \beta_S$ , and the  $\omega_k$  are the unique positive solutions such that

$$\int_{S_k} |\omega_k| dS = 1.$$

If  $p \geq 2$ , we have

$$\begin{aligned} \int_{S_k} |\nabla' \omega_k|^p dS &\leq \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega_k|^2 dS \\ &= (p-1) \beta_{S_k} (\beta_{S_k} - \beta_0) \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k^2 dS \\ &\leq 2^{\frac{(p-4)_+}{2}} (p-1) \beta_{S_k} (\beta_{S_k} - \beta_0) \int_{S_k} \left( \beta_{S_k}^{p-2} \omega_k^p + |\nabla' \omega_k|^{p-2} \omega_k^2 \right) dS \\ &\leq c_7(N, p, \beta_{S_k}) \int_{S_k} \omega_k^p dS + \frac{1}{2} \int_{S_k} |\nabla' \omega_k|^p dS. \end{aligned}$$

Since  $\beta_{S_k} \leq \beta_1$ , we derive

$$\int_{S_k} |\nabla' \omega_k|^p dS \leq c_8, \tag{2.13}$$

from the normalization assumption with  $c_8 = 2c_7(N, p, \beta_1)$ .

If  $1 < p < 2$ , we have

$$\begin{aligned} \int_{S_k} |\nabla' \omega_k|^p dS &\leq \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p}{2}} dS \\ &\leq \beta_{S_k} (p\beta_{S_k} + (p-1)\beta_0) \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k^2 dS \\ &\leq \beta_k^{p-1} (p\beta_{S_k} + (p-1)\beta_0) \int_{S_k} \omega_k^p dS, \end{aligned}$$

and we obtain (2.13) with  $c_8 = \beta_1^{p-1}(p\beta_1 + (p-1)\beta_0)$ .

Next we extend  $\omega_k$  by 0 in  $S_k^c$ . Then there exists  $\omega \in W_0^{1,p}(S)$  such that  $\omega_k \rightharpoonup \omega$  weakly in  $W_0^{1,p}(S)$ , up to subsequence that we still denote  $\{\omega_k\}$ , and  $\omega_k \rightarrow \omega$  in  $L^p(S)$ .

*Step 1: We claim that  $\nabla'\omega_k$  converges to  $\nabla'\omega$  locally in  $L^p(S)$ .*

Let  $a \in S$  and  $r > 0$  such that  $B_{4r}(a) \subset S$ . Then for  $k \geq k_0$ ,  $\overline{B_{2r}(a)} \subset S_k$ . Let  $\zeta \in C_0^\infty(B_{2r}(a))$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta = 1$  in  $B_r(a)$ . For test function we choose  $\eta_k = \zeta(\omega - \omega_k)$ , then

$$\int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \langle \nabla'\omega_k, \nabla'\eta_k \rangle dS = (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \eta_k dS.$$

By the above inequality, we have

$$\begin{aligned} & \int_{B_{2r}(a)} \left\langle \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'\eta_k \right\rangle dS \\ &= \int_{B_{2r}(a)} \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla'\omega, \nabla'\eta_k \rangle dS \\ & \quad - (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \eta_k dS. \end{aligned}$$

Using the weak convergence of the gradient, we have

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla'\omega, \nabla'\eta_k \rangle dS = 0.$$

Since  $\omega_k$  is uniformly bounded in  $W_0^{1,p}(S)$  and  $\omega_k \rightarrow \omega$  in  $L^p(S)$ , we have

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \eta_k dS = 0,$$

and

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} (\omega - \omega_k) \left\langle \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'\zeta \right\rangle dS = 0.$$

Combining the above relations we infer

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} \zeta \left\langle \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'(\omega - \omega_k) \right\rangle dS = 0. \quad (2.14)$$

Next we write

$$\begin{aligned} & \int_{B_{2r}(a)} \zeta \left\langle \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'(\omega - \omega_k) \right\rangle dS \\ &= \frac{1}{2} \int_{B_{2r}(a)} \zeta \left( \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} + \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \right) |\nabla'(\omega - \omega_k)|^2 dS \\ & \quad + \frac{1}{2} \int_{B_{2r}(a)} \zeta \left( \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \right) \\ & \quad \quad \quad \times \left( |\nabla'\omega|^2 + \beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2 - |\nabla'\omega_k|^2 \right) dS \\ & \quad - \frac{1}{2} \int_{B_{2r}(a)} \zeta \left( \left( \beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS. \end{aligned} \quad (2.15)$$

If  $p \geq 2$ , we have from (2.4),

$$\begin{aligned}
& \int_{B_{2r}(a)} \zeta \left\langle \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\
& \geq \frac{1}{2} \int_{B_{2r}(a)} \zeta \left( |\nabla' \omega|^{p-2} + |\nabla' \omega_k|^{p-2} \right) |\nabla' (\omega - \omega_k)|^2 dS \\
& \quad - \frac{1}{2} \int_{B_{2r}(a)} \zeta \left( \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS \\
& \geq \min\{2^{-1}, 2^{2-p}\} \int_{B_{2r}(a)} \zeta |\nabla' (\omega - \omega_k)|^p dS \\
& \quad - \frac{1}{2} \int_{B_{2r}(a)} \zeta \left( \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS.
\end{aligned} \tag{2.16}$$

Since  $\omega_k \rightarrow \omega$  in  $L^p(S)$ ,  $\beta_{S_k} \rightarrow \beta$  and  $\omega_k, \omega$  are uniformly bounded in  $W_0^{1,p}(S)$ , we derive

$$\int_{B_{2r}(a)} \zeta \left( \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS \rightarrow 0$$

as  $k \rightarrow \infty$ . Jointly with (2.14) we infer that

$$\lim_{k \rightarrow \infty} \int_{B_r(a)} |\nabla' (\omega - \omega_k)|^p dS = 0. \tag{2.17}$$

