A REMARK ON THE INTERSECTION OF PLANE CURVES

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C. CILIBERTO, F. FLAMINI, M. ZAIDENBERG

Abstract. Let D be a very general curve of degree $d = 2\ell - \varepsilon$ in $\mathbb{P}^2$, with $\varepsilon \in \{0, 1\}$. Let $\Gamma \subset \mathbb{P}^2$ be an integral curve of geometric genus g and degree m, $\Gamma \neq D$, and let $\nu : C \to \Gamma$ be the normalization. Let δ be the degree of the *reduction modulo 2* of the divisor $\nu^*(D)$ of C (see § 2.1). In this paper we prove the inequality $4g + \delta \geqslant m(d - 8 + 2\varepsilon) + 5$. We compare this with similar inequalities due to Geng Xu ([10, 11]) and Xi Chen ([1, 2]).

Contents

Introduction

Given an effective divisor $D \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ and an integral (i.e., reduced and irreducible) projective curve Γ of degree m in $\mathbb{P}^n$, which is not contained in supp(D), let $j(D, \Gamma)$ be the order of $\Gamma \cap D$. Assume D is very general and set

$$j(n, d, m) := \min\{j(D, \Gamma) \,|\, \Gamma \subset \mathbb{P}^n \text{ as above}\} \quad \text{and} \quad j(n, d) := \min_{m \geqslant 1}\{j(n, d, m)\}.$$

Similarly, with Γ and D as before, let $i(D, \Gamma)$ stand for the *number of places* of Γ on D, that is, the number of centers of local branches of the curve Γ on D. Then, set

$$i(n, d, m) := \min\{i(D, \Gamma) \,|\, \Gamma \subset \mathbb{P}^n \text{ as above}\} \quad \text{and} \quad i(n, d) := \min_{m \geqslant 1}\{i(n, d, m)\}.$$

The problem of computing $j(n, d)$ and $i(n, d)$ has been considered in [1, 10, 11] (basically devoted to $n = 2$ case) and [2] (where the case $n \geqslant 2$ is considered). The relations of this with the famous Kobayashi problem on hyperbolicity of the complement of a very general hypersurface in $\mathbb{P}^n$ is well known and we do not dwell on this here (see, e.g., [2]).

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Geng Xu ([10, Thm. 1]) proved that

$$j(2,d) = d-2, \text{ for any } d \geqslant 3,$$

where the equality is attained either by a bitangent line or by an inflectional tangent line of D, i.e. the minimum is achieved by $m=1$. Moreover, for $d=3$, he also proved in [11, Corollary] that, for any integer $m \geqslant 1$, the number of rational curves of degree m which meet set-theoretically a given (arbitrary) smooth plane cubic curve D at exactly one point is finite and positive. Therefore, for $d=3$ the minimum $j(2,3)=1$ is achieved by any integer $m \geqslant 1$.

Xi Chen ([1, Thm. 1.2]) proved that, for $d>m$, one has

$$j(2,d,m) \geqslant \min\left\{dm - \frac{m(m+3)}{2},\ 2dm - 2m^2 - 2\right\}.$$

Furthermore (cf. [1, Cor. 1.1]), for $d \geqslant \max\{\frac{3m}{2}-1,\ 3\}$ one has

$$j(2,d,m) = dm - \dim(|\mathcal{O}_D(m)|) = dm - \frac{m(m+3)}{2}\ .$$

In addition, he conjectured (see [1, Conj. 1.1]) that

$$j(2,d,m) = dm - \dim(|\mathcal{O}_D(m)|) \quad \text{if} \quad d > \max\{m,2\}.$$

In arbitrary dimension $n \geqslant 2$, Xi Chen ([2, Thm 1.7]) proved that, for D very general and Γ as above, one has

$$2g-2+i(D,\Gamma) \geqslant (d-2n)m\,, \tag{1}$$

where g is the *geometric genus* of Γ, i.e., the arithmetic genus of its normalization.

In this paper we obtain a new inequality of type (1), although only in the case $n=2$ (see Theorem 3.1). Indeed, let D be a very general curve of degree $d = 2\ell - \varepsilon$ in $\mathbb{P}^2$, with $\varepsilon \in \{0,1\}$. Let Γ be an integral curve in $\mathbb{P}^2$ of geometric genus g and degree m, $\Gamma \neq D$, and let $\nu : C \to \Gamma$ be the normalization. Let $\delta(D,\Gamma)$ be the degree of the *reduction modulo 2* of the divisor $\nu^*(D)$ on C (cf. § 2.1). In Theorem 3.1 we prove that

$$4g + \delta(D,\Gamma) \geqslant m(d+2\varepsilon-8)+5. \tag{2}$$

Note that $\delta(D,\Gamma) \leqslant i(D,\Gamma)$, and the equality holds if and only if at any place p of Γ on D, the local intersection multiplicity of D and Γ at p is odd. This happens, for instance, if Γ intersects D transversely. In the latter case $\delta(D,\Gamma) = i(D,\Gamma) = md$ and both (1) and (2) are uninteresting. On the other hand, (1) and (2) become interesting when $\delta(D,\Gamma)$ and $i(D,\Gamma)$ are small. Though the difference between the two quantities is a priori unpredictable, one may expect that, generally speaking, $\delta(D,\Gamma)$ is strictly smaller than $i(D,\Gamma)$. Unfortunately, the genus g works against us in (2); however, for $g=0,1$ and d even, (2) is better than (1). Further related problems have been recently considered in [3, 7, 8].

As a final additional remark, note that (2) is more useful than (1) if one looks, as we do in this paper, at the geometric genera of curves contained in a double cover of $\mathbb{P}^2$ branched along a very general curve of even degree. For example, letting $g=0$, $\delta(D,\Gamma)=0,2$ and d even, which corresponds to looking at rational curves on a *double plane*, that is, the double cover of $\mathbb{P}^2$ branched along a very general curve D of degree d. By (2) we see that such a rational curve over Γ might exist, as expected, only for $d \leqslant 6$ (for low m one has even smaller bounds on d). The case $d=6$ corresponds to a $K3$ surface, which always contains infinitely many rational curves. By contrast, the double planes with very general branching curves of even degree $\geqslant 8$ do not carry any rational curves.

