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The spherical p -harmonic eigenvalue problem in non-smooth domains

Konstantinos Gkikas*

Laurent Véron†

Abstract

We prove the existence of p -harmonic functions under the form $u(r, \sigma) = r^{-\beta}\omega(\sigma)$ in any cone C_S generated by a spherical domain S and vanishing on ∂C_S . We prove the uniqueness of the exponent β and of the normalized function ω under a Lipschitz condition on S .

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1 Introduction

Let $p > 1$, S a domain of the unit sphere S^{N-1} of \mathbb{R}^N and $C_S := \{(r, \sigma) : r > 0, \sigma \in S\}$ the positive cone generated by S . If one looks for p -harmonic functions in C_S under the form $u(x) = u(r, \sigma) = r^{-\beta}\omega(\sigma)$ vanishing on $\partial C_S \setminus \{0\}$, then ω satisfies the *spherical p -harmonic eigenvalue problem* on S

$$\begin{aligned}
 -\operatorname{div}' \left(\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega \right) &= (p-1)\beta(\beta - \beta_0) \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega && \text{in } S \\
 \omega &= 0 && \text{in } \partial S
 \end{aligned}
 \tag{1.1}$$

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with $\beta_0 = \frac{N-p}{p-1}$ and were div' and ∇' denote the divergence operator and the covariant gradient on S^{N-1} endowed with the metric induced by its isometric inbedding into \mathbb{R}^N . Separable solutions play a key role for describing the boundary behaviour and the singularities of solutions of a large variety of quasilinear equations. When $N = 2$ the equation is completely integrable and has been solved by Kroll in the regular case $\beta < 0$ and Kichenassamy and Véron in the the singular case $\beta > 0$. In higher dimension, Tolksdorff [15] proved the following:

Theorem A *If S is a smooth spherical domain, there exist two couples (β_S, ω_S) and (β'_S, ω'_S) where $\beta_S > 0$ and $\beta'_S < 0$, ω_S and ω'_S are positive $C^2(\bar{S})$ -functions vanishing on ∂S which solve (1.1) with $(\beta, \omega) = (\beta_S, \omega_S)$ or $(\beta, \omega) = (\beta'_S, \omega'_S)$. Furthermore β_S and β'_S are unique, and ω_S and ω'_S are unique up to an homothety.*

A more general and transparent proof has been obtained by Porretta and Véron [13], but always in the case of a smooth spherical domain. The aim of this article is to extend Theorem A to a general spherical domain. If we consider an increasing sequence of smooth domains $\{S_k\}$ such that $S_k \subset \bar{S}_k \subset S_{k+1}$ and $\cup_k S_k = S$ we prove the following:

Theorem B *Assume that S^c is not polar. Then the sequence of the $\beta_{S_k} > 0$ from Theorem A is decreasing and converges to $\beta_S > 0$. There exists $\omega_S \in W_0^{1,p}(S) \cap L^\infty(S)$ weak solution of (1.1) with $\beta = \beta_S$. Furthermore $\beta_S > 0$ is the largest exponent β such that (1.1) admits a positive solution $\omega_S \in W_0^{1,p}(S)$.*

Under a mild assumption on S it is possible to approximate it by a decreasing sequence of smooth domains S'_k such that $S'_k \subset \bar{S}'_k \subset S'_{k-1}$ and $\cap_k S'_k = \bar{S}$

Theorem C *Assume that $S = \bar{S}$. Then the sequence $\beta_{S'_k} > 0$ is increasing and converges to $\hat{\beta}_S > 0$ and there exists $\hat{\omega}_S \in W_0^{1,p}(S) \cap L^\infty(S)$ weak solution of (1.1) with $\beta = \hat{\beta}_S$. Furthermore $\hat{\beta}_S$ is the smallest exponent β such that (1.1) admits a positive solution $\omega_S \in W_0^{1,p}(S)$.*

We prove the uniqueness of the exponent β , under a Lipschitz assumption on S .

Theorem D *Assume that S is a Lipschitz domain, then $\beta_S = \hat{\beta}_S$ and if ω and ω' are two positive solutions of (1.1) in $W_0^{1,p}(S)$, there exists a constant $c > 0$ such that $c^{-1}\omega' \leq \omega \leq c\omega'$.*

The proof of Theorem C is based upon a sharp form of boundary Harnack inequality proved in [10],

$$\left| \ln \frac{\omega(\sigma_1)}{\omega'(\sigma_1)} - \ln \frac{\omega(\sigma_2)}{\omega'(\sigma_2)} \right| \leq c_1 |\sigma_1 - \sigma_2|^\alpha \quad \forall \sigma_1, \sigma_2 \in S, \quad (1.2)$$

for some $c_1 = c_1(N, p, S) > 0$ and $\alpha \in (0, 1)$. Actually we have a stronger result, much more delicate to obtain.

Theorem E *Let S be a Lipschitz subdomain of S^{N-1} . Then two positive solutions of (1.1) in $W_0^{1,p}(S)$ are proportional.*

The proof is based upon a non trivial adaptation of a series of deep results of Lewis and Nyström [10] concerning the p -Martin boundary of domains.

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2 Existence

2.1 Estimates

Through this article we assume that S^c is not polar, or equivalently that it has positive $c_{1,p}^{S^{N-1}}$ -capacity.

Lemma 2.1. Assume $p > 1$. Then any solution $\omega \in W_0^{1,p}(S)$ of (1.1) satisfies

$$\|\omega\|_{C^\gamma(S)} \leq c_1 \|\omega\|_{L^p(S)}, \quad (2.1)$$

if $p > N - 1$ where $\gamma = 1 - \frac{N-1}{p}$ if $p > N - 1$ and

$$\|\omega\|_{L^\infty(S)} \leq c_1 \|\omega\|_{L^p(S)}, \quad (2.2)$$

if $1 < p \leq N - 1$, where $c_1 > 0$ depends on p, N, β .

Proof. Multiplying the equation by ω and using Hölder's inequality, we derive

$$\begin{aligned} (i) \quad & \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{2}} dS \leq (\beta(p\beta - (p-1)\beta_0))^{\frac{p}{2}} \int_S |\omega|^p dS \quad \text{if } p \geq 2, \\ (ii) \quad & \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{2}} dS \leq \beta^{p-1} (p\beta - (p-1)\beta_0) \int_S |\omega|^p dS \quad \text{if } 1 < p < 2. \end{aligned} \quad (2.3)$$

Notice that these inequalities hold for all $p > 1$. If $p > N - 1$ (2.1) follows by Morrey's inequality. Here after we assume $1 < p \leq N - 1$. Let $\alpha \geq 1$ and $k > 0$. Then $\zeta = \min\{|\omega|, k\}^{\alpha-1} \omega$ is an admissible test function, hence

1- If $p \geq 2$,

$$\begin{aligned} & \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega, \nabla' \zeta \rangle dS = (p-1)\beta(\beta - \beta_0) \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega \zeta dS \\ & \leq c_2 \int_S |\nabla' \omega|^{p-2} \omega^2 \min\{|\omega|, k\}^{\alpha-1} dS + c_2 \beta^p \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \\ & \leq c_2 \left(\int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \right)^{\frac{p-2}{p}} \left(\int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \right)^{\frac{2}{p}} \\ & \quad + c_2 \beta^p \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \end{aligned} \quad (2.4)$$

where $c_2 = c_2(N, p, \beta) > 0$. Since

$$\int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega, \nabla' \zeta \rangle dS \geq c_3(p) \int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS,$$

it implies that there exists $c_4 = c_4(N, p, \beta)$ such that

$$\int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \leq c_4 \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \quad (2.5)$$

which yields

$$\int_S |\nabla' j(\omega)|^p dS \leq c_4 \int_S |j(\omega)|^p dS, \quad (2.6)$$

where $j(\omega) = \min\{|\omega|, k\}^{\frac{\alpha-1}{p}} \omega$.