If  $1 < p < 2$ , then

$$\begin{aligned}
& \int_{B_{2r}(a)} \zeta \left\langle \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\
& = \int_{B_{2r}(a)} \zeta \left\langle \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\
& \quad + \int_{B_{2r}(a)} \zeta \left\langle \left( \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \right) \nabla' \omega \cdot \nabla' (\omega - \omega_k) \right\rangle dS.
\end{aligned} \tag{2.18}$$

Up to extracting a subsequence, we have that  $\omega_k \rightarrow \omega$  a.e. in  $S$  and that there exists  $\Phi \in L^1(S)$  such that

$$|\omega_k|^p + |\omega|^p \leq \Phi \quad \text{a.e. in } S \quad \text{and } \forall k \geq 1. \tag{2.19}$$

Since

$$\left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla \omega| \leq \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-1}{2}} \leq \beta_{S_k}^{p-1} \omega_k^{p-1} + |\nabla' \omega|^{p-1},$$

and

$$\left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla \omega| \leq \beta^{p-1} \omega^{p-1} + |\nabla' \omega|^{p-1},$$

we derive that

$$\left| \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \right| |\nabla' \omega| \leq 2 \left( \beta^{p-1} \Phi^{p-1} + |\nabla' \omega|^{p-1} \right),$$

which implies that

$$\zeta \left( \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \right) \nabla' \omega \rightarrow 0 \quad \text{in } L^p(S)$$



where  $p'$  is the conjugate of  $p$ , and finally

$$\int_{B_{2r}(a)} \zeta \left\langle \left( (\beta^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} - (\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \right) \nabla' \omega \cdot \nabla' (\omega - \omega_k) \right\rangle dS \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.20)$$

For the last term on the right-hand side of (2.18), we have, for  $\gamma \in \mathbb{R}_+$  and  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^N$ ,

$$\begin{aligned} (\gamma + |\mathbf{B}|^2)^{\frac{p-2}{2}} \mathbf{B} - (\gamma + |\mathbf{A}|^2)^{\frac{p-2}{2}} \mathbf{A} &= \int_0^1 \frac{d}{dt} \left( (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} (t\mathbf{B} + (1-t)\mathbf{A}) \right) dt \\ &= \left( \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt \right) (\mathbf{B} - \mathbf{A}) \\ &\quad + (p-2) \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-4}{2}} \langle t\mathbf{B} + (1-t)\mathbf{A} \cdot \mathbf{B} - \mathbf{A} \rangle (t\mathbf{B} + (1-t)\mathbf{A}) dt. \end{aligned}$$

This implies

$$\begin{aligned} \langle (\gamma + |\mathbf{B}|^2)^{\frac{p-2}{2}} \mathbf{B} - (\gamma + |\mathbf{A}|^2)^{\frac{p-2}{2}} \mathbf{A} \cdot \mathbf{B} - \mathbf{A} \rangle &= \left( \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt \right) |\mathbf{B} - \mathbf{A}|^2 \\ &\quad + (p-2) \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-4}{2}} \langle t\mathbf{B} + (1-t)\mathbf{A} \cdot \mathbf{B} - \mathbf{A} \rangle^2 dt. \end{aligned}$$

We observe that

$$\begin{aligned} \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-4}{2}} \langle t\mathbf{B} + (1-t)\mathbf{A} \cdot \mathbf{B} - \mathbf{A} \rangle^2 dt \\ \leq |\mathbf{B} - \mathbf{A}|^2 \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt, \end{aligned}$$

and since  $1 < p < 2$ , we finally obtain

$$\begin{aligned} \langle (\gamma + |\mathbf{B}|^2)^{\frac{p-2}{2}} \mathbf{B} - (\gamma + |\mathbf{A}|^2)^{\frac{p-2}{2}} \mathbf{A} \cdot \mathbf{B} - \mathbf{A} \rangle \\ \geq (p-1) \left( \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt \right) |\mathbf{B} - \mathbf{A}|^2 \quad (2.21) \\ \geq (p-1) |\mathbf{B} - \mathbf{A}|^2 (\gamma + 1 + |\mathbf{B}|^2 + |\mathbf{A}|^2)^{\frac{p-2}{2}}. \end{aligned}$$

We plug this estimate into (2.18) with  $\gamma = \beta_k^2 \omega_k^2$ ,  $\mathbf{A} = \nabla' \omega$  and  $\mathbf{B} = \nabla' \omega_k$ , then

$$\begin{aligned} \int_{B_{2r}(a)} \zeta \left\langle (\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \nabla' \omega - (\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\ \geq \int_{B_{2r}(a)} \zeta |\nabla' (\omega - \omega_k)|^2 \left( \beta_k^2 \omega_k^2 + 1 + |\nabla' \omega_k|^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} dS. \end{aligned} \quad (2.22)$$

Set  $\phi(\cdot) = \beta_k^2 \omega_k^2 + 1 + |\nabla' \omega_k|^2 + |\nabla' \omega|^2$ , then

$$\begin{aligned} \int_{B_r(a)} |\nabla' \omega - \nabla' \omega_k|^p dS &= \int_{B_r(a)} |\nabla' \omega - \nabla' \omega_k|^p \phi^{\frac{p(p-2)}{4}} \phi^{-\frac{p(p-2)}{4}} dS \\ &\leq \left( \int_{B_r(a)} |\nabla' \omega - \nabla' \omega_k|^2 \phi^{\frac{p-2}{2}} dS \right)^{\frac{p}{2}} \left( \int_{B_r(a)} \phi^{\frac{p}{2}} dS \right)^{\frac{2-p}{2}}. \end{aligned} \quad (2.23)$$

Jointly with (2.14) and (2.22) we conclude that (2.17). Step 1 follows by a standard covering argument.

