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CHARACTERISTIC FUNCTIONS AS BOUNDED MULTIPLIERS ON ANISOTROPIC SPACES

VIVIANE BALADI

Abstract. We show that characteristic functions of domains with boundaries transversal to stable cones are bounded multipliers on a recently introduced scale $\mathcal{U}_p^{t,s}$ ($s < 0 < t$ and $1 \le p \le \infty$) of anisotropic Banach spaces, under the conditions $-1 + 1/p < s < -s < t$, with $p \in (1, \infty)$.

1. Introduction

A function $g : M \to \mathbb{C}$ is a called bounded multiplier on a Banach space $\mathcal{B}$ of distributions on a d-dimensional Riemann manifold M if there exists a finite constant C_g so that $\|g \cdot \varphi\| \le C_g \|\varphi\|$ for all φ, where $\|\cdot\|$ is the norm of $\mathcal{B}$. One interesting special case is when g is the characteristic function 1_Λ of an open domain $\Lambda \subset M$: Half a century ago, Strichartz [16] proved that for any $d \ge 1$, if $M = \mathbb{R}^d$ and $\mathcal{B}$ is the Sobolev[1] space $H_p^t(\mathbb{R}^d)$ for $p \in (1, \infty)$ and $t \in \mathbb{R}$, then the characteristic function 1_Λ of a half (hyper)plane, is a bounded multiplier on $H_p^t(\mathbb{R}^d)$ if and only if $-1 + 1/p < t < 1/p$.

In the present work, we consider a newly introduced space of anisotropic distributions $\mathcal{B}$ on a manifold M, adapted to smooth hyperbolic dynamics, and we prove the bounded multiplier property for characteristic functions of suitable subsets $\Lambda \subset M$.

Fix $r > 1$, and suppose from now on that M is connected and compact. The simplest hyperbolic maps on M are transitive C^r Anosov diffeomorphisms T. The Ruelle transfer operator associated to such a map T and to a C^{r-1} function h on M (for example, $h = 1/|\det DT|$) is defined on C^{r-1} functions φ by

$$\mathcal{L}_h \varphi = (h \cdot \varphi) \circ T^{-1} . \tag{1}$$

Blank–Keller–Liverani [7] were the first to study the spectrum of such transfer operators on a suitable Banach space $\mathcal{B}$ of *anisotropic distributions* and to exploit this spectrum to get information on the Sinai–Ruelle–Bowen (physical) measure: The spectral radius of $\mathcal{L}_{1/|\det DT|}$ is equal to 1, there is a simple positive maximal eigenvalue, whose eigenvector is in fact a Radon measure μ, which is just the physical measure of T. Finally, the rest of the spectrum lies in a disc of radius strictly smaller than 1, which implies exponential decay of correlations $\int \varphi(\psi \circ T^n) \mathrm{d}\mu - \int \varphi \mathrm{d}\mu \int \psi \mathrm{d}\mu$

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[1]Recall that $\|\varphi\|_{H_p^t} = \|(\mathrm{id} + \Delta)^{t/2} \varphi\|_{L_p} = \|\mathbb{F}^{-1}(1 + |\xi|^2)^{t/2} \mathbb{F}\varphi\|_{L_p}$, with Δ the Laplacian and $\mathbb{F}$ the Fourier transform.

for Hölder observables ψ and φ as $n \to \infty$. (The first step in this analysis is to show the bound $\rho_{ess} < 1$ for the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ on $\mathcal{B}$.)

Some natural dynamical systems originating from physics (such as Sinai billiards) enjoy uniform hyperbolicity, but are only *piecewise smooth.* Letting $M = \cup_i \Lambda_i$ be a (finite or countable) partition of M into domains where the dynamics is smooth, one can often reduce to the smooth hyperbolic case via the decomposition

$$\mathcal{L}_{1/|\det DT|}\varphi = \sum_i \frac{(1_{\Lambda_i} \cdot \varphi)}{|\det DT|} \circ T^{-1} . \tag{2}$$

This motivates our study of bounded multiplier properties of characteristic functions.

In the 15 years since the publication of [7], dynamicists and semi-classical analysts have created a rich jungle of spaces of anisotropic distributions for hyperbolic dynamics (here, $d = d_s + d_u$ with $d_s \geq 1$ and $d_u \geq 1$). These spaces are usually scaled by two real numbers $v < 0$ and $t > 0$. Leaving aside the classical foliated anisotropic spaces of Triebel [17] (which are limited to "bunched" cases [4], and seem to fail for Sinai billiards), they come in two groups:

In the first, "geometric" group [7, 13], a class of d_s-dimensional "admissible" leaves Γ (having tangent vectors in stable cones for T) is introduced, and the norm of φ is obtained by fixing an integer $t \geq 1$ and taking a supremum, over all admissible leaves Γ, of the partial derivatives of φ of total order at most t, integrated against $C^{|v|}$ test functions on Γ. Modifications of this space, for suitable noninteger $0 < t < 1$ and $|v| < 1$, were introduced to work with piecewise smooth systems [8, 9] (only in dimension two). A version of these spaces for piecewise smooth hyperbolic flows in dimension three recently allowed to prove exponential mixing for Sinai billiard flows [3].

In the[2] second, "microlocal," group [5], a third parameter $p \in [1, \infty)$ is present, and the norm (in charts) of φ is the L_p average of $\Delta^{t,v}(\varphi)$, where the operator $\Delta^{t,v}$ interpolates smoothly between $(\mathrm{id} + \Delta)^{v/2}$ in *stable cones* in the cotangent space, and $(\mathrm{id} + \Delta)^{t/2}$ in *unstable cones* in the cotangent space. Powerful tools are available for this microlocal approach, allowing in particular to study the dynamical determinants and zeta functions[3] much more efficiently than for the geometric spaces. Variants of these microlocal spaces (usually in the Hilbert setting $p = 2$) have also been studied by the semi-classical community, starting from [10]. However, S. Gouëzel pointed out over ten years ago that *characteristic functions cannot be bounded multipliers* on spaces defined by conical wave front sets as in [5] or [10] (Gouëzel's counterexamples are presented in [2, App. 1]). The microlocal spaces of the type defined in [5, 6] or [10] thus appear *unsuitable* to study piecewise smooth dynamics.

In order to overcome this limitation of the microlocal approach, we recently introduced [2] a new scale $\mathcal{U}_p^{t,s}$ of microlocal anisotropic spaces, obtained by mimicking the geometric construction of the geometric spaces $\mathcal{B}^{t,|s+t|}$ (for integer t) of Gouëzel–Liverani [13]. We showed in [2] the expected bound on the essential spectral radius of the transfer operator of a C^r Anosov diffeomorphism acting on $\mathcal{U}_p^{t,s}$ (if $-(r-1) < s < -t < 0$), and we conjectured that characteristic functions

[2]This group could also be called pseudodifferential, or semi-classical, or Sobolev.

[3]The "kneading determinants" of by Milnor and Thurston from the 70's are revisited as "nuclear decompositions" in [1].

of domains with piecewise smooth boundaries everywhere transversal to the stable cones should be bounded multipliers on $\mathcal{U}_p^{t,s}$, if s and t satisfy additional constraints depending on $p \in (0,1)$. *The main result of the present paper, Theorem 3.1, is a proof of this bounded multiplier property if* $\max\{-(r-1), -1+1/p\} < s < -t < 0$.

This result opens the door to the spectral study, not only of hyperbolic maps with discontinuities in arbitrary dimensions, but also (using nuclear power decompositions [1, 2]) of the hitherto unexplored topic of the dynamical zeta functions of piecewise expanding and piecewise hyperbolic maps in any dimensions. This should include billiards maps [9] and their dynamical zeta functions in arbitrary dimensions. We also hope that the spaces $\mathcal{U}_p^{t,s}$ will allow to extend the scope of the renewal methods introduced in [14] to dynamical systems with infinite invariant measures. (The induction procedure used there introduces discontinuities in the dynamics.) Finally, it goes without saying that suitable version of the spaces $\mathcal{U}_p^{t,s}$ will be useful to study flows.