2- If $1 < p < 2$, then

$$\begin{aligned} & \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega, \nabla' \zeta \rangle dS = \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega|^2 \min\{|\omega|, k\}^{\alpha-1} dS \\ & \quad + (\alpha - 1) \int_{S \cap \{|\omega| < k\}} \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega|^2 |\omega|^{\alpha-1} dS. \end{aligned} \quad (2.7)$$

Since

$$\begin{aligned} & \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega|^2 \min\{|\omega|, k\}^{\alpha-1} dS = \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{2}} \min\{|\omega|, k\}^{\alpha-1} dS \\ & \quad - \beta^2 \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \min\{|\omega|, k\}^{\alpha-1} \omega^2 dS \\ & \geq \int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS - \beta^2 \int_S \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \min\{|\omega|, k\}^{\alpha-1} \omega^2 dS, \end{aligned}$$

we derive

$$\int_S |\nabla' \omega|^p \min\{|\omega|, k\}^{\alpha-1} dS \leq \beta^{p-1} (p\beta - (p-1)\beta_0) \int_S |\omega|^p \min\{|\omega|, k\}^{\alpha-1} dS, \quad (2.8)$$

which leads to (2.6). Letting $k \rightarrow \infty$ we infer by Fatou's lemma,

$$\int_S \left| \nabla' |\omega|^{\frac{\alpha-1}{p}+1} \right|^p dS \leq c_4 \int_S |\omega|^{\alpha-1+p} dS. \quad (2.9)$$

If $p < N - 1$ we derive from Sobolev inequality and putting $q = \alpha - 1 + p$ and $s = \frac{N-1}{N-1-p} > 1$

$$\left(\int_S |\omega|^{sq} dS \right)^{\frac{1}{s}} \leq c_5 \int_S |\omega|^q dS, \quad (2.10)$$

and $c_5 > 0$ depends on N, p and β . Iterating this estimate by Moser's method we derive (2.10).

If $p = N - 1$ we have for $1 \leq m < p - 1$ and $m^* = \frac{m(N-1)}{N-1-m}$

$$c_6 \left(\int_S |\omega|^{(\frac{\alpha-1}{p}+1)m^*} dS \right)^{\frac{pm}{m^*}} \leq \left(\int_S \left| \nabla' |\omega|^{\frac{\alpha-1}{p}+1} \right|^m dS \right)^{\frac{p}{m}} \leq |S|^{\frac{p}{m}-1} c_4 \int_S |\omega|^{\alpha-1+p} dS,$$

and $c_6 = c_6(N, p)$, hence

$$\left(\int_S |\omega|^{tq} dS \right)^{\frac{1}{t}} \leq c_5 \int_S |\omega|^q dS, \quad (2.11)$$

with $t = \frac{m(N-1)}{p(N-1-m)} = \frac{m}{N-1-m}$. The proof follows again by Moser's iterative scheme. \square

Proposition 2.2. *Let S_1 and S_2 be two subdomains of S^{N-1} such that $S_1 \subset \overline{S_1} \subset S_2$ and S_2 not polar. Let $\beta_j > 0, j=1,2$, such that there exist positive solutions $\omega_j \in W_0^{1,p}(S_j)$ solutions of*

$$\begin{aligned} -\operatorname{div}' \left(\left(\beta_j^2 \omega_j^2 + |\nabla' \omega_j|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_j \right) &= (p-1)\beta_j(\beta_j - \beta_0) \left(\beta_j^2 \omega_j^2 + |\nabla' \omega_j|^2 \right)^{\frac{p-2}{2}} \omega_j && \text{in } S_j \\ \omega_j &= 0 && \text{in } \partial S_j. \end{aligned} \quad (2.12)$$

Then $\beta_1 \geq \beta_2$.

Proof. Set $u_j(r, \sigma) = r^{-\beta_j} \omega_j(\sigma)$, $C_{S_j} = (0, \infty) \times S_j$ and assume $\beta_1 < \beta_2$. By Harnack inequality $\omega_2 \geq c > 0$ on S_1 , thus

$$u_2(r, \sigma) \geq cr^{-\beta_2} \quad \text{a.e. in } C_{S_1}.$$

For $\epsilon > 0$ there exist $r_\epsilon > 0$ such that

$$\epsilon u_2(x) \geq u_1(x) \quad \forall x \in C_{S_1} \cap \overline{B_{r_\epsilon}}.$$

Let $\delta > 0$, there exists $R_\delta > 0$ such that

$$u_1(x) \leq \delta \quad \forall x \in C_{S_1} \cap B_{R_\delta}^c.$$

Hence $\zeta = (u_1 - \epsilon u_2 - \delta)_+ \in W_0^{1,p}(Q_{S_1}^{r_\epsilon, R_\delta})$, where $Q_{S_1}^{r_\epsilon, R_\delta} = \{x \in C_{S_1} : r_\epsilon < |x| < R_\delta\}$. This implies

$$\begin{aligned} 0 &= \int_{Q_{S_1}^{r_\epsilon, R_\delta}} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla(\epsilon u_1)|^{p-2} \nabla(\epsilon u_1), \nabla \zeta \rangle dx \\ &= \int_{Q_{S_1}^{r_\epsilon, R_\delta} \cap \{u_1 - \epsilon u_2 \geq \delta\}} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla(\epsilon u_1)|^{p-2} \nabla(\epsilon u_1), \nabla(u_1 - u_2) \rangle dx. \end{aligned}$$

Therefore $\nabla(u_1 - \epsilon u_2 - \delta)_+ = 0$ a.e. in $Q_{S_1}^{r_\epsilon, R_\delta}$, which leads to $u_1 - \epsilon u_2 \leq \delta$ in the same set. Letting $\delta \rightarrow 0$ yields $R_\delta \rightarrow \infty$, thus we obtain $u_1 \leq \epsilon u_2$ in $C_{S_1} \cap \overline{B}_{r_\epsilon}^c$ hence $u_1 \leq 0$ in C_{S_1} , contradiction. \square

2.2 Approximations from inside

Proof of Theorem B. Let $\{S_k\}$ be an increasing sequence of smooth domains such that $S_k \subset \overline{S}_k \subset S_{k+1}$. We denote by $\{(\beta_{S_k}, \omega_k)\}$ the corresponding sequence of solutions of (1.1) with $\beta = \beta_{S_k}$ and $\omega = \omega_k$. The sequence $\{\beta_{S_k}\}$ is uniquely determined by [15], it admits a limit $\beta := \beta_S$, and the ω_k are the unique positive solutions such that

$$\int_{S_k} |\omega_k| dS = 1.$$

If $p \geq 2$, we have

$$\begin{aligned} \int_{S_k} |\nabla' \omega_k|^p dS &\leq \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega_k|^2 dS \\ &= (p-1) \beta_{S_k} (\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k^2 dS \\ &\leq 2^{\frac{(p-4)_+}{2}} (p-1) \beta_{S_k} (\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^{p-2} \omega_k^p + |\nabla' \omega_k|^{p-2} \omega_k^2 \right) dS \\ &\leq c_7(N, p, \beta_{S_k}) \int_{S_k} \omega_k^p dS + \frac{1}{2} \int_{S_k} |\nabla' \omega_k|^p dS. \end{aligned}$$

Since $\beta_{S_k} \leq \beta_1$, we derive

$$\int_{S_k} |\nabla' \omega_k|^p dS \leq c_8, \tag{2.13}$$

from the normalization assumption with $c_8 = 2c_7(N, p, \beta_1)$.