The proof of Theorem 3.1 presented in §3 follows, with minor variations due to the different setting, the basic ideas exploited in [5] (and later in [6]). These are based on a smart use of the theory of *focal loci*, see e.g. [4]. For the reader's convenience, we recall in § 1 the basic notions and results of this theory. We apply this technique to families of double covers of $\mathbb{P}^2$ branched along a very general plane curve D or along D plus a general line, according to whether the degree of D is even (see § 2.3 and § 3.2.1) or odd (see § 2.4 and § 3.2.2).

1. Focal loci

For the reader's convenience, we recall here some basic notions from [4, 5].

Let X be a smooth projective variety of dimension $n+1$. Assume we have a flat, projective family $\mathcal{D} \stackrel{p}{\longrightarrow} \mathcal{B}$ of effective divisors on X over a smooth, irreducible, quasiprojective base $\mathcal{B}$, with irreducible general fiber. Up to shrinking $\mathcal{B}$ to a suitable Zariski dense, open subset, we may suppose that for any closed point $b \in \mathcal{B}$ the fiber D_b of $\mathcal{D} \stackrel{p}{\longrightarrow} \mathcal{B}$ over b is irreducible.

Assume we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{C} & \stackrel{i}{\hookrightarrow} & \mathcal{D} \\ & {}_{\mathfrak{q}}\searrow & \downarrow p \\ & & \mathcal{B} \end{array} \tag{3}$$

where $\mathfrak{q} : \mathfrak{C} \to \mathcal{B}$ is a flat projective family such that, for all $b \in \mathcal{B}$, the fiber Γ_b over b is a reduced curve of *geometric genus* g, and where i is an inclusion: so, for any $b \in \mathcal{B}$, one has $\Gamma_b \subset D_b$ via the inclusion i_b.

By a result of Tessier (see [9, Théorème 1]), there is a simultaneous normalization

$$\begin{array}{ccc} \mathcal{C} & \stackrel{\nu}{\longrightarrow} & \mathfrak{C} \\ & {}_{q}\searrow & \downarrow \mathfrak{q} \\ & & \mathcal{B} \end{array} \tag{4}$$

such that $\mathcal{C}$ is smooth and, for every $b \in \mathcal{B}$, the fiber C_b of $q : \mathcal{C} \to \mathcal{B}$ is the normalization $\nu_b : C_b \to \Gamma_b$ of Γ_b. For any $b \in \mathcal{B}$, the curve C_b is smooth of *(arithmetic) genus* g.

Composing with the inclusion $\mathcal{D} \stackrel{j}{\hookrightarrow} \mathcal{B} \times X$, we get the commutative diagram

$$\begin{array}{ccccccc} \mathcal{C} & \stackrel{\nu}{\longrightarrow} & \mathfrak{C} & \stackrel{i}{\hookrightarrow} & \mathcal{D} & \stackrel{j}{\hookrightarrow} & \mathcal{B} \times X \\ & {}_{q}\searrow & \downarrow \mathfrak{q} & & \downarrow p & \swarrow_{\mathrm{pr}_1} & \downarrow \mathrm{pr}_2 \\ & & \mathcal{B} & \xrightarrow[\mathrm{id}]{} & \mathcal{B} & & X \end{array} \tag{5}$$

where pr_i is the projection onto the ith factor, for $i = 1, 2$.

We set

$$s := j \circ i \circ \nu \colon \mathcal{C} \to \mathcal{B} \times X \,,$$

and let $\mathcal{N} := \mathcal{N}_s$ be the normal sheaf to s, defined by the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{C}} \stackrel{ds}{\longrightarrow} s^*(\mathcal{T}_{\mathcal{B} \times X}) \longrightarrow \mathcal{N} \longrightarrow 0 \,,$$

where $\mathcal{T}_Y$ stands for the tangent sheaf of a smooth variety Y.

For $b \in \mathcal{B}$ we set

$$N_b := \mathcal{N}|_{C_b} = \mathcal{N} \otimes \mathcal{O}_{C_b} \quad \text{and} \quad s_b = s|_{C_b} \colon C_b \to \{b\} \times X = X \,.$$

In addition, we let

$$\varphi := \mathrm{pr}_2 \circ s : \mathcal{C} \to X \,.$$

Then $\varphi_b = \varphi|_{C_b}$ coincides with s_b for any $b \in \mathcal{B}$, that is,

$$\varphi_b = s_b : \; C_b \xrightarrow{\;\nu_b\;} \Gamma_b \stackrel{i_b}{\hookrightarrow} D_b \stackrel{(\mathrm{pr}_2 \circ j)_b}{\hookrightarrow} X \,.$$

As in [5, § 2], we set

$$z(\mathcal{C}) := \dim\,(\varphi(\mathcal{C})), \tag{6}$$

so that $z(\mathcal{C}) \leqslant n+1 = \dim(X)$. If $z(\mathcal{C}) = n+1$ one says that $\mathfrak{C} \stackrel{\mathfrak{q}}{\longrightarrow} \mathcal{B}$, or $\mathcal{C} \stackrel{q}{\longrightarrow} \mathcal{B}$, is a *covering family.*

Proposition 1.1 (See [4, Prop. 1.4 and p. 98])**.** *In the above setting, we have:*

(a) *for any* $b \in \mathcal{B}$*, the sheaf* N_b *fits into the exact sequence*

$$0 \longrightarrow \mathcal{T}_{C_b} \xrightarrow{ds_b} s_b^*(\mathcal{T}_X) \longrightarrow N_b \longrightarrow 0$$

and $\mathcal{C} \xrightarrow{q} \mathcal{B}$ *induces on* C_b *a* characteristic map

$$\chi_b \colon T_{\mathcal{B},b} \otimes \mathcal{O}_{C_b} \longrightarrow N_b\,,$$

where $T_{\mathcal{B},b}$ *denotes the tangent space to* $\mathcal{B}$ *at* b*;*

(b) *if* $b \in \mathcal{B}$ *and* $x \in C_b$ *are general points, then*

$$\dim(N_{b,x}) = \dim(s_b^*(\mathcal{T}_X)_x) - \dim(\mathcal{T}_{C_b,x}) = n \quad \textit{and} \quad \mathrm{rk}\,(\chi_{b,x}) = z(\mathcal{C}) - 1\,.$$

Definition 1.2 (See [4, Def. (1.5)])**.** Given $b \in \mathcal{B}$, the *focal set* of $\mathcal{C} \xrightarrow{q} \mathcal{B}$ on C_b is the closed subscheme Φ_b of C_b defined as

$$\Phi_b := \{x \in C_b |\, \mathrm{rk}(\chi_{b,x}) < z(\mathcal{C}) - 1\}.$$

If $b \in \mathcal{B}$ is general, then Φ_b is a proper subscheme of C_b. The points in Φ_b are called *focal points* of $\mathcal{C} \xrightarrow{q} \mathcal{B}$ on C_b. We denote by Φ_b^{sm} the open subset of Φ_b consisting of the points $x \in \Phi_b$ which map to smooth points of Γ_b via the normalization morphism $\nu_b \colon C_b \to \Gamma_b$.