*Step 2: We claim that  $\omega_k$  converges to  $\omega$  in  $W_0^{1,p}(S)$ .*

Up to a subsequence that we denote again by  $\{k\}$ , we can assume that  $\omega_k \rightarrow \omega$  and  $\nabla' \omega_k \rightarrow \nabla' \omega$  a.e. in  $S$ . Let  $\zeta \in C_0^\infty(S)$ , then there exists  $k_\epsilon \in \mathbb{N}$  such that the support  $K$  of  $\zeta$  is a compact subset of  $S_k$  for all  $k \geq k_\epsilon$ . If  $1 < p < 2$ ,

$$\left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega_k| \leq |\nabla' \omega_k|^{p-1},$$

which bounded in  $L^{p'}(K)$ , then uniformly integrable in  $K$  and by Vitali's convergence theorem

$$\left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \rightarrow \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega,$$

in  $L_{loc}^1(S)$ . Similarly

$$\left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \rightarrow \left( \beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega,$$

in  $L_{loc}^1(S)$ . If  $p \geq 2$

$$\left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega_k| \leq c \left( |\omega_k|^{p-1} + |\nabla' \omega_k|^{p-1} \right),$$

and we conclude again by Vitali's convergence theorem that the previous convergences hold. Since

$$\int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega_k, \nabla' \zeta \rangle dS = (p-1) \beta_{S_k} (\beta_{S_k} - \beta_0) \int_{S_k} \left( \beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \zeta dS$$

we conclude that  $\omega$  is a weak solution of (1.1) with  $\beta = \beta_S$ .  $\square$

### 2.3 Approximations from outside

*Proof of Theorem C.* Since  $\overline{S}^c$  has a non-empty interior, the existence of a sequence  $\{\omega'_k\}$  corresponding to solutions of (1.1) in  $S'_k$  with  $\beta = \beta_{S'_k}$  is the consequence of [13]. The fact that  $\{\beta_{S'_k}\}$  is increasing follows from Proposition 2.2. We denote by  $\hat{\beta} := \hat{\beta}_S$  its limit, and it is smaller or equal to  $\beta_S$ . Estimates (2.4) are valid with  $S'_k$ ,  $\omega'_k$  and  $\beta_{S'_k}$  instead of  $S$ ,  $\omega$  and  $\beta$ . If we extend  $\omega'_k$  by 0 in  $S_k^c$  these estimates are valid with  $S^{N-1}$  instead of  $S'_k$ . Then up to a subsequence there exists  $\omega \in W^{1,p}(S^{N-1})$  and a subsequence still denoted by  $\{k\}$  such that  $\omega'_k \rightharpoonup \omega$  weakly in  $W^{1,p}(S^{N-1})$ , strongly in  $L^p(S^{N-1})$  and a.e. in  $S^{N-1}$ . Furthermore, as in the proof of Theorem A, for any compact set  $K \subset S$ ,  $\nabla' \omega'_k \rightarrow \nabla' \omega'$  in  $L^p(K)$ . This is sufficient to assert that  $\omega$  is a weak solution of

$$-div' \left( \left( \hat{\beta}^2 \omega'^2 + |\nabla' \omega'|^2 \right)^{\frac{p-2}{2}} \nabla' \omega' \right) = (p-1) \hat{\beta} (\hat{\beta} - \beta_0) \left( \hat{\beta}^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega' \quad \text{in } S.$$

Moreover  $\omega'|_{S'_k}$  belongs to  $W_0^{1,p}(S'_k)$  for all  $k$ . Since  $\omega'_k = 0$  in  $S_k^c$  and converges a.e. to  $\omega$ , this last function vanishes a.e. in  $\cup_k S_k^c = (\cap_k S_k)^c = \overline{S}^c$ . Therefore  $\omega$  vanishes a.e. in  $\overline{S}^c$  and since it is quasi continuous, it vanishes,  $(1-p)$ -quasi everywhere in  $\overline{S}^c$ . From Netrusov's theorem (see [1, Th 10.1.1]- (iii)) there exists a sequence  $\{\eta_n\} \subset C_0^\infty(S)$  which converges to  $\omega$  in  $W^{1,p}(S)$ , thus  $\omega \in W_0^{1,p}(S)$ .  $\square$

### 3 Uniqueness

#### 3.1 Uniqueness of exponent $\beta$

*Proof of Theorem D.* If  $S$  is Lipschitz,  $C_S$  is also Lipschitz. We fix  $z \in S \approx S^{N-1} \cap \partial C_S$  and we apply [10, Th 2] in  $G_z = C_S \cap B_{\frac{1}{2}}(z)$  to two separable  $p$ -harmonic functions  $u(r, \sigma) = r^{-\beta} \omega(\sigma)$  and  $u'(r, \sigma) = r^{-\beta'} \omega'(\sigma)$ . There exist  $\gamma \in (0, \frac{1}{2})$ ,  $c_{10} > 0$  and  $\alpha \in (0, 1)$  such that

$$\left| \ln \frac{u(y_1)}{u'(y_1)} - \ln \frac{u(y_2)}{u'(y_2)} \right| \leq c_{10} |y_1 - y_2|^\alpha \quad \forall y_1, y_2 \in C_S \cap B_\gamma(z). \quad (3.24)$$

Assume  $|y_1| = |y_2| = 1$ , then

$$\left| \ln \frac{\omega(y_1)}{\omega'(y_1)} - \ln \frac{\omega(y_2)}{\omega'(y_2)} \right| \leq c_{10} |y_1 - y_2|^\alpha \quad \forall y_1, y_2 \in S \cap B_\gamma(z). \quad (3.25)$$