If $1 < p < 2$, we have

$$\begin{aligned} \int_{S_k} |\nabla' \omega_k|^p dS &\leq \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p}{2}} dS \\ &\leq \beta_{S_k} (p\beta_{S_k} + (p-1)\beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k^2 dS \\ &\leq \beta_k^{p-1} (p\beta_{S_k} + (p-1)\beta_0) \int_{S_k} \omega_k^p dS, \end{aligned}$$

and we obtain (2.13) with $c_8 = \beta_1^{p-1}(p\beta_1 + (p-1)\beta_0)$.

Next we extend ω_k by 0 in S_k^c . Then there exists $\omega \in W_0^{1,p}(S)$ such that $\omega_k \rightharpoonup \omega$ weakly in $W_0^{1,p}(S)$, up to subsequence that we still denote $\{\omega_k\}$, and $\omega_k \rightarrow \omega$ in $L^p(S)$.

Step 1: We claim that $\nabla'\omega_k$ converges to $\nabla'\omega$ locally in $L^p(S)$.

Let $a \in S$ and $r > 0$ such that $B_{4r}(a) \subset S$. Then for $k \geq k_0$, $\overline{B_{2r}(a)} \subset S_k$. Let $\zeta \in C_0^\infty(B_{2r}(a))$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_r(a)$. For test function we choose $\eta_k = \zeta(\omega - \omega_k)$, then

$$\int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \langle \nabla'\omega_k, \nabla'\eta_k \rangle dS = (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \eta_k dS.$$

By the above inequality, we have

$$\begin{aligned} & \int_{B_{2r}(a)} \left\langle \left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'\eta_k \right\rangle dS \\ &= \int_{B_{2r}(a)} \left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla'\omega, \nabla'\eta_k \rangle dS \\ & \quad - (p-1)\beta_{S_k}(\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \eta_k dS. \end{aligned}$$

Using the weak convergence of the gradient, we have

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} \left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \langle \nabla'\omega, \nabla'\eta_k \rangle dS = 0.$$

Since ω_k is uniformly bounded in $W_0^{1,p}(S)$ and $\omega_k \rightarrow \omega$ in $L^p(S)$, we have

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \eta_k dS = 0,$$

and

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} (\omega - \omega_k) \left\langle \left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'\zeta \right\rangle dS = 0.$$

Combining the above relations we infer

$$\lim_{k \rightarrow \infty} \int_{B_{2r}(a)} \zeta \left\langle \left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'(\omega - \omega_k) \right\rangle dS = 0. \quad (2.14)$$

Next we write

$$\begin{aligned} & \int_{B_{2r}(a)} \zeta \left\langle \left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} \nabla'\omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \nabla'\omega_k, \nabla'(\omega - \omega_k) \right\rangle dS \\ &= \frac{1}{2} \int_{B_{2r}(a)} \zeta \left(\left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} + \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \right) |\nabla'(\omega - \omega_k)|^2 dS \\ & \quad + \frac{1}{2} \int_{B_{2r}(a)} \zeta \left(\left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \right) \\ & \quad \quad \quad \times \left(|\nabla'\omega|^2 + \beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2 - |\nabla'\omega_k|^2 \right) dS \\ & \quad - \frac{1}{2} \int_{B_{2r}(a)} \zeta \left(\left(\beta^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla'\omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS. \end{aligned} \quad (2.15)$$

If $p \geq 2$, we have from (2.4),

$$\begin{aligned}
& \int_{B_{2r}(a)} \zeta \left\langle \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\
& \geq \frac{1}{2} \int_{B_{2r}(a)} \zeta \left(|\nabla' \omega|^{p-2} + |\nabla' \omega_k|^{p-2} \right) |\nabla' (\omega - \omega_k)|^2 dS \\
& \quad - \frac{1}{2} \int_{B_{2r}(a)} \zeta \left(\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS \\
& \geq \min\{2^{-1}, 2^{2-p}\} \int_{B_{2r}(a)} \zeta |\nabla' (\omega - \omega_k)|^p dS \\
& \quad - \frac{1}{2} \int_{B_{2r}(a)} \zeta \left(\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS.
\end{aligned} \tag{2.16}$$

Since $\omega_k \rightarrow \omega$ in $L^p(S)$, $\beta_{S_k} \rightarrow \beta$ and ω_k, ω are uniformly bounded in $W_0^{1,p}(S)$, we derive

$$\int_{B_{2r}(a)} \zeta \left(\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \right) (\beta^2 \omega^2 - \beta_{S_k}^2 \omega_k^2) dS \rightarrow 0$$

as $k \rightarrow \infty$. Jointly with (2.14) we infer that

$$\lim_{k \rightarrow \infty} \int_{B_r(a)} |\nabla' (\omega - \omega_k)|^p dS = 0. \tag{2.17}$$

If $1 < p < 2$, then

$$\begin{aligned}
& \int_{B_{2r}(a)} \zeta \left\langle \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\
& = \int_{B_{2r}(a)} \zeta \left\langle \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\
& \quad + \int_{B_{2r}(a)} \zeta \left\langle \left(\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \right) \nabla' \omega \cdot \nabla' (\omega - \omega_k) \right\rangle dS.
\end{aligned} \tag{2.18}$$

Up to extracting a subsequence, we have that $\omega_k \rightarrow \omega$ a.e. in S and that there exists $\Phi \in L^1(S)$ such that

$$|\omega_k|^p + |\omega|^p \leq \Phi \quad \text{a.e. in } S \quad \text{and } \forall k \geq 1. \tag{2.19}$$

Since

$$\left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla \omega| \leq \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-1}{2}} \leq \beta_{S_k}^{p-1} \omega_k^{p-1} + |\nabla' \omega|^{p-1},$$

and

$$\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} |\nabla \omega| \leq \beta^{p-1} \omega^{p-1} + |\nabla' \omega|^{p-1},$$

we derive that

$$\left| \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \right| |\nabla' \omega| \leq 2 \left(\beta^{p-1} \Phi^{p-1} + |\nabla' \omega|^{p-1} \right),$$

which implies that

$$\zeta \left(\left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} - \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \right) \nabla' \omega \rightarrow 0 \quad \text{in } L^p(S)$$

where p' is the conjugate of p , and finally

$$\int_{B_{2r}(a)} \zeta \left\langle \left((\beta^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} - (\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \right) \nabla' \omega \cdot \nabla' (\omega - \omega_k) \right\rangle dS \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.20)$$