F. Faure and M. Tsujii [11] announced (SNS, Pisa, June 2016) new microlocal anisotropic spaces, for which the wave front set is more narrowly constrained than in the cones used in previous microlocal spaces suitable for hyperbolic dynamics. It would be interesting to check whether characteristic functions are bounded multipliers on these new spaces. (Note however that, contrary to the spaces $\mathcal{U}_p^{t,s}$ or the spaces of [10, 5, 13, 9], the new spaces of [11] do not appear suitable for perturbations of hyperbolic maps or flows.)

2. $\mathcal{U}_p^{t,s}$: A Fourier version of the Demers–Gouëzel–Liverani spaces

We recall the "microlocal" spaces $\mathcal{U}_p^{t,s}$, for real numbers s and t (in the application, $s < -t < 0$) and $1 \le p \le \infty$, introduced in [2] and inspired by the "geometric" spaces $\mathcal{B}^{t,|s+t|}$ (for $s < -t$ and integer t) from [13].

2.1. **Basic notation.** Suppose that $d = d_s + d_u$ with $d_u \ge 1$ and $d_s \ge 1$. For $\ell \ge 1$ and $x \in \mathbb{R}^\ell$, $\xi \in \mathbb{R}^\ell$, we write $x\xi$ for the scalar product of x and ξ. The Fourier transform $\mathbb{F}$ and its inverse $\mathbb{F}^{-1}$ are defined on rapidly decreasing functions φ, ψ by

$$\mathbb{F}(\varphi)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi}\varphi(x)\mathrm{d}x\,, \quad \xi \in \mathbb{R}^d\,, \tag{3}$$

$$\mathbb{F}^{-1}(\psi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}\psi(\xi)\mathrm{d}\xi\,, \quad x \in \mathbb{R}^d\,, \tag{4}$$

and extended to the space of temperate distributions φ, ψ as usual [15]. For suitable functions $a : \mathbb{R}^d \to \mathbb{R}$ (called "symbols"), we define an operator a^{Op} acting on suitable $\varphi : \mathbb{R}^d \to \mathbb{C}$, by

$$a^{Op}(\varphi) = \mathbb{F}^{-1}(a(\cdot)\cdot\mathbb{F}(\varphi)) = (\mathbb{F}^{-1}a) * \varphi\,. \tag{5}$$

Note that $\|a^{Op}\varphi\|_{L_p} \le \|\mathbb{F}^{-1}a\|_1\|\varphi\|_{L_p}$ for each $1 \le p \le \infty$, by Young's inequality in L_p.

Fix a C^∞ function $\chi : \mathbb{R}_+ \to [0,1]$ with $\chi(x) = 1$for $x \le 1$, and $\chi(x) = 0$ for $x \ge 2$. For $\ell \ge 1$, define $\psi_n^{(\ell)} : \mathbb{R}^\ell \to [0,1]$ for $n \in \mathbb{Z}_+$, by $\psi_0^{(\ell)}(\xi) = \chi(\|\xi\|)$, and

$$\psi_n^{(\ell)}(\xi) = \chi(2^{-n}\|\xi\|) - \chi(2^{-n+1}\|\xi\|)\,, \quad n \ge 1\,. \tag{6}$$

We set $\psi_n = \psi_n^{(d)}$. Note that

$$\mathbb{F}^{-1}\psi_n^{(\ell)} = 2^{d(n-1)}\mathbb{F}^{-1}\psi_1^{(\ell)}(2^{n-1}x) \text{ and } (\sum_{k\le n}\mathbb{F}^{-1}\psi_k^{[\ell)})(x) = 2^{dn}\mathbb{F}^{-1}\chi(2^n x)\,,$$

so that, for any ℓ,

$$\sup_n \|\mathbb{F}^{-1}\psi_n^{(\ell)}\|_{L_1(\mathbb{R}^\ell)} < \infty\,, \qquad \sup_n \|\sum_{k\le n}\mathbb{F}^{-1}\psi_k^{(\ell)}\|_{L_1(\mathbb{R}^\ell)} < \infty\,, \tag{7}$$

and for every multi-index β, there exists a constant C_β such that

$$\|\partial^\beta \psi_n^{(\ell)}\|_{L_\infty} \le C_\beta 2^{-n|\beta|}\,, \qquad \forall\, n \ge 0\,. \tag{8}$$

We shall work with the following operators $(\psi_n^{(\ell)})^{Op}$ (putting $\psi_n^{Op} = (\psi_n^{(d)})^{Op}$):

$$(\psi_n^{(\ell)})^{Op}(\varphi)(x) = \frac{1}{(2\pi)^d}\int_{y\in\mathbb{R}^d}\int_{\eta\in\mathbb{R}^d} e^{i(x-y)\eta}\psi_n^{(\ell)}(\eta)\varphi(y)\mathrm{d}\eta\mathrm{d}y\,.$$

Note finally the following almost orthogonality property

$$(\psi_n^{(\ell)})^{Op}\circ(\psi_m^{(\ell)})^{Op} \equiv 0 \quad \text{ if } |n-m|\ge 2\,. \tag{9}$$

2.2. **The local anisotropic spaces $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ for compact $K\subset\mathbb{R}^d$.** Recall that a cone is a subset of $\mathbb{R}^d$ invariant under scalar multiplication. For two cones $\mathbf{C}$ and $\mathbf{C}'$ in $\mathbb{R}^d$, we write $\mathbf{C}\Subset\mathbf{C}'$ if $\overline{\mathbf{C}}\subset$ interior$(\mathbf{C}')\cup\{0\}$. We say that a cone $\mathbf{C}$ is d'-dimensional if $d'\ge 1$ is the maximal dimension of a linear subset of $\mathbf{C}$.

Definition 2.1. A *cone pair* is $\mathbf{C}_\pm = (\mathbf{C}_+,\mathbf{C}_-)$, where $\mathbf{C}_+$ and $\mathbf{C}_-$ are closed cones in $\mathbb{R}^d$, with nonempty interiors, of respective dimensions d_u and d_s and so that $\mathbf{C}_+\cap\mathbf{C}_- = \{0\}$. We always assume that $\mathbb{R}^{d_s}\times\{0\}$ is included in $\mathbf{C}_-$.

Recall that $r>1$. The next key ingredient is adapted from [6]:

Definition 2.2 (Admissible (or fake) stable leaves)**.** Let $\mathbf{C}_+$ be a cone, and let $C_\mathcal{F}>1$. Then $\mathcal{F}(\mathbf{C}_+, C_\mathcal{F})$ (noted simply $\mathcal{F}$ when the meaning is clear) is the set of all C^r (embedded) submanifolds $\Gamma\subset\mathbb{R}^d$, of dimension d_s, with C^r norms of submanifold charts bounded by $C_\mathcal{F}$, and so that the straight line connecting any two distinct points in Γ is normal to a d_u-dimensional subspace contained in $\mathbf{C}_+$.

Denote by π_- the orthogonal projection from $\mathbb{R}^d$ to the quotient $\mathbb{R}^{d_s}$ and by π_Γ its restriction to Γ. Our assumption on $\mathcal{F}$ implies that $\pi_\Gamma:\Gamma\to\mathbb{R}^{d_s}$ is a C^r diffeomorphism onto its image with a C^r inverse, whose C^r norm is bounded by a universal scalar multiple of $\mathcal{C}_F$. We replace $\mathcal{C}_F$ by this larger constant in the sequel.