We denote by  $\ell(x, y)$  the geodesic distance on  $S^{N-1}$  and by  $\ell(x, K)$  the geodesic distance from a point  $x \in S^{N-1}$  to a subset  $K$ . Since the set  $S_\gamma = \{\sigma \in S : \ell(\sigma, \partial S) \leq \frac{\gamma}{2}\}$  can be covered by a finite number of balls with center on  $\partial S$ , we infer that

$$\left| \ln \frac{\omega(y_1)}{\omega'(y_1)} - \ln \frac{\omega(y_2)}{\omega'(y_2)} \right| \leq c_{11} \quad \forall y_1, y_2 \in S_\gamma. \quad (3.26)$$

In  $S \setminus \bar{S}_{\frac{\gamma}{2}}$  we can use Harnack inequality to obtain

$$-c_{12} \leq \ln \frac{\omega(y_1)}{\omega(y_2)} \leq c_{12} \quad \forall y_1, y_2 \in S \setminus \bar{S}_{\frac{\gamma}{2}} \text{ s.t. } \ell(y_1, y_2) \leq \frac{\gamma}{2}. \quad (3.27)$$

Hence there exists a constant  $c_{13} > 0$  such that (3.27) holds for any  $y_1, y_2 \in S \setminus \bar{S}_{\frac{\gamma}{2}}$ , with  $c_{12}$  replaced by  $c_{13}$ . Furthermore  $\omega'$  satisfies the same inequality in  $S \setminus \bar{S}_{\frac{\gamma}{2}}$ . Combining the two inequalities we obtain

$$-2c_{13} \leq \ln \frac{\omega(y_1)}{\omega(y_2)} - \ln \frac{\omega'(y_1)}{\omega'(y_2)} \leq 2c_{13} \quad \forall y_1, y_2 \in S \setminus \bar{S}_{\frac{\gamma}{2}}. \quad (3.28)$$

Combining this estimate with (3.25) we derive that it holds for all  $y_1, y_2 \in S$ . This implies

$$e^{-2c_{13}} \frac{\omega(y_2)}{\omega'(y_2)} \leq \frac{\omega(y_1)}{\omega'(y_1)} \leq e^{2c_{13}} \frac{\omega(y_2)}{\omega'(y_2)} \quad \forall y_1, y_2 \in S. \quad (3.29)$$

Assume now that there exist two exponents  $\beta > \beta' > 0$  such that  $r^{-\beta} \omega(\cdot)$  and  $r^{-\beta'} \omega'(\cdot)$  are  $p$ -harmonic and positive in the cone  $C_S$  and vanishes on  $\partial C_S$ . Put  $\theta = \frac{\beta}{\beta'}$ ,  $\eta = \omega'^{\theta}$  and

$$\mathcal{T}(\eta) = -\operatorname{div}' \left( \left( \beta^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{p-2}{2}} \nabla' \eta \right) - (p-1) \beta (\beta - \beta_0) \left( \beta^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{p-2}{2}} \eta,$$

then

$$\mathcal{T}(\eta) = -\theta^{p-2} \left( \beta'^2 \omega'^2 + |\nabla' \omega'|^2 \right)^{\frac{p-2}{2}} \left( (\beta - \beta') \omega'^2 + (p-1) \theta (\theta - 1) |\nabla' \omega'|^2 \right) \leq 0.$$

Up to multiplying  $\omega'$  by  $\lambda$ , we can assume that  $\eta \leq \omega$  and that the graphs of  $\eta$  and  $\omega$  are tangent in  $\bar{S}$ . Since  $\omega' \leq c\omega$ ,  $\eta = o(\omega)$  near  $\partial S$ . Hence there exists  $\sigma_0 \in S$  such that  $\omega(\sigma_0) = \eta(\sigma_0)$  and the

coincidence set of  $\eta$  and  $\omega$  is a compact subset of  $S$ . We put  $w = \omega - \eta$ , since  $\nabla\omega(\sigma_0) = \nabla\eta(\sigma_0)$  we proceed as in [14, Th 4.1] (see also [4] in the flat case) and derive that  $w$  satisfies, in a system of local coordinates  $(\sigma_1, \dots, \sigma_{N-1})$  near  $\sigma_0$ ,

$$\mathcal{L}w := - \sum_{\ell, j} \frac{\partial}{\partial \sigma_\ell} \left( A_{j, \ell} \frac{\partial w}{\partial \sigma_j} \right) + \sum_j C_j \frac{\partial w}{\partial \sigma_\ell} + Cw \geq 0,$$

where the matrix  $(A_{j, \ell})$  is smooth, symmetric and positive near  $\sigma_0$  and the  $C_j$  and  $C$  are bounded. Hence  $w$  is locally zero. By a standard argument of connectedness, this implies that the zero set of  $w$  must be empty, contradiction. Hence  $\beta = \beta'$ .  $\square$

## 3.2 Uniqueness of eigenfunction

The proof is based upon a delicate adaptation of the characterisation of the  $p$ -Martin boundary obtained in [10], but we first give a proof in the convex case.