For the last term on the right-hand side of (2.18), we have, for $\gamma \in \mathbb{R}_+$ and $\mathbf{A}, \mathbf{B} \in \mathbb{R}^N$,

$$\begin{aligned} (\gamma + |\mathbf{B}|^2)^{\frac{p-2}{2}} \mathbf{B} - (\gamma + |\mathbf{A}|^2)^{\frac{p-2}{2}} \mathbf{A} &= \int_0^1 \frac{d}{dt} \left((\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} (t\mathbf{B} + (1-t)\mathbf{A}) \right) dt \\ &= \left(\int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt \right) (\mathbf{B} - \mathbf{A}) \\ &\quad + (p-2) \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-4}{2}} \langle t\mathbf{B} + (1-t)\mathbf{A}, \mathbf{B} - \mathbf{A} \rangle (t\mathbf{B} + (1-t)\mathbf{A}) dt. \end{aligned}$$

This implies

$$\begin{aligned} \left\langle (\gamma + |\mathbf{B}|^2)^{\frac{p-2}{2}} \mathbf{B} - (\gamma + |\mathbf{A}|^2)^{\frac{p-2}{2}} \mathbf{A}, \mathbf{B} - \mathbf{A} \right\rangle &= \left(\int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt \right) |\mathbf{B} - \mathbf{A}|^2 \\ &\quad + (p-2) \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-4}{2}} \langle t\mathbf{B} + (1-t)\mathbf{A}, \mathbf{B} - \mathbf{A} \rangle^2 dt. \end{aligned}$$

We observe that

$$\begin{aligned} \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-4}{2}} \langle t\mathbf{B} + (1-t)\mathbf{A}, \mathbf{B} - \mathbf{A} \rangle^2 dt \\ \leq |\mathbf{B} - \mathbf{A}|^2 \int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt, \end{aligned}$$

and since $1 < p < 2$, we finally obtain

$$\begin{aligned} \left\langle (\gamma + |\mathbf{B}|^2)^{\frac{p-2}{2}} \mathbf{B} - (\gamma + |\mathbf{A}|^2)^{\frac{p-2}{2}} \mathbf{A}, \mathbf{B} - \mathbf{A} \right\rangle \\ \geq (p-1) \left(\int_0^1 (\gamma + |t\mathbf{B} + (1-t)\mathbf{A}|^2)^{\frac{p-2}{2}} dt \right) |\mathbf{B} - \mathbf{A}|^2 \quad (2.21) \\ \geq (p-1) |\mathbf{B} - \mathbf{A}|^2 (\gamma + 1 + |\mathbf{B}|^2 + |\mathbf{A}|^2)^{\frac{p-2}{2}}. \end{aligned}$$

We plug this estimate into (2.18) with $\gamma = \beta_k^2 \omega_k^2$, $\mathbf{A} = \nabla' \omega$ and $\mathbf{B} = \nabla' \omega_k$, then

$$\begin{aligned} \int_{B_{2r}(a)} \zeta \left\langle (\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega|^2)^{\frac{p-2}{2}} \nabla' \omega - (\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2)^{\frac{p-2}{2}} \nabla' \omega_k \cdot \nabla' (\omega - \omega_k) \right\rangle dS \\ \geq \int_{B_{2r}(a)} \zeta |\nabla' (\omega - \omega_k)|^2 \left(\beta_k^2 \omega_k^2 + 1 + |\nabla' \omega_k|^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} dS. \end{aligned} \quad (2.22)$$

Set $\phi(\cdot) = \beta_k^2 \omega_k^2 + 1 + |\nabla' \omega_k|^2 + |\nabla' \omega|^2$, then

$$\begin{aligned} \int_{B_r(a)} |\nabla' \omega - \nabla' \omega_k|^p dS &= \int_{B_r(a)} |\nabla' \omega - \nabla' \omega_k|^p \phi^{\frac{p(p-2)}{4}} \phi^{-\frac{p(p-2)}{4}} dS \\ &\leq \left(\int_{B_r(a)} |\nabla' \omega - \nabla' \omega_k|^2 \phi^{\frac{p-2}{2}} dS \right)^{\frac{p}{2}} \left(\int_{B_r(a)} \phi^{\frac{p}{2}} dS \right)^{\frac{2-p}{2}}. \end{aligned} \quad (2.23)$$

Jointly with (2.14) and (2.22) we conclude that (2.17). Step 1 follows by a standard covering argument.

Step 2: We claim that ω_k converges to ω in $W_0^{1,p}(S)$.

Up to a subsequence that we denote again by $\{k\}$, we can assume that $\omega_k \rightarrow \omega$ and $\nabla' \omega_k \rightarrow \nabla' \omega$ a.e. in S . Let $\zeta \in C_0^\infty(S)$, then there exists $k_\epsilon \in \mathbb{N}$ such that the support K of ζ is a compact subset of S_k for all $k \geq k_\epsilon$. If $1 < p < 2$,

$$\left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega_k| \leq |\nabla' \omega_k|^{p-1},$$

which bounded in $L^{p'}(K)$, then uniformly integrable in K and by Vitali's convergence theorem

$$\left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_k \rightarrow \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \nabla' \omega,$$

in $L_{loc}^1(S)$. Similarly

$$\left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \rightarrow \left(\beta^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega,$$

in $L_{loc}^1(S)$. If $p \geq 2$

$$\left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} |\nabla' \omega_k| \leq c \left(|\omega_k|^{p-1} + |\nabla' \omega_k|^{p-1} \right),$$

and we conclude again by Vitali's convergence theorem that the previous convergences hold. Since

$$\int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \langle \nabla' \omega_k, \nabla' \zeta \rangle dS = (p-1) \beta_{S_k} (\beta_{S_k} - \beta_0) \int_{S_k} \left(\beta_{S_k}^2 \omega_k^2 + |\nabla' \omega_k|^2 \right)^{\frac{p-2}{2}} \omega_k \zeta dS$$

we conclude that ω is a weak solution of (1.1) with $\beta = \beta_S$. \square

2.3 Approximations from outside

Proof of Theorem C. Since \overline{S}^c has a non-empty interior, the existence of a sequence $\{\omega'_k\}$ corresponding to solutions of (1.1) in S'_k with $\beta = \beta_{S'_k}$ is the consequence of [13]. The fact that $\{\beta_{S'_k}\}$ is increasing follows from Proposition 2.2. We denote by $\hat{\beta} := \hat{\beta}_S$ its limit, and it is smaller or equal to β_S . Estimates (2.4) are valid with S'_k , ω'_k and $\beta_{S'_k}$ instead of S , ω and β . If we extend ω'_k by 0 in S_k^c these estimates are valid with S^{N-1} instead of S'_k . Then up to a subsequence there exists $\omega \in W^{1,p}(S^{N-1})$ and a subsequence still denoted by $\{k\}$ such that $\omega'_k \rightharpoonup \omega$ weakly in $W^{1,p}(S^{N-1})$, strongly in $L^p(S^{N-1})$ and a.e. in S^{N-1} . Furthermore, as in the proof of Theorem A, for any compact set $K \subset S$, $\nabla' \omega'_k \rightarrow \nabla' \omega'$ in $L^p(K)$. This is sufficient to assert that ω is a weak solution of

$$-div' \left(\left(\hat{\beta}^2 \omega'^2 + |\nabla' \omega'|^2 \right)^{\frac{p-2}{2}} \nabla' \omega' \right) = (p-1) \hat{\beta} (\hat{\beta} - \beta_0) \left(\hat{\beta}^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p-2}{2}} \omega' \quad \text{in } S.$$