Proposition 1.3 ([5, Prop. 2.3 and Prop. 2.4])**.** *Let* $\mathcal{C} \longrightarrow \mathcal{B}$ *be a covering family. Then the following hold.*

(i) *Suppose that for* $x \in C_b$ *the point* $s(x)$ *is smooth in both* Γ_b *and* D_b*. Assume also that* $s(x)$ *is a* fundamental point *of the family* $\mathcal{D} \xrightarrow{p} \mathcal{B}$*, i.e. it is a base point of the family. Then* $x \in \Phi_b^{\mathrm{sm}}$*.*

(ii) *For a general point* $b \in \mathcal{B}$ *one has*

$$\deg(\Phi_b^{\mathrm{sm}}) \leqslant 2g - 2 - K_X \cdot \Gamma_b\,. \tag{7}$$

2. Double planes

In this section we collect useful material for the proof of our main result. The result itself is stated and proven in §3. The contents of this section, which suffice for our applications, can be easily adapted to the higher dimensional case.

2.1. **The δ–invariant.** Let C be any smooth, irreducible, projective curve, and let $\Delta = \sum_i m_i p_i$ be an effective divisor on C. We set $\Delta_2 := \sum_i \overline{m}_i p_i$, where $\overline{m}_i \in \{0, 1\}$ is the residue of the integer m_i modulo 2. We also set $\delta_2(\Delta) := \deg(\Delta_2)$.

For any smooth curve $D \subset \mathbb{P}^2$ and any integral curve $\Gamma \subset \mathbb{P}^2$, $\Gamma \neq D$, with normalization $\nu \colon C \to \Gamma$, we set

$$\delta(D, \Gamma) := \delta_2(\nu^*(D))\,. \tag{8}$$

We notice that

$$\delta(D, \Gamma) \leqslant i(D, \Gamma).$$

2.2. **Basics on a certain weighted projective 3-space.** For any positive integer ℓ, we denote by $\mathcal{L}_\ell$ the linear system $|\mathcal{O}_{\mathbb{P}^2}(\ell)|$ of plane curves of degree ℓ, and by $\mathcal{U}_\ell$ its open dense subset of points corresponding to smooth curves. We let $N_\ell = \dim(\mathcal{L}_\ell) = \frac{\ell(\ell+3)}{2}$. We denote by $\mathcal{D}_\ell \to \mathcal{L}_\ell$ the universal curve, and we use the same notation $\mathcal{D}_\ell \to \mathcal{U}_\ell$ for its restriction to $\mathcal{U}_\ell$.

The linear system $\mathcal{L}_\ell$ determines the ℓth *Veronese embedding* $\mathbb{P}^2 \overset{v_\ell}{\hookrightarrow} \mathbb{P}^{N_\ell}$, whose image is the ℓ*–Veronese surface* V_ℓ in $\mathbb{P}^{N_\ell}$. Let $[x] = [x_0, x_1, x_2]$ be homogeneous coordinates in $\mathbb{P}^2$, and let

$$[x^I], \text{ where } I = (i_0, i_1, i_2) \text{ is a multiindex such that } |I| = i_0 + i_1 + i_2 = \ell,$$

be homogeneous coordinates in $\mathbb{P}^{N_\ell}$. In these coordinates the Veronese map is given by

$$\mathbb{P}^2 \ni [x] \xrightarrow{v_\ell} [x^I]_{|I|=\ell} \in \mathbb{P}^{N_\ell}, \text{ where } x^I := x_0^{i_0} x_1^{i_1} x_2^{i_2}.$$

We equip the weighted projective 3-space $\mathbb{P}(1, 1, 1, \ell)$ with weigthed homogeneous coordinates $[x, z] := [x_0, x_1, x_2, z]$, where x_0, x_1, x_2 [resp. z] have *weigth* 1 [resp. has weight ℓ]. We introduce as well coordinates $[x^I, z]_{|I|=\ell}$ in $\mathbb{P}^{N_\ell+1}$ and embed $\mathbb{P}^{N_\ell}$ in $\mathbb{P}^{N_\ell+1}$ as the hyperplane Π with equation $z = 0$. Then $\mathbb{P}(1, 1, 1, \ell)$

can be identified with the cone $W_\ell \subset \mathbb{P}^{N_\ell+1}$ over the l–Veronese surface V_ℓ with vertex $P = [0, \dots, 0, 1]$. Blowing P up yields a minimal resolution

$$\rho : Z_\ell \to W_\ell \cong \mathbb{P}(1, 1, 1, \ell),$$

with exceptional divisor $E \cong V_\ell \cong \mathbb{P}^2$. The projection from P induces a $\mathbb{P}^1$–bundle structure

$$\pi \colon Z_\ell \to V_\ell \cong \mathbb{P}^2 .$$

Let f be the class of a fiber of π. One has

$$Z_\ell \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(\ell) \oplus \mathcal{O}_{\mathbb{P}^2}) \quad \text{and} \quad \mathcal{O}_{Z_\ell}(1) = \rho^*(\mathcal{O}_{W_\ell}(1)).$$

For every integer m, we set

$$\mathcal{O}_\ell(m) := \pi^*(\mathcal{O}_{\mathbb{P}^2}(m)) \quad \text{and} \quad \mathcal{L}_\ell(m) := |\mathcal{O}_\ell(m)| . \tag{9}$$

Note that

$$\mathcal{O}_{Z_\ell}(1) \cong \mathcal{O}_\ell(\ell) \otimes \mathcal{O}_{Z_\ell}(E). \tag{10}$$

Since

$$E \cong \mathbb{P}^2, \;\; \mathcal{O}_{Z_\ell}(1) \otimes \mathcal{O}_E \cong \mathcal{O}_E, \;\text{ and }\; \mathcal{O}_\ell(\ell) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^2}(\ell),$$

we deduce

$$\mathcal{O}_{Z_\ell}(E) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^2}(-\ell). \tag{11}$$

Finally, we denote by K_ℓ the canonical sheaf of Z_ℓ.