### 3.2.1 The convex case

**Theorem 3.1.** *Assume  $S$  is a convex spherical subdomain. Then two positive solutions of (1.1) are proportional.*

*Proof.* We recall that a domain  $S$  is (geodesically) convex if a minimal geodesic joining two points of  $S$  is contained in  $S$ . If  $S \subset S^{N-1}$  is convex, the cone  $C_S$  is convex too. Since  $S$  is convex, it is Lipschitz and by Theorem D,  $\beta_S = \hat{\beta}_S := \beta$ . Let  $\omega$  and  $\omega'$  be two positive solutions of (1.1) satisfying  $\sup_S \omega = \sup_S \omega' = 1$ . We denote by  $u_\omega(x) = |x|^{-\beta} \omega(\cdot)$  and  $u_{\omega'}(x) = |x|^{-\beta} \omega'(\cdot)$  the corresponding separable  $p$ -harmonic functions defined in  $C_S$ . If  $0 < a < b$ , we set  $C_S^{a, b} = C_S \cap (B_b \setminus \overline{B}_a)$ . Then for  $0 < \epsilon < 1$  we denote by  $u_\epsilon$  the unique function which satisfies

$$\begin{aligned} -\Delta_p u_\epsilon &= 0 && \text{in } C_S^{\epsilon, 1} \\ u_\epsilon &= \epsilon^{-\beta} \omega && \text{in } C_S \cap \partial B_\epsilon \\ u_\epsilon &= 0 && \text{in } (C_S \cap \partial B_1) \cup (\partial C_S \cap (\overline{B}_1 \setminus B_\epsilon)). \end{aligned} \quad (3.30)$$

Then

$$(u_\omega - 1)_+ \leq u_\epsilon \leq u_\omega \quad \text{in } C_S^{\epsilon, 1}. \quad (3.31)$$

Furthermore  $\epsilon \mapsto u_\epsilon$  is increasing. When  $\epsilon \downarrow 0$ ,  $u_\epsilon \uparrow u_0$  where  $u_0$  is positive and  $p$ -harmonic in  $C_S^{1, 0}$ , vanishes on  $\partial C_S^{1, 0} \setminus \{0\}$  and satisfies (3.30) with  $\epsilon = 0$ . In particular

$$\lim_{r \rightarrow 0} r^\beta u_0(r, \sigma) = \omega(\sigma) \quad \text{locally uniformly in } S. \quad (3.32)$$

We construct the same approximation  $u'_\epsilon$  in  $C_S^{\epsilon, 1}$  with  $\omega'$  instead of  $\omega$ . Mutadis mutandis (3.31) holds and  $u'_\epsilon \uparrow u'_0$  which is positive and  $p$ -harmonic in  $C_S^1$ , satisfies

$$(u_{\omega'} - 1)_+ \leq u'_0 \leq u_{\omega'} \quad \text{in } C_S^{1, 0},$$

and thus

$$\lim_{r \rightarrow 0} r^\beta u'_0(r, \sigma) = \omega'(\sigma) \quad \text{locally uniformly in } S. \quad (3.33)$$

However, by [10, Th 4]  $u_0$  and  $u'_0$  are proportional. Combined with (3.32), (3.33) it implies the claim.

### 3.2.2 Proof of Theorem E

In what follows we borrow most of our construction from [10] that we adapt to the case of an infinite cone and make explicit for the sake of completeness. The next *nondegeneracy property* of positive  $p$ -harmonic functions is proved in [10, Lemma 4.28].

**Proposition 3.2.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain and  $1 < p < \infty$ . Then there exist constants  $\rho > 0$ ,  $c_{14}, c_{15} > 0$  depending respectively on  $\Omega$  (for  $\rho$ ), and  $p$ ,  $N$  and the Lipschitz norm  $M$  of  $\partial\Omega$  (for  $c_{14}$  and  $c_{15}$ ) with the property that for any  $w \in \partial\Omega$  and any positive  $p$ -harmonic function  $u$  in  $\Omega$ , continuous in  $\overline{\Omega} \cap \overline{B}_{2\rho}(w)$  and vanishing on  $\partial\Omega \cap B_\rho(w)$ , one can find  $\xi \in S^{N-1}$ , independent of  $u$ , such that*

$$c_{14}^{-1} \frac{u(y)}{\text{dist}(y, \partial\Omega)} \leq \langle \nabla u(y), \xi \rangle \leq |\nabla u(y)| \leq c_{14} \frac{u(y)}{\text{dist}(y, \partial\Omega)}, \quad (3.34)$$

for all  $y \in C_S \cap \overline{B}_{\frac{\rho|w|}{c_{15}}}(w)$ .

If  $\Omega$  is replaced by a cone  $C_S$ , the nondegeneracy property still holds uniformly on  $\partial C_S \setminus \{0\}$ .

**Corollary 3.3.** *Let  $1 < p < \infty$ ,  $S \subset S^{N-1}$  is a Lipschitz domain and  $C_S$  the cone generated by  $S$ .*

(i) *Then there exist constants  $\rho < \frac{1}{2}$ ,  $c_{14}, c_{15} > 0$  depending respectively on  $S$  (for  $\rho$ ), and  $p$ ,  $N$  and the Lipschitz norm  $M$  of  $\partial S$  and  $\text{diam}(S)$  (for  $c_{14}$  and  $c_{15}$ ) with the property that for any  $w \in \partial C_S$  and any positive  $p$ -harmonic function  $u$  in  $C_S$ , continuous in  $\overline{C_S} \cap \overline{B}_{2\rho|w|}(w)$  and vanishing on  $\partial C_S \cap \overline{B}_{\rho|w|}(w)$  continuous, one can find  $\xi \in S^{N-1}$ , independent of  $u$ , such that*

$$c_{14}^{-1} \frac{u(y)}{\text{dist}(y, \partial C_S)} \leq \langle \nabla u(y), \xi \rangle \leq |\nabla u(y)| \leq c_{14} \frac{u(y)}{\text{dist}(y, \partial C_S)}, \quad (3.35)$$

for all  $y \in B_{\frac{\rho}{c_{15}}}(w) \cap C_S$ .