Definition 2.3 (Isotropic norm on stable leaves)**.** Fix a cone pair $\mathbf{C}_\pm$. Let $\Gamma\in\mathcal{F}(\mathbf{C}_+, C_\mathcal{F})$ and let $\varphi\in C^0(\Gamma)$. For $w\in\Gamma\subset\mathbb{R}^d$, we set

$$\psi_{\ell_s}^{Op(\Gamma)}(\varphi)(w) = \frac{1}{(2\pi)^{d_s}}\int_{z\in\mathbb{R}^{d_s}}\int_{\eta_s\in\mathbb{R}^{d_s}} e^{i(\pi_\Gamma(w)-z)\eta_s}\psi_{\ell_s}^{(d_s)}(\eta_s)\varphi(\pi_\Gamma^{-1}(z))\mathrm{d}\eta_s\mathrm{d}z\,, \tag{10}$$

where $\psi_k^{(d_s)}:\mathbb{R}^{d_s}\to[0,1]$ is defined in (6). For every real numbers $1\le p\le\infty$, and $-(r-1)<s<r-1$, define an auxiliary isotropic norm on $C^0(\Gamma)$ as

$$\|\varphi\|_{p,\Gamma}^s = \sup_{\ell_s\in\mathbb{Z}_+} 2^{\ell_s s}\|\psi_{\ell_s}^{Op(\Gamma)}(\varphi)\|_{L_p(\mu_\Gamma)}\,, \tag{11}$$

where μ_Γ is the Riemann volume on Γ induced by the standard metric on $\mathbb{R}^d$.

Note that (11) is just ([15, §2.1, Def. 2]) the classical d_s-dimensional Besov norm $B^s_{p,\infty}$ of $\varphi|$ in the chart given by π_Γ^{-1}:

$$\|\varphi\|^s_{p,\Gamma} = \|\varphi \circ \pi_\Gamma^{-1}\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\,.$$

We next recall the definition of the local space given in [2]:

Definition 2.4 (The local space $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$)**.** Let $K \subset \mathbb{R}^d$ be a non-empty compact set. For a cone pair $\mathbf{C}_\pm = (\mathbf{C}_+, \mathbf{C}_-)$, a constant $C_{\mathcal{F}} \geq 1$, and real numbers, $1 \leq p \leq \infty$, and $-(r-1) < s < -t < 0$, define for a C^∞ function φ supported in K,

$$\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} = \sup_{\Gamma \in \mathcal{F}(\mathbf{C}_+, C_{\mathcal{F}})} \sup_{\ell \in \mathbb{Z}_+} 2^{\ell t} \|\psi_\ell^{Op}(\varphi)\|^s_{p,\Gamma}\,. \tag{12}$$

Set $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ to be the completion of $C^\infty(K)$ with respect to $\|\cdot\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}}$.

Remark 2.5. The definition in [2] was more general, allowing also Besov spaces $B^s_{p,q}$ for $1 \leq q < \infty$, where ([15, §2.1, Def. 2])

$$\|\varphi\|_{B^s_{p,q}(\mathbb{R}^{d_s})} = \left(\sum_{\ell_s \in \mathbb{Z}_+} \left(2^{\ell_s s} \|(\psi_{\ell_s}^{(d_s)})^{Op}(\varphi)\|_{L_p(\mathbb{R}^{d_s})}\right)^q \right)^{1/q}. \tag{13}$$

The following lemma was proved in [2]:

Lemma 2.6 (Comparing $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ with classical spaces)**.** *Assume $-(r-1) < s < -t < 0$. For any $u > t$, there exists a constant $C = C(u, K)$ such that $\|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} \leq C\|\varphi\|_{C^u}$ for all $\varphi \in C^\infty(K)$. For any $u > |t+s|$, the space $\mathcal{U}_p^{\mathbf{C}_\pm,t,s}(K)$ is contained in the space of distributions of order u supported on K.*

2.3. **The global spaces $\mathcal{U}_p^{t,s}$ of anisotropic distributions.** To introduce the global spaces $\mathcal{U}_p^{t,s}$ of distributions on a compact manifold M, we need one last definition:

Definition 2.7. An *admissible chart system and partition of unity* is a finite system of local charts $\{(V_\omega, \kappa_\omega)\}_{\omega\in\Omega}$, with open subsets $V_\omega \subset M$, and C^∞ diffeomorphisms $\kappa_\omega : U_\omega \to V_\omega$ such that $M \subset \cup_\omega V_\omega$, and U_ω is a bounded open subset of $\mathbb{R}^d$, together with a C^∞ partition of unity $\{\theta_\omega\}_{\omega\in\Omega}$ for M, subordinate to the cover $\mathcal{V} = \{V_\omega\}$.

Definition 2.8 (Anisotropic spaces $\mathcal{U}_p^{t,s}$ on M)**.** Fix an admissible chart system and partition of unity, fix $\mathcal{C}_F \geq 1$ and fix cone pairs $\{\mathbf{C}_{\omega,\pm} = (\mathbf{C}_{\omega,+}, \mathbf{C}_{\omega,-})\}_{\omega\in\Omega}$. Let $1 \leq p \leq \infty$, and fix real numbers $-(r-1) < s < -t < 0$. The Banach space $\mathcal{U}_p^{t,s} = \mathcal{U}_p^{t,s,\mathbf{C}}$ is the completion of $C^\infty(M)$ for the norm

$$\|\varphi\|_{\mathcal{U}_p^{t,s}(T)} := \max_{\omega\in\Omega} \|(\theta_\omega \cdot \varphi) \circ \kappa_\omega\|_{\mathcal{U}_p^{\mathbf{C}_{\omega,\pm},t,s}}\,.$$

Remark 2.9 (Admissible cone pairs). To get a a spectral gap for the transfer operator $\mathcal{L}_{1/|\det DT}$ associated to a C^r Anosov diffeomorphism T for $r > 1$, one must require $s < -t < 0$, and, in addition, one must restrict to families of *admissible* cone pairs $\{\mathbf{C}_\omega, \pm\}$, i.e. which satisfy the following conditions (see [2]). Let E^s and E^u be the stable, respectively unstable, bundles of T, then:

a) If $x \in V_\omega$, the cone $(D\kappa_\omega^{-1})^*_x(\mathbf{C}_{\omega,+})$ contains the (d_u-dimensional) normal subspace of $E^s(x)$, and the cone $(D\kappa_\omega^{-1})^*_x(\mathbf{C}_{\omega,-})$ contains the (d_s-dimensional) normal subspace of $E^u(x)$.

b) If $V_{\omega'\omega} = T(V_\omega) \cap V_{\omega'} \neq \emptyset$, the C^r map corresponding to T^{-1} in charts,

$$F = F_{\omega'\omega} = \kappa_\omega^{-1} \circ T^{-1} \circ \kappa_{\omega'} : \kappa_{\omega'}^{-1}(V_{\omega'\omega}) \to U_\omega \,,$$

extends to a bilipschitz C^1 diffeomorphism of $\mathbb{R}^d$ so that

$$DF_x^{tr}(\mathbb{R}^d \setminus \mathbf{C}_{\omega,+}) \Subset \mathbf{C}_{\omega',-} \,, \qquad \forall x \in \mathbb{R}^d \,.$$

(One then says that F is *cone hyperbolic* from $\mathbf{C}_{\omega,\pm}$ to $\mathbf{C}_{\omega',\pm}$.)

c) Furthermore, there exists, for each x, y, a linear transformation $\mathbb{L}_{xy}$ so that

$$(\mathbb{L}_{xy})^{tr}(\mathbb{R}^d \setminus \mathbf{C}_{\omega,+}) \Subset \mathbf{C}_{\omega',-} \text{ and } \mathbb{L}_{xy}(x-y) = F(x) - F(y) \,.$$

(One then says that F is *regular cone hyperbolic* from $\mathbf{C}_{\omega,\pm}$ to $\mathbf{C}_{\omega',\pm}$.)