Lemma 2.1. *One has*

$$K_\ell \cong \mathcal{O}_\ell(\ell - 3) \otimes \mathcal{O}_{Z_\ell}(-2) \cong \mathcal{O}_\ell(-\ell - 3) \otimes \mathcal{O}_{Z_\ell}(-2E) .$$

Proof. The Picard group $\mathrm{Pic}(Z_\ell)$ is freely generated by the classes $\mathcal{O}_\ell(1)$ and $\mathcal{O}_{Z_\ell}(E)$, and also by $\mathcal{O}_\ell(1)$ and $\mathcal{O}_{Z_\ell}(1)$, see (10). Let H [resp. L] be a general member of $|\mathcal{O}_{Z_\ell}(1)|$ [resp. of $\mathcal{L}_\ell(1)$]. Write $K_\ell \sim \alpha H + \beta L$, where $\alpha, \beta \in \mathbb{Z}$. From the relations $K_\ell \cdot f = -2$, $H \cdot f = 1$, and $L \cdot f = 0$ one gets $\alpha = -2$.

By adjunction formula and (11) we obtain

$$\mathcal{O}_{\mathbb{P}^2}(-3) \cong K_E = (K_\ell + E)|_E \cong (-2H + \beta L + E)|_E \cong \mathcal{O}_{\mathbb{P}^2}(\beta - \ell) .$$

So, $\beta = \ell - 3$, as desired. □

Finally, let $\mathbb{G}_\ell$ be the group of all automorphisms of $\mathbb{P}(1, 1, 1, \ell)$ which stabilize the divisor with equation $z = 0$. This group is naturally isomorphic to the automorphism group of the pair (W_ℓ, V_ℓ), i.e. automorphisms of W_ℓ stabilizing V_ℓ, where V_ℓ is cut out on W_ℓ by Π. In turn, the latter group is isomorphic to the automorphism group of the pair $(Z_\ell, \rho^*(V_\ell))$. One has the exact sequence

$$1 \to \mathbb{C}^* \to \mathbb{G}_\ell \to \mathrm{PGL}(3, \mathbb{C}) \to 1. \tag{12}$$

2.3. **The even degree case.** Let D be a smooth curve in $\mathbb{P}^2$ of even degree $d = 2\ell \geqslant 2$ which, in the homogeneous coordinate system fixed in §2.2, is given by equation $f(x_0, x_1, x_2) = 0$, where f is a homogeneous polynomial of degree d. Viewed as a hypersurface of V_ℓ, D is cut out on V_ℓ by a quadric with equation $Q(x^I)_{|I|=\ell} = 0$, where Q is a homogeneous polynomial of degree 2 in the variables $\{x^I\}_{|I|=\ell}$.

The *double plane* associated with D is the double cover $\psi : \mathbb{D}^* \to \mathbb{P}^2$ branched along D. It can be embedded in $\mathbb{P}(1, 1, 1, \ell)$ as a hypersurface $\mathbb{D}^*_a$ defined by a (weighted homogeneous) equation of the form $az^2 = f(x_0, x_1, x_2)$, for any $a \in \mathbb{C}^*$. Under the identification of $\mathbb{P}(1, 1, 1, \ell)$ with W_ℓ, we see that $\mathbb{D}^*_a$ is cut out on W_ℓ by a quadric in $\mathbb{P}^{N_\ell+1}$ of the form $az^2 = Q(x^I)_{|I|=\ell}$.

Consider the sublinear system $\mathcal{Q}_\ell$ of $|\mathcal{O}_{W_\ell}(2)|$ of surfaces cut out on W_ℓ by the quadrics of $\mathbb{P}^{N_\ell+1}$ with equation of the form $az^2 = Q(x^I)_{|I|=\ell}$.

When $a \neq 0$, the quadrics in question are such that their polar hyperplane with respect to P has equation $z = 0$. When $a = 0$, such a quadric is singular at P, it represents the cone with vertex at P over the quadric in $\Pi = \{z = 0\} \cong \mathbb{P}^{N_\ell}$ with equation $Q(x^I)_{|I|=\ell} = 0$ and it cuts out on W_ℓ a cone, with vertex at P, over a quadric section of V_ℓ.

In particular, $\mathcal{Q}_\ell$ contains the codimension 1 sublinear system $\mathcal{Q}_\ell^c$ of all such cones with vertex at P over a quadric section of V_ℓ, thus $\dim(\mathcal{Q}_\ell) = N_d + 1$. Moreover $\mathcal{Q}_\ell$ is stable under the action of $\mathbb{G}_\ell$ on W_ℓ.

We set $\widetilde{\mathcal{Q}}_\ell := \rho^*(\mathcal{Q}_\ell)$, which is a sublinear system of $|\mathcal{O}_{Z_\ell}(2)|$. Note that $\widetilde{\mathcal{Q}}_\ell$ contains the sublinear system $\widetilde{\mathcal{Q}}_\ell^c = \rho^*(\mathcal{Q}_\ell^c)$ of all divisors of the form $2E$ plus a divisor in $\mathcal{L}_\ell(d)$.

We denote by $\mathcal{Q}_\ell^*$ the dense open subset of $\mathcal{Q}_\ell$ of points corresponding to smooth surfaces. Since no surface in $\mathcal{Q}_\ell^*$ passes through P, we may and will identify $\mathcal{Q}_\ell^*$ with its pull–back via ρ on Z_ℓ, which is a dense open subset of $\widetilde{\mathcal{Q}}_\ell$ sitting in the complement of $\widetilde{\mathcal{Q}}_\ell^c$. We denote by $\mathcal{D}_\ell^* \to \mathcal{Q}_\ell^*$ the universal family.