(ii) *Then there exist positive constants  $\kappa$  and  $c_{16}, c_{17}$  depending on  $S$  (for  $\kappa$ ), and  $p$ ,  $N$  and the Lipschitz norm  $M$  of  $\partial S$  and  $\text{diam}(S)$  (for  $c_{16}, c_{17}$  such that for any  $a > 0$  and any positive  $p$ -harmonic function  $u$  in  $C_S^a$  vanishing on  $\partial C_S \cap B_a^c$ , there holds*

$$c_{16}^{-1} \frac{u(y)}{\text{dist}(y, \partial C_S)} \leq |\nabla u(y)| \leq c_{16} \frac{u(y)}{\text{dist}(y, \partial C_S)} \quad \forall y \in C_S^{c_{17}a} \text{ s.t. } \text{dist}(y, \partial C_S) \leq \kappa|y|. \quad (3.36)$$

Let  $\omega, \omega' \in W_0^{1,p}(S) \cap C(\overline{S})$  be positive solutions (1.1). Since  $\frac{\omega}{\omega'}$  is bounded from above and from below in  $S$  by positive constants, we can assume, as in the proof of Theorem D, that  $\omega \geq \omega'$  in  $S$  and that the graphs of  $\omega$  and  $\omega'$  are tangent. hence, if  $\omega \neq \omega'$ , then  $\omega > \omega'$  in  $S$  and there exists a sequence  $\{\sigma_n\}$  converging to  $\sigma_0 \in \partial S$  as  $n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} \frac{\omega'(\sigma_n)}{\omega(\sigma_n)} = 1.$$

We define  $\delta_1 = \sup\{\delta > 0 : \delta\omega < \omega'\}$ . For  $t \in (\delta_1, 1)$ , we set

$$\phi_t = \sup\{\omega', t\omega\} \quad \text{and} \quad \psi_t = \inf\left\{\frac{t}{\delta_1}\omega', \omega\right\} \quad (3.37)$$

We also set

$$v_{\phi_t}(r, \sigma) = r^{-\beta}\phi_t(\sigma) \quad \text{and} \quad v_{\psi_t}(r, \sigma) = r^{-\beta}\psi_t(\sigma) \quad \forall (r, \sigma) \in (0, \infty) \times S. \quad (3.38)$$

**Lemma 3.4.** *The functions  $\phi_t$  and  $\psi_t$  are respectively a subsolution and a supersolution of (1.1) in  $W_0^{1,p}(S)$ ,  $v_{\phi_t}$  and  $v_{\psi_t}$  are respectively a subsolution and a supersolution of  $-\Delta_p$  in  $C_S$ , and there exists  $\eta \in W_0^{1,p}(S)$  solution of (1.1) such that*

$$\omega' \leq \phi_t \leq \eta \leq \psi_t \leq \omega \quad \forall t \in (\delta_1, 1). \quad (3.39)$$

*If  $S_t$  is the subset of  $\eta \in W_0^{1,p}(S)$  solutions of (1.1) and satisfying (3.39), then  $\omega_t = \sup\{\eta : \eta \in S_t\}$  belongs to  $S_t$ . It is increasing with respect to  $t$  with uniform limits  $\omega'$  when  $t \downarrow \delta_1$  and  $\omega$  when  $t \uparrow 1$ . Finally, if  $\theta_t = \frac{t-\delta_1}{1-\delta_1}$ , there holds*

$$\phi_t \leq \theta_t \omega + (1 - \theta_t) \omega' \leq \psi_t. \quad (3.40)$$

*Proof.* Clearly  $\phi_t$  and  $\psi_t$  are respectively a subsolution and a supersolution of the operator  $\mathcal{T}$ , they belong to  $W_0^{1,p}(S) \cap L^\infty(S)$  and they satisfy  $\omega' \leq \phi_t \leq \psi_t \leq \omega$ . Furthermore, by Dini convergence theorem

$$\lim_{t \uparrow 1} \phi_t = \omega = \lim_{t \uparrow 1} \psi_t \quad \text{and} \quad \lim_{t \downarrow \delta_1} \phi_t = \omega' = \lim_{t \downarrow \delta_1} \psi_t,$$

uniformly in  $\bar{S}$ . Moreover, in spherical coordinates,

$$\begin{aligned} -\Delta_p u(r, \sigma) = & \left( \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \right)_r - \frac{N-1}{r} \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \\ & - \frac{1}{r^2} \operatorname{div}' \left( \left( u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} \nabla' u \right). \end{aligned}$$

Hence, if  $u(r, \sigma) = r^{-\beta} \eta(\sigma)$ ,

$$-\Delta_p u(r, \sigma) = \beta^{p-2} r^{-(p-1)(\beta+1)-1} \mathcal{T}(\eta).$$

Thus  $v_{\phi_t}$  is a subsolution  $-\Delta_p$  in  $C_S$  and  $v_{\psi_t}$  is a supersolution. Since the operator  $\mathcal{T}$  is a Leray-Lions operator, it follows by [3] that there exists  $\eta \in W_0^{1,p}(S) \cap L^\infty(S)$  satisfying  $\mathcal{T}(\eta) = 0$  and  $\phi_t \leq \eta \leq \psi_t$  in  $S$ . We denote by  $S_t$  the set of  $\eta \in W_0^{1,p}(S) \cap L^\infty(S)$  satisfying  $\mathcal{T}(\eta) = 0$  and  $\phi_t \leq \eta \leq \psi_t$  in  $S$ . Then there exists a sequence  $\{\eta_n\} \subset S_t$  and  $\omega_t \in W_0^{1,p}(S) \cap L^\infty(S)$  such that  $\eta_n(\sigma) \uparrow \omega_t(\sigma)$  for all  $\sigma \in \Sigma$ , where  $\Sigma$  is a countable dense subset of  $S$ . By Lemma 2.1  $\{\eta_n\}$  is bounded in  $L^p(S)$ , hence in  $C^\gamma(S)$  for some  $\gamma \in (0, 1)$ . By the estimates of the proof of Theorem B-Step 2,  $\{\eta_n\}$  is bounded in  $W_0^{1,p}(S)$ . By standard regularity theory, we can also assume that  $\eta_n \rightarrow \omega_t$  in the  $C_{loc}^1(S)$ -topology. Hence  $\omega_t$  is a weak solution of (1.1), it belongs to  $W_0^{1,p}(S) \cap L^\infty(S)$  and satisfies  $\phi_t \leq \omega_t \leq \psi_t$ . Therefore it is the maximal element of  $S_t$ . The monotony of  $\omega_t$  is a consequence of the monotonicity of  $\phi_t$  and  $\psi_t$  and the last statement (3.40) is a straightforward computation.  $\square$