Finally, note that if F is regular cone hyperbolic from $\mathbf{C}_{\omega,\pm}$ to $\mathbf{C}_{\omega',\pm}$ and if one assumes in addition (this is always possible, up to taking smaller charts) that the extension of F to $\mathbb{R}^d$ is C^r, then there exists $C_{\mathcal{F}} < \infty$ so that this extension maps each element of $\mathcal{F}(\mathbf{C}_{\omega,+}, C_{\mathcal{F}})$ to an element of $\mathcal{F}(\mathbf{C}_{\omega',+}, C_{\mathcal{F}})$.

The anisotropic spaces $\mathcal{U}_1^{t,s}$ (with $p = 1$) are then analogues of the Blank–Keller–Gouëzel–Liverani [7, 13] spaces $\mathcal{B}^{t,|s+t|}$ for integer t and $s < -t$. The spaces $\mathcal{U}_p^{t,s}$ are somewhat similar to the Demers–Liverani spaces [8] when $p > 1$ and $-1 + 1/p < s < -t < 0$. See [2].

3. Characteristic functions as bounded multipliers

3.1. **Statement of the main result.** Let $\Lambda \subset M$ be an open set so that $\partial\Lambda$ is a finite union of C^2 curves, everywhere transversal to the stable cones of T. We claim that if $1 < p < \infty$ and $-1 + 1/p < s < -t < 0$ there exists $C_\Lambda < \infty$ so that

$$\|1_\Lambda \varphi\|_{\mathcal{U}_p^{t,s}} \le C_\Lambda \|1_\Lambda \varphi\|_{\mathcal{U}_p^{t,s}} \,, \quad \forall \varphi \,.$$

By using suitable C^2 coordinates,[4] we can reduce to the following bounded multiplier statement on $\mathcal{U}_p^{t,s}(K)$ for the characteristic function of a strip Λ:

Theorem 3.1. *Fix $r > 1$. Let $K \subset \mathbb{R}^d$ be compact, and let Λ be the strip*

$$\Lambda = \{x \in \mathbb{R}^d \mid 0 < x_1 < 1\} \,,$$

(in particular the characteristic function $1_\Lambda(x)$ only depends on $x_1 \in \mathbb{R}$). For any

$$1 < p < \infty \text{ and } \max\{-(r-1), -1 + \frac{1}{p}\} < s < -t < 0 \,,$$

there exists $C < \infty$ so that for any $\varphi \in \mathcal{U}_p^{t,s}(K)$ we have,

$$\|1_\Lambda \varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} \le C \|\varphi\|_{\mathcal{U}_p^{\mathbf{C}_\pm,t,s}} \,. \tag{14}$$

The conditions in the theorem imply $t < 1 - 1/p$. This does not imply $t < 1/p$ if $p > 2$.

Remark 3.2 (Heuristic proof via interpolation: $t < 1/p$ vs. $t < |s|$). A heuristic argument for the bounded multiplier property (14) under the conditions $-1+1/p < s < 0 < t < 1/p$ was sketched in [2, Remark 3.9], exploiting via interpolation the fact that ([15, Thm 4.6.3/1]) the characteristic function of a half-plane in $\mathbb{R}^n$ is a bounded multiplier on the Besov space $B^\tau_{p,\infty}(\mathbb{R}^n)$ if $\frac{1}{p} - 1 < \tau < \frac{1}{p}$. It does not

[4] In order to change charts, one must work with an additional set of cone pairs satisfying $\mathbb{R}^d \setminus \mathbf{C}'_{\omega,+} \Subset \mathbf{C}_{\omega,-}$. This is not a problem since the dynamics is cone-hyperbolic, in view of the Lasota–Yorke estimate proved in [2, Lemma 4.2].

seem easy to fill in details of this argument, and we shall prove Theorem 3.1 using paraproduct decompositions instead of interpolation. The restriction $s < -t$ is in any case necessary for applications to hyperbolic dynamics, and the bound for the essential spectral radius in [2] improves as $p \to 1$ (see the Lasota–Yorke estimate in [2]). If one is only interested in the bounded multiplier property, it may be worthwile to investigate the case $-1 + 1/p < s < -s \le t < 1/p$. The only part of the proof of Theorem 3.1 which requires modification is the contribution of Π_1. For $d_s = 1$, it seems that [15, Thm 4.4.3.2 (ii), p. 173] (for $n = 1$, $q = q_1 = \infty$) allows to implement this modification. If $d_s \ge 2$, the problem is more difficult.

3.2. **Basic toolbox (Nikol'skij and Young bounds, paraproduct decomposition, and a crucial trivial observation on functions of a single variable).** The proofs below use the *Nikol'skij inequality* (see e.g. [15, Remark 2.2.3.4, p. 32]) which says, in dimension $D \ge 1$, that for any $p > p_1 > 0$ there exists C so that for any $M > 1$, and any f with $\operatorname{supp} \mathbb{F}(f) \subset \{|\xi| \le M\}$,

$$\|f\|_{L_p(\mathbb{R}^D)} \le C M^{D(1/p_1 - 1/p)} \|f\|_{L_{p_1}(\mathbb{R}^D)} \,. \tag{15}$$

We shall also use the following *leafwise version of Young's inequality* (which can be proved like [6, Lemma 4.2], see also [1, Chapter 5], by using that any translation $\Gamma + x$ of $\Gamma \in \mathcal{F}$ also belongs to $\mathcal{F}$):

$$\|\tilde\psi * \varphi\|^s_{p,\Gamma} \le \|\tilde\psi\|_{L_1(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} \|\varphi\|^s_{p,\Gamma+x} \le \|\tilde\psi\|_{L_1} \sup_{\tilde\Gamma \in \mathcal{F}} \|\varphi\|^s_{p,\tilde\Gamma} \,. \tag{16}$$

Write $S_k\varphi = \psi_k^{Op}(\varphi)$ for $k \ge 0$, set $S_{-1}\varphi \equiv 0$, and put $S^j\varphi = \sum_{k=0}^j S_k\varphi$ for any integer $j \ge 0$. The *paraproduct decomposition* (see [15, §4.4]) is then given by

$$\begin{aligned}
\varphi \cdot \upsilon &= \lim_{j\to\infty} (S^j\varphi)\cdot(S^j\upsilon) \\
&= \sum_{k=2}^\infty \sum_{j=0}^{k-2} S_j\varphi \cdot S_k\upsilon + \sum_{k=0}^\infty \sum_{j=k-1}^{k+1} S_j\varphi \cdot S_k\upsilon + \sum_{j=2}^\infty \sum_{k=0}^{j-2} S_j\varphi \cdot S_k\upsilon \\
&= \Pi_1(\varphi,\upsilon) + \Pi_2(\varphi,\upsilon) + \Pi_3(\varphi,\upsilon)\,, \qquad (17)
\end{aligned}$$

where we put

$$\Pi_1(\varphi,\upsilon) = \sum_{k=2}^\infty S^{k-2}\varphi \cdot S_k\upsilon\,, \qquad \Pi_2(\varphi,\upsilon) = \sum_{k=0}^\infty (S_{k-1}\varphi + S_k\varphi + S_{k+1}\varphi)\cdot S_k\upsilon\,,$$

$$\text{and} \qquad \Pi_3(\varphi,\upsilon) = \sum_{j=2}^\infty S_j\varphi \cdot S^{j-2}\upsilon = \Pi_1(\upsilon,\varphi)\,.$$

The two key facts motivating the decomposition (17) are

$$\operatorname{supp} \mathbb{F}(S^{k-2}\varphi \cdot S_k\upsilon) \subset \{2^{k-3} \le \|\xi\| \le 2^{k+1}\}\,, \quad \forall k \ge 2\,, \tag{18}$$

and

$$\operatorname{supp} \mathbb{F}\,(\sum_{j=k-1}^{k+1} S_j\varphi \cdot S_k\upsilon) \subset \{\|\xi\| \le 5\cdot 2^k\}\,, \quad \forall k \ge 0\,. \tag{19}$$