A surface $\mathbb{D}^* \in \mathcal{Q}_\ell^*$ cuts out on V_ℓ a smooth curve $D \in \mathcal{U}_\ell$ and conversely; indeed the projection from P realizes $\mathbb{D}^*$ as the double cover of $\mathbb{P}^2$ branched along D. This yields a surjective morphism

$$\mathcal{Q}_\ell^* \ni \mathbb{D}^* \xrightarrow{\beta} \mathbb{D}^* \cap V_\ell := D \in \mathcal{U}_d\,,$$

which sends the double plane $\mathbb{D}^*$ to its branching divisor D. This morphism is equivariant under the actions of $\mathbb{G}_\ell$ on both $\mathcal{Q}_\ell^*$ and $\mathcal{U}_d$, where $\mathbb{G}_\ell$ acts on $\mathcal{U}_d$ via the natural action of the quotient group $\mathrm{PGL}(3,\mathbb{C})$, see (12). The morphism β is not injective, its fibers being isomorphic to $\mathbb{C}^*$.

As an immediate consequence of Lemma 2.1, we have:

Lemma 2.2. *Let D be a smooth curve in $\mathbb{P}^2$ of even degree $d = 2\ell \geqslant 2$, let $\psi : \mathbb{D}^* \to \mathbb{P}^2$ be the double cover branched along D, let $\Gamma \subset \mathbb{P}^2$ be a projective curve of degree m not containing D, and let Γ^* be its pull–back via ψ considered as a curve in Z_ℓ. One has*

$$K_\ell \cdot \Gamma^* = -m(d+6)\,. \tag{13}$$

In the setting of Lemma 2.2, consider the diagram

$$\begin{array}{ccc} C^* & \xrightarrow{\ \nu^*\ } & \Gamma^* \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} \\ C & \xrightarrow[\ \nu\ ]{} & \Gamma \end{array} \tag{14}$$

where ν and ν^* are the normalization morphisms and ψ and ψ' have degree 2 (to ease notation, here we have identified ψ with its restriction to Γ^*).

Let $\delta := \delta(D,\Gamma)$. It could be that C^* splits into two components both isomorphic to C; in this case $\delta = 0$. If $\delta = 0$ and the genus of C is zero, then C^* certainly splits. Suppose that C^* is irreducible, and let g and g^* be the geometric genera of Γ and Γ^* (i.e. the arithmetic genera of C and C^*, respectively). Since ψ' has exactly δ ramification points, the Riemann-Hurwitz formula yields

$$2(g^*-1) = 4(g-1) + \delta\,. \tag{15}$$

2.4. **The odd degree case.** Fix a line $h \in |\mathcal{O}_{\mathbb{P}^2}(1)|$, and let D be a smooth curve in $\mathbb{P}^2$ of degree $d = 2\ell - 1 \geqslant 1$, which intersects h transversely. We denote by $\mathcal{U}_d^h$ the open subset of $\mathcal{U}_d$ consisting of such curves.

For each $D \in \mathcal{U}_d^h$, we consider the reducible curve of degree $d + 1 = 2\ell$

$$\Delta := D + h \in |\mathcal{O}_{\mathbb{P}^2}(d+1)|$$

and the double cover $\psi\colon \mathbb{D}^* \to \mathbb{P}^2$, branched along Δ. The difference with the even degree case is that $\mathbb{D}^*$ is no longer smooth, but it has double points at the d points in $D \cap h$. In any event, as in the even degree case, we can consider the set $\mathcal{Q}_{\ell;h}^* \subset |\mathcal{O}_{Z_\ell}(2)|$ of all such surfaces $\mathbb{D}^*$, with its universal family $\mathcal{D}_{\ell;h}^* \to \mathcal{Q}_{\ell;h}^*$ which parametrizes all double planes $\psi\colon \mathbb{D}^* \to \mathbb{P}^2$ as above. We still have the morphism

$$\beta : \mathcal{Q}_{\ell;h}^* \to \mathcal{U}_d^h$$

associating to $\mathbb{D}^*$ the branching divisor Δ of $\varphi\colon \mathbb{D}^* \to \mathbb{P}^2$ minus h.

The group acting here is no longer the full group $\mathbb{G}_\ell$ but its subgroup $\mathbb{G}_{\ell;h}$ which fits in the exact sequence

$$1 \to \mathbb{C}^* \to \mathbb{G}_{\ell;h} \to \mathrm{Aff}(2,\mathbb{C}) \to 1,$$

where $\mathrm{Aff}(2,\mathbb{C})$ is the *affine group* of all projective transformations in $\mathrm{PGL}(3,\mathbb{C})$ stabilizing h.

Keeping the setting and notation of §2.3, Lemma 2.2 still holds, as well as diagram (14). If Γ intersects h at m distinct points which are off D, then the double cover $\psi' : C^* \to C$ has $\delta + m \geqslant m > 0$ ramification points, where $\delta = \delta(D, \Gamma)$ as above. In particular, C^* is irreducible, and (15) is replaced by

$$2(g^* - 1) = 4(g - 1) + \delta + m. \tag{16}$$

3. The main result

In this section we prove the following:

Theorem 3.1. *Let $\delta \geqslant 0$ be an integer such that, for a very general curve D in $\mathbb{P}^2$ of degree $d = 2\ell - \varepsilon$, where $\varepsilon \in \{0, 1\}$, there exists an integral curve $\Gamma \subset \mathbb{P}^2$, $\Gamma \neq D$, of geometric genus g and degree m with $\delta(D, \Gamma) = \delta$. Then*

$$4g + \delta \geqslant m(d + 2\varepsilon - 8) + 5. \tag{17}$$

The proof of Theorem 3.1 will be done in §3.2. First we need some more preliminaries, which we collect in the next subsection. We keep all notation and conventions introduced so far.

3.1. **Constructing appropriate families.** Fix integers $m \geqslant 1$ and $g \geqslant 0$. Let $\mathcal{H}$ be the locally closed subset of $\mathcal{L}_m$, whose points correspond to integral curves $\Gamma \subset \mathbb{P}^2$ of degree m and geometric genus g; $\mathcal{H}$ is a quasiprojective variety. We let $\mathcal{U} \to \mathcal{H}$ be the universal curve.

3.1.1. *The even degree case.* Fix an even positive integer $d = 2\ell$ and a non-negative integer δ. Consider the locally closed subset $\mathcal{I}$ of $\mathcal{H} \times \mathcal{Q}_\ell^*$ of pairs $(\Gamma, \mathbb{D}^*)$ such that Γ does not coincide with the branch curve D of $\psi : \mathbb{D}^* \to \mathbb{P}^2$ and $\delta(D, \Gamma) = \delta$. Remember that we may equivalently interpret $\mathbb{D}^*$ as a surface in W_ℓ or in Z_ℓ. Each irreducible component of $\mathcal{I}$ is fixed by the obvious action of $\mathbb{G}_\ell$ on $\mathcal{H} \times \mathcal{Q}_\ell^*$.