Moreover $\omega'|_{S'_k}$ belongs to $W_0^{1,p}(S'_k)$ for all k . Since $\omega'_k = 0$ in S_k^c and converges a.e. to ω , this last function vanishes a.e. in $\cup_k S_k^c = (\cap_k S_k)^c = \overline{S}^c$. Therefore ω vanishes a.e. in \overline{S}^c and since it is quasi continuous, it vanishes, $(1-p)$ -quasi everywhere in \overline{S}^c . From Netrusov's theorem (see [1, Th 10.1.1]- (iii)) there exists a sequence $\{\eta_n\} \subset C_0^\infty(S)$ which converges to ω in $W^{1,p}(S)$, thus $\omega \in W_0^{1,p}(S)$. \square

3 Uniqueness

3.1 Uniqueness of exponent β

Proof of Theorem D. If S is Lipschitz, C_S is also Lipschitz. We fix $z \in S \approx S^{N-1} \cap \partial C_S$ and we apply [10, Th 2] in $G_z = C_S \cap B_{\frac{1}{2}}(z)$ to two separable p -harmonic functions $u(r, \sigma) = r^{-\beta} \omega(\sigma)$ and $u'(r, \sigma) = r^{-\beta'} \omega'(\sigma)$. There exist $\gamma \in (0, \frac{1}{2})$, $c_{10} > 0$ and $\alpha \in (0, 1)$ such that

$$\left| \ln \frac{u(y_1)}{u'(y_1)} - \ln \frac{u(y_2)}{u'(y_2)} \right| \leq c_{10} |y_1 - y_2|^\alpha \quad \forall y_1, y_2 \in C_S \cap B_\gamma(z). \quad (3.24)$$

Assume $|y_1| = |y_2| = 1$, then

$$\left| \ln \frac{\omega(y_1)}{\omega'(y_1)} - \ln \frac{\omega(y_2)}{\omega'(y_2)} \right| \leq c_{10} |y_1 - y_2|^\alpha \quad \forall y_1, y_2 \in S \cap B_\gamma(z). \quad (3.25)$$

We denote by $\ell(x, y)$ the geodesic distance on S^{N-1} and by $\ell(x, K)$ the geodesic distance from a point $x \in S^{N-1}$ to a subset K . Since the set $S_\gamma = \{\sigma \in S : \ell(\sigma, \partial S) \leq \frac{\gamma}{2}\}$ can be covered by a finite number of balls with center on ∂S , we infer that

$$\left| \ln \frac{\omega(y_1)}{\omega'(y_1)} - \ln \frac{\omega(y_2)}{\omega'(y_2)} \right| \leq c_{11} \quad \forall y_1, y_2 \in S_\gamma. \quad (3.26)$$

In $S \setminus \bar{S}_{\frac{\gamma}{2}}$ we can use Harnack inequality to obtain

$$-c_{12} \leq \ln \frac{\omega(y_1)}{\omega(y_2)} \leq c_{12} \quad \forall y_1, y_2 \in S \setminus \bar{S}_{\frac{\gamma}{2}} \text{ s.t. } \ell(y_1, y_2) \leq \frac{\gamma}{2}. \quad (3.27)$$

Hence there exists a constant $c_{13} > 0$ such that (3.27) holds for any $y_1, y_2 \in S \setminus \bar{S}_{\frac{\gamma}{2}}$, with c_{12} replaced by c_{13} . Furthermore ω' satisfies the same inequality in $S \setminus \bar{S}_{\frac{\gamma}{2}}$. Combining the two inequalities we obtain

$$-2c_{13} \leq \ln \frac{\omega(y_1)}{\omega(y_2)} - \ln \frac{\omega'(y_1)}{\omega'(y_2)} \leq 2c_{13} \quad \forall y_1, y_2 \in S \setminus \bar{S}_{\frac{\gamma}{2}}. \quad (3.28)$$

Combining this estimate with (3.25) we derive that it holds for all $y_1, y_2 \in S$. This implies

$$e^{-2c_{13}} \frac{\omega(y_2)}{\omega'(y_2)} \leq \frac{\omega(y_1)}{\omega'(y_1)} \leq e^{2c_{13}} \frac{\omega(y_2)}{\omega'(y_2)} \quad \forall y_1, y_2 \in S. \quad (3.29)$$

Assume now that there exist two exponents $\beta > \beta' > 0$ such that $r^{-\beta} \omega(\cdot)$ and $r^{-\beta'} \omega'(\cdot)$ are p -harmonic and positive in the cone C_S and vanishes on ∂C_S . Put $\theta = \frac{\beta}{\beta'}$, $\eta = \omega'^{\theta}$ and

$$\mathcal{T}(\eta) = -\operatorname{div}' \left(\left(\beta^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{p-2}{2}} \nabla' \eta \right) - (p-1) \beta (\beta - \beta_0) \left(\beta^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{p-2}{2}} \eta,$$

then

$$\mathcal{T}(\eta) = -\theta^{p-2} \left(\beta'^2 \omega'^2 + |\nabla' \omega'|^2 \right)^{\frac{p-2}{2}} \left((\beta - \beta') \omega'^2 + (p-1) \theta (\theta - 1) |\nabla' \omega'|^2 \right) \leq 0.$$

Up to multiplying ω' by λ , we can assume that $\eta \leq \omega$ and that the graphs of η and ω are tangent in \bar{S} . Since $\omega' \leq c\omega$, $\eta = o(\omega)$ near ∂S . Hence there exists $\sigma_0 \in S$ such that $\omega(\sigma_0) = \eta(\sigma_0)$ and the

coincidence set of η and ω is a compact subset of S . We put $w = \omega - \eta$, since $\nabla\omega(\sigma_0) = \nabla\eta(\sigma_0)$ we proceed as in [14, Th 4.1] (see also [4] in the flat case) and derive that w satisfies, in a system of local coordinates $(\sigma_1, \dots, \sigma_{N-1})$ near σ_0 ,

$$\mathcal{L}w := - \sum_{\ell, j} \frac{\partial}{\partial \sigma_\ell} \left(A_{j, \ell} \frac{\partial w}{\partial \sigma_j} \right) + \sum_j C_j \frac{\partial w}{\partial \sigma_\ell} + Cw \geq 0,$$

where the matrix $(A_{j, \ell})$ is smooth, symmetric and positive near σ_0 and the C_j and C are bounded. Hence w is locally zero. By a standard argument of connectedness, this implies that the zero set of w must be empty, contradiction. Hence $\beta = \beta'$. \square

3.2 Uniqueness of eigenfunction

The proof is based upon a delicate adaptation of the characterisation of the p -Martin boundary obtained in [10], but we first give a proof in the convex case.