Finally, the proof of Theorem 3.1 hinges on the fact that the singular set of a characteristic function is co-dimension one: The characteristic function 1_Λ only

depends on the first coordinate x_1 of $x \in \mathbb{R}^d$. We shall use below the fact that (see [15, Lemma 4.6.3.2 (ii), p. 209, Lemma 2.3.1/3, p. 48]) for all $p \in (1, \infty)$

$$\|1_\Lambda\|_{B^s_{p,q}(\mathbb{R}^d)} < \infty\,, \text{ if } 0 < t < 1/p \text{ and } 0 < q < \infty \text{ or } t = 1/p \text{ and } q = \infty\,. \tag{20}$$

We also note for further use the *trivial but absolutely essential fact* that if a function $\upsilon(x)$ only depends on x_1 then $S_k\upsilon = (\mathbb{F}^{-1}\psi_k) * \upsilon$ also only depends on x_1 for all k, and, more precisely,

$$S_k\upsilon(x) := (\mathbb{F}^{-1}\psi_k) * \upsilon(x) = (\mathbb{F}^{-1}\psi_k^{(1)}) * \upsilon(x_1)\,. \tag{21}$$

Indeed

$$(\mathbb{F}^{-1}\psi_k) * \upsilon(x) = \int (\mathbb{F}^{-1}\psi_k)(y)\mathrm{d}y_2 \dots \mathrm{d}y_d\, \upsilon(x_1 - y_1)\mathrm{d}y_1\,,$$

and, since $(2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} e^{i(y_2,\dots,y_d)(\xi_2,\dots,\xi_d)} dy_2 \dots dy_d$ (the inverse Fourier transform of the constant function) is the Dirac mass at $(\xi_2, \dots, \xi_d) = 0$, we get,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} (\mathbb{F}^{-1}\psi_k)(y_1, y_2, \dots, y_d)\mathrm{d}y_2 \dots \mathrm{d}y_d \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} e^{iy_1\xi_1} \psi_k(\xi)\mathrm{d}\xi_1 \mathrm{d}\xi_2 \dots \mathrm{d}\xi_d e^{i(y_2,\dots,y_d)(\xi_2,\dots,\xi_d)} \mathrm{d}y_2 \dots \mathrm{d}y_d \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iy_1\xi_1} \psi_k(\xi_1, 0)\mathrm{d}\xi_1 = (\mathbb{F}^{-1}\psi_k^{(1)})(y_1)\,,
\end{aligned}$$

where we used that $\psi_k^{(d)}(\xi_1, 0) = \psi_k^{(1)}(\xi_1)$.

3.3. **Multipliers depending on a single coordinate.** This subsection is devoted to a classical property of multipliers depending only on a single coordinate, which is instrumental in the proof of Theorem 3.1. If $1 \le p \le \infty$ we let $1 \le p' \le \infty$ be so that $1/p + 1/p' = 1$, i.e.,

$$p' = \frac{p}{p-1}\,. \tag{22}$$

Lemma 3.3. *Let $d_s \ge 1$. Let $1 < p < \infty$ and let $-1 + \frac{1}{p} < s < 0$. Then there exists $C < \infty$ so that for all $f, g : \mathbb{R}^{d_s} \to \mathbb{C}$ with $g(x) = g(x_1)$,*

$$\|fg\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le C\|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}(\|g\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} + \|g\|_{L_\infty(\mathbb{R})})\,. \tag{23}$$

Remark 3.4. The bound (23) is a special case of a much more general result (see e.g. [15, Cor 4.6.2.1 (40)]) which also implies that if $g(x) = g(x_1)$ then

$$\|fg\|_{B^t_{p,\infty}(\mathbb{R}^{d_s})} \le C\|f\|_{B^t_{p,\infty}(\mathbb{R}^{d_s})}(\limsup_{q\to p} \|g\|_{B^{1/q}_{q,\infty}(\mathbb{R})} + \|g\|_{L_\infty(\mathbb{R})}) \text{ if } 0 < t < \frac{1}{p}\,, \tag{24}$$

for a constant C, which may depend on p and t, but not on f or g.

For the convenience of the reader, and as a warmup in the use of paraproducts, we include a proof of Lemma 3.3.

Proof of Lemma 3.3. The proof uses the decomposition $\tilde\Pi_1(f,g) + \tilde\Pi_2(f,g) + \tilde\Pi_3(f,g)$ obtained from (17) by replacing S_k and S^k by the d_s-dimensional operators

$$\tilde S_k := (\psi_k^{(d_s)})^{Op} f\,, \qquad \tilde S^k := \sum_{j=0}^k (\psi_j^{(d_s)})^{Op} f = \sum_{j=0}^k \tilde S_j f\,. \tag{25}$$

The bound for the contribution of $\tilde{\Pi}_3(f,g)$ is easy and does not require any condition on s or g: Indeed, by (18), we have

$$\|\sum_{j=2}^{\infty}\tilde{S}_j f\tilde{S}^{j-2}g\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\le C\sup_{k\ge 2}2^{ks}\sum_{\ell=-1}^{+1}\|\tilde{S}_{k+\ell}f\tilde{S}^{k+\ell-2}g\|_{L_p(\mathbb{R}^{d_s})}\,.$$

We focus on the term for $\ell=0$ (the others are similar) and get

$$\begin{aligned}\sup_{k\ge 2}2^{ks}\|\tilde{S}_k f\tilde{S}^{k-2}g\|_{L_p(\mathbb{R}^{d_s})}&\le C\sup_{k}2^{ks}\|\tilde{S}_k f\|_{L_p(\mathbb{R}^{d_s})}\sup_k\|\tilde{S}^k g\|_{L_\infty}\\&\le C\|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\|g\|_{L_\infty}\,,\end{aligned}\tag{26}$$

where we used the Hölder inequality and then the Young inequality, together with the second claim of (7).

For $\tilde{\Pi}_1(f,g)$, we do not require any condition on g, and the condition on s is limited to $s<0$: Indeed, exploiting again (18), we get

$$\|\sum_{j=2}^{\infty}\tilde{S}^{j-2}f\tilde{S}_j g\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\le C\sup_{k\ge 2}2^{ks}\sum_{\ell=-1}^{+1}\|\tilde{S}^{k+\ell-2}f\tilde{S}_{k+\ell}g\|_{L_p(\mathbb{R}^{d_s})}\,.$$

Focusing again on the terms for $\ell=0$, we find

$$\begin{aligned}\sup_{k\ge 2}2^{ks}\|\tilde{S}^{k-2}f\tilde{S}_k g\|_{L_p(\mathbb{R}^{d_s})}&\le C\sup_{k}2^{ks}\|\sum_{j=0}^{k-2}\tilde{S}_j f\|_{L_p(\mathbb{R}^{d_s})}\sup_k\|\tilde{S}_k g\|_{L_\infty}\\&\le C\sup_k\Big(\sum_{j=0}^{k-2}2^{(k-j)s}\Big)\sup_j 2^{js}\|\tilde{S}_j f\|_{L_p(\mathbb{R}^{d_s})}\|g\|_{L_\infty}\\&\le 2C\|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\|g\|_{L_\infty}\,,\end{aligned}\tag{27}$$

where we used the Hölder inequality and then the Young inequality, together with the first claim of (7).