For any $(\Gamma, \mathbb{D}^*) \in \mathcal{I}$, the pull–back $\Gamma^* \subset \mathbb{D}^*$ of Γ via ψ is a reduced curve in Z_ℓ. Hence there is a morphism $\mu : \mathcal{I} \to \mathcal{K}$, where $\mathcal{K}$ is the Hilbert scheme of curves of Z_ℓ. We let $\mathcal{V} \to \mathcal{K}$ be the corresponding universal family. The map μ is equivariant under the actions of $\mathbb{G}_\ell$ on both $\mathcal{I}$ and $\mathcal{K}$.

Let $\pi_1 : \mathcal{I} \to \mathcal{H}$ and $\pi_2 : \mathcal{I} \to \mathcal{Q}_\ell^*$ be the two projections. Under the hypotheses of Theorem 3.1 and with notation as in § 2.3, the following holds.

Lemma 3.2. *There exists an irreducible component I of $\mathcal{I}$ which dominates $\mathcal{Q}_\ell$ via π_2. Hence I dominates also $\mathcal{U}_d \subset \mathcal{L}_d$ via $\beta \circ \pi_2$.*

Given I as in Lemma 3.2, we choose an irreducible, smooth subvariety $\mathcal{B}$ of I, such that π_2 restricts to an étale morphism of $\mathcal{B}$ onto its image, which is dense in $\mathcal{Q}_\ell$. To place our objects in the context of §1, consider the universal family $\mathbb{D}_\ell^* \to \mathcal{Q}_\ell^*$ (cf. § 2.3) of double planes $\mathbb{D}^*$ [resp. $\mathcal{V} \to \mathcal{K}$ of curves $\Gamma^* \subset \mathbb{D}^*$]. Up to possibly shrinking $\mathcal{B}$ and performing an étale cover of it, the morphisms $\mathcal{B} \xrightarrow{\pi_2} \mathcal{Q}_\ell^*$ and $\mathcal{B} \xrightarrow{\mu} \mathcal{K}$ give rise to families

$$\mathcal{D} := \pi_2^*(\mathbb{D}_\ell^*) \xrightarrow{p} \mathcal{B} \quad \text{and} \quad \mathfrak{C} := \mu^*(\mathcal{V}) \xrightarrow{\mathfrak{q}} \mathcal{B}\,.$$

over $\mathcal{B}$ fitting in diagram (3). We may assume that there exists a simultaneous normalization ν and a family $\mathcal{C} \xrightarrow{q} \mathcal{B}$ as in (4), with $\mathcal{C}$ smooth fitting in (5), where $X = Z_\ell$.

3.1.2. *The odd degree case.* Fix now an odd positive integer $d = 2\ell - 1$ and a non-negative integer δ, and fix a line h in $\mathbb{P}^2$. We consider the locally closed subset $\mathcal{I}$ of $\mathcal{H} \times \mathcal{Q}_{\ell;h}$ consisting of pairs $(\Gamma, \mathbb{D}^*) \in \mathcal{H} \times \mathcal{Q}_\ell^*$ such that Γ is not contained in the branch divisor Δ of $\psi : \mathbb{D}^* \to \mathbb{P}^2$, $\delta(D, \Gamma) = \delta$, and Γ intersects h at m distinct points which are off D.

For any point $(\Gamma, \mathbb{D}^*) \in \mathcal{I}$, the pull–back $\Gamma^* \subset \mathbb{D}^*$ of Γ via ψ is an integral curve in Z_ℓ. So, we still have the morphisms $\mu : \mathcal{I} \to \mathcal{K}$, $\pi_1 : \mathcal{I} \to \mathcal{H}$ and $\pi_2 : \mathcal{I} \to \mathcal{Q}_{\ell;h}^*$ equivariant under actions of $\mathbb{G}_{\ell;h}$.

As before, we have the following

Lemma 3.3. *There exists an irreducible component I of $\mathcal{I}$ which dominates $\mathcal{Q}_{\ell;h}^*$ via π_2.*

As in the even case, given I as in Lemma 3.3, we may construct a smooth $\mathcal{B}$ having an étale, dominant morphism to $\mathcal{Q}_{\ell;h}$, together with families

$$\mathcal{D} := \pi_2^*(\mathbb{D}^*_{\ell,h}) \xrightarrow{p} \mathcal{B}, \quad \mathfrak{C} := \mu^*(\mathcal{V}) \xrightarrow{\mathfrak{q}} \mathcal{B},$$

fitting in diagram (3). Consider a simultaneous normalization ν and a family $\mathcal{C} \xrightarrow{q} \mathcal{B}$ as in (4), with $\mathcal{C}$ smooth. In view of Lemmata 3.2 and 3.3, the constructed families fit in diagram (5), with $X = Z_\ell$.

In both cases, the next lemma allows to apply Proposition 1.3 in our setting.

Lemma 3.4. *For any $d > 0$, $\mathcal{C} \xrightarrow{q} \mathcal{B}$ is a covering family, i.e. $z(\mathcal{C}) = 3$.*

Proof. By the discussion in §3.1, for d even $\varphi(\mathcal{C})$ is stable under the action of $\mathbb{G}_\ell$ on Z_ℓ, which is transitive; for d odd $\varphi(\mathcal{C})$ is stable under the action of $\mathbb{G}_{\ell;h}$, which is transitive on the dense open subset of Z_ℓ whose complement is $\pi^{-1}(h) \cup E$. This proves the assertion. □

3.2. **Proof of Theorem 3.1.** Our proof follows the one of Theorem (1.2) in [5]. First we recall the following useful fact.

Lemma 3.5 (See [5, Lemma (3.1)]). *Let $g : V \to W$ be a linear map of finite dimensional vector spaces. Suppose that $\dim(g(V)) > k$. Let $\{V_i\}_{i\in I}$ be a family of vector subspaces of V, such that $\bigcup_{i\in I} V_i$ spans V, and for any pair $(i,j) \in I \times I$, with $i \neq j$, there is a finite sequence $i_1 = i, i_2, \dots, i_{t-1}, i_t = j$ of distinct elements of I with $\dim(g(V_{i_h} \cap V_{i_{h+1}})) \geqslant k$, for all $h \in \{1,\dots,t-1\}$. Then there is an index $i \in I$ such that $\dim(g(V_i)) > k$.*

3.2.1. *The even degree case.* We need to construct a suitable subfamily of $\mathfrak{C} \to \mathcal{B}$ with the covering property.