Next we recall the deformation of  $p$ -harmonic functions already used in [10]. If  $\tau \in (0, 1)$  and  $0 < a < b$ , we denote by  $v_{\tau,a,b}$  the  $p$ -harmonic function defined in  $C_S^{a,b}$  satisfying

$$v_{\tau,a,b}(x) = \begin{cases} a^{-\beta}(\tau\omega + (1-\tau)\omega')\left(\frac{x}{|x|}\right) & \text{if } x \in C_S \cap \partial B_a \\ 0 & \text{if } x \in C_S \cap \partial B_b \\ 0 & \text{if } x \in \partial C_S \cap (\bar{B}_b \setminus B_a). \end{cases} \quad (3.41)$$

**Lemma 3.5.** *The mapping  $(\tau, b) \mapsto v_{\tau,a,b}$  is continuous and increasing. If  $v_{\tau,a} = \lim_{b \rightarrow \infty} v_{\tau,a,b}$ , then it is a positive  $p$ -harmonic function in  $C_S^{a,\infty}$  vanishing on  $\partial S \cap B_a^c$ , and there holds*

$$u_{\omega'}(x) \leq v_{\phi_{\tau^*}}(x) \leq v_{\tau,a}(x) \leq v_{\psi_{\tau^*}}(x) \leq u_\omega(x) \quad \forall x \in C_S^{a,\infty}, \quad (3.42)$$

where  $\tau^* = (1 - \delta_1)\tau + \delta_1$  and as a consequence

$$\limsup_{\tau \uparrow 1} \sup_{|x| \geq a} |x|^\beta (u_\omega(x) - v_{\tau,a}(x)) = 0 \quad \text{and} \quad \limsup_{\tau \downarrow 0} \sup_{|x| \geq a} |x|^\beta (v_{\tau,a}(x) - u_{\omega'}(x)) = 0 \quad (3.43)$$

Furthermore

$$0 \leq \frac{v_{\tau',a} - v_{\tau,a}}{\tau' - \tau} \leq \left( \frac{1}{\delta_1} - 1 \right) v_{\tau',a} \quad \forall 0 \leq \tau < \tau' \leq 1. \quad (3.44)$$

*Proof.* The uniqueness and the (strict) monotonicity of  $(\tau, b) \mapsto v_{\tau,a,b}$  follow from the monotonicity of  $\tau \mapsto \tau\omega + (1 - \tau)\omega'$  and the strong maximum principle. The continuity is a consequence of uniqueness and regularity theory for  $p$ -harmonic functions. It follows from (3.40) with  $t = \tau^*$  and the fact that  $v_{\phi_{\tau^*}}$  and  $v_{\psi_{\tau^*}}$  are respectively a subsolution and a supersolution of  $-\Delta_p$ , that we have

$$u_{\omega'}(x) \leq v_{\phi_{\tau^*}}(x) \leq v_{\tau,a,b}(x) \leq v_{\psi_{\tau^*}}(x) \leq u_\omega(x) \quad \forall x \in C_S^{a,b},$$

which yields (3.42). Similarly, we have on  $\partial C_S^{a,b}$

$$0 \leq \frac{v_{\tau',a,b} - v_{\tau,a,b}}{\tau' - \tau} = u_\omega - u_{\omega'} \leq (\delta_1^{-1} - 1)u_{\omega'} \leq (\delta_1^{-1} - 1)v_{\tau,a,b}, \quad (3.45)$$

equivalently

$$0 \leq v_{\tau',a,b} \leq (1 + (\tau' - \tau)(\delta_1^{-1} - 1)) v_{\tau,a,b}. \quad (3.46)$$

By the maximum principle (3.45) holds in  $C_S^{a,b}$ . This implies (3.44).  $\square$

As a consequence of (3.44),  $\partial_\tau v_{\tau,a}$  exists for almost all  $\tau \in (0, 1)$  in  $W_0^{1,p}(C_S^{a,b})$  for all  $b > a$  and it is a solution of

$$\begin{aligned} \mathbb{L}w &= \nabla \cdot \left( (p-2) |\nabla v_{\tau,a}|^{p-4} \langle \nabla v_{\tau,a}, \nabla Z \rangle \nabla v_{\tau,a} \right) \\ &= \sum_{i,j} \frac{\partial}{\partial x_j} \left( b_{i,j}(x) \frac{\partial w}{\partial x_i} \right) = 0 \end{aligned} \quad (3.47)$$

where

$$b_{i,j}(x) = |\nabla v_{\tau,a}|^{p-4} \left( (p-2) \frac{\partial v_{\tau,a}}{\partial x_j} \frac{\partial v_{\tau,a}}{\partial x_i} + \delta_{ij} |\nabla v_{\tau,a}|^2 \right).$$

$\mathbb{L}$  satisfies the following ellipticity condition

$$\min\{1, p-1\} |\nabla v_{\tau,a}|^2 |\xi|^2 \leq \sum_{i,j} b_{i,j}(x) \xi_i \xi_j \leq \max\{1, p-1\} |\nabla v_{\tau,a}|^2 |\xi|^2 \quad \forall \xi \in \mathbb{R}^N. \quad (3.48)$$