The computation for $\tilde{\Pi}_2(f,g)$ is trickier and will use the assumption $s>-1+1/p$ together with the Nikol'skij inequality (15). For $\ell\in\{0,\pm1\}$, by (19), we get

$$\|\sum_{j=0}^{\infty}\tilde{S}_{j+\ell}f\tilde{S}_j g\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\le C\sum_{j=0}^{\infty}\sup_{k\ge 0}2^{ks}\|\tilde{S}_k(\tilde{S}_{k+j+\ell}f\tilde{S}_{k+j}g)\|_{L_p(\mathbb{R}^{d_s})}\,.\tag{28}$$

In the sequel, we consider the terms with $\ell=0$ (the other terms are almost identical). Setting $y=(x_2,\dots,x_{d_s})$ and applying the one-dimensional Nikol'skij inequality (15) for $1<p_1<p$, we have,

$$\begin{aligned}2^{ks}\|\tilde{S}_k\upsilon\|_{L_p(\mathbb{R}^{d_s})}&=\Big(\int\Big[\Big(\int 2^{ksp}|\tilde{S}_k\upsilon(x_1,y)|^p\mathrm{d}x_1\Big)^{1/p}\Big]^p\mathrm{d}y\Big)^{1/p}\\&\le\Big(\int\Big[\Big(\int 2^{k(s+\frac{1}{p_1}-\frac{1}{p})p_1}|\tilde{S}_k\upsilon(x_1,y)|^{p_1}\mathrm{d}x_1\Big)^{1/p_1}\Big]^p\mathrm{d}y\Big)^{1/p}\\&=2^{k(s+\frac{1}{p_1}-\frac{1}{p})}A(p,p_1,\tilde{S}_k\upsilon)\,,\end{aligned}\tag{29}$$

where

$$A(p,p_1,\tilde{S}_k\upsilon)=\Big(\int\Big[\Big(\int|\tilde{S}_k\upsilon(x_1,y)|^{p_1}\mathrm{d}x_1\Big)^{1/p_1}\Big]^p\mathrm{d}y\Big)^{1/p}\,.\tag{30}$$

Since $s > -1 + 1/p$, we may choose $p_1 \in (1, p)$ close enough to 1 so that

$$s_1 = s + \frac{1}{p_1} - \frac{1}{p} > 0\,. \tag{31}$$

Then, the right-hand side of (28) can be bounded as follows, using (29),

$$\begin{aligned}\sum_{j=0}^{\infty} \sup_{k\ge 0} 2^{ks} \|\tilde S_k(\tilde S_{k+j} f \tilde S_{k+j} g)\|_{L_p} &\le \sum_{j=0}^{\infty} \sup_k 2^{ks_1} A(p, p_1, \tilde S_k(\tilde S_{k+j} f \tilde S_{k+j} g)) \\ &\le \Big(\sum_{j=0}^{\infty} 2^{-js_1}\Big) \sup_{k,j} 2^{(k+j)s_1} A(p, p_1, \tilde S_k(\tilde S_{k+j} f \tilde S_{k+j} g)) \\ &\le C \sup_{m\ge 0} 2^{ms_1} A(p, p_1, \tilde S_m f \tilde S_m g)\,.\end{aligned} \tag{32}$$

In the last line we used (19) to exploit the that there exists $C < \infty$, depending on $p > 1$ and $p_1 > 1$, so that, for any $\{\upsilon_k\}_{k\ge 0}$ so that $\mathrm{supp}\,(\mathbb{F}(\upsilon_k)) \subset \{|\xi| \le 5 \cdot 2^k\}$,

$$A(p, p_1, \tilde S_k(\upsilon_{k+j})) \le C A(p, p_1, \upsilon_{k+j})\,, \quad \forall k \ge 0\,,\ j \ge 0\,.$$

(The above basically follows from Young's inequality, see [15, Thm 2.6.3, (5), p. 96], noting that $p > 1$ and $p_1 > 1$, so that $\max\{0, 1/p - 1, 1/p_1 - 1\} = 0$, and noting that f_j in the right-hand side of [15, (5), p. 96] should be replaced by $f_{j+\ell}$, see [12, Thm 2.4.1.(II) and (III)].)

Next, recalling that g only depends on x_1, using (21), and applying the Hölder inequality in dx_1 for $1/p_1 = 1/p + 1/q$, we find C so that for all k

$$\begin{aligned}A(p, p_1, \tilde S_k f \tilde S_k g) &= \Big(\int \Big[\Big(\int |\tilde S_k g(x_1) \tilde S_k f(x_1, y)|^{p_1} \mathrm{d}x_1\Big)^{1/p_1}\Big]^p \mathrm{d}y\Big)^{1/p} \\ &\le C\Big(\int \Big[\Big(\int |S_k g(x_1)|^q \mathrm{d}x_1\Big)^{1/q} \Big(\int |\tilde S_k f(x_1, y)|^p \mathrm{d}x_1\Big)^{1/p}\Big]^p \mathrm{d}y\Big)^{1/p} \\ &\le C\Big(\int |\tilde S_k g(x_1)|^q \mathrm{d}x_1\Big)^{1/q} \Big(\int \Big[\Big(\int |\tilde S_k f(x_1, y)|^p \mathrm{d}x_1\Big)^{1/p}\Big]^p \mathrm{d}y\Big)^{1/p} \\ &= C \|\tilde S_k g\|_{L_q(\mathbb{R})} \|\tilde S_k f\|_{L_p(\mathbb{R}^{d_s})}\,.\end{aligned}$$

Note that (21) implies $\tilde S_k g = (\psi_k^{(1)})^{Op} g$. Finally, putting together (28) and (32), we find, recalling (31) and (22),

$$\begin{aligned}\Big\|\sum_{j=0}^{\infty} \tilde S_j f \tilde S_j g\Big\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} &\le C \sup_{k\ge 0}\big(2^{ks_1} \|\tilde S_k g\|_{L_q(\mathbb{R})} \|\tilde S_k f\|_{L_p(\mathbb{R}^{d_s})}\big) \\ &= C \sup_{k\ge 0}\big(2^{k\frac{1}{q}} \|\tilde S_k g\|_{L_q(\mathbb{R})}\big) \sup_{j\ge 0}\big(2^{ks} \|\tilde S_k f\|_{L_p(\mathbb{R}^{d_s})}\big) \\ &\le C \sup_{k\ge 0}\big(2^{k\frac{1}{q}} 2^{k(\frac{1}{p'} - \frac{1}{q})} \|\tilde S_k g\|_{L_{p'}(\mathbb{R})}\big) \|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \qquad (33) \\ &\le C \|g\|_{B^{1/p'}_{p'}(\mathbb{R})} \|f\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\,, \qquad (34)\end{aligned}$$

where we used the one-dimensional Nikol'skij inequality for $q > p' > 1$ in (33) (recalling (19)). Together, (26), (27), and (34) give (23), concluding the proof of Lemma 3.3. □

3.4. **Proof of Theorem 3.1.** To prove the theorem, we need one last lemma. The point is that if Γ is horizontal, i.e. $\Gamma = \mathbb{R}^{d_s} \times \{0\}$, then (9) implies

$$\tilde{S}_{k_s}((S^k\varphi) \circ \Pi_\Gamma^{-1}|_{\mathbb{R}^{d_s}}) \equiv 0\,, \quad \forall k_s > k+2 \geq 2\,. \tag{35}$$

If Γ is an arbitrary admissible stable leaf, then we must work harder. To state the bound replacing the trivial decoupling property (35), we need notation: Defining $b : \mathbb{R}^d \to \mathbb{R}_+$ by $b(x) = 1$ if $\|x\| \leq 1$ and $b(x) = \|x\|^{-d-1}$ if $\|x\| > 1$, we set $b_k(x) = 2^{dk} \cdot b(2^k x)$ for $k \geq 0$. (Note that $\|b_k\|_{L_1(\mathbb{R}^d)} = \|b\|_{L_1(\mathbb{R}^d)} < \infty$.)