Fix a general point $b_0 \in \mathcal{B}$, and let Γ_0^* and $\mathbb{D}_0^*$ be the corresponding elements of the families $\mathfrak{C} \to \mathcal{B}$ and $\mathcal{D} \to \mathcal{B}$, respectively.

Let $\mathcal{L}$ be the open subset of the linear system $\mathcal{L}_\ell(d-1)$ as in (9) consisting of the surfaces $F \in \mathcal{L}_\ell(d-1)$ which do not contain Γ_0^*. A general such surface F meets Γ_0^* transversely. By genericity, we may suppose that all surfaces F defined by the pull–back via $\pi : Z_\ell \to V_\ell \cong \mathbb{P}^2$ of degree $d-1$ monomials in the variables x_0, x_1, x_2 belong to $\mathcal{L}$. For a given $F \in \mathcal{L}$, we denote by $\mathcal{B}_F$ the subvariety of $\mathcal{B}$ parameterizing all double planes in $\mathcal{D} \to \mathcal{B}$ containing the complete intersection curve of F and $\mathbb{D}_0^*$. In addition, for a general point $\xi \in \Gamma_0^*$ we let $\mathcal{B}_{F,\xi}$ denote the subvariety of $\mathcal{B}_F$ parameterizing all surfaces in $\mathcal{D} \to \mathcal{B}_F$ which pass through ξ.

Lemma 3.6. *For $F \in \mathcal{L}$ and $\xi \in \Gamma_0^*$ as above one has*

$$\dim(\mathcal{B}_F) = 3 \ \textit{and} \ \dim(\mathcal{B}_{F,\xi}) = 2\,.$$

Furthermore, b_0 is a smooth point of both $\mathcal{B}_F$ and $\mathcal{B}_{F,\xi}$.

Proof. Consider the sublinear system Λ_F of $\widetilde{\mathcal{Q}}_\ell = \rho^*(\mathcal{Q}_\ell)$ on Z_ℓ consisting of all surfaces containing the complete intersection curve $F \cap \mathbb{D}_0^*$. Imposing to the surfaces in Λ_F the condition to contain a general point of F, the divisor $F + 2E$ splits off, and the residual surface sits in $\mathcal{L}_\ell(1)$. Hence Λ_F contains a codimension 1 sublinear system consisting of surfaces of the form $2E + F + L$, with L varying in $\mathcal{L}_\ell(1)$, which has dimension 2. Hence $\dim(\Lambda_F) = 3$. Since $\mathcal{B}$ dominates $\mathcal{Q}_\ell$ via π_2, which is finite on $\mathcal{B}$, and $\mathcal{B}_F$ is the inverse image of Λ_F, one has $\dim(\mathcal{B}_F) = \dim(\Lambda_F) = 3$. The proof is similar for $\mathcal{B}_{F,\xi}$. The final assertion follows by the genericity assumptions. □

We denote by T_0 the tangent space to $\mathcal{B}$ at b_0, and by T_F and $T_{F,\xi}$ the 3 and 2–dimensional subspaces of T_0 tangent to $\mathcal{B}_F$ and to $\mathcal{B}_{F,\xi}$ at b_0, respectively.

Lemma 3.7. *One has:*

(a) *$\bigcup_{F\in\mathcal{L}} T_F$ spans T_0;*
(b) *given $F \in \mathcal{L}$, the union $\bigcup_{\xi\in\Gamma_0^*} T_{F,\xi}$ spans T_F.*

Proof. (a) Since π_2 is étale on $\mathcal{B}$, T_0 is isomorphic to the tangent space to $\mathcal{Q}_\ell$ at $\mathbb{D}_0^*$. Remember that, by §2.3, the double plane $\mathbb{D}_0^*$, considered in W_ℓ, is cut out by a quadric with equation $z^2 = Q(x^I)_{|I|=\ell}$. So T_0 can be identified with the vector space of homogeneous quadratic polynomials of the form $az^2 - G(x^I)_{|I|=\ell}$ modulo the one-dimensional linear space spanned by $z^2 - Q(x^I)_{|I|=\ell}$ and by the linear space of quadratic polynomials in $\{x^I\}_{|I|=\ell}$ defining V_ℓ. [1] Hence T_0 can be identified with the vector space of quadratic polynomials in $\{x^I\}_{|I|=\ell}$, modulo the vector space of quadratic polynomials in $\{x^I\}_{|I|=\ell}$ defining V_ℓ. This, in turn, can be identified with the vector space S_d of homogeneous polynomials of degree d in x_0, x_1, x_2.

Now T_F can be identified with the vector subspace $S_d(f)$ of $T_0 \cong S_d$ of polynomials of the form fh, where f is a fixed homogeneous polynomial of degree $d-1$ (determined by F), and h is any linear form. By assumption on $\mathcal{L}$, $\bigcup_{F\in\mathcal{L}} T_F$ contains all monomials of degree d, which do span S_d.

(b) Given F, $T_{F,\xi}$ can be identified with the vector space of homogeneous polynomials of the form fh, where h vanishes at $\pi(\xi) \in \mathbb{P}^2$. These polynomials do span $T_F \cong S_d(f)$. □

Next we consider the restrictions

$$\mathcal{D}_F \xrightarrow{p} \mathcal{B}_F, \quad \mathfrak{C}_F \xrightarrow{\mathfrak{q}} \mathcal{B}_F, \text{ and } \mathcal{D}_{F,\xi} \xrightarrow{p} \mathcal{B}_{F,\xi}, \quad \mathfrak{C}_{F,\xi} \xrightarrow{\mathfrak{q}} \mathcal{B}_{F,\xi} \quad \text{of} \quad \mathcal{D} \xrightarrow{p} \mathcal{B} \quad \text{and } \mathfrak{C} \xrightarrow{\mathfrak{q}} \mathcal{B}.$$

Proposition 3.8. *For general $F \in \mathcal{L}$ and $\xi \in \Gamma_0^*$, the families*

$$\mathfrak{C}_F \xrightarrow{\mathfrak{q}} \mathcal{B}_F \text{ and } \mathfrak{C}_{F,\xi} \xrightarrow{\mathfrak{q}} \mathcal{B}_{F,\xi}$$

have the covering property.