3.2.1 The convex case

Theorem 3.1. *Assume S is a convex spherical subdomain. Then two positive solutions of (1.1) are proportional.*

Proof. We recall that a domain S is (geodesically) convex if a minimal geodesic joining two points of S is contained in S . If $S \subset S^{N-1}$ is convex, the cone C_S is convex too. Since S is convex, it is Lipschitz and by Theorem D, $\beta_S = \hat{\beta}_S := \beta$. Let ω and ω' be two positive solutions of (1.1) satisfying $\sup_S \omega = \sup_S \omega' = 1$. We denote by $u_\omega(x) = |x|^{-\beta} \omega(\cdot)$ and $u_{\omega'}(x) = |x|^{-\beta} \omega'(\cdot)$ the corresponding separable p -harmonic functions defined in C_S . If $0 < a < b$, we set $C_S^{a, b} = C_S \cap (B_b \setminus \overline{B}_a)$. Then for $0 < \epsilon < 1$ we denote by u_ϵ the unique function which satisfies

$$\begin{aligned} -\Delta_p u_\epsilon &= 0 && \text{in } C_S^{\epsilon, 1} \\ u_\epsilon &= \epsilon^{-\beta} \omega && \text{in } C_S \cap \partial B_\epsilon \\ u_\epsilon &= 0 && \text{in } (C_S \cap \partial B_1) \cup (\partial C_S \cap (\overline{B}_1 \setminus B_\epsilon)). \end{aligned} \quad (3.30)$$

Then

$$(u_\omega - 1)_+ \leq u_\epsilon \leq u_\omega \quad \text{in } C_S^{\epsilon, 1}. \quad (3.31)$$

Furthermore $\epsilon \mapsto u_\epsilon$ is increasing. When $\epsilon \downarrow 0$, $u_\epsilon \uparrow u_0$ where u_0 is positive and p -harmonic in $C_S^{1, 0}$, vanishes on $\partial C_S^{1, 0} \setminus \{0\}$ and satisfies (3.30) with $\epsilon = 0$. In particular

$$\lim_{r \rightarrow 0} r^\beta u_0(r, \sigma) = \omega(\sigma) \quad \text{locally uniformly in } S. \quad (3.32)$$

We construct the same approximation u'_ϵ in $C_S^{\epsilon, 1}$ with ω' instead of ω . Mutadis mutandis (3.31) holds and $u'_\epsilon \uparrow u'_0$ which is positive and p -harmonic in C_S^1 , satisfies

$$(u_{\omega'} - 1)_+ \leq u'_0 \leq u_{\omega'} \quad \text{in } C_S^{1, 0},$$

and thus

$$\lim_{r \rightarrow 0} r^\beta u'_0(r, \sigma) = \omega'(\sigma) \quad \text{locally uniformly in } S. \quad (3.33)$$

However, by [10, Th 4] u_0 and u'_0 are proportional. Combined with (3.32), (3.33) it implies the claim.

3.2.2 Proof of Theorem E

In what follows we borrow most of our construction from [10] that we adapt to the case of an infinite cone and make explicit for the sake of completeness. The next *nondegeneracy property* of positive p -harmonic functions is proved in [10, Lemma 4.28].

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $1 < p < \infty$. Then there exist constants $\rho > 0$, $c_{14}, c_{15} > 0$ depending respectively on Ω (for ρ), and p , N and the Lipschitz norm M of $\partial\Omega$ (for c_{14} and c_{15}) with the property that for any $w \in \partial\Omega$ and any positive p -harmonic function u in Ω , continuous in $\overline{\Omega} \cap \overline{B}_{2\rho}(w)$ and vanishing on $\partial\Omega \cap B_\rho(w)$, one can find $\xi \in S^{N-1}$, independent of u , such that*

$$c_{14}^{-1} \frac{u(y)}{\text{dist}(y, \partial\Omega)} \leq \langle \nabla u(y), \xi \rangle \leq |\nabla u(y)| \leq c_{14} \frac{u(y)}{\text{dist}(y, \partial\Omega)}, \quad (3.34)$$

for all $y \in C_S \cap \overline{B}_{\frac{\rho|w|}{c_{15}}}(w)$.

If Ω is replaced by a cone C_S , the nondegeneracy property still holds uniformly on $\partial C_S \setminus \{0\}$.

Corollary 3.3. *Let $1 < p < \infty$, $S \subset S^{N-1}$ is a Lipschitz domain and C_S the cone generated by S .*

(i) *Then there exist constants $\rho < \frac{1}{2}$, $c_{14}, c_{15} > 0$ depending respectively on S (for ρ), and p , N and the Lipschitz norm M of ∂S and $\text{diam}(S)$ (for c_{14} and c_{15}) with the property that for any $w \in \partial C_S$ and any positive p -harmonic function u in C_S , continuous in $\overline{C_S} \cap \overline{B}_{2\rho|w|}(w)$ and vanishing on $\partial C_S \cap \overline{B}_{\rho|w|}(w)$ continuous, one can find $\xi \in S^{N-1}$, independent of u , such that*

$$c_{14}^{-1} \frac{u(y)}{\text{dist}(y, \partial C_S)} \leq \langle \nabla u(y), \xi \rangle \leq |\nabla u(y)| \leq c_{14} \frac{u(y)}{\text{dist}(y, \partial C_S)}, \quad (3.35)$$

for all $y \in B_{\frac{\rho}{c_{15}}}(w) \cap C_S$.

(ii) *Then there exist positive constants κ and c_{16}, c_{17} depending on S (for κ), and p , N and the Lipschitz norm M of ∂S and $\text{diam}(S)$ (for c_{16}, c_{17} such that for any $a > 0$ and any positive p -harmonic function u in C_S^a vanishing on $\partial C_S \cap B_a^c$, there holds*

$$c_{16}^{-1} \frac{u(y)}{\text{dist}(y, \partial C_S)} \leq |\nabla u(y)| \leq c_{16} \frac{u(y)}{\text{dist}(y, \partial C_S)} \quad \forall y \in C_S^{c_{17}a} \text{ s.t. } \text{dist}(y, \partial C_S) \leq \kappa|y|. \quad (3.36)$$

Let $\omega, \omega' \in W_0^{1,p}(S) \cap C(\overline{S})$ be positive solutions (1.1). Since $\frac{\omega}{\omega'}$ is bounded from above and from below in S by positive constants, we can assume, as in the proof of Theorem D, that $\omega \geq \omega'$ in S and that the graphs of ω and ω' are tangent. hence, if $\omega \neq \omega'$, then $\omega > \omega'$ in S and there exists a sequence $\{\sigma_n\}$ converging to $\sigma_0 \in \partial S$ as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{\omega'(\sigma_n)}{\omega(\sigma_n)} = 1.$$

We define $\delta_1 = \sup\{\delta > 0 : \delta\omega < \omega'\}$. For $t \in (\delta_1, 1)$, we set

$$\phi_t = \sup\{\omega', t\omega\} \quad \text{and} \quad \psi_t = \inf\left\{\frac{t}{\delta_1}\omega', \omega\right\} \quad (3.37)$$

We also set

$$v_{\phi_t}(r, \sigma) = r^{-\beta}\phi_t(\sigma) \quad \text{and} \quad v_{\psi_t}(r, \sigma) = r^{-\beta}\psi_t(\sigma) \quad \forall (r, \sigma) \in (0, \infty) \times S. \quad (3.38)$$