Lemma 3.5 (Decoupled wave packets in $\mathbb{R}^d$ and the cotangent space of Γ)**.** *There exists $C_0 \in [2,\infty)$ (depending on $\mathcal{C}_\mathcal{F}$) so that for any $k_s > k + C_0 \geq C_0$ and any $\Gamma \in \mathcal{F}$, the kernel $V(x,y)$ defined by $\tilde{S}_{k_s}((S^k\varphi) \circ \Pi_\Gamma^{-1})(x) = \int_{y\in\mathbb{R}^d} V(x,y)\varphi(y)\mathrm{d}y$ for $x \in \mathbb{R}^{d_s}$ and φ supported in K satisfies*

$$|V(x,y)| \leq C_0 2^{-k_s r} b_k(\pi_\Gamma^{-1}(x) - y)\,, \quad \forall x \in \mathbb{R}^{d_s}\,, \forall y \in \mathbb{R}^d\,. \tag{36}$$

The lemma implies that $\int_{y\in\mathbb{R}^d} V(x,y)\varphi(y)\mathrm{d}y$ is bounded by a convolution with a function in $L_1(\mathbb{R}^d)$, so that (16) holds.

Proof. The kernel $V(x,y)$ is given by the formula

$$\frac{1}{(2\pi)^{d_s+d}} \int_{z\in\mathbb{R}^{d_s}} \int_{\eta\in\mathbb{R}^d} \int_{\eta_s\in\mathbb{R}^{d_s}} e^{i(\pi_\Gamma^{-1}(z)-y)\eta} e^{i(x-z)\eta_s} \sum_{j=0}^{k} \psi_j(\eta)\psi_{k_s}^{(d_s)}(\eta_s)\mathrm{d}\eta_s\mathrm{d}\eta\mathrm{d}z\,.$$

As a warmup, let us prove (35) if Γ is horizontal or, more generally, affine: Letting $\eta = (\eta_-, \eta_+)$ with $\eta_- = \pi_-(\eta) \in \mathbb{R}^{d_s}$, we have $\pi_\Gamma^{-1}(z) = (z, A(z) + A_0)$ with $A_0 \in \mathbb{R}^{d_u}$ and $A : \mathbb{R}^{d_s} \to \mathbb{R}^{d_u}$ linear ($A \equiv 0$ if Γ is horizontal), so that (using like in (21) that $\mathbb{F}^{-1}(1)$ is the Dirac at 0), $V(x,y)$ can be rewritten as

$$\begin{aligned}
&\frac{1}{(2\pi)^{d+d_s}} \int_{\mathbb{R}^{2d_s+d}} e^{-iy\eta} e^{ix\eta_-} e^{iA_0\eta_+} e^{iz(-\eta_s+\eta_-+A^{tr}\eta_+)} \sum_{j=0}^{k} \psi_j(\eta)\psi_{k_s}^{(d_s)}(\eta_s)\mathrm{d}\eta_s\mathrm{d}\eta\mathrm{d}z \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iy\eta} e^{ix\eta_-} e^{iA_0\eta_+} \sum_{j=0}^{k} \psi_j(\eta)\psi_{k_s}^{(d_s)}(\eta_- + A^{tr}\eta_+)\mathrm{d}\eta \equiv 0\,,
\end{aligned}$$

by (9), if $k_s > k + C_0$, where $C_0 \geq 2$ depends on $\|A\| \leq \mathcal{C}_\mathcal{F}$.

More generally, $\Gamma \in \mathcal{F}$ is the graph of a C^r map γ (with $\|\gamma\|_{C^r} \leq \mathcal{C}_\mathcal{F}$), i.e.,

$$\pi_\Gamma^{-1}(z) = (z, \gamma(z))\,, \qquad z \in \mathbb{R}^{d_s} \cap \pi_-(\Gamma)\,. \tag{37}$$

The lemma is thus obtained integrating by parts r times (in the sense of [2, App. C] if r is not an integer) with respect to z in the kernel $V(x,y)$, using (8), and proceeding as in the end of the proof of [1, Lemma 2.34], mutatis mutandis (using that $\|y - \Pi_\Gamma^{-1}(x)\| > 2^{-k}$ implies that either $\|y - \Pi_\Gamma^{-1}(z)\| > 2^{-k+1}$ or $\|\Pi_\Gamma^{-1}(z) - \Pi_\Gamma^{-1}(x)\| > 2^{-k+1}$, choosing C_0 depending on $\mathcal{C}_\mathcal{F}$, so that $\|\Pi_\Gamma^{-1}(z) - \Pi_\Gamma^{-1}(x)\| > 2^{-k+1}$ implies $\|z - x\| \geq 2^{-k+1}/C_0$). □

Proof of Theorem 3.1. Our starting point is the decomposition (17) applied to $\upsilon = 1_\Lambda$. We consider first the term $\Pi_3(\varphi, 1_\Lambda)$. We will bootstrap from Lemma 3.3: By

(18) and (23), there exists a constant C so that any $\ell \geq 0$, since $-1+1/p<s<0$,

$$\begin{aligned}
2^{\ell t}\|S_\ell(\Pi_3(\varphi,1_\Lambda))\|^s_{p,\Gamma} &\leq 2^{\ell t}\sum_{k=\ell-3}^{\ell+1}\|S_k\varphi\cdot S^{k-2}1_\Lambda\|^s_{p,\Gamma}\\
&\leq 2^{\ell t}\sum_{k=\ell-3}^{\ell+1}\|S_k\varphi\|^s_{p,\Gamma}(\|1_\Lambda^{k-2,\Gamma}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})}+\|1_\Lambda^{k-2,\Gamma}\|_{L_\infty(\mathbb{R})})\\
&\leq C\sup_n 2^{nt}\|S_n\varphi\|^s_{p,\Gamma}\leq C\|\varphi\|_{\mathcal{U}^{t,s}_p}\,,
\end{aligned}$$

where we used (23) from Lemma 3.3 for $f(x_-)=S_k\varphi(x_-,\gamma(x_-))$ with $\gamma=\gamma(\Gamma)$ from (37), and $g(x_-)=1_\Lambda^{k-2,\Gamma}(x_-)$ with

$$1_\Lambda^{k-2,\Gamma}(x_-)=(S^{k-2}1_\Lambda)(x_-,\gamma(x_-))=\sum_{j=0}^{k-2}(\mathbb{F}^{-1}\psi_j*1_\Lambda)(x_-,\gamma(x_-))\,. \tag{38}$$

Indeed, this implies that $1_\Lambda^{k-2,\Gamma}(x_-)$ is a function of x_1 alone (recalling (21)), and the leafwise Young inequality (16), together with the second claim of (7) and the fact that $\|1_\Lambda\|_{B^{1/t}_{t,\infty}(\mathbb{R})}<\infty$ (for any $1<t<\infty$, see e.g. [15, Lemma 2.3.1/3(ii), Lemma 2.3.5]), give that both $\|1_\Lambda^{k-2,\Gamma}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})}$ and $\|1_\Lambda^{k-2,\Gamma}\|_{L_\infty(\mathbb{R})}$ are finite, uniformly in Γ and k. This concludes the bound for $\Pi_3(\varphi,1_\Lambda)$.