Proof. We prove the assertion for $\mathfrak{C}_F \xrightarrow{\mathfrak{q}} \mathcal{B}_F$. The proof for $\mathfrak{C}_{F,\xi} \xrightarrow{\mathfrak{q}} \mathcal{B}_{F,\xi}$ is similar (and analogous to the proof of the corresponding statement in [5, Theorem (1.2)]), hence it can be left to the reader.

Let $\mathbb{M}$ be the set of all monomials of degree $d-1$ in x_0, x_1, x_2. Consider the family $\{F_M\}_{M\in\mathbb{M}}$, where $F_M \in \mathcal{L}$ is defined by the pull–back via $\pi\colon Z_\ell \to V_\ell \cong \mathbb{P}^2$ of the monomial M. Take two monomials M', M'' which differ only in degree 1, i.e., their lowest common multiple U has degree d. Then $\mathcal{B}_{F_{M'}} \cap \mathcal{B}_{F_{M''}}$ contains the pull–back via π_2 of an open, dense subset of the pencil $\langle \mathbb{D}_t^* \rangle$ spanned by $\mathbb{D}_0^*$ and F_U, where F_U is the pull–back via π of the monomial U. The base locus of this pencil does not contain Γ_0^*. Therefore, Γ_0^* varies in a non-trivial one-parameter family $\langle \Gamma_t^* \rangle$ together with members $\mathbb{D}_t^*$ varying in the pencil $\langle \mathbb{D}_t^* \rangle$.

Next we apply Lemma 3.5 with

- $V = T_0$;
- $W = H^0(\Gamma_0^*, N_{\Gamma_0^*|Z_\ell})$;
- the linear map g induced by the characteristic map (see Proposition 1.1 (a));
- the family of subspaces $\{V_i\}_{i\in I}$ given by $\{T_{F_M}\}_{M\in\mathbb{M}}$.

For each pair of monomials M', M'', there is a sequence of monomials $M_1 = M', M_2, \ldots, M_{t-1}, M_t = M''$, such that for all $i = 1, \ldots, t-1$, the lowest common multiple of M_i and M_{i+1} has degree d. The above argument implies that $g(T_{F_{M_i}} \cap T_{F_{M_{i+1}}})$ has dimension at least 1, for all $i = 1, \ldots, m-1$. Furthermore, one has $\dim(g(T_0)) \geqslant 2$, because $\mathfrak{C} \to \mathcal{B}$ is a covering family (see (b) of Proposition 1.1 and Lemma 3.4). By Lemma 3.5 there is a monomial $M \in \mathbb{M}$ such that $\dim(g(T_{F_M})) \geqslant 2$; by virtue of Lemma 3.6, this implies that $\mathfrak{C}_{F_M} \to \mathfrak{B}_{F_M}$ is a covering family. This proves the assertion. □

To finish the proof of Theorem 3.1 in this case, consider the covering family $\mathfrak{C}_{F,\xi} \xrightarrow{\mathfrak{q}} \mathcal{B}_{F,\xi}$, with $F \in \mathcal{L}$ and $\xi \in \Gamma_0^*$ general. Using (7), (13), and (15), for $b = (\Gamma_b^*, \mathbb{D}_b^*) \in \mathcal{B}_{F,\xi}$ general (see §3.1.1) we deduce

$$\deg(\Phi_b^{\mathrm{sm}}) \leqslant 4(g-1) + \delta + 2m(\ell+3) = 4(g-1) + \delta + m(d+6)\,. \tag{18}$$

On the other hand, by construction and by (a) of Proposition 1.3,

$$\deg(\Phi_b^{\mathrm{sm}}) \geqslant 1 + \deg(\Gamma_b^* \cap F) = 1 + 2(d-1)m\,. \tag{19}$$

Comparing (18) and (19) gives (17).

[1] An explanation is in order. Consider a vector space V and a nonzero vector $v \in V$, along with the associated projective space $\mathbb{P}(V)$ and the corresponding point $[v] \in \mathbb{P}(V)$. Then the tangent space $T_{[v]}\mathbb{P}(V)$ can be canonically identified with $\mathrm{Hom}(\langle v\rangle, V/\langle v\rangle) \cong V/\langle v\rangle$.

3.2.2. *The odd degree case.* The proof runs exactly as in the case of even d, so we will be brief and leave the details to the reader.

Fix again $b_0 \in \mathcal{B}$, Γ_0^* and $\mathbb{D}_0^*$ as in the even degree case. Following what we did in §3.1.2, we replace D_b by $D_b + h$, where $h \subset \mathbb{P}^2$ is a general line. In the present setting we let $\mathcal{L}$ be the open subset of $\mathcal{L}_\ell(d)$ consisting of the surfaces $F \in \mathcal{L}_\ell(d)$ which do not contain Γ_0^*. Again we may assume that all surfaces F defined by the pull–back via π of degree d monomials in the variables x_0, x_1, x_2 belong to $\mathcal{L}$. Given $F \in \mathcal{L}$, we define $\mathcal{B}_F$ and $\mathcal{B}_{F,\xi}$ as in the even degree case, and the analogue of Lemma 3.6 still holds. Then, with the usual meaning for T_0, T_F and $T_{F,\xi}$, the analogue of Lemma 3.7 holds. Similarly as in Proposition 3.8, the covering property holds for the restricted families

$$\mathcal{D}_F \xrightarrow{p} \mathcal{B}_F, \quad \mathfrak{C}_F \xrightarrow{\mathfrak{q}} \mathcal{B}_F, \quad \text{and} \quad \mathcal{D}_{F,\xi} \xrightarrow{p} \mathcal{B}_{F,\xi}, \quad \mathfrak{C}_{F,\xi} \xrightarrow{\mathfrak{q}} \mathcal{B}_{F,\xi}\,.$$

We conclude finally as in the even degree case: (18) holds with no change, whereas (19) has to be replaced by

$$\deg(\Phi_b^{\mathrm{sm}}) \geqslant 1 + \deg(\Gamma_b \cap F) = 1 + 2dm\,,$$

and again, (17) follows.

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