It is important to notice that  $\mathbb{L}v_{\tau,a} = (p-1)\Delta_p v_{\tau,a} = 0$ . The estimate (3.48) combined with (3.36) and the decay of  $v_{\tau,a}$  and  $\partial_\tau v_{\tau,a}$  implies that they satisfy Harnack inequality and boundary Harnack inequality in  $C_S^a$ . There exists a constant  $\hat{c} > c_{17} > 1$  (see 3.36) such that

$$\frac{1}{\hat{c}} \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} \leq \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \leq \hat{c} \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} \quad \forall x \in C_S^{\hat{c}a}, \quad (3.49)$$

where  $x_a = (\hat{c}a, \sigma_0)$  for some  $\sigma_0 \in S$  fixed. We set

$$M(t) = \sup_{x \in C_S^t} \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \quad \text{and} \quad m(t) = \inf_{x \in C_S^t} \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \quad \forall t > a \quad (3.50)$$

**Lemma 3.6.** For  $t > \hat{c}a$  there holds

$$M(\hat{c}t) - m(\hat{c}t) \leq \frac{\hat{c}^2 - 1}{\hat{c}^2 + 1} (M(t) - m(t)). \quad (3.51)$$

*Proof.* There holds

$$\partial_\tau v_{\tau,a} - m(t)v_{\tau,a} \geq 0 \quad \text{and} \quad M(t)v_{\tau,a} - \partial_\tau v_{\tau,a} \geq 0 \quad \in C_S^t.$$

Estimate (3.49) is valid for any couple of positive solutions  $(h_1, h_2)$  of  $\mathbb{L}h = 0$  in  $C_S^a$  vanishing on  $\partial C_S^a \cap B_a^c$ , in particular for  $(\partial_\tau v_{\tau,a} - m(t)v_{\tau,a}, v_{\tau,a})$  and  $(M(t)v_{\tau,a} - \partial_\tau v_{\tau,a}, v_{\tau,a})$ . Hence

$$\frac{1}{\hat{c}} \left( \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t) \right) \leq \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} - m(t) \leq \hat{c} \left( \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t) \right) \quad \forall x \in C_S^t. \quad (3.52)$$

This implies

$$\frac{1}{\hat{c}} \left( \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t) \right) \leq m(\hat{c}t) - m(t),$$

and

$$\frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} - m(t) \leq \hat{c}^2 (m(\hat{c}t) - m(t)) \quad \forall x \in C_S^t.$$

Finally

$$M(\hat{c}t) - m(t) \leq \hat{c}^2 (m(\hat{c}t) - m(t)). \quad (3.53)$$

Similarly

$$M(t) - m(\hat{c}t) \leq \hat{c}^2 (M(t) - M(\hat{c}t)). \quad (3.54)$$

Summing the two inequalities we get

$$(M(t) - m(t)) + (M(\hat{c}t) - m(\hat{c}t)) \leq \hat{c}^2 ((M(t) - m(t)) - (M(\hat{c}t) - m(\hat{c}t))),$$

which yields (3.52).  $\square$

*End of the proof.* By the differentiability property of  $v_{\tau,a}$  with respect to  $\tau$ , there exists two countable dense sets  $\{r_\nu\} \subset [a, \infty)$  and  $\{\sigma_\mu\} \subset [a, \infty)$  such that  $\partial_\tau v_{\tau,a}(r_\nu, \sigma_\mu)$  exists for almost all  $\tau$ . We put  $x_{\nu,\mu} = (r_\nu, \sigma_\mu)$ , hence

$$\begin{aligned} \ln \left( \frac{\omega(\sigma_\mu)}{\omega'(\sigma_\mu)} \right) - \ln \left( \frac{\omega(\sigma_{\mu'})}{\omega'(\sigma_{\mu'})} \right) &= \ln \left( \frac{v_{1,a}(x_{\nu,\mu})}{v_{0,a}(x_{\nu,\mu})} \right) - \ln \left( \frac{v_{1,a}(x_{\nu,\mu'})}{v_{0,a}(x_{\nu,\mu'})} \right) \\ &= \int_0^1 \left( \frac{\partial_\tau v_{\tau,a}(x_{\nu,\mu})}{v_{\tau,a}(x_{\nu,\mu})} - \frac{\partial_\tau v_{\tau,a}(x_{\nu,\mu'})}{v_{\tau,a}(x_{\nu,\mu'})} \right) d\tau. \end{aligned} \quad (3.55)$$

Using the continuity of  $\frac{\omega}{\omega'}$  and the density of  $\{\sigma_m\}$  we derive

$$\left| \ln \left( \frac{\omega(\sigma)}{\omega'(\sigma)} \right) - \ln \left( \frac{\omega(\sigma')}{\omega'(\sigma')} \right) \right| \leq M(r_\nu) - m(r_\nu) \quad \forall (\sigma, \sigma') \in S \times S. \quad (3.56)$$

We can assume that  $r_\nu \geq \hat{c}^{\nu} a$  for some sequence  $\{\nu_n\}$  tending to infinity with  $n$ , hence

$$\left| \ln \left( \frac{\omega(\sigma)}{\omega'(\sigma)} \right) - \ln \left( \frac{\omega(\sigma')}{\omega'(\sigma')} \right) \right| \leq \theta^n (M(\hat{c}^{\nu_n}) - m(\hat{c}^{\nu_n})) \quad \forall (\sigma, \sigma') \in S \times S \quad \forall n \in \mathbb{N}^*, \quad (3.57)$$

where  $\theta = \frac{\hat{c}^2 - 1}{\hat{c}^2 + 1} < 1$ . Letting  $n \rightarrow \infty$  implies the claim.  $\square$



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