Lemma 3.4. *The functions ϕ_t and ψ_t are respectively a subsolution and a supersolution of (1.1) in $W_0^{1,p}(S)$, v_{ϕ_t} and v_{ψ_t} are respectively a subsolution and a supersolution of $-\Delta_p$ in C_S , and there exists $\eta \in W_0^{1,p}(S)$ solution of (1.1) such that*

$$\omega' \leq \phi_t \leq \eta \leq \psi_t \leq \omega \quad \forall t \in (\delta_1, 1). \quad (3.39)$$

If S_t is the subset of $\eta \in W_0^{1,p}(S)$ solutions of (1.1) and satisfying (3.39), then $\omega_t = \sup\{\eta : \eta \in S_t\}$ belongs to S_t . It is increasing with respect to t with uniform limits ω' when $t \downarrow \delta_1$ and ω when $t \uparrow 1$. Finally, if $\theta_t = \frac{t-\delta_1}{1-\delta_1}$, there holds

$$\phi_t \leq \theta_t \omega + (1 - \theta_t) \omega' \leq \psi_t. \quad (3.40)$$

Proof. Clearly ϕ_t and ψ_t are respectively a subsolution and a supersolution of the operator \mathcal{T} , they belong to $W_0^{1,p}(S) \cap L^\infty(S)$ and they satisfy $\omega' \leq \phi_t \leq \psi_t \leq \omega$. Furthermore, by Dini convergence theorem

$$\lim_{t \uparrow 1} \phi_t = \omega = \lim_{t \uparrow 1} \psi_t \quad \text{and} \quad \lim_{t \downarrow \delta_1} \phi_t = \omega' = \lim_{t \downarrow \delta_1} \psi_t,$$

uniformly in \bar{S} . Moreover, in spherical coordinates,

$$\begin{aligned} -\Delta_p u(r, \sigma) = & \left(\left(u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \right)_r - \frac{N-1}{r} \left(u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} u_r \\ & - \frac{1}{r^2} \operatorname{div}' \left(\left(u_r^2 + r^{-2} |\nabla' u|^2 \right)^{\frac{p-2}{2}} \nabla' u \right). \end{aligned}$$

Hence, if $u(r, \sigma) = r^{-\beta} \eta(\sigma)$,

$$-\Delta_p u(r, \sigma) = \beta^{p-2} r^{-(p-1)(\beta+1)-1} \mathcal{T}(\eta).$$

Thus v_{ϕ_t} is a subsolution $-\Delta_p$ in C_S and v_{ψ_t} is a supersolution. Since the operator \mathcal{T} is a Leray-Lions operator, it follows by [3] that there exists $\eta \in W_0^{1,p}(S) \cap L^\infty(S)$ satisfying $\mathcal{T}(\eta) = 0$ and $\phi_t \leq \eta \leq \psi_t$ in S . We denote by S_t the set of $\eta \in W_0^{1,p}(S) \cap L^\infty(S)$ satisfying $\mathcal{T}(\eta) = 0$ and $\phi_t \leq \eta \leq \psi_t$ in S . Then there exists a sequence $\{\eta_n\} \subset S_t$ and $\omega_t \in W_0^{1,p}(S) \cap L^\infty(S)$ such that $\eta_n(\sigma) \uparrow \omega_t(\sigma)$ for all $\sigma \in \Sigma$, where Σ is a countable dense subset of S . By Lemma 2.1 $\{\eta_n\}$ is bounded in $L^p(S)$, hence in $C^\gamma(S)$ for some $\gamma \in (0, 1)$. By the estimates of the proof of Theorem B-Step 2, $\{\eta_n\}$ is bounded in $W_0^{1,p}(S)$. By standard regularity theory, we can also assume that $\eta_n \rightarrow \omega_t$ in the $C_{loc}^1(S)$ -topology. Hence ω_t is a weak solution of (1.1), it belongs to $W_0^{1,p}(S) \cap L^\infty(S)$ and satisfies $\phi_t \leq \omega_t \leq \psi_t$. Therefore it is the maximal element of S_t . The monotony of ω_t is a consequence of the monotonicity of ϕ_t and ψ_t and the last statement (3.40) is a straightforward computation. \square

Next we recall the deformation of p -harmonic functions already used in [10]. If $\tau \in (0, 1)$ and $0 < a < b$, we denote by $v_{\tau,a,b}$ the p -harmonic function defined in $C_S^{a,b}$ satisfying

$$v_{\tau,a,b}(x) = \begin{cases} a^{-\beta}(\tau\omega + (1-\tau)\omega')\left(\frac{x}{|x|}\right) & \text{if } x \in C_S \cap \partial B_a \\ 0 & \text{if } x \in C_S \cap \partial B_b \\ 0 & \text{if } x \in \partial C_S \cap (\bar{B}_b \setminus B_a). \end{cases} \quad (3.41)$$

Lemma 3.5. *The mapping $(\tau, b) \mapsto v_{\tau,a,b}$ is continuous and increasing. If $v_{\tau,a} = \lim_{b \rightarrow \infty} v_{\tau,a,b}$, then it is a positive p -harmonic function in $C_S^{a,\infty}$ vanishing on $\partial S \cap B_a^c$, and there holds*

$$u_{\omega'}(x) \leq v_{\phi_{\tau^*}}(x) \leq v_{\tau,a}(x) \leq v_{\psi_{\tau^*}}(x) \leq u_\omega(x) \quad \forall x \in C_S^{a,\infty}, \quad (3.42)$$

where $\tau^* = (1 - \delta_1)\tau + \delta_1$ and as a consequence

$$\limsup_{\tau \uparrow 1} \sup_{|x| \geq a} |x|^\beta (u_\omega(x) - v_{\tau,a}(x)) = 0 \quad \text{and} \quad \limsup_{\tau \downarrow 0} \sup_{|x| \geq a} |x|^\beta (v_{\tau,a}(x) - u_{\omega'}(x)) = 0 \quad (3.43)$$

Furthermore

$$0 \leq \frac{v_{\tau',a} - v_{\tau,a}}{\tau' - \tau} \leq \left(\frac{1}{\delta_1} - 1 \right) v_{\tau',a} \quad \forall 0 \leq \tau < \tau' \leq 1. \quad (3.44)$$

Proof. The uniqueness and the (strict) monotonicity of $(\tau, b) \mapsto v_{\tau,a,b}$ follow from the monotonicity of $\tau \mapsto \tau\omega + (1 - \tau)\omega'$ and the strong maximum principle. The continuity is a consequence of uniqueness and regularity theory for p -harmonic functions. It follows from (3.40) with $t = \tau^*$ and the fact that $v_{\phi_{\tau^*}}$ and $v_{\psi_{\tau^*}}$ are respectively a subsolution and a supersolution of $-\Delta_p$, that we have

$$u_{\omega'}(x) \leq v_{\phi_{\tau^*}}(x) \leq v_{\tau,a,b}(x) \leq v_{\psi_{\tau^*}}(x) \leq u_\omega(x) \quad \forall x \in C_S^{a,b},$$

which yields (3.42). Similarly, we have on $\partial C_S^{a,b}$

$$0 \leq \frac{v_{\tau',a,b} - v_{\tau,a,b}}{\tau' - \tau} = u_\omega - u_{\omega'} \leq (\delta_1^{-1} - 1)u_{\omega'} \leq (\delta_1^{-1} - 1)v_{\tau,a,b}, \quad (3.45)$$

equivalently

$$0 \leq v_{\tau',a,b} \leq (1 + (\tau' - \tau)(\delta_1^{-1} - 1)) v_{\tau,a,b}. \quad (3.46)$$