We move to $\Pi_2(\varphi,1_\Lambda)$. Using (19), and applying (23) from Lemma 3.3 again, we find for any $\ell\geq 0$, since $t>0$,

$$\begin{aligned}
2^{\ell t}\|S_\ell(\Pi_2(\varphi,1_\Lambda))\|^s_{p,\Gamma} &\leq 2^{\ell t}3\sum_{k\geq\ell-1}\|S_k\varphi\cdot S_k1_\Lambda\|^s_{p,\Gamma}\\
&\leq 3\sup_k 2^{kt}\|S_k\varphi\|^s_{p,\Gamma}(\|1^\Gamma_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})}+\|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})})\sum_{k\geq\ell-1}2^{(\ell-k)t}\\
&\leq C\sup_k 2^{kt}\|S_k\varphi\|^s_{p,\Gamma}\leq C\|\varphi\|_{\mathcal{U}^{t,s}_p}\,,
\end{aligned}$$

where

$$1^\Gamma_{\Lambda,k}(x_-)=(S_k1_\Lambda)(x_-,\gamma(x_-))=(\mathbb{F}^{-1}\psi_k*1_\Lambda)(x_-,\gamma(x_-))\,, \tag{39}$$

so that $1^\Gamma_{\Lambda,k}(x_-)=1^\Gamma_{\Lambda,k}(x_1)$. Indeed, the leafwise Young inequality (16), together with the first claim of (7), give that

$$\sup_{k,\Gamma}\|1^\Gamma_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})}<\infty\,,\qquad \sup_{k,\Gamma}\|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})}\,. \tag{40}$$

It remains to bound the contribution of $\Pi_1(\varphi,1_\Lambda)$. This is the trickiest estimate. It will use Lemma 3.5 and our assumption that $s+t<0$. For any $\ell\geq 0$, we have

$$2^{\ell t}\|\psi_\ell^{Op}(\Pi_1(\varphi,1_\Lambda))\|^s_{p,\Gamma}\leq\sum_{k=\ell-3}^{\ell+1}2^{\ell t}\|S^{k-2}\varphi\cdot S_k1_\Lambda\|^s_{p,\Gamma}\,. \tag{41}$$

We may focus on the term $k=\ell$, as the others are almost identical. We will use the paraproduct decomposition $\tilde\Pi_1+\tilde\Pi_2+\tilde\Pi_3$ and the operators $\tilde S_j$ and $\tilde S^j$ (see (25)).

Put $(S^{k-2}\varphi)^\Gamma = (S^{k-2}\varphi)\circ\pi_\Gamma^{-1}$. By (21) and (18), we have

$$\|S^{k-2}\varphi\cdot S_k 1_\Lambda\|^s_{p,\Gamma} \le \sum_{i=1}^{2}\|\tilde\Pi_i((S^{k-2}\varphi)^\Gamma, 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}} + \mathcal{R}^\Gamma_{k,s,p,\Lambda}(\varphi) \tag{42}$$

$$+\sum_{j=k+2}^{k-2+C_0}\|\tilde S_j((S^{k-2}(\varphi))^\Gamma)(\tilde S_k 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}}\,, \tag{43}$$

where $C_0\ge 2$ is given by Lemma 3.5. Then, Lemma 3.5, Lemma 3.3, together with (40) and the leafwise Young inequality (16) imply (since $0<t<r-1<r$)

$$\sup_{k\ge k_0,\Gamma} 2^{kt}\mathcal{R}^\Gamma_{k,s,p,\Lambda}(\varphi) \le C_0\sup_{k,\Gamma}2^{k(t-r))}\|S^k\varphi\|^s_{p,\Gamma}\le C\|\varphi\|_{\mathcal{U}_p^{t,s}}\,. \tag{44}$$

The finite sum in (43) is easy to bound and left to the reader. For the contribution of $\tilde\Pi_1$ in (42), using again (21) and (18), we find

$$2^{kt}\|\tilde\Pi_1((S^{k-2}\varphi)^\Gamma, 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\le 2^{kt}\|(\tilde S^{k-2}(S^{k-2}\varphi)^\Gamma)\cdot 1^\Gamma_{\Lambda,k})\|_{B^s_{p,\infty}}$$

Set $(S_j\varphi)^\Gamma = (S_j\varphi)\circ\pi_\Gamma^{-1}$. By Lemma 3.5, up to a remainder which can be handled similarly as in (44), the right-hand side above is bounded by

$$\begin{aligned}
&2^{kt}2^{ks}\|[\sum_{j=0}^{k-2}\sum_{m=0}^{j}\tilde S_m(S_j\varphi)^\Gamma)]\cdot(1^\Gamma_{\Lambda,k})\|_{L_p(\mathbb{R}^{d_s})}\\
&\quad\le(\sup_{0\le j\le k-2}\sum_{m=0}^{j}2^{(j-m)s})2^{kt}\sum_{j=0}^{k-2}\sup_{0\le m\le j}2^{(k-j+m)s}\|[\tilde S_m(S_j\varphi)^\Gamma]\cdot(1^\Gamma_{\Lambda,k})\|_{L_p}\\
&\quad\le C\sum_{j=0}^{k-2}\sup_{0\le m\le j}2^{(k-j)(t+s)}2^{ms}2^{jt}\|\tilde S_k([\tilde S_m(S_j\varphi)^\Gamma]\cdot(1^\Gamma_{\Lambda,k}))\|_{L_p(\mathbb{R}^{d_s})}\,,
\end{aligned}$$

using that $s<0$. Now, since $s+t<0$, we get

$$\begin{aligned}
&\sum_{j=0}^{k-2}\sup_{0\le m\le j}2^{(k-j)(t+s)}2^{ms}2^{jt}\|\tilde S_k([\tilde S_m(S_j\varphi)^\Gamma]\cdot(1^\Gamma_{\Lambda,k}))\|_{L_p(\mathbb{R}^{d_s})}\\
&\quad\le C\sup_m\sup_j 2^{ms}2^{jt}\|\tilde S_m(S_j\varphi)^\Gamma\|_{L_p(\mathbb{R}^{d_s})}\|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})}\\
&\quad\le C\sup_j 2^{jt}\|S_j\varphi\|^s_{p,\Gamma}\le C\|\varphi\|_{\mathcal{U}_p^{t,s}}\,.
\end{aligned}$$

Finally, using (21) once more, we bound the contribution of $\tilde\Pi_2$ in (42):

$$\begin{aligned}
&2^{kt}\|\tilde\Pi_2((S^{k-2}\varphi)^\Gamma, 1^\Gamma_{\Lambda,k}\|_{B^s_{p,\infty}}\le 2^{kt}\sum_{\ell=-1}^{1}\|(\tilde S_{k+\ell}(S^{k-2}\varphi)^\Gamma)\cdot 1^\Gamma_{\Lambda,k}\|_{B^s_{p,\infty}}\\
&\le 2^{kt}\tilde\mathcal{R}^\Gamma_{k,p,s,\Lambda}(\varphi)+2^{kt}\sum_{\ell=-1}^{1}\sum_{\tilde\ell=2}^{7}\|(\tilde S_{k+\ell}(S_{k-\tilde\ell}\varphi)^\Gamma)\cdot 1^\Gamma_{\Lambda,k}\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})}\,,
\end{aligned}$$

where $2^{kt}\tilde\mathcal{R}^\Gamma_{k,p,s,\Lambda}(\varphi)$ can be bounded similarly as (44), using Lemma 3.5. For the double sum, we focus on the contributions with $\ell=0$ and $\tilde\ell=2$, the others being

similar. Then, applying Lemma 3.3, we find

$$\sup_{k,\Gamma} 2^{kt} \|(\tilde{S}_k(S_{k-2}\varphi)^\Gamma) \cdot 1^\Gamma_{\Lambda,k}\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} \le \sup_{k,\Gamma} 2^{kt} \|(\tilde{S}_k(S_{k-2}\varphi)^\Gamma) \cdot 1^\Gamma_{\Lambda,k}\|_{B^s_{p,\infty}}$$
$$\le \sup_{k,\Gamma} 2^{kt} \|(\tilde{S}_k(S_{k-2}\varphi)^\Gamma)\|_{B^s_{p,\infty}(\mathbb{R}^{d_s})} (\|1^\Gamma_{\Lambda,k}\|_{B^{1/p'}_{p',\infty}(\mathbb{R})} + \|1^\Gamma_{\Lambda,k}\|_{L_\infty(\mathbb{R})}) \le C\|\varphi\|_{\mathcal{U}^{t,s}_p} ,$$

using (40) once more. This ends the proof of Theorem 3.1. □

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