By the maximum principle (3.45) holds in $C_S^{a,b}$. This implies (3.44). \square

As a consequence of (3.44), $\partial_\tau v_{\tau,a}$ exists for almost all $\tau \in (0, 1)$ in $W_0^{1,p}(C_S^{a,b})$ for all $b > a$ and it is a solution of

$$\begin{aligned} \mathbb{L}w &= \nabla \cdot \left((p-2) |\nabla v_{\tau,a}|^{p-4} \langle \nabla v_{\tau,a}, \nabla Z \rangle \nabla v_{\tau,a} \right) \\ &= \sum_{i,j} \frac{\partial}{\partial x_j} \left(b_{i,j}(x) \frac{\partial w}{\partial x_i} \right) = 0 \end{aligned} \quad (3.47)$$

where

$$b_{i,j}(x) = |\nabla v_{\tau,a}|^{p-4} \left((p-2) \frac{\partial v_{\tau,a}}{\partial x_j} \frac{\partial v_{\tau,a}}{\partial x_i} + \delta_{ij} |\nabla v_{\tau,a}|^2 \right).$$

\mathbb{L} satisfies the following ellipticity condition

$$\min\{1, p-1\} |\nabla v_{\tau,a}|^2 |\xi|^2 \leq \sum_{i,j} b_{i,j}(x) \xi_i \xi_j \leq \max\{1, p-1\} |\nabla v_{\tau,a}|^2 |\xi|^2 \quad \forall \xi \in \mathbb{R}^N. \quad (3.48)$$

It is important to notice that $\mathbb{L}v_{\tau,a} = (p-1)\Delta_p v_{\tau,a} = 0$. The estimate (3.48) combined with (3.36) and the decay of $v_{\tau,a}$ and $\partial_\tau v_{\tau,a}$ implies that they satisfy Harnack inequality and boundary Harnack inequality in C_S^a . There exists a constant $\hat{c} > c_{17} > 1$ (see 3.36) such that

$$\frac{1}{\hat{c}} \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} \leq \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \leq \hat{c} \frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} \quad \forall x \in C_S^{\hat{c}a}, \quad (3.49)$$

where $x_a = (\hat{c}a, \sigma_0)$ for some $\sigma_0 \in S$ fixed. We set

$$M(t) = \sup_{x \in C_S^t} \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \quad \text{and} \quad m(t) = \inf_{x \in C_S^t} \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} \quad \forall t > a \quad (3.50)$$

Lemma 3.6. For $t > \hat{c}a$ there holds

$$M(\hat{c}t) - m(\hat{c}t) \leq \frac{\hat{c}^2 - 1}{\hat{c}^2 + 1} (M(t) - m(t)). \quad (3.51)$$

Proof. There holds

$$\partial_\tau v_{\tau,a} - m(t)v_{\tau,a} \geq 0 \quad \text{and} \quad M(t)v_{\tau,a} - \partial_\tau v_{\tau,a} \geq 0 \quad \in C_S^t.$$

Estimate (3.49) is valid for any couple of positive solutions (h_1, h_2) of $\mathbb{L}h = 0$ in C_S^a vanishing on $\partial C_S^a \cap B_a^c$, in particular for $(\partial_\tau v_{\tau,a} - m(t)v_{\tau,a}, v_{\tau,a})$ and $(M(t)v_{\tau,a} - \partial_\tau v_{\tau,a}, v_{\tau,a})$. Hence

$$\frac{1}{\hat{c}} \left(\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t) \right) \leq \frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} - m(t) \leq \hat{c} \left(\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t) \right) \quad \forall x \in C_S^t. \quad (3.52)$$

This implies

$$\frac{1}{\hat{c}} \left(\frac{\partial_\tau v_{\tau,a}(x_a)}{v_{\tau,a}(x_a)} - m(t) \right) \leq m(\hat{c}t) - m(t),$$

and

$$\frac{\partial_\tau v_{\tau,a}(x)}{v_{\tau,a}(x)} - m(t) \leq \hat{c}^2 (m(\hat{c}t) - m(t)) \quad \forall x \in C_S^t.$$

Finally

$$M(\hat{c}t) - m(t) \leq \hat{c}^2 (m(\hat{c}t) - m(t)). \quad (3.53)$$

Similarly

$$M(t) - m(\hat{c}t) \leq \hat{c}^2 (M(t) - M(\hat{c}t)). \quad (3.54)$$

Summing the two inequalities we get

$$(M(t) - m(t)) + (M(\hat{c}t) - m(\hat{c}t)) \leq \hat{c}^2 ((M(t) - m(t)) - (M(\hat{c}t) - m(\hat{c}t))),$$

which yields (3.52). \square

End of the proof. By the differentiability property of $v_{\tau,a}$ with respect to τ , there exists two countable dense sets $\{r_\nu\} \subset [a, \infty)$ and $\{\sigma_\mu\} \subset [a, \infty)$ such that $\partial_\tau v_{\tau,a}(r_\nu, \sigma_\mu)$ exists for almost all τ . We put $x_{\nu,\mu} = (r_\nu, \sigma_\mu)$, hence

$$\begin{aligned} \ln \left(\frac{\omega(\sigma_\mu)}{\omega'(\sigma_\mu)} \right) - \ln \left(\frac{\omega(\sigma_{\mu'})}{\omega'(\sigma_{\mu'})} \right) &= \ln \left(\frac{v_{1,a}(x_{\nu,\mu})}{v_{0,a}(x_{\nu,\mu})} \right) - \ln \left(\frac{v_{1,a}(x_{\nu,\mu'})}{v_{0,a}(x_{\nu,\mu'})} \right) \\ &= \int_0^1 \left(\frac{\partial_\tau v_{\tau,a}(x_{\nu,\mu})}{v_{\tau,a}(x_{\nu,\mu})} - \frac{\partial_\tau v_{\tau,a}(x_{\nu,\mu'})}{v_{\tau,a}(x_{\nu,\mu'})} \right) d\tau. \end{aligned} \quad (3.55)$$

Using the continuity of $\frac{\omega}{\omega'}$ and the density of $\{\sigma_m\}$ we derive

$$\left| \ln \left(\frac{\omega(\sigma)}{\omega'(\sigma)} \right) - \ln \left(\frac{\omega(\sigma')}{\omega'(\sigma')} \right) \right| \leq M(r_\nu) - m(r_\nu) \quad \forall (\sigma, \sigma') \in S \times S. \quad (3.56)$$

We can assume that $r_\nu \geq \hat{c}^{\nu} a$ for some sequence $\{\nu_n\}$ tending to infinity with n , hence

$$\left| \ln \left(\frac{\omega(\sigma)}{\omega'(\sigma)} \right) - \ln \left(\frac{\omega(\sigma')}{\omega'(\sigma')} \right) \right| \leq \theta^n (M(\hat{c}^{\nu_n}) - m(\hat{c}^{\nu_n})) \quad \forall (\sigma, \sigma') \in S \times S \quad \forall n \in \mathbb{N}^*, \quad (3.57)$$

where $\theta = \frac{\hat{c}^2 - 1}{\hat{c}^2 + 1} < 1$. Letting $n \rightarrow \infty$ implies the claim. \